International Journal of Applied Mathematics

Volume 31 No. 4 2018, 483-525
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v31i4.1

WELL-POSEDNESS OF NONLINEAR FRACTIONAL SCHRÖDINGER AND WAVE EQUATIONS IN SOBOLEV SPACES

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Abstract: In this paper, we establish the local well-posedness results in sub-critical and critical cases for the pure power-type nonlinear fractional Schrödinger and wave equations on $\mathbb{R}^d$, namely

$$i\partial_t u + \Lambda^\sigma u + \mu|u|^{\nu-1}u = 0, \quad u_{|t=0} = \varphi,$$

and

$$\partial_t^2 v + \Lambda^{2\sigma} v + \mu|v|^{\nu-1}v = 0, \quad v_{|t=0} = \varphi, \quad \partial_t v_{|t=0} = \phi,$$

where $\sigma \in (0,\infty)\backslash\{1\}, \nu > 1, \mu \in \{\pm 1\}$ and $\Lambda = \sqrt{-\Delta}$ is the Fourier multiplier by $|\xi|$. For the nonlinear fractional Schrödinger equation, we extend the previous results in [22] for $\sigma \geq 2$. These results cover the well-known results for Schrödinger equation $\sigma = 2$ given in [4]. In the case $\sigma \in (0,2)\backslash\{1\}$, we show the local well-posedness in the sub-critical case for $\nu > 1$ in contrast to $\nu \geq 2$ when $d = 1$, and $\nu \geq 3$ when $d \geq 2$ of [22]. These results also generalize the ones of [11] when $d = 1$ and of [18] when $d \geq 2$, where the authors considered the cubic fractional Schrödinger equation with $\sigma \in (1,2)$. To our knowledge, the nonlinear fractional wave equation does not seem to have been much considered, up to [37] on the scattering operator with σ an even integer and [6], [7] in the context of the damped fractional wave equation.

AMS Subject Classification: 35A01, 35Q55
Key Words: fractional Schrödinger equation; fractional wave equation; Strichartz estimate; local well-posedness

Received: February 27, 2017

1. Introduction and Main Results

Let $\sigma \in (0,\infty)\backslash\{1\}$. We consider the Cauchy fractional Schrödinger and wave equations posed on $\mathbb{R}^d, d \geq 1$, namely

$$i\partial_t u + \Lambda^\sigma u + \mu|u|^{\nu-1}u = 0, \quad u_{|t=0} = \varphi, \tag{NLFS}$$

and

$$\partial_t^2 v + \Lambda^{2\sigma} v + \mu|v|^{\nu-1}v = 0, \quad v_{|t=0} = \varphi, \quad \partial_t v_{|t=0} = \phi, \tag{NLFW}$$

where $\nu > 1$ and $\mu \in \{\pm 1\}$. The operator $\Lambda^\sigma = (\sqrt{-\Delta})^\sigma$ is the Fourier multiplier by $|\xi|^\sigma$ where $\Delta = \sum_{j=1}^d \partial_j^2$ is the free Laplace operator on $\mathbb{R}^d$. The number $\mu = 1$ (resp. $\mu = -1$) corresponds to the defocusing case (resp. focusing case). When $\sigma \in (0,2)\backslash\{1\}$, the fractional Schrödinger equation was discovered by N. Laskin (see [26], [27]) owing to the extension of the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. This equation also appears in the water wave models (see [23]). When $\sigma \in [2,\infty)$, the (NLFS) generalizes the well-known nonlinear Schrödinger equation $\sigma = 2$ or the fourth-order nonlinear Schrödinger equation $\sigma = 4$ (see e.g. [30] and references therein). In the case $\sigma \in (0,2)\backslash\{1\}$, the fractional wave equation, introduced in [8], reflects the Lévy stable process and the fractional Brownian motion. In the other side, when $\sigma \in [2,\infty)$, the (NLFW) can be seen as a generalization of the fourth-order nonlinear wave equation (see e.g. [31]).

The study of nonlinear fractional Schrödinger and wave equations has attracted a lot of interest in the past several years (see [9], [10], [11], [17], [18], [20], [22], [23], [29], [30], [37] and references therein). It is well known that (see [15], [24], [5] or [34]) the (NLFS) and (NLFW) are locally well-posed in H^γ with $\gamma \geq d/2$ provided the nonlinearity is sufficiently regular. The main purpose of this note is to prove the local well-posedness for (NLFS) and (NLFW) for $\gamma \in [0, d/2)$. For the (NLFS), we extend the previous results in [22] for $\sigma \geq 2$. These results cover the well-known results for Schrödinger equation $\sigma = 2$ given in [4]. In the case $\sigma \in (0,2)\backslash\{1\}$, we show the local well-posedness in the sub-critical case for $\nu > 1$ in contrast to $\nu \geq 2$ when $d = 1$, and $\nu \geq 3$ when $d \geq 2$ of [22]. These results generalize the ones of [11] when $d = 1$, and of [18] when $d \geq 2$, where the authors considered the cubic fractional Schrödinger equation with $\sigma \in (1,2)$. We also give the global existence in energy space, namely $H^{\sigma/2}$ under some assumptions. Moreover, in the critical case, we prove the global existence and scattering with small homogeneous data instead of inhomogeneous one in [22]. To our knowledge, the (NLFW) does not seem to have been much considered, up to [37] on the scattering operator with σ an even integer and [6], [7] in the context of the damped fractional wave equation.

Let us introduce some standard notations (see the appendix of [16], Chapter 5 of [36] or Chapter 6 of [2]). Let $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi_0(\xi) = 1$ for $|\xi| \leq 1$ and $\operatorname{supp}(\chi_0) \subset \{\xi \in \mathbb{R}^d, |\xi| \leq 2\}$. We set $\chi(\xi) := \chi_0(\xi) - \chi_0(2\xi)$. It is easy to see that $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\operatorname{supp}(\chi) \subset \{\xi \in \mathbb{R}^d, 1/2 \leq |\xi| \leq 2\}$. We denote the Littlewood-Paley projections by $P_0 := \chi_0(D), P_N := \chi(N^{-1}D)$ with $N = 2^k, k \in \mathbb{Z}$ where $\chi_0(D), \chi(N^{-1}D)$ are the Fourier multipliers by $\chi_0(\xi)$ and $\chi(N^{-1}\xi)$ respectively. Given $\gamma \in \mathbb{R}$ and $1 \leq q \leq \infty$, one defines the Sobolev and Besov spaces as

$$H_q^\gamma := \Big\{u \in \mathscr{S}' \mid \|u\|_{H_q^\gamma} := \| \langle\Lambda\rangle^\gamma u\|_{L^q} < \infty\Big\}, \quad \langle\Lambda\rangle := \sqrt{1+\Lambda^2},$$
$$B_q^\gamma := \Big\{u \in \mathscr{S}' \mid \|u\|_{B_q^\gamma} := \|P_0 u\|_{L^q} + \Big(\sum_{N\in 2^{\mathbb{N}}} N^{2\gamma}\|P_N u\|^2_{L^q}\Big)^{1/2} < \infty\Big\},$$

where $\mathscr{S}'$ is the space of tempered distributions. Now, let $\mathscr{S}_0$ be a subspace of the Schwartz space $\mathscr{S}$ consisting of functions ϕ satisfying $D^\alpha\hat{\phi}(0) = 0$ for all $\alpha \in \mathbb{N}^d$ where $\hat{\cdot}$ is the Fourier transform on $\mathscr{S}$ and $\mathscr{S}_0'$ its topological dual space. One can see $\mathscr{S}_0'$ as $\mathscr{S}'/\mathscr{P}$ where $\mathscr{P}$ is the set of all polynomials on $\mathbb{R}^d$. The homogeneous Sobolev and Besov spaces are defined by

$$\dot{H}_q^\gamma := \Big\{u \in \mathscr{S}_0' \mid \|u\|_{\dot{H}_q^\gamma} := \|\Lambda^\gamma u\|_{L^q} < \infty\Big\},$$
$$\dot{B}_q^\gamma := \Big\{u \in \mathscr{S}_0' \mid \|u\|_{\dot{B}_q^\gamma} := \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2\gamma}\|P_N u\|^2_{L^q}\Big)^{1/2} < \infty\Big\}.$$

We again refer the reader to Appendix of [16], Chapter 5 of [36] or Chapter 6 of [2] for various properties of these function spaces. We note that $H_q^\gamma, B_q^\gamma, \dot{H}_q^\gamma$ and $\dot{B}_q^\gamma$ are Banach spaces with the norms $\|u\|_{H_q^\gamma}, \|u\|_{B_q^\gamma}, \|u\|_{\dot{H}_q^\gamma}$ and $\|u\|_{\dot{B}_q^\gamma}$ respectively. In the sequel, we shall use $H^\gamma := H_2^\gamma$, $\dot{H}^\gamma := \dot{H}_2^\gamma$. We also have that if $2 \leq q < \infty$, then $\dot{B}_q^\gamma \subset \dot{H}_q^\gamma$ with the reverse inclusion for $1 < q \leq 2$. In particular, $\dot{B}_2^\gamma = \dot{H}^\gamma$ and $\dot{B}_2^0 = \dot{H}_2^0 = L^2$. Moreover, if $\gamma > 0$, then $H_q^\gamma = L^q \cap \dot{H}_q^\gamma$ and $B_q^\gamma = L^q \cap \dot{B}_q^\gamma$.

Before stating main results, we recall some facts on (NLFS) and (NLFW). By a standard approximation argument, the following quantities are conserved by the flow of (NLFS),

$$M_{\mathrm{s}}(u) = \int |u(t,x)|^2 dx,$$

$$E_{\rm s}(u)=\int \frac{1}{2}|\Lambda^{\sigma/2}u(t,x)|^2+\frac{\mu}{\nu+1}|u(t,x)|^{\nu+1}dx.$$

Moreover, if we set for $\lambda>0$,

$$u_\lambda(t,x)=\lambda^{-\frac{\sigma}{\nu-1}}u(\lambda^{-\sigma}t,\lambda^{-1}x),$$

then the (NLFS) is invariant under this scaling that is for $T\in(0,+\infty]$,

$$u \text{ solves (NLFS) on } (-T,T) \Longleftrightarrow u_\lambda \text{ solves (NLFS) on } (-\lambda^\sigma T,\lambda^\sigma T).$$

We also have

$$\|u_\lambda(0)\|_{\dot{H}^\gamma}=\lambda^{\frac{d}{2}-\frac{\sigma}{\nu-1}-\gamma}\|\varphi\|_{\dot{H}^\gamma}.$$

From this, we define the critical regularity exponent for (NLFS) by

$$\gamma_{\rm s}=\frac{d}{2}-\frac{\sigma}{\nu-1}. \tag{1.1}$$

Similarly, the following energy is conserved under the flow of (NLFW),

$$E_{\rm w}(v)=\int \frac{1}{2}|\partial_t v(t,x)|^2+\frac{1}{2}|\Lambda^\sigma v(t,x)|^2+\frac{\mu}{\nu+1}|v(t,x)|^{\nu+1}dx,$$

and the (NLFW) is invariant under the following scaling

$$v_\lambda(t,x)=\lambda^{-\frac{2\sigma}{\nu-1}}v(\lambda^{-\sigma}t,\lambda^{-1}x).$$

Using

$$\|v_\lambda(0)\|_{\dot{H}^\gamma}=\lambda^{\frac{d}{2}-\frac{2\sigma}{\nu-1}-\gamma}\|\varphi\|_{\dot{H}^\gamma},$$
$$\|\partial_t v_\lambda(0)\|_{\dot{H}^{\gamma-\sigma}}=\lambda^{\frac{d}{2}-\frac{2\sigma}{\nu-1}-\gamma}\|\phi\|_{\dot{H}^{\gamma-\sigma}},$$

we define the critical regularity exponent for (NLFW) by

$$\gamma_{\rm w}=\frac{d}{2}-\frac{2\sigma}{\nu-1}. \tag{1.2}$$

By the standard argument (see e.g [28]), it is easy to see that the (NLFS) (resp. (NLFW)) is ill-posed if $\varphi\in\dot{H}^\gamma$ with $\gamma<\gamma_{\rm s}$ (resp. $v_0\in\dot{H}^\gamma, v_1\in\dot{H}^{\gamma-\sigma}$ with $\gamma<\gamma_{\rm w}$). Indeed if u solves the (NLFS) with initial data $\varphi\in\dot{H}^\gamma$ and the lifespan T, then the norm $\|u_\lambda(0)\|_{\dot{H}^\gamma}$ and the lifespan of u_λ go to zero as $\lambda\to 0$. Thus we can expect the well-posedness results for (NLFS) (resp. (NLFW)) when $\gamma\geq\gamma_{\rm s}$ (resp. $\gamma\geq\gamma_{\rm w}$).

Throughout this note, a pair (p,q) is said to be admissible if

$$(p,q) \in [2,\infty]^2, \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}. \tag{1.3}$$

We also denote for $(p,q) \in [1,\infty]^2$,

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{\sigma}{p}. \tag{1.4}$$

Note that when $\sigma \in (0,2)\backslash\{1\}$, then $\gamma_{p,q} > 0$ for all admissible pairs except $(p,q) = (\infty, 2)$. Therefore, it is convenient to separate two cases $\sigma \in (0,2)\backslash\{1\}$ and $\sigma \in [2,\infty)$. Moreover, since we are working in spaces of fractional order γ, γ_{s} or γ_{w}, we need the nonlinearity $F(z) = -\mu|z|^{\nu-1}z$ to have enough regularity. When ν is an odd integer, $F \in C^\infty(\mathbb{C},\mathbb{C})$ (in the real sense). When ν is not an odd integer, we need the following assumption

$$\lceil \gamma \rceil, \lceil \gamma_{\mathrm{s}} \rceil \text{ or } \lceil \gamma_{\mathrm{w}} \rceil \leq \nu, \tag{1.5}$$

where $\lceil \gamma \rceil$ is the smallest integer greater than or equal to γ, similarly for $\lceil \gamma_{\mathrm{s}} \rceil$ and $\lceil \gamma_{\mathrm{w}} \rceil$. Our first result is the following local well-posedness for (NLFS) in the sub-critical case.

Theorem 1. *Given $\sigma \in (0,2)\backslash\{1\}$ and $\nu > 1$. Let $\gamma \in [0, d/2)$ be such that*

$$\begin{cases} \gamma > 1/2 - \sigma/\max(\nu-1, 4) & \text{when } d = 1, \\ \gamma > d/2 - \sigma/\max(\nu-1,2) & \text{when } d \geq 2, \end{cases} \tag{1.6}$$

and also, if ν is not an odd integer, (1.5). *Then for all $\varphi \in H^\gamma$, there exist $T^* \in (0,\infty]$ and a unique solution to* (NLFS) *satisfying*

$$u \in C([0,T^*), H^\gamma) \cap L^p_{\mathrm{loc}}([0,T^*), L^\infty),$$

for some $p > \max(\nu-1,4)$ when $d=1$ and some $p > \max(\nu-1,2)$ when $d \geq 2$. Moreover, the following properties hold:

i. *If $T^* < \infty$, then $\|u(t)\|_{H^\gamma} \to \infty$ as $t \to T^*$,*

ii. *u depends continuously on φ in the following sense. There exists $0 < T < T^*$ such that if $\varphi_n \to \varphi$ in H^γ and if u_n denotes the solution of* (NLFS) *with initial data φ_n, then $0 < T < T^*(\varphi_n)$ for all n sufficiently large and u_n is bounded in $L^a([0,T], H_b^{\gamma-\gamma_{a,b}})$ for any admissible pair (a,b) with $b < \infty$. Moreover, $u_n \to u$ in $L^a([0,T], H_b^{-\gamma_{a,b}})$ as $n \to \infty$. In particular, $u_n \to u$ in $C([0,T], H^{\gamma-\epsilon})$ for all $\epsilon > 0$.*

Remark 2. If we assume that $\nu > 1$ is an odd integer or

$$\lceil \gamma \rceil \leq \nu - 1 \tag{1.7}$$

otherwise, then the continuous dependence holds in $C([0,T], H^\gamma)$ (see Remark 24).

As mentioned before, this result improves the one in [22] at the point that Hong-Sire only give the local well-posedness for $\nu \geq 2$ when $d = 1$ and $\nu \geq 3$ when $d \geq 2$. This result also covers the one in [11] when $d = 1$ and in [18] when $d \geq 2$, where the authors considered the cubic fractional Schrödinger equation with $\sigma \in (1,2)$. When $\sigma \geq 2$, we have the following better result which generalizes the case $\sigma = 2$ given in [4].

Theorem 3. *Given $\sigma \geq 2$ and $\nu > 1$. Let $\gamma \in [0, d/2)$ be such that $\gamma > \gamma_{\rm s}$, and also, if ν is not an odd integer,* (1.5)*. Let (p,q) be the admissible pair defined by*

$$p = \frac{2\sigma(\nu+1)}{(\nu-1)(d-2\gamma)}, \quad q = \frac{d(\nu+1)}{d+(\nu-1)\gamma}. \tag{1.8}$$

Then for all $\varphi \in H^\gamma$, there exist $T^* \in (0,\infty]$ and a unique solution to (NLFS) *satisfying*

$$u \in C([0,T^*), H^\gamma) \cap L^p_{\rm loc}([0,T^*), H^\gamma_q).$$

Moreover, the following properties hold:

i. *If $T^* < \infty$, then $\|u(t)\|_{\dot{H}^\gamma} \to \infty$ as $t \to T^*$,*

ii. *u depends continuously on φ in the following sense. There exists $0 < T < T^*$ such that if $\varphi_n \to \varphi$ in H^γ and if u_n denotes the solution of* (NLFS) *with initial data φ_n, then $0 < T < T^*(\varphi_n)$ for all n sufficiently large and u_n is bounded in $L^a([0,T], H^\gamma_b)$ for any admissible pair (a,b) with $\gamma_{a,b} = 0$ and $b < \infty$. Moreover, $u_n \to u$ in $L^a([0,T], L^b)$ as $n \to \infty$. In particular, $u_n \to u$ in $C([0,T], H^{\gamma-\epsilon})$ for all $\epsilon > 0$.*

Thanks to the conservation of mass, we immediately have the following global well-posedness in L^2 when $\sigma \geq 2$.

Corollary 4. *Let $\sigma \geq 2$ and $\nu \in (1, 1+2\sigma/d)$. Then for all $\varphi \in L^2$, there exists a unique global solution to* (NLFS) *satisfying $u \in C(\mathbb{R}, L^2) \cap L^p_{\rm loc}(\mathbb{R}, L^q)$, where (p,q) given in* (1.8)*.*

Proposition 5. *Let*

$$\begin{cases} \sigma \in (2/3, 1) & \text{when } d = 1, \\ \sigma \in (1, 2) & \text{when } d = 2, \\ \sigma \in (3/2, 3) & \text{when } d = 3, \\ \sigma \in [2, d) & \text{when } d \geq 4, \end{cases} \tag{1.9}$$

and $\nu > 1$ *be such that* $\sigma/2 > \gamma_{\rm s}$*, and also, if* ν *is not an odd integer,* $\lceil \sigma/2 \rceil \leq \nu$*. Then for any* $\varphi \in H^{\sigma/2}$*, the solution to* (NLFS) *given in* Theorem 1 *and* Theorem 7 *can be extended to the whole* $\mathbb{R}$ *if one of the following is satisfied:*

i. $\mu = 1$,

ii. $\mu = -1, \nu < 1 + 2\sigma/d$,

iii. $\mu = -1, \nu = 1 + 2\sigma/d$ *and* $\|\varphi\|_{L^2}$ *is small,*

iv. $\mu = -1$ *and* $\|\varphi\|_{H^{\sigma/2}}$ *is small.*

We now turn to the local well-posedness and scattering with small data for (NLFS) in the critical case.

Theorem 6. *Let* $\sigma \in (0, 2)\backslash\{1\}$ *and*

$$\begin{cases} \nu > 5 \text{ when } d = 1, \\ \nu > 3 \text{ when } d \geq 2, \end{cases} \tag{1.10}$$

be such that $\gamma_{\rm s} \geq 0$*, and also, if* ν *is not an odd integer,* (1.5)*. Then for all* $\varphi \in H^{\gamma_{\rm s}}$*, there exist* $T^* \in (0, \infty]$ *and a unique solution to* (NLFS) *satisfying*

$$u \in C([0, T^*), H^{\gamma_{\rm s}}) \cap L^p_{\rm loc}([0, T^*), B^{\gamma_{\rm s} - \gamma_{p,q}}_q),$$

where $p = 4, q = \infty$ *when* $d = 1$*;* $2 < p < \nu - 1, q = p^\star = 2p/(p-2)$ *when* $d = 2$ *and* $p = 2, q = 2^\star = 2d/(d-2)$ *when* $d \geq 3$*. Moreover, if* $\|\varphi\|_{\dot{H}^{\gamma_{\rm s}}} < \varepsilon$ *for some* $\varepsilon > 0$ *small enough, then* $T^* = \infty$ *and the solution is scattering in* $H^{\gamma_{\rm s}}$*, i.e. there exists* $\varphi^+ \in H^{\gamma_{\rm s}}$ *such that*

$$\lim_{t \to +\infty} \|u(t) - e^{it\Lambda^\sigma} \varphi^+\|_{H^{\gamma_{\rm s}}} = 0.$$

This theorem is a modification of Theorem 1.2 and Theorem 1.3 in [22] where the authors proved the global well-posedness and scattering for small inhomogeneous data. Note that Strichartz estimate is not sufficient to give the local existence in the critical case. It needs a delicate estimate on $L^{\nu-1}_t L^\infty_x$ (see Lemma 3.5 in [22]). The range $\nu \in (1,5]$ when $d=1$ and $\nu \in (1,3]$ still remains open, and it requires another technique rather than Strichartz estimate. The situation becomes better when $\sigma \geq 2$, and we have the following result.

Theorem 7. *Let $\sigma \geq 2$ and $\nu > 1$ such that $\gamma_{\mathrm{s}} \geq 0$, and also, if ν is not an odd integer,* (1.5). *Let*

$$p = \nu + 1, \quad q = \frac{2d(\nu+1)}{d(\nu+1) - 2\sigma}. \tag{1.11}$$

Then for any $\varphi \in H^{\gamma_{\mathrm{s}}}$, there exist $T^* \in (0,\infty]$ and a unique solution to (NLFS) *satisfying*

$$u \in C([0,T^*), H^{\gamma_{\mathrm{s}}}) \cap L^p_{\mathrm{loc}}([0,T^*), H^{\gamma_{\mathrm{s}}}_q).$$

Moreover, if $\|\varphi\|_{\dot{H}^{\gamma_{\mathrm{s}}}} < \varepsilon$ for some $\varepsilon > 0$ small enough, then $T^* = \infty$ and the solution is scattering in $H^{\gamma_{\mathrm{s}}}$.

We next give the local well-posedness results for the (NLFW). Let us start with the local well-posedness in the sub-critical case.

Theorem 8. *Given $\sigma \in (0,\infty)\backslash\{1\}$ and $\nu > 1$. Let $\gamma \in [0, d/2)$ be as in* (1.6) *and also, if ν is not an odd integer,* (1.5). *Then for all $(\varphi,\phi) \in H^\gamma \times H^{\gamma-\sigma}$, there exist $T^* \in (0,\infty]$ and a unique solution to* (NLFW) *satisfying*

$$v \in C([0,T^*), H^\gamma) \cap C^1([0,T^*), H^{\gamma-\sigma}) \cap L^p_{\mathrm{loc}}([0,T^*), L^\infty),$$

for some $p > \max(\nu-1, 4)$ when $d=1$ and some $p > \max(\nu-1,2)$ when $d \geq 2$. Moreover, the following properties hold:

i. *If $T^* < \infty$, then $\|[v](t)\|_{\mathcal{H}^\gamma} \to \infty$ as $t \to T^*$,*

ii. *v depends continuously on (φ,ϕ) in the following sense. There exists $0 < T < T^*$ such that if $(\varphi_n,\phi_n) \to (\varphi,\phi)$ in $H^\gamma \times H^{\gamma-\sigma}$ and if v_n denotes the solution of* (NLFW) *with initial data (φ_n,ϕ_n), then $0 < T < T^*(\varphi_n,\phi_n)$ for all n sufficiently large and v_n is bounded in $L^a([0,T], H^{\gamma-\gamma_{a,b}}_b)$ for any admissible pair (a,b) with $b < \infty$. Moreover, $v_n \to v$ in $L^a(I, H^{-\gamma_{a,b}}_b)$ as $n \to \infty$. In particular, $v_n \to v$ in $C([0,T], H^{\gamma-\epsilon}) \cap C^1([0,T], H^{\gamma-\sigma-\epsilon})$ for all $\epsilon > 0$.*

We note that (1.6) is necessary to use the Sobolev embedding, but it produces a gap between γ_{w} and $1/2 - \sigma/\max(\nu-1, 4)$ when $d=1$ and

$d/2 - \sigma/\max(\nu-1,2)$ when $d \geq 2$. Moreover, if we assume that $\nu > 1$ is an odd integer or (1.7) otherwise, then the continuous dependence holds in $C([0,T], H^\gamma) \cap C^1([0,T], H^{\gamma-\sigma})$.

The following result gives the local well-posedness for (NLFW) in the σ-sub-critical case.

Theorem 9. *1. Assume for $d = 1,2,3,4$,*

$$\sigma \in \Big(0, \frac{d}{d+2}\Big), \ \nu \in \Big(\frac{d}{d-2\sigma}, \frac{2d-d\sigma}{2d-(d+4)\sigma}\Big] \\ \text{or } \sigma \in \Big[\frac{d}{d+2}, \frac{d}{2}\Big)\backslash\{1\}, \ \nu \in \Big(\frac{d}{d-2\sigma}, \frac{d+2\sigma}{d-2\sigma}\Big); \quad (1.12)$$

for $d = 5, ..., 11$,

$$\sigma \in \Big(0, \frac{2}{3}\Big), \ \nu \in \Big(\frac{d}{d-2\sigma}, \frac{2d-d\sigma}{2d-(d+4)\sigma}\Big] \\ \text{or } \sigma \in \Big[\frac{2}{3}, \frac{d}{6}\Big)\backslash\{1\}, \ \nu \in \Big(\frac{d}{d-2\sigma}, \frac{d}{d-3\sigma}\Big] \\ \text{or } \sigma \in \Big[\frac{d}{6}, 2\Big)\backslash\{1\}, \ \nu \in \Big(\frac{d}{d-2\sigma}, \frac{d+2\sigma}{d-2\sigma}\Big); \quad (1.13)$$

and for $d \geq 12$,

$$\sigma \in \Big(0, \frac{2}{3}\Big), \ \nu \in \Big(\frac{d}{d-2\sigma}, \frac{2d-d\sigma}{2d-(d+4)\sigma}\Big] \\ \text{or } \sigma \in \Big[\frac{2}{3}, 2\Big)\backslash\{1\}, \ \nu \in \Big(\frac{d}{d-2\sigma}, \frac{d}{d-3\sigma}\Big]. \quad (1.14)$$

Let (p,q) be an admissible pair defined by

$$p = \frac{2\sigma\nu}{(d-2\sigma)\nu - d}, \quad q = 2\nu. \quad (1.15)$$

Then for all $(\varphi, \phi) \in \dot{H}^\sigma \times L^2$, there exist $T^* \in (0,\infty]$ and a unique solution to (NLFW) satisfying

$$v \in C([0,T^*), \dot{H}^\sigma) \cap C^1([0,T^*), L^2) \cap L^p_{\mathrm{loc}}([0,T^*), L^q).$$

Moreover, the following properties hold:

i. *If $T^* < \infty$, then $\|[v](t)\|_{\dot{H}^\sigma} \to \infty$ as $t \to T^*$,*

ii. *v depends continuously on (φ, ϕ) in the following sense. There exists $0 < T < T^*$ such that if $(\varphi_n, \phi_n) \to (\varphi, \phi)$ in $\dot{H}^\sigma \times L^2$ and if v_n denotes the solution of* (NLFW) *with initial data (φ_n, ϕ_n), then $0 < T < T^*(\varphi_n, \phi_n)$ for all n sufficiently large and $v_n \to v$ in $C([0,T], \dot{H}^\sigma) \cap C^1([0,T], L^2)$.*

2. Let

$$\sigma \in \Big[2, \frac{d}{2}\Big), \quad \nu \in \Big[\frac{d\sigma^*}{d+\sigma}, \sigma^*\Big), \tag{1.16}$$

where $\sigma^* := (d+2\sigma)/(d-2\sigma)$. Let (p,q) be an admissible pair defined by

$$p = 2\sigma^*, \quad p = \frac{2d\sigma^*}{d+\sigma}. \tag{1.17}$$

Then the same conclusion as in Item 1 *holds true.*

This theorem and the conservation of energy imply the following global well-posedness for the defocusing (NLFW).

Corollary 10. *Under the assumptions of* Theorem 9, *for all $(\varphi, \phi) \in \dot{H}^\sigma \times L^2$, there exists a unique global solution to the defocusing* (NLFW) *satisfying*

$$v \in C(\mathbb{R}, \dot{H}^\sigma) \cap C^1(\mathbb{R}, L^2) \cap L^p_{\mathrm{loc}}(\mathbb{R}, L^q),$$

where (p,q) are as in Theorem 9.

The next result gives the local well-posedness with small data scattering for (NLFW) in the critical case.

Theorem 11. *1. Assume for $d \geq 1$ that*

$$\sigma \in \Big[\frac{d}{d+1}, d\Big) \backslash \{1\}, \quad \nu \in \Big[1 + \frac{4\sigma}{d-\sigma}, \infty\Big), \tag{1.18}$$

and also, if ν is not an odd integer,

$$\lceil \gamma_{\mathrm{w}} \rceil - \frac{\sigma}{2} \leq \nu - 1. \tag{1.19}$$

Let p, a be defined by

$$p = \frac{(d+\sigma)(\nu-1)}{2\sigma}, \quad a = \frac{2(d+\sigma)}{d-\sigma}. \tag{1.20}$$

Then for all $(\varphi, \phi) \in \dot{H}^{\gamma_w} \times \dot{H}^{\gamma_w - \sigma}$, *there exist* $T^* \in (0, \infty]$ *and a unique solution to* (NLFW) *satisfying*

$$v \in C([0, T^*), \dot{H}^{\gamma_w}) \cap C^1([0, T^*), \dot{H}^{\gamma_w - \sigma}) \cap L^p_{\mathrm{loc}}([0, T^*), L^p) \cap L^a_{\mathrm{loc}}([0, T^*), \dot{H}_a^{\gamma_w - \frac{\sigma}{2}}).$$

Moreover, if $\|[v](0)\|_{\dot{H}^{\gamma_w}} < \varepsilon$ *for some* $\varepsilon > 0$ *small enough, then* $T^* = \infty$ *and the solution is scattering in* $\dot{H}^{\gamma_w} \times \dot{H}^{\gamma_w - \sigma}$, *i.e. there exist* $(\varphi^+, \phi^+) \in \dot{H}^{\gamma_w} \times \dot{H}^{\gamma_w - \sigma}$ *such that the (weak) solution to the linear fractional wave equation*

$$\begin{cases} \partial_t^2 v^+(t,x) + \Lambda^{2\sigma} v^+(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ v^+(0,x) = \varphi^+(x), & \partial_t v^+(0,x) = \phi^+(x), \quad x \in \mathbb{R}^d, \end{cases}$$

satisfies

$$\lim_{t \to +\infty} \|[v(t) - v^+(t)]\|_{\dot{H}^{\gamma_w}} = 0.$$

2. *Assume for* $d \geq 1$ *that*

$$\sigma \in \Big[\frac{d^2 + 4d}{3d + 4}, \infty\Big) \backslash \{1\}, \quad \nu \in \Big[1 + \frac{4\sigma(d+2)}{d(d+\sigma)}, \infty\Big)$$
$$\text{or } \sigma \in \Big[\frac{d}{d+1}, \frac{d^2+4d}{3d+4}\Big) \backslash \{1\},$$
$$\nu \in \Big[1 + \frac{4\sigma(d+2)}{d(d+\sigma)}, 1 + \frac{4\sigma(d+2)}{d^2 - 3d\sigma + 4d - 4\sigma}\Big]. \quad (1.21)$$

Then for all $(\varphi, \phi) \in \dot{H}^{\gamma_w} \times \dot{H}^{\gamma_w - \sigma}$, *there exist* $T^* \in (0, \infty]$ *and a unique solution to* (NLFW) *satisfying*

$$v \in C([0, T^*), \dot{H}^{\gamma_w}) \cap C^1([0, T^*), \dot{H}^{\gamma_w - \sigma}) \cap L^p_{\mathrm{loc}}([0, T^*), L^p),$$

where p *is as above. Moreover, if* $\|[v](0)\|_{\dot{H}^{\gamma_w}} < \varepsilon$ *for some* $\varepsilon > 0$ *small enough, then* $T^* = \infty$ *and the solution is scattering in* $\dot{H}^{\gamma_w} \times \dot{H}^{\gamma_w - \sigma}$.

Finally, we have the following local well-posedness and scattering with small data for (NLFW) in the σ-critical case.

Theorem 12. *Let*

$$\begin{cases} \sigma \in \Big[\frac{d}{d+2}, \frac{d}{2}\Big) \backslash \{1\} & \text{when } d = \{1, 2, 3, 4\}, \\ \sigma \in \Big[\frac{d}{6}, \frac{d}{2}\Big) \backslash \{1\} & \text{when } d \geq 5, \end{cases} \quad (1.22)$$

and $\nu = 1 + 4\sigma/(d-2\sigma)$. *Then for all* $(\varphi, \phi) \in \dot{H}^\sigma \times L^2$, *there exist* $T^* \in (0, \infty]$ *and a unique solution to* (NLFW) *satisfying*

$$v \in C([0,T^*), \dot{H}^\sigma) \cap C^1([0,T^*), L^2) \cap L^\nu_{\text{loc}}([0,T^*), L^{2\nu}).$$

Moreover, if $\|[v](0)\|_{\dot{H}^\sigma} < \varepsilon$ *for some* $\varepsilon > 0$ *small enough, then* $T^* = \infty$ *and the solution is scattering in* $\dot{H}^\sigma \times L^2$.

The rest of this note is organized as follows. In Section 2, we prove Strichartz estimates for the fractional Schrödinger and wave equations. In Section 3, we recall the fractional derivatives of the nonlinearity. Section 4 is devoted to the proofs of the local well-posedness in the sub-critical case and the local well-posedness with small data scattering in the critical case for the (NLFS). We finally prove the local well-posedness in the sub-critical case and the local well-posedness with small data scattering in the critical case for the (NLFW) in Section 5.

2. Strichartz Estimates

In this section, we recall Strichartz estimates for the linear fractional Schrödinger and wave equations.

Theorem 13 (Strichartz estimates [12]). *Let* $d \geq 1, \sigma \in (0,\infty)\backslash\{1\}, \gamma \in \mathbb{R}$ *and a (weak) solution to the linear fractional Schrödinger equation, namely*

$$u(t) = e^{it\Lambda^\sigma}\varphi + \int_0^t e^{i(t-s)\Lambda^\sigma} F(s)ds,$$

for some data φ, F. *Then for all* (p,q) *and* (a,b) *admissible pairs,*

$$\|u\|_{L^p(\mathbb{R},\dot{B}^\gamma_q)} \lesssim \|\varphi\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},\dot{B}^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-\sigma}_{b'})}, \tag{2.1}$$

where $\gamma_{p,q}$ *and* $\gamma_{a',b'}$ *are as in* (1.4). *In particular,*

$$\|u\|_{L^p(\mathbb{R},\dot{B}^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\varphi\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R},\dot{H}^\gamma)}, \tag{2.2}$$

and

$$\|u\|_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{p,q}}_2)} + \|u\|_{L^p(\mathbb{R},\dot{B}^0_q)} \lesssim \|\varphi\|_{\dot{H}^{\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},\dot{B}^0_{b'})}, \tag{2.3}$$

provided that

$$\gamma_{p,q} = \gamma_{a',b'} + \sigma. \tag{2.4}$$

Here (a, a') *is a conjugate pair.*

Sketch of Proof. We firstly note this theorem is proved if we establish

$$\|e^{it\Lambda^\sigma} P_1\varphi\|_{L^p(\mathbb{R},L^q)} \lesssim \|P_1\varphi\|_{L^2}, \tag{2.5}$$

$$\Big\| \int_0^t e^{i(t-s)\Lambda^\sigma} P_1 F(s) ds \Big\|_{L^p(\mathbb{R},L^q)} \lesssim \|P_1 F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{2.6}$$

for all (p, q), (a, b) admissible pairs. Indeed, by change of variables, we see that

$$\|e^{it\Lambda^\sigma} P_N\varphi\|_{L^p(\mathbb{R},L^q)} = N^{-(d/q+\sigma/p)} \|e^{it\Lambda^\sigma} P_1\varphi_N\|_{L^p(\mathbb{R},L^q)},$$
$$\|P_1\varphi_N\|_{L^2} = N^{d/2}\|P_N\varphi\|_{L^2},$$

where $\varphi_N(x) = \varphi(N^{-1}x)$. The estimate (2.5) implies that

$$\|e^{it\Lambda^\sigma} P_N\varphi\|_{L^p(\mathbb{R},L^q)} \lesssim N^{\gamma_{p,q}} \|P_N\varphi\|_{L^2}, \tag{2.7}$$

for all $N \in 2^{\mathbb{Z}}$. Similarly,

$$\Big\| \int_0^t e^{i(t-s)\Lambda^\sigma} P_N F(s) ds \Big\|_{L^p(\mathbb{R},L^q)} = N^{-(d/q+\sigma/p+\sigma)} \Big\| \int_0^t e^{i(t-s)\Lambda^\sigma} P_1 F_N(s) ds \Big\|_{L^p(\mathbb{R},L^q)},$$

where $F_N(t, x) = F(N^{-\sigma}t, N^{-1}x)$. We also have from (2.6) and the fact

$$\|P_1 F_N\|_{L^{a'}(\mathbb{R},L^{b'})} = N^{(d/b'+\sigma/a')} \|P_N F\|_{L^{a'}(\mathbb{R},L^{b'})}$$

that

$$\Big\| \int_0^t e^{i(t-s)\Lambda^\sigma} P_N F(s) ds \Big\|_{L^p(\mathbb{R},L^q)} \lesssim N^{\gamma_{p,q}-\gamma_{a',b'}-\sigma} \|P_N F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{2.8}$$

for all $N \in 2^{\mathbb{Z}}$. We see from (2.7) and (2.8) that

$$N^\gamma \|P_N u\|_{L^p(\mathbb{R},L^q)} \lesssim N^{\gamma+\gamma_{p,q}} \|P_N\varphi\|_{L^2} + N^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-\sigma} \|P_N F\|_{L^{a'}(\mathbb{R},L^{b'})}.$$

By taking the $\ell^2(2^{\mathbb{Z}})$ norm both sides and using the Minkowski inequality, we get (2.1). The estimates (2.2) and (2.3) follow easily from (2.1). It remains to prove (2.5) and (2.6). By the TT^*-criterion (see [25] or [1]), we need to show

$$\|T(t)\|_{L^2\to L^2} \lesssim 1, \tag{2.9}$$

$$\|T(t)\|_{L^1\to L^\infty} \lesssim (1+|t|)^{-d/2}, \tag{2.10}$$

for all $t\in\mathbb{R}$ where $T(t) := e^{it\Lambda^\sigma}P_1$. The energy estimate (2.9) is obvious by using the Plancherel theorem. The dispersive estimate (2.10) follows by the standard stationary phase theorem. The proof is complete. □

Corollary 14. *Let $d\geq 1, \sigma\in(0,\infty)\backslash\{1\}, \gamma\in\mathbb{R}$. If u is a (weak) solution to the linear fractional Schrödinger equation for some data φ, F, then for all (p,q) and (a,b) admissible with $q<\infty$ and $b<\infty$ satisfying* (2.4),

$$\|u\|_{L^p(\mathbb{R},\dot{H}^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\varphi\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R},\dot{H}^\gamma)}, \tag{2.11}$$

$$\|u\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}})} + \|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\varphi\|_{\dot{H}^{\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}. \tag{2.12}$$

Corollary 15. *Let $d\geq 1$, $\sigma\in(0,\infty)\backslash\{1\}$, $\gamma\geq 0$ and I a bounded interval. If u is a (weak) solution to the linear fractional Schrödinger equation for some data φ, F, then for all (p,q) admissible satisfying $q<\infty$,*

$$\|u\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\varphi\|_{H^\gamma} + \|F\|_{L^1(I,H^\gamma)}. \tag{2.13}$$

Proof. We firstly note that when $\gamma_{p,q}\geq 0$ (or at least $\sigma\in(0,2]\backslash\{1\}$), we can obtain (2.13) for any $\gamma\in\mathbb{R}$ and $I=\mathbb{R}$. To see this, we write $\|u\|_{L^p(\mathbb{R},H^{\gamma-\gamma_{p,q}}_q)} = \|\langle\Lambda\rangle^{\gamma-\gamma_{p,q}} u\|_{L^p(\mathbb{R},L^q)}$ and use (2.11) with $\gamma=\gamma_{p,q}$ to obtain

$$\|u\|_{L^p(\mathbb{R},H^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\langle\Lambda\rangle^{\gamma-\gamma_{p,q}}\varphi\|_{\dot{H}^{\gamma_{p,q}}} + \|\langle\Lambda\rangle^{\gamma-\gamma_{p,q}}F\|_{L^1(\mathbb{R},\dot{H}^{\gamma_{p,q}})}.$$

This gives the claim since $\|v\|_{\dot{H}^{\gamma_{p,q}}} \leq \|v\|_{H^{\gamma_{p,q}}}$ using that $\gamma_{p,q}\geq 0$. It remains to treat the case $\gamma_{p,q}<0$. By the Minkowski inequality and the unitary of $e^{it\Lambda^\sigma}$ in L^2, the estimate (2.13) is proved if we can show for $\gamma\geq 0$, $I\subset\mathbb{R}$ a bounded interval and all (p,q) admissible with $q<\infty$ that

$$\|e^{it\Lambda^\sigma}\varphi\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\varphi\|_{H^\gamma}. \tag{2.14}$$

Indeed, if we have (2.14), then

$$\Big\|\int_0^t e^{i(t-s)\Lambda^\sigma}F(s)ds\Big\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)}$$

$$\leq \int_I \|\mathbb{1}_{[0,t]}(s) e^{i(t-s)\Lambda^\sigma} F(s)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} ds$$
$$\leq \int_I \|e^{i(t-s)\Lambda^\sigma} F(s)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} ds$$
$$\lesssim \int_I \|F(s)\|_{H^\gamma} ds = \|F\|_{L^1(I,H^\gamma)}.$$

We now prove (2.14). To do so, we write

$$\langle\Lambda\rangle^{\gamma-\gamma_{p,q}} e^{it\Lambda^\sigma} \varphi = \psi(D) \langle\Lambda\rangle^{\gamma-\gamma_{p,q}} e^{it\Lambda^\sigma} \varphi + (1-\psi)(D) \langle\Lambda\rangle^{\gamma-\gamma_{p,q}} e^{it\Lambda^\sigma} \varphi,$$

for some $\psi \in C_0^\infty(\mathbb{R}^d)$ valued in $[0,1]$ and equal to 1 near the origin. For the first term, the Sobolev embedding implies

$$\|\psi(D) \langle\Lambda\rangle^{\gamma-\gamma_{p,q}} e^{it\Lambda^\sigma} \varphi\|_{L^q} \lesssim \|\psi(D) \langle\Lambda\rangle^{\gamma-\gamma_{p,q}} e^{it\Lambda^\sigma} \varphi\|_{H^\delta},$$

for some $\delta > d/2 - d/q$. Thanks to the support of ψ and the unitary property of $e^{it\Lambda^\sigma}$ in L^2, we get

$$\|\psi(D) \langle\Lambda\rangle^{\gamma-\gamma_{p,q}} e^{it\Lambda^\sigma} \varphi\|_{L^p(I,L^q)} \lesssim \|\varphi\|_{L^2} \lesssim \|\varphi\|_{H^\gamma}.$$

Here the boundedness of I is crucial to have the first estimate. For the second term, using (2.12), we obtain

$$\begin{aligned}\|(1-\psi)(D) \langle\Lambda\rangle^{\gamma-\gamma_{p,q}} e^{it\Lambda^\sigma} \varphi\|_{L^p(I,L^q)} &\lesssim \|(1-\psi)(D) \langle\Lambda\rangle^{\gamma-\gamma_{p,q}} \varphi\|_{\dot H^{\gamma_{p,q}}} \\ &\lesssim \|\varphi\|_{H^\gamma}.\end{aligned}$$

Combining the two terms, we have (2.14). This completes the proof. □

Corollary 16. *Let $d \geq 1, \sigma \in (0,\infty)\backslash\{1\}, \gamma \in \mathbb{R}$ and a (weak) solution to the linear fractional wave equation, namely*

$$v(t) = \cos(t\Lambda^\sigma)\varphi + \frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\phi + \int_0^t \frac{\sin((t-s)\Lambda^\sigma)}{\Lambda^\sigma} G(s) ds,$$

for some data φ, ϕ, G. Then for all (p,q) and (a,b) admissible pairs,

$$\|[v]\|_{L^p(\mathbb{R},\dot B_q^\gamma)} \lesssim \|[v](0)\|_{\dot H^{\gamma+\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},\dot B_{b'}^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-2\sigma})}, \tag{2.15}$$

where

$$\|[v]\|_{L^p(\mathbb{R},\dot B_q^\gamma)} := \|v\|_{L^p(\mathbb{R},\dot B_q^\gamma)} + \|\partial_t v\|_{L^p(\mathbb{R},\dot B_q^{\gamma-\sigma})};$$

$$\|[v](0)\|_{\dot{H}^{\gamma+\gamma_{p,q}}} := \|\varphi\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|\phi\|_{\dot{H}^{\gamma+\gamma_{p,q}-\sigma}}.$$

In particular,

$$\|[v]\|_{L^p(\mathbb{R},\dot{B}_q^{\gamma-\gamma_{p,q}})} \lesssim \|[v](0)\|_{\dot{H}^\gamma} + \|G\|_{L^1(\mathbb{R},\dot{H}^{\gamma-\sigma})}, \tag{2.16}$$

and

$$\|[v]\|_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_{p,q}})} + \|[v]\|_{L^p(\mathbb{R},\dot{B}_q^0)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},\dot{B}_{b'}^0)}, \tag{2.17}$$

provided that

$$\gamma_{p,q} = \gamma_{a',b'} + 2\sigma. \tag{2.18}$$

Proof. It follows easily from Theorem 13 and the fact that

$$\cos(t\Lambda^\sigma) = \frac{e^{it\Lambda^\sigma} + e^{-it\Lambda^\sigma}}{2}, \quad \sin(t\Lambda^\sigma) = \frac{e^{it\Lambda^\sigma} - e^{-it\Lambda^\sigma}}{2i}.$$

□

As in Corollary 14, we have the following usual Strichartz estimates for the fractional wave equation.

Corollary 17. *Let $d \geq 1, \sigma \in (0,\infty)\backslash\{1\}, \gamma \in \mathbb{R}$. If v is a (weak) solution to the linear fractional wave equation for some data φ, ϕ, G, then for all (p,q) and (a,b) admissible satisfying $q < \infty, b < \infty$ and (2.18),*

$$\|v\|_{L^p(\mathbb{R},\dot{H}_q^{\gamma-\gamma_{p,q}})} \lesssim \|[v](0)\|_{\dot{H}^\gamma} + \|G\|_{L^1(\mathbb{R},\dot{H}^{\gamma-\sigma})}, \tag{2.19}$$

$$\|[v]\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}})} + \|v\|_{L^p(\mathbb{R},L^q)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},L^{b'})}. \tag{2.20}$$

The following result, which is similar to Corollary 15, gives the local Strichartz estimates for the fractional wave equation.

Corollary 18. *Let $d \geq 1$, $\sigma \in (0,\infty)\backslash\{1\}$, $\gamma \geq 0$ and $I \subset \mathbb{R}$ a bounded interval. If v is a (weak) solution to the linear fractional wave equation for some data φ, ϕ, G, then for all (p,q) admissible satisfying $q < \infty$,*

$$\|v\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|[v](0)\|_{H^\gamma} + \|G\|_{L^1(I,H^{\gamma-\sigma})}. \tag{2.21}$$

Proof. The proof is similar to the one of Corollary 15. Thanks to the Minkowski inequality, it suffices to prove for all $\gamma \geq 0$, all $I \subset \mathbb{R}$ bounded interval and all (p,q) admissible pair with $q < \infty$,

$$\| \cos(t\Lambda^\sigma)\varphi\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\varphi\|_{H^\gamma}, \tag{2.22}$$

$$\Big\|\frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\phi\Big\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\phi\|_{H^{\gamma-\sigma}}. \tag{2.23}$$

The estimate (2.22) follows from the ones of $e^{\pm it\Lambda^\sigma}$. We will give the proof of (2.23). To do this, we write

$$\begin{aligned}\langle\Lambda\rangle^{\gamma-\gamma_{p,q}} & \frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma} \\ &= \psi(D)\langle\Lambda\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma} + (1-\psi)(D)\langle\Lambda\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma},\end{aligned}$$

for some ψ as in the proof of Corollary 15. For the first term, the Sobolev embedding and the fact $\Big\|\frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\Big\|_{L^2\to L^2} \leq |t|$ imply

$$\Big\|\psi(D)\langle\Lambda\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\phi\Big\|_{L^q} \lesssim |t|\|\psi(D)\langle\Lambda\rangle^{\gamma+\delta-\gamma_{p,q}}\phi\|_{L^2},$$

for some $\delta > d/2 - d/q$. This gives

$$\Big\|\psi(D)\langle\Lambda\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\phi\Big\|_{L^p(I,L^q)} \lesssim \|\phi\|_{H^{\gamma-\sigma}}.$$

Here we use the fact that $\|\psi(D)\langle\Lambda\rangle^{\delta+\sigma-\gamma_{p,q}}\|_{L^2\to L^2} \lesssim 1$. For the second term, we apply (2.14) with the fact $\sin(t\Lambda^\sigma) = (e^{it\Lambda^\sigma} - e^{-it\Lambda^\sigma})/2i$ and get

$$\begin{aligned}\Big\|(1-\psi)(D)\langle\Lambda\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\phi\Big\|_{L^p(I,L^q)} &\lesssim \|(1-\psi)(D)\Lambda^{-\sigma}\phi\|_{H^\gamma} \\ &\lesssim \|\phi\|_{H^{\gamma-\sigma}}.\end{aligned}$$

Here we use that $\|(1-\psi)(D)\langle\Lambda\rangle^\sigma\Lambda^{-\sigma}\|_{L^2\to L^2} \lesssim 1$ by functional calculus. Combining two terms, we have (2.23). The proof is complete. □

3. Nonlinear Estimates

In this section, we recall some estimates related to the fractional derivatives of nonlinear operators. Let us start with the following Kato-Ponce inequality (or fractional Leibniz rule).

Proposition 19. *Let $\gamma \geq 0, 1 < r < \infty$ and $1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then there exists $C = C(d, \gamma, r, p_1, q_1, p_2, q_2) > 0$ such that for all $u, v \in \mathscr{S}$,*

$$\|\Lambda^\gamma (uv)\|_{L^r} \leq C\Big(\|\Lambda^\gamma u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \|\Lambda^\gamma v\|_{L^{q_2}}\Big), \tag{3.1}$$

$$\| \langle\Lambda\rangle^\gamma (uv)\|_{L^r} \leq C\Big(\| \langle\Lambda\rangle^\gamma u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \| \langle\Lambda\rangle^\gamma v\|_{L^{q_2}}\Big). \tag{3.2}$$

We refer to [21] (and references therein) for the proof of above inequalities and more general results. We also have the following fractional chain rule.

Proposition 20. *Let $F \in C^1(\mathbb{C}, \mathbb{C})$ and $G \in C(\mathbb{C}, \mathbb{R}^+)$ such that $F(0) = 0$ and*

$$|F'(\theta z + (1-\theta)\zeta)| \leq \mu(\theta)(G(z) + G(\zeta)), \quad z, \zeta \in \mathbb{C}, \quad 0 \leq \theta \leq 1,$$

where $\mu \in L^1((0,1))$. Then for $\gamma \in (0,1)$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, there exists $C = C(d, \mu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,

$$\|\Lambda^\gamma F(u)\|_{L^r} \leq C\|F'(u)\|_{L^q} \|\Lambda^\gamma u\|_{L^p}, \tag{3.3}$$

$$\| \langle\Lambda\rangle^\gamma F(u)\|_{L^r} \leq C\|F'(u)\|_{L^q} \| \langle\Lambda\rangle^\gamma u\|_{L^p}. \tag{3.4}$$

We refer to [13] (see also [32]) for the proof of (3.3) and Proposition 5.1 of [35] for (3.4). A direct consequence of the fractional Leibniz rule and the fractional chain rule is the following fractional derivatives estimates.

Corollary 21. *Let $F \in C^k(\mathbb{C}, \mathbb{C}), k \in \mathbb{N}\backslash\{0\}$. Assume that there is $\nu \geq k$ such that*

$$|D^i F(z)| \leq C|z|^{\nu - i}, \quad z \in \mathbb{C}, \quad i = 1, 2, ..., k.$$

Then for $\gamma \in [0, k]$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$, there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,

$$\|\Lambda^\gamma F(u)\|_{L^r} \leq C\|u\|^{\nu-1}_{L^q} \|\Lambda^\gamma u\|_{L^p}, \tag{3.5}$$

$$\| \langle\Lambda\rangle^\gamma F(u)\|_{L^r} \leq C\|u\|^{\nu-1}_{L^q}\| \langle\Lambda\rangle^\gamma u\|_{L^p}. \tag{3.6}$$

The reader can find the proof of (3.5) in Lemma A.3 of [24]. The one of (3.6) follows from (3.5), the Hölder inequality and the fact that

$$\| \langle\Lambda\rangle^\gamma u\|_{L^r} \sim \|u\|_{L^r} + \|\Lambda^\gamma u\|_{L^r},$$

for $1 < r < \infty, \gamma > 0$. Another consequence of the fractional Leibniz rule given in Proposition 19 is the following result.

Corollary 22. *Let $F(z)$ be a homogeneous polynomial in $z, \overline{z}$ of degree $\nu \geq 1$. Then* (3.5) *and* (3.6) *hold true for any $\gamma \geq 0$ and r, p, q as in* Corollary 21.

Corollary 23. *Let $F(z) = |z|^{\nu-1}z$ with $\nu > 1$, $\gamma \geq 0$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$.*

i. *If ν is an odd integer or* [1] *$\lceil\gamma\rceil \leq \nu$ otherwise, then there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,*

$$\|F(u)\|_{\dot{H}^\gamma_r} \leq C\|u\|^{\nu-1}_{L^q}\|u\|_{\dot{H}^\gamma_p}.$$

A similar estimate holds with $\dot{H}^\gamma_r, \dot{H}^\gamma_p$-norms are replaced by H^γ_r, H^γ_p-norms, respectively.

ii. *If ν is an odd integer or $\lceil\gamma\rceil \leq \nu - 1$ otherwise, then there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u, v \in \mathscr{S}$,*

$$\begin{aligned}\|F(u) - F(v)\|_{\dot{H}^\gamma_r} \leq C\Big(&(\|u\|^{\nu-1}_{L^q} + \|v\|^{\nu-1}_{L^q})\|u - v\|_{\dot{H}^\gamma_p} \\ &+ (\|u\|^{\nu-2}_{L^q} + \|v\|^{\nu-2}_{L^q})(\|u\|_{\dot{H}^\gamma_p} + \|v\|_{\dot{H}^\gamma_p})\|u - v\|_{L^q}\Big).\end{aligned}$$

A similar estimate holds with $\dot{H}^\gamma_r, \dot{H}^\gamma_p$-norms are replaced by H^γ_r, H^γ_p-norms respectively.

Proof. Item 1 is an immediate consequence of Corollary 21 and Corollary 22. For Item 2, we firstly write

[1] see (1.5) for the definition of $\lceil\cdot\rceil$.

$$F(u) - F(v) = \int_0^1 \Big(\partial_z F(v + t(u-v))(u-v) \\ + \partial_{\overline{z}} F(v + t(u-v))\overline{(u-v)} \Big) dt,$$

and use the fractional Leibniz rule given in Proposition 19. Then the results follows by applying the fractional derivatives given in Corollary 21 and Corollary 22. □

4. Nonlinear Fractional Schrödinger Equations

4.1. Local Well-Posedness in Sub-Critical Cases

In this subsection, we give the proofs of Theorem 1, Theorem 3 and Proposition 5.

Proof of Theorem 1. We follow the standard process (see e.g. Chapter 4 of [5] or [3]) by using the fixed point argument in a suitable Banach space. We firstly choose $p > \max(\nu - 1, 4)$ when $d = 1$ and $p > \max(\nu - 1, 2)$ when $d \geq 2$ such that $\gamma > d/2 - \sigma/p$ and then choose $q \in [2, \infty)$ such that

$$\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}.$$

Step 1. *Existence.* Let us consider

$$X := \Big\{ u \in L^\infty(I, H^\gamma) \cap L^p(I, H_q^{\gamma - \gamma_{p,q}}) \ : \\ \|u\|_{L^\infty(I,H^\gamma)} + \|u\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \leq M \Big\},$$

equipped with the distance

$$d(u,v) := \|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^p(I,H_q^{-\gamma_{p,q}})},$$

where $I = [0, T]$ and $M, T > 0$ to be chosen later. The persistence of regularity (see e.g. Theorem 1.2.5 of [5]) shows that (X, d) is a complete metric space. By the Duhamel formula, it suffices to prove that the functional

$$\Phi(u)(t) = e^{it\Lambda^\sigma}\varphi + i\mu \int_0^t e^{i(t-s)\Lambda^\sigma} |u(s)|^{\nu-1} u(s) ds \tag{4.1}$$

is a contraction on (X, d). The local Strichartz estimate (2.13) gives

$$\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\varphi\|_{H^\gamma} + \|F(u)\|_{L^1(I,H^\gamma)},$$

and

$$\|\Phi(u) - \Phi(v)\|_{L^\infty(I,L^2)} + \|\Phi(u) - \Phi(v)\|_{L^p(I,H_q^{-\gamma_{p,q}})} \lesssim \|F(u) - F(v)\|_{L^1(I,L^2)},$$

where $F(u) = |u|^{\nu-1}u$. By our assumptions on ν, Corollary 23 gives

$$\begin{aligned}\|F(u)\|_{L^1(I,H^\gamma)} &\lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\|u\|_{L^\infty(I,H^\gamma)} \\ &\lesssim T^{1-\frac{\nu-1}{p}}\|u\|^{\nu-1}_{L^p(I,L^\infty)}\|u\|_{L^\infty(I,H^\gamma)},\end{aligned} \tag{4.2}$$

and

$$\|F(u) - F(v)\|_{L^1(I,L^2)} \tag{4.3}$$
$$\lesssim \left(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\right)\|u - v\|_{L^\infty(I,L^2)}$$
$$\lesssim T^{1-\frac{\nu-1}{p}}\left(\|u\|^{\nu-1}_{L^p(I,L^\infty)} + \|v\|^{\nu-1}_{L^p(I,L^\infty)}\right)\|u - v\|_{L^\infty(I,L^2)}. \tag{4.4}$$

Using the fact $\gamma - \gamma_{p,q} > d/q$, the Sobolev embedding implies $L^p(I, H_q^{\gamma-\gamma_{p,q}}) \subset L^p(I, L^\infty)$. Thus, we get

$$\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\varphi\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}}\|u\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}\|u\|_{L^\infty(I,H^\gamma)},$$

and

$$d(\Phi(u), \Phi(v)) \lesssim T^{1-\frac{\nu-1}{p}}\left(\|u\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} + \|v\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}\right)\|u - v\|_{L^\infty(I,L^2)}.$$

This shows that for all $u, v \in X$, there exists $C > 0$ independent of $\varphi \in H^\gamma$ such that

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} &\leq C\|\varphi\|_{H^\gamma} + CT^{1-\frac{\nu-1}{p}}M^\nu, \\ d(\Phi(u), \Phi(v)) &\leq CT^{1-\frac{\nu-1}{p}}M^{\nu-1}d(u, v).\end{aligned}$$

Therefore, if we set $M = 2C\|\varphi\|_{H^\gamma}$ and choose $T > 0$ small enough so that $CT^{1-\frac{\nu-1}{p}} M^{\nu-1} \leq \frac{1}{2}$, then X is stable by Φ and Φ is a contraction on X. By the fixed point theorem, there exists a unique $u \in X$ so that $\Phi(u) = u$.
Step 2. *Uniqueness.* Consider $u, v \in C(I, H^\gamma) \cap L^p(I, L^\infty)$ two solutions of (NLFS). Since the uniqueness is a local property (see Chapter 4 of [5]), it suffices to show $u = v$ for T is small. We have from (4.4) that

$$d(u,v) \leq CT^{1-\frac{\nu-1}{p}} \Big(\|u\|^{\nu-1}_{L^p(I,L^\infty)} + \|v\|^{\nu-1}_{L^p(I,L^\infty)} \Big) d(u,v).$$

Since $\|u\|_{L^p(I,L^\infty)}$ is small if T is small and similarly for v, we see that if $T > 0$ small enough,

$$d(u,v) \leq \frac{1}{2} d(u,v) \text{ or } u = v.$$

Step 3. *Item i.* Since the time of existence constructed in Step 1 only depends on $\|\varphi\|_{H^\gamma}$. The blowup alternative follows by standard argument (see again Chapter 4 of [5]).
Step 4. *Item ii.* Let $\varphi_n \to \varphi$ in H^γ and $C, T = T(\varphi)$ be as in Step 1. Set $M = 4C\|\varphi\|_{H^\gamma}$. It follows that $2C\|\varphi_n\|_{H^\gamma} \leq M$ for sufficiently large n. Thus the solution u_n constructed in Step 1 belongs to X with $T = T(\varphi)$ for n large enough. We have from Strichartz estimate (2.13) and (4.2) that

$$\|u\|_{L^a(I,H_b^{\gamma-\gamma_{a,b}})} \lesssim \|\varphi\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}} \|u\|^{\nu-1}_{L^p(I,L^\infty)} \|u\|_{L^\infty(I,H^\gamma)},$$

provided (a,b) is admissible and $b < \infty$. This shows the boundedness of u_n in $L^a(I, H_b^{\gamma-\gamma_{a,b}})$. We also have from (4.4) and the choice of T that

$$d(u_n,u) \leq C\|\varphi_n - \varphi\|_{L^2} + \frac{1}{2} d(u_n,u) \text{ or } d(u_n,u) \leq 2C\|\varphi_n - \varphi\|_{L^2}.$$

This yields that $u_n \to u$ in $L^\infty(I, L^2) \cap L^p(I, H_q^{-\gamma_{p,q}})$. Strichartz estimate (2.13) again implies that $u_n \to u$ in $L^a(I, H_b^{-\gamma_{a,b}})$ for any admissible pair (a,b) with $b < \infty$. The convergence in $C(I, H^{\gamma-\epsilon})$ follows from the boundedness in $L^\infty(I, H^\gamma)$, the convergence in $L^\infty(I, L^2)$ and that $\|u\|_{H^{\gamma-\epsilon}} \leq \|u\|^{1-\frac{\epsilon}{\gamma}}_{H^\gamma} \|u\|^{\frac{\epsilon}{\gamma}}_{L^2}$. □

Remark 24. If we assume that $\nu > 1$ is an odd integer or

$$\lceil \gamma \rceil \leq \nu - 1$$

otherwise, then the continuous dependence holds in $C(I, H^\gamma)$. To see this, we consider X as above equipped with the following metric

$$d(u,v) := \|u - v\|_{L^\infty(I,H^\gamma)} + \|u - v\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}.$$

Using Item ii of Corollary 23, we have

$$\begin{aligned}\|F(u)-F(v)\|_{L^1(I,H^\gamma)} &\lesssim \Big(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}+\|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\Big)\|u-v\|_{L^\infty(I,H^\gamma)} \\ &\quad+\Big(\|u\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)}+\|v\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)}\Big) \\ &\quad\times\Big(\|u\|_{L^\infty(I,H^\gamma)}+\|v\|_{L^\infty(I,H^\gamma)}\Big)\|u-v\|_{L^{\nu-1}(I,L^\infty)}.\end{aligned}$$

Using the Sobolev embedding, we see that for all $u,v\in X$,

$$d(\Phi(u),\Phi(v))\lesssim T^{1-\frac{\nu-1}{p}}M^{\nu-1}d(u,v).$$

Therefore, the continuity in $C(I,H^\gamma)$ follows as in Step 4.

Proof of Theorem 3. Let (p,q) be as in (1.8). It is easy to see that (p,q) is admissible and $\gamma_{p,q}=0=\gamma_{p',q'}+\sigma$. We next choose (m,n) so that

$$\frac{1}{p'}=\frac{1}{p}+\frac{\nu-1}{m},\quad \frac{1}{q'}=\frac{1}{q}+\frac{\nu-1}{n}. \tag{4.5}$$

It is easy to see that

$$\frac{\nu-1}{m}-\frac{\nu-1}{p}=1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}>0,\quad q\leq n=\frac{dq}{d-\gamma q}.$$

The Sobolev embedding implies

$$\|u\|^{\nu-1}_{L^m(I,L^n)}\lesssim |I|^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}}\|u\|^{\nu-1}_{L^p(I,\dot H^\gamma_q)}. \tag{4.6}$$

Step 1. *Existence.* Let us consider

$$X:=\Big\{u\in L^p(I,H^\gamma_q)\mid \|u\|_{L^p(I,\dot H^\gamma_q)}\leq M\Big\},$$

equipped with the distance

$$d(u,v)=\|u-v\|_{L^p(I,L^q)},$$

where $I=[0,T]$ and $M,T>0$ to be determined. One can easily verify that (X,d) is a complete metric space (see e.g. [4]). The Strichartz estimate (2.12) implies

$$\|\Phi(u)\|_{L^p(I,\dot H^\gamma_q)}\lesssim\|\varphi\|_{\dot H^\gamma}+\|F(u)\|_{L^{p'}(I,\dot H^\gamma_{q'})},$$

$$\|\Phi(u) - \Phi(v)\|_{L^p(I,L^q)} \lesssim \|F(u) - F(v)\|_{L^{p'}(I,L^{q'})}.$$

It follows from Corollary 23, (4.5) and (4.6) that

$$\|\Phi(u)\|_{L^p(I,\dot{H}^\gamma_q)} \lesssim \|\varphi\|_{\dot{H}^\gamma} + T^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}} \|u\|^\nu_{L^p(I,\dot{H}^\gamma_q)}, \tag{4.7}$$

and

$$\begin{aligned} &\|\Phi(u) - \Phi(v)\|_{L^p(I,L^q)} \\ &\qquad \lesssim T^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}} \Big(\|u\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)} + \|v\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)}\Big) \|u - v\|_{L^p(I,L^q)}. \end{aligned} \tag{4.8}$$

This implies for all $u, v \in X$, there exists C independent of $\varphi \in H^\gamma$ such that

$$\begin{aligned} \|\Phi(u)\|_{L^p(I,\dot{H}^\gamma_q)} &\leq C\|\varphi\|_{\dot{H}^\gamma} + CT^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}} M^\nu, \\ d(\Phi(u), \Phi(v)) &\leq CT^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}} M^{\nu-1} d(u, v). \end{aligned}$$

If we set $M = 2C\|\varphi\|_{\dot{H}^\gamma}$ and choose $T > 0$ small enough so that

$$CT^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}} M^{\nu-1} \leq \frac{1}{2},$$

then Φ is a strict contraction on X. Thus Φ has a unique fixed point in X. Since $\varphi \in H^\gamma$ and $u \in L^p(I, H^\gamma_q)$, the continuity in H^γ follows easily from Strichartz estimates (see e.g. [4]). This proves the existence of solution $u \in C(I, H^\gamma) \cap L^p(I, H^\gamma_q)$ to (NLFS).

Step 2. *Uniqueness.* The uniqueness is similar to Step 2 of the proof of Theorem 1 using (4.8). Note that $\|u\|_{L^p(I,\dot{H}^\gamma_q)}$ can be small if T is taken small enough.

Step 3. Item i. The blowup alternative is easy since the time of existence depends only on $\|\varphi\|_{\dot{H}^\gamma}$.

Step 4. Item ii. The continuous dependence is similar to that of Theorem 1. We have from Strichartz estimate (2.12) and (4.7) that

$$\begin{aligned} \|u\|_{L^a(I,\dot{H}^\gamma_b)} &\lesssim \|\varphi\|_{\dot{H}^\gamma} + T^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}} \|u\|^\nu_{L^p(I,\dot{H}^\gamma_q)}, \\ \|u\|_{L^a(I,L^b)} &\lesssim \|\varphi\|_{L^2} + T^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}} \|u\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)} \|u\|_{L^p(I,L^q)}, \end{aligned}$$

provided that (a, b) is admissible, $b < \infty$ and $\gamma_{a,b} = 0$. This gives the boundedness of u_n in $L^a(I, H^\gamma_b)$. The convergence in $L^a(I, L^b)$ and $H^{\gamma-\epsilon}$ follows similarly as in Step 4 of Theorem 1 using (4.8). □

Proof of Proposition 5. The assumption (1.9) allows us to apply Theorem 1 and Theorem 3 with $\gamma = \sigma/2$ and obtain the local well-posedness in $H^{\sigma/2}$. We now prove the global extension using the blowup alternative. Item i follows from the conservation of mass and energy. For Item ii and Item iii, we firstly use Gagliardo-Nirenberg's inequality (see e.g. Appendix of [34]) with the fact that

$$\frac{1}{\nu+1} = \frac{1}{2} - \frac{\theta\sigma}{2d} \text{ or } \theta = \frac{d(\nu-1)}{\sigma(\nu+1)}$$

and the conservation of mass to get

$$\begin{aligned}\|u(t)\|_{L^{\nu+1}}^{\nu+1} &\lesssim \|\Lambda^{\sigma/2}u(t)\|_{L^2}^{\frac{d(\nu-1)}{\sigma}} \|u(t)\|_{L^2}^{\nu+1-\frac{d(\nu-1)}{\sigma}} \\ &= \|u(t)\|_{\dot{H}^{\sigma/2}}^{\frac{d(\nu-1)}{\sigma}} \|\varphi\|_{L^2}^{\nu+1-\frac{d(\nu-1)}{\sigma}}.\end{aligned}$$

Note that here the assumption $\nu \leq 1 + 2\sigma/d$ ensures that $\theta \in (0,1)$. The conservation of mass then gives

$$\begin{aligned}\frac{1}{2}\|u(t)\|_{\dot{H}^{\sigma/2}}^2 &= E_{\mathrm{s}}(u(t)) - \frac{\mu}{\nu+1}\|u(t)\|_{L^{\nu+1}}^{\nu+1} \\ &\lesssim E_{\mathrm{s}}(\varphi) - \frac{\mu}{\nu+1}\|u(t)\|_{\dot{H}^{\sigma/2}}^{\frac{d(\nu-1)}{\sigma}} \|\varphi\|_{L^2}^{\nu+1-\frac{d(\nu-1)}{\sigma}}.\end{aligned}$$

If $\nu \in (1, 1+2\sigma/d)$ or $\frac{d(\nu-1)}{\sigma} \in (0,2)$, then $\|u(t)\|_{\dot{H}^{\sigma/2}} \leq C$. This together with the conservation of mass implies the boundedness of $\|u(t)\|_{H^{\sigma/2}}$ and Item ii follows. Item iii is treated similarly with $\|\varphi\|_{L^2}$ is small. It remains to show Item iv. By Sobolev embedding with $\frac{1}{2} \leq \frac{1}{\nu+1} + \frac{\sigma}{2d}$, we have

$$\|\varphi\|_{L^{\nu+1}} \leq C\|\varphi\|_{H^{\sigma/2}}.$$

This shows that $E(\varphi)$ is small if $\|\varphi\|_{H^{\sigma/2}}$ is small. Similarly,

$$\frac{1}{2}\|u(t)\|_{\dot{H}^{\sigma/2}}^2 = E_{\mathrm{s}}(u(t)) - \frac{\mu}{\nu+1}\|u(t)\|_{L^{\nu+1}}^{\nu+1} \leq E_{\mathrm{s}}(\varphi) + C\|u(t)\|_{H^{\sigma/2}}^{\nu+1},$$

with $\nu+1 > 2$. This again implies that $\|u(t)\|_{H^{\sigma/2}}$ is bounded provided $\|\varphi\|_{H^{\sigma/2}}$ is small. This completes the proof. □

4.2. Local Well-Posedness in Critical Cases

In this subsection, we give the proofs of Theorem 6 and Theorem 7.

Proof of Theorem 6. Let us recall the following result which gives a good control for the nonlinear term.

Lemma 25 ([22]). *Let $\sigma \in (0,2)\backslash\{1\}$, ν be as in (1.10), $\gamma_{\rm s}$ as in (1.1). Then we have*

$$\|u\|^{\nu-1}_{L^{\nu-1}(\mathbb{R},L^\infty)} \lesssim \begin{cases} \|u\|^4_{L^4(\mathbb{R},\dot{B}^{\gamma_{\rm s}-\gamma_{4,\infty}}_\infty} \|u\|^{\nu-5}_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{\rm s}}_2)} & \text{when } d=1, \\ \|u\|^p_{L^p(\mathbb{R},\dot{B}^{\gamma_{\rm s}-\gamma_{p,p^\star}}_{p^\star})} \|u\|^{\nu-1-p}_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{\rm s}}_2)} & \text{when } d=2, \\ \|u\|^2_{L^2(\mathbb{R},\dot{B}^{\gamma_{\rm s}-\gamma_{2,2^\star}}_{2^\star})} \|u\|^{\nu-3}_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{\rm s}}_2)} & \text{when } d\geq 3, \end{cases}$$

where $2<p<\nu-1$, $p^\star = 2p/(p-2)$ and $2^\star = 2d/(d-2)$.

This result is a slight modification of Lemma 3.5 in [22] which generalizes Lemma 3.1 in [14]. The main difference is the exponent power in $\mathbb{R}^2$. The proof is similar to the one given there, thus we omit it.

Step 1. *Existence.* We only treat for $d \geq 3$, the ones for $d=1, d=2$ are completely similar. Let us consider

$$X := \Big\{u \in L^\infty(I,H^{\gamma_{\rm s}}) \cap L^2(I, B^{\gamma_{\rm s}-\gamma_{2,2^\star}}_{2^\star}) \ : \ \|u\|_{L^\infty(I,\dot{H}^{\gamma_{\rm s}})} \leq M, \|u\|_{L^2(I,\dot{B}^{\gamma_{\rm s}-\gamma_{2,2^\star}}_{2^\star})} \leq N\Big\},$$

equipped with the distance

$$d(u,v) := \|u-v\|_{L^\infty(I,L^2)} + \|u-v\|_{L^2(I,\dot{B}^{-\gamma_{2,2^\star}}_{2^\star})},$$

where $I=[0,T]$ and $T,M,N>0$ will be chosen later. One can check (see again [4] or Chapter 4 of [5]) that (X,d) is a complete metric space. Using the Duhamel formula

$$\begin{aligned} \Phi(u)(t) &= e^{it\Lambda^\sigma}\varphi + i\mu\int_0^t e^{i(t-s)\Lambda^\sigma}|u(s)|^{\nu-1}u(s)ds \\ &=: u_{\rm hom}(t) + u_{\rm inh}(t), \end{aligned} \tag{4.9}$$

the Strichartz estimate (2.2) yields

$$\|u_{\rm hom}\|_{L^2(I,\dot{B}^{\gamma_{\rm s}-\gamma_{2,2^\star}}_{2^\star})} \lesssim \|\varphi\|_{\dot{H}^{\gamma_{\rm s}}}.$$

A similar estimate holds for $\|u_{\rm hom}\|_{L^\infty(I,\dot{H}^{\gamma_{\rm s}})}$. It is easy to see that

$$\|u_{\rm hom}\|_{L^2(I,\dot{B}^{\gamma_{\rm s}-\gamma_{2,2^\star}}_{2^\star})} \leq \varepsilon$$

for some $\varepsilon > 0$ small enough which will be chosen later provided either $\|\varphi\|_{\dot{H}^{\gamma_s}}$ is small or it is satisfied for some $T > 0$ small enough by the dominated convergence theorem. Therefore, we can take $T = \infty$ in the first case and T be this small time in the second. On the other hand, using again (2.2), we have

$$\|u_{\text{inh}}\|_{L^2(I,\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})} \lesssim \|F(u)\|_{L^1(I,\dot{H}^{\gamma_s})}.$$

A same estimate holds for $\|u_{\text{inh}}\|_{L^\infty(I,\dot{H}^{\gamma_s})}$. Corollary 23 and Lemma 25 give

$$\begin{aligned} \|F(u)\|_{L^1(I,\dot{H}^{\gamma_s})} &\lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\|u\|_{L^\infty(I,\dot{H}^{\gamma_s})} \\ &\lesssim \|u\|^2_{L^2(I,\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-2}_{L^\infty(I,\dot{H}^{\gamma_s})}. \end{aligned} \tag{4.10}$$

Similarly, we have

$$\begin{aligned} &\|F(u)-F(v)\|_{L^1(I,L^2)} \\ &\quad\lesssim \Big(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\Big)\|u-v\|_{L^\infty(I,L^2)} \\ &\quad\lesssim \Big(\|u\|^2_{L^2(I,\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-3}_{L^\infty(I,\dot{H}^{\gamma_s})} \\ &\qquad + \|v\|^2_{L^2(I,\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})}\|v\|^{\nu-3}_{L^\infty(I,\dot{H}^{\gamma_s})}\Big)\|u-v\|_{L^\infty(I,L^2)}. \end{aligned} \tag{4.11}$$

This implies for all $u, v \in X$, there exists $C > 0$ independent of $\varphi \in H^{\gamma_s}$ such that

$$\begin{aligned} \|\Phi(u)\|_{L^2(I,\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})} &\leq \varepsilon + CN^2M^{\nu-2}, \\ \|\Phi(u)\|_{L^\infty(I,\dot{H}^{\gamma_s})} &\leq C\|\varphi\|_{\dot{H}^{\gamma_s}} + CN^2M^{\nu-2}, \\ d(\Phi(u),\Phi(v)) &\leq CN^2M^{\nu-3}d(u,v). \end{aligned}$$

Now by setting $N = 2\varepsilon$ and $M = 2C\|\varphi\|_{\dot{H}^{\gamma_s}}$ and choosing $\varepsilon > 0$ small enough such that $CN^2M^{\nu-3} \leq \min\{1/2, \varepsilon/M\}$, we see that X is stable by Φ and Φ is a contraction on X. By the fixed point theorem, there exists a unique solution $u \in X$ to (NLFS). Note that when $\|\varphi\|_{\dot{H}^{\gamma_s}}$ is small enough, we can take $T = \infty$.

Step 2. *Uniqueness.* The uniqueness in $C^\infty(I, H^{\gamma_s}) \cap L^2(I, B^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})$ follows as in Step 2 of the proof of Theorem 1 using (4.11). Here $\|u\|_{L^2(I,\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})}$ can be small as T is small.

Step 3. *Scattering.* The global existence when $\|\varphi\|_{\dot{H}^{\gamma_s}}$ is small is given in Step 1. It remains to show the scattering property. Thanks to (4.10), we see that

$$\|e^{-it_2\Lambda^\sigma}u(t_2) - e^{-it_1\Lambda^\sigma}u(t_1)\|_{\dot{H}^{\gamma_s}}$$

$$
\begin{aligned}
&= \Big\| i\mu \int_{t_1}^{t_2} e^{-is\Lambda^\sigma} (|u|^{\nu-1}u)(s)ds \Big\|_{\dot{H}^{\gamma_s}} \\
&\leq \|F(u)\|_{L^1([t_1,t_2],\dot{H}^{\gamma_s})} \\
&\lesssim \|u\|^2_{L^2([t_1,t_2],\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})} \|u\|^{\nu-2}_{L^\infty([t_1,t_2],\dot{H}^{\gamma_s})} \to 0 \qquad (4.12)
\end{aligned}
$$

as $t_1, t_2 \to +\infty$. We have from (4.11) that

$$
\begin{aligned}
&\|e^{-it_2\Lambda^\sigma}u(t_2) - e^{-it_1\Lambda^\sigma}u(t_1)\|_{L^2} \\
&\qquad \lesssim \|u\|^2_{L^2([t_1,t_2],\dot{B}^{\gamma_s-\gamma_{2,2^\star}}_{2^\star})} \|u\|^{\nu-3}_{L^\infty([t_1,t_2],\dot{H}^{\gamma_s})} \|u\|_{L^\infty([t_1,t_2],L^2)}, \quad (4.13)
\end{aligned}
$$

which also tends to zero as $t_1, t_2 \to +\infty$. This implies that the limit

$$\varphi^+ := \lim_{t\to+\infty} e^{-it\Lambda^\sigma}u(t)$$

exists in H^{γ_s}. Moreover, we have

$$u(t) - e^{it\Lambda^\sigma}\varphi^+ = -i\mu \int_t^{+\infty} e^{i(t-s)\Lambda^\sigma} F(u(s))ds.$$

The unitary property of $e^{it\Lambda^\sigma}$ in L^2, (4.12) and (4.13) imply that $\|u(t) - e^{it\Lambda^\sigma}\varphi^+\|_{H^{\gamma_s}} \to 0$ when $t \to +\infty$. This completes the proof of Theorem 6. □

Proof of Theorem 7. The proof is similar to the one of Theorem 6. Thus, we only give the main steps. It is easy to check that the admissible pair (p, q) given in (1.11) satisfies $\gamma_{p,q} = 0 = \gamma_{p',q'} + \sigma$. We next choose n so that

$$\frac{1}{q'} = \frac{1}{q} + \frac{\nu-1}{n} \text{ or } n = \frac{dq}{d - \gamma_s q}.$$

The Sobolev embedding gives

$$\|u\|_{L^p(I,L^n)} \lesssim \|u\|_{L^p(I,\dot{H}^{\gamma_s})}. \qquad (4.14)$$

Step 1. *Existence.* We will show that the functional Φ given in (4.9) is a contraction on

$$X := \Big\{ u \in L^p(I, H^{\gamma_s}_q) \mid \|u\|_{L^p(I,\dot{H}^{\gamma_s}_q)} \leq M \Big\},$$

which equipped with the distance

$$d(u, v) = \|u - v\|_{L^p(I,L^q)},$$

where $I = [0,T]$ and $M, T > 0$ to be determined. The Strichartz estimate (2.12) implies

$$\|u_{\rm hom}\|_{L^p(I,\dot H^{\gamma_s}_q)} \lesssim \|\varphi\|_{\dot H^{\gamma_s}}.$$

This shows that $\|u_{\rm hom}\|_{L^p(I,\dot H^{\gamma_s}_q)} \leq \varepsilon$ for some $\varepsilon>0$ small enough provided that T is small or $\|\varphi\|_{\dot H^{\gamma_s}}$ is small. Similarly, we have

$$\|u_{\rm inh}\|_{L^p(I,\dot H^{\gamma_s}_q)} \lesssim \|F(u)\|_{L^{p'}(I,\dot H^{\gamma_s}_{q'})}.$$

It follows from Corollary 23, the choice of n and (4.14) that

$$\|F(u)\|_{L^{p'}(I,\dot H^{\gamma_s}_{q'})} \lesssim \|u\|^{\nu}_{L^p(I,\dot H^{\gamma_s}_q)}, \tag{4.15}$$

$$\|F(u)-F(v)\|_{L^{p'}(I,L^{q'})} \lesssim \Big(\|u\|^{\nu-1}_{L^p(I,\dot H^{\gamma_s}_q)} + \|v\|^{\nu-1}_{L^p(I,\dot H^{\gamma_s}_q)}\Big)\|u-v\|_{L^p(I,L^q)}. \tag{4.16}$$

Thus, the Strichartz estimate (2.12) implies for all $u,v\in X$, there exists C independent of $\varphi \in H^{\gamma_s}$ such that

$$\begin{aligned}\|\Phi(u)\|_{L^p(I,\dot H^{\gamma_s}_q)} &\leq \varepsilon + CM^{\nu},\\ d(\Phi(u),\Phi(v)) &\leq CM^{\nu-1}d(u,v).\end{aligned}$$

If we choose $\varepsilon, M>0$ small so that

$$CM^{\nu-1}\leq \frac{1}{2}, \quad \varepsilon+\frac{M}{2}\leq M,$$

then X is stable by Φ and Φ a contraction on X. Using the argument as in Step 1 of the proof of Theorem 3, we obtain the existence of solution $u \in C(I,H^{\gamma_s})\cap L^p(I,H^{\gamma_s}_q)$ to (NLFS). Note that when $\|\varphi\|_{\dot H^{\gamma_s}}$ is small, we can take $T=\infty$.

Step 2. *Uniqueness.* It follows easily from (4.16) by the same argument given in Step 2 of the proof of Theorem 1 using (4.16).

Step 3. *Scattering.* The global existence when $\|\varphi\|_{\dot H^{\gamma_s}}$ is small follows from Step 1. The scattering is treated similarly as in Step 3 of the proof of Theorem 6. The main point is to show

$$\|e^{-it_2\Lambda^\sigma}u(t_2) - e^{-it_1\Lambda^\sigma}u(t_1)\|_{H^{\gamma_s}} \to 0 \tag{4.17}$$

as $t_1,t_2\to+\infty$. To do so, we use the adjoint estimate to the homogeneous Strichartz estimate, namely $\varphi\in L^2 \mapsto e^{it\Lambda^\sigma}\varphi \in L^p(\mathbb{R},L^q)$ to get

$$\|e^{-it_2\Lambda^\sigma}u(t_2) - e^{-it_1\Lambda^\sigma}u(t_1)\|_{\dot H^{\gamma_s}}$$

$$
\begin{aligned}
&= \Big\| i\mu \int_{t_1}^{t_2} e^{-is\Lambda^\sigma} (|u|^{\nu-1}u)(s)ds \Big\|_{\dot{H}^{\gamma_s}} \\
&= \Big\| \int_{\mathbb{R}} \Lambda^{\gamma_s} e^{-is\Lambda^\sigma} (\mathbb{1}_{[t_1,t_2]}|u|^{\nu-1}u)(s)ds \Big\|_{L^2} \\
&\lesssim \|F(u)\|_{L^{p'}([t_1,t_2],\dot{H}^{\gamma_s}_{q'})}.
\end{aligned}
$$

Similarly,

$$
\|e^{-it_2\Lambda^\sigma}u(t_2) - e^{-it_1\Lambda^\sigma}u(t_1)\|_{L^2} \lesssim \|F(u)\|_{L^{p'}([t_1,t_2],L^{q'})}.
$$

Using (4.15) and (4.16), we get (4.17). The proof is complete. □

5. Nonlinear Fractional Wave Equations

5.1. Local Well-Posedness in Subcritical Cases

In this subsection, we will give the proofs of Theorem 8 and Theorem 9.

Proof of Theorem 8. The proof is very close to the one of Theorem 1. Let (p,q) be the fractional pair in the proof of Theorem 1.
Step 1. *Existence.* We will solve (NLFW) in

$$
Y := \Big\{ v \in C(I,H^\gamma) \cap C^1(I,H^{\gamma-\sigma}) \cap L^p(I,H_q^{\gamma-\gamma_{p,q}}) \ : \\
\|[v]\|_{L^\infty(I,H^\gamma)} + \|v\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \leq M \Big\},
$$

equipped with the distance

$$
d(v,w) := \|[v-w]\|_{L^\infty(I,L^2)} + \|v-w\|_{L^p(I,H_q^{-\gamma_{p,q}})},
$$

where $I = [0,T]$ and $T, M > 0$ will be chosen later. The persistence of regularity implies that (Y,d) is a complete metric space. By the Duhamel formula, it suffices to prove that the functional

$$
\Psi(v)(t) \\
= \cos(t\Lambda^\sigma)\varphi + \frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\phi - \mu \int_0^t \frac{\sin((t-s)\Lambda^\sigma)}{\Lambda^\sigma}|v(s)|^{\nu-1}v(s)ds \quad (5.1)
$$

is a contraction on (Y, d). The local Strichartz estimates (2.21) imply

$$\begin{aligned}\|[\Psi(v)]\|_{L^\infty(I,H^\gamma)} + \|\Psi(v)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} &\lesssim \|[v](0)\|_{H^\gamma} + \|F(v)\|_{L^1(I,H^{\gamma-\sigma})} \\ &\lesssim \|[v](0)\|_{H^\gamma} + \|F(v)\|_{L^1(I,H^\gamma)},\end{aligned}$$

where $F(v) = |v|^{\nu-1}v$. As in the proof of Theorem 1, Corollary 23 implies

$$\|F(v)\|_{L^1(I,H^\gamma)} \lesssim T^{1-\frac{\nu-1}{p}} \|v\|^{\nu-1}_{L^p(I,L^\infty)} \|v\|_{L^\infty(I,H^\gamma)}.$$

Similarly,

$$\|F(v) - F(w)\|_{L^1(I,L^2)} \lesssim T^{1-\frac{\nu-1}{p}} \Big(\|v\|^{\nu-1}_{L^p(I,L^\infty)} + \|w\|^{\nu-1}_{L^p(I,L^\infty)}\Big) \|v-w\|_{L^\infty(I,L^2)}. \quad (5.2)$$

The Sobolev embedding $L^p(I, H_q^{\gamma-\gamma_{p,q}}) \subset L^p(I, L^\infty)$ then implies that

$$\|[\Psi(v)]\|_{L^\infty(I,H^\gamma)} + \|\Psi(v)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|[v](0)\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}} \|v\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \|v\|_{L^\infty(I,H^\gamma)},$$

and

$$d(\Psi(v), \Psi(w)) \lesssim T^{1-\frac{\nu-1}{p}} \Big(\|v\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} + \|w\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}\Big) d(v,w).$$

Therefore, for all $v, w \in Y$, there exists a constant $C > 0$ independent of φ, ϕ such that

$$\|[\Psi(v)]\|_{L^\infty(I,H^\gamma)} + \|\Psi(v)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \le C\|[v](0)\|_{H^\gamma} + CT^{1-\frac{\nu-1}{p}} M^\nu,$$

and

$$d(\Psi(v), \Psi(w)) \le CT^{1-\frac{\nu-1}{p}} M^{\nu-1} d(v,w).$$

Setting $M = 2C\|[v](0)\|_{H^\gamma}$ and choosing $T > 0$ small enough so that

$$CT^{1-\frac{\nu-1}{p}} M^{\nu-1} \le \frac{1}{2},$$

we see that Y is stable by Ψ and Ψ is a contraction on Y. By the fixed point theorem, there exists a unique solution $v \in Y$ to (NLFW).

Step 2. *Uniqueness.* The uniqueness of solution $v \in C(I, H^\gamma) \cap C^1(I, H^{\gamma-\sigma}) \cap L^p(I, L^\infty)$ follows as in the proof of Theorem 1 using (5.2).

Step 3. The blowup alternative follows easily since the time of existence depends only on $\|[v](0)\|_{H^\gamma}$.

Step 4. The continuous dependence is similar to that of Theorem 1. □

Proof of Theorem 9. 1. Let us firstly consider *Item 1.* We note (see Remark 26) that under the assumptions (1.12), (1.13) and (1.14) (see Remark 26), the pair (p,q) given in (1.15) is admissible satisfying $\gamma_{p,q} = \sigma = \gamma_{1,2} + 2\sigma$ and $1 - \nu/p > 0$. Consider now

$$Y := \Big\{ v \in C(I, \dot{H}^\sigma) \cap C^1(I, L^2) \cap L^p(I, L^q) \ : \\ \|[v]\|_{L^\infty(I,\dot{H}^\sigma)} + \|v\|_{L^p(I,L^q)} \leq M \Big\},$$

equipped with the distance

$$d(v,w) := \|[v-w]\|_{L^\infty(I,\dot{H}^\sigma)} + \|v-w\|_{L^p(I,L^q)},$$

where $I = [0,T]$ and $M > 0$ will be chosen later. We will prove that the functional (5.1) is a contraction on Y. The Strichartz estimate (2.20) implies

$$\begin{aligned} \|[\Psi(v)]\|_{L^\infty(I,\dot{H}^\sigma)} + \|\Psi(v)\|_{L^p(I,L^q)} &\lesssim \|[v](0)\|_{\dot{H}^\sigma} + \|F(v)\|_{L^1(I,L^2)} \\ &= \|[v](0)\|_{\dot{H}^\sigma} + \|v\|^\nu_{L^\nu(I,L^{2\nu})} \\ &\lesssim \|[v](0)\|_{\dot{H}^\sigma} + T^{1-\frac{\nu}{p}} \|v\|^\nu_{L^p(I,L^q)}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\|F(v) - F(w)\|_{L^1(I,L^2)} \\ &\quad\lesssim \Big(\|v\|^{\nu-1}_{L^\nu(I,L^{2\nu})} + \|w\|^{\nu-1}_{L^\nu(I,L^{2\nu})} \Big) \|v-w\|_{L^\nu(I,L^{2\nu})} \\ &\quad\lesssim T^{1-\frac{\nu}{p}} \Big(\|v\|^{\nu-1}_{L^p(I,L^q)} + \|v\|^{\nu-1}_{L^p(I,L^q)} \Big) \|v-w\|_{L^p(I,L^q)}. \end{aligned} \tag{5.3}$$

This implies that for all $v,w \in Y$, there exists $C > 0$ independent of $(\varphi,\phi) \in \dot{H}^\sigma \times L^2$ such that,

$$\begin{aligned} \|[\Psi(v)]\|_{L^\infty(I,\dot{H}^\sigma)} + \|\Psi(v)\|_{L^p(I,L^q)} &\leq C\|[v](0)\|_{\dot{H}^\sigma} + CT^{1-\frac{\nu}{p}} M^\nu, \\ d(\Psi(v),\Psi(w)) &\leq CT^{1-\frac{\nu}{p}} M^{\nu-1} d(v,w). \end{aligned}$$

By setting $M = 2C\|[v](0)\|_{\dot{H}^\sigma}$, choosing $T > 0$ small enough so that

$$CT^{1-\frac{\nu}{p}} M^{\nu-1} \leq \frac{1}{2}$$

and arguing as in the proof of Theorem 8, we have the existence and uniqueness of solution $v \in C(I, \dot{H}^\sigma) \cap C^1(I, L^2) \cap L^p(I, L^q)$. The blowup alternative is immediate since the time of existence only depends on $\|[v](0)\|_{\dot{H}^\sigma}$. Finally, the continuous dependence is proved by using (5.3).

2. The proof of *Item 2* is similar, thus we only give the main steps. It is easy to see that under the assumption (1.16), the pair (p,q) defined in (1.17) is admissible and $\gamma_{p,q}=\sigma$. Since $\nu\in[d\sigma^*/(d+\sigma),\sigma^*)$, we see that $q/\nu\in(1,2]$. This allows to choose $b\in[2,\infty]$ so that $b'=q/\nu$. We next choose $a\in[2,\infty]$ such that (a,b) is admissible and $\gamma_{a,b}=-\gamma_{a',b'}-\sigma=0$ or $\gamma_{a',b'}+2\sigma=\sigma$. Thanks to the fact that $\nu<\sigma^*$, we see that

$$\frac{1}{a'}-\frac{\nu}{p}>0.$$

This shows that $\frac{1}{a'}=\frac{1}{p}+\frac{\nu-1}{m}$ with

$$\frac{\nu-1}{m}>\frac{\nu-1}{p}.$$

We will prove that Ψ is a contraction on

$$Y:=\Big\{v\in v\in C(I,\dot{H}^\sigma)\cap C^1(I,L^2)\cap L^p(I,L^q)\ :\ \|[v]\|_{L^\infty(I,\dot{H}^\sigma)}+\|v\|_{L^p(I,L^q)}\leq M\Big\},$$

equipped with the distance

$$d(v,w):=\|[v-w]\|_{L^\infty(I,\dot{H}^\sigma)}+\|v-w\|_{L^p(I,L^q)}.$$

The Strichartz estimate (2.20) implies

$$\begin{aligned}\|[\Psi(v)]\|_{L^\infty(I,\dot{H}^\sigma)}&+\|\Psi(v)\|_{L^p(I,L^q)}\\&\lesssim\|[v](0)\|_{\dot{H}^\sigma}+\|F(v)\|_{L^{a'}(I,L^{b'})}\\&=\|[v](0)\|_{\dot{H}^\sigma}+\|v\|^{\nu-1}_{L^m(I,L^q)}\|v\|_{L^p(I,L^q)}\\&\lesssim\|[v](0)\|_{\dot{H}^\sigma}+T^{\frac{\nu-1}{m}-\frac{\nu-1}{p}}\|v\|^{\nu}_{L^p(I,L^q)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|F(v)&-F(w)\|_{L^{a'}(I,L^{b'})}\\&\lesssim\Big(\|v\|^{\nu-1}_{L^m(I,L^q)}+\|w\|^{\nu-1}_{L^m(I,L^q)}\Big)\|v-w\|_{L^p(I,L^q)}\\&\lesssim T^{\frac{\nu-1}{m}-\frac{\nu-1}{p}}\Big(\|v\|^{\nu-1}_{L^p(I,L^q)}+\|v\|^{\nu-1}_{L^p(I,L^q)}\Big)\|v-w\|_{L^p(I,L^q)}.\end{aligned}$$

This implies that for all $v, w \in Y$, there exists $C > 0$ independent of $(\varphi, \phi) \in \dot{H}^\sigma \times L^2$ such that,

$$\|[\Psi(v)]\|_{L^\infty(I,\dot{H}^\sigma)} + \|\Psi(v)\|_{L^p(I,L^q)} \leq C\|[v](0)\|_{\dot{H}^\sigma} + CT^{\frac{\nu-1}{m}-\frac{\nu-1}{p}} M^\nu,$$
$$d(\Psi(v), \Psi(w)) \leq CT^{\frac{\nu-1}{m}-\frac{\nu-1}{p}} M^{\nu-1} d(v, w).$$

The conclusion is similar as in Item 1. The proof is now complete. □

Remark 26. Let us give some comments on the assumptions (1.12), (1.13) and (1.14). In order to make (p, q) defined in (1.15) to be admissible satisfying $\gamma_{p,q} = \sigma = \gamma_{1,2} + 2\sigma$ and $1 - \nu/p > 0$, we need the following conditions:
- A first condition is $(d - 2\sigma)\nu > d$ which ensures p is a positive number.
- The next one is $p \geq 4$ when $d = 1$ and $p \geq 2$ when $d \geq 2$. Thus $(2 - 5\sigma)\nu \leq 2$ when $d = 1$ and $(d - 3\sigma)\nu \leq d$ when $d \geq 2$.
- We also need $\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}$ which implies $(2d - 4\sigma - d\sigma)\nu \leq 2d - d\sigma$. When $d = 1$, we have $(2 - 5\sigma)\nu \leq 2 - \sigma$.
- Condition $\gamma_{p,q} = \sigma = \gamma_{1,2} + 2\sigma$ is easy to check.
- Finally, we have $(d - 2\sigma)\nu < d + 2\sigma$ which yields $1 - \nu/p > 0$.

Therefore, we need

$$\begin{cases} (1-2\sigma)\nu > 1 \\ (1-2\sigma)\nu < 1 + 2\sigma \\ (2-5\sigma)\nu \leq 2 - \sigma \end{cases} \quad \text{when } d = 1$$

$$\text{and} \begin{cases} (d-2\sigma)\nu > d \\ (d-2\sigma)\nu < d + 2\sigma \\ (d-3\sigma)\nu \leq d \\ (2d - 4\sigma - d\sigma)\nu \leq 2d - d\sigma \end{cases} \quad \text{when } d \geq 2.$$

One can solve easily the above systems of inequalities and obtain (1.12), (1.13) and (1.14).

5.2. Local Well-Posedness in Critical Cases

In this subsection, we will give the proofs of Theorem 11 and Theorem 12.

Proof of Theorem 11. 1. Let us treat the first case (1.18). Consider

$$Y := \Big\{ v \in C(I, \dot{H}^{\gamma_{\mathrm{w}}}) \cap C^1(I, \dot{H}^{\gamma_{\mathrm{w}}-\sigma}) \cap L^p(I, L^p) \cap L^a(I, \dot{H}_a^{\gamma_{\mathrm{w}}-\frac{\sigma}{2}}) \ :$$

$$\|[v]\|_{L^\infty(I,\dot H^{\gamma_w})} \leq M, \|v\|_{L^p(I,L^p)} + \|v\|_{L^a(I,\dot H_a^{\gamma_w-\frac{\sigma}{2}})} \leq N\Big\}$$

equipped with the distance

$$d(v,w) := \|[v-w]\|_{L^\infty(I,\dot H^{\gamma_w})} + \|v-w\|_{L^p(I,L^p)} + \|v-w\|_{L^a(I,\dot H_a^{\gamma_w-\frac{\sigma}{2}})},$$

where (p,a) is given in (1.20), $I=[0,T]$ and $T,M,N>0$ will be chosen later. Using the Duhamel's formula, it suffices to show that the functional

$$\begin{aligned}\Psi(v)(t) &= \cos(t\Lambda^\sigma)\varphi + \frac{\sin(t\Lambda^\sigma)}{\Lambda^\sigma}\phi - \mu\int_0^t \frac{\sin((t-s)\Lambda^\sigma)}{\Lambda^\sigma}|v(s)|^{\nu-1}v(s)ds \\ &=: v_{\mathrm{hom}}(t) + v_{\mathrm{inh}}(t),\end{aligned}$$

is a contraction on Y, where $v_{\mathrm{hom}}(t)$ is the sum of the first two terms and $v_{\mathrm{inh}}(t)$ is the last term. It is easy to check that under the assumptions (1.18), (p,p) and (a,a) are admissible with $\gamma_{p,p}=\gamma_w$ and $\gamma_{a,a}=\sigma/2$. The Strichartz estimate (2.20) then implies

$$\|v_{\mathrm{hom}}\|_{L^p(I,L^p)} + \|v_{\mathrm{hom}}\|_{L^a(I,\dot H_a^{\gamma_w-\frac{\sigma}{2}})} \lesssim \|[v](0)\|_{\dot H^{\gamma_w}}. \tag{5.4}$$

Thus the left hand side of (5.4) can be taken smaller than ε for some $\varepsilon>0$ small enough provided that either $\|[v](0)\|_{\dot H^{\gamma_w}}$ is small or it is true for some $T>0$ small enough by the dominated convergence. On the other hand, the homogeneous Sobolev embedding with the fact that $\gamma_w-\sigma/2\geq 0$ implies $L^p(I,\dot H_q^{\gamma_w-\frac{\sigma}{2}}) \subset L^p(I,L^p)$ where $d/q = d/p + (\gamma_w-\sigma/2)$. For such q, we see that (p,q) is admissible satisfying

$$\gamma_{p,q} = \frac{\sigma}{2} = \gamma_{a,a} = \gamma_{a',a'} + 2\sigma.$$

The Sobolev embedding and Strichartz estimate (2.20) then yield

$$\|v_{\mathrm{inh}}\|_{L^p(I,L^p)} + \|v_{\mathrm{inh}}\|_{L^a(I,\dot H_a^{\gamma_w-\frac{\sigma}{2}})} \lesssim \|F(v)\|_{L^{a'}(I,\dot H_{a'}^{\gamma_w-\frac{\sigma}{2}})}.$$

Using (1.19) and the fact that $\frac{1}{a'}=\frac{1}{a}+\frac{\nu-1}{p}$, Corollary 23 gives

$$\|F(v)\|_{L^{a'}(I,\dot H_{a'}^{\gamma_w-\frac{\sigma}{2}})} \lesssim \|v\|^{\nu-1}_{L^p(I,L^p)}\|v\|_{L^a(I,\dot H_a^{\gamma_w-\frac{\sigma}{2}})}.$$

Similarly,

$$\|F(v)-F(w)\|_{L^{a'}(I,\dot H_{a'}^{\gamma_w-\frac{\sigma}{2}})}$$

$$\begin{aligned}&\lesssim \Big(\|v\|^{\nu-1}_{L^p(I,L^p)}+\|w\|^{\nu-1}_{L^p(I,L^p)}\Big)\|u-v\|_{L^a(I,\dot H_a^{\gamma_{\rm w}-\frac{\sigma}{2}})}\\&\quad+\Big(\|v\|^{\nu-2}_{L^p(I,L^p)}+\|w\|^{\nu-2}_{L^p(I,L^p)}\Big)\\&\quad\times\Big(\|v\|_{L^a(I,\dot H_a^{\gamma_{\rm w}-\frac{\sigma}{2}})}+\|w\|_{L^a(I,\dot H_a^{\gamma_{\rm w}-\frac{\sigma}{2}})}\Big)\|v-w\|_{L^p(I,L^p)}. \qquad (5.5)\end{aligned}$$

Similarly, by rewriting $\gamma_{\rm w}=\gamma_{\rm w}-\frac{\sigma}{2}+\gamma_{a,a}$, the Strichartz estimate (2.20) also gives

$$\|[\Psi(v)]\|_{L^\infty(I,\dot H^{\gamma_{\rm w}})}\lesssim\|[v](0)\|_{\dot H^{\gamma_{\rm w}}}+\|v\|^{\nu-1}_{L^p(I,L^p)}\|v\|_{L^a(I,\dot H_a^{\gamma_{\rm w}-\frac{\sigma}{2}})}.$$

This implies for all $v,w\in Y$, there exists $C>0$ independent of $(\varphi,\phi)\in\dot H^{\gamma_{\rm w}}\times\dot H^{\gamma_{\rm w}-\sigma}$ such that

$$\begin{aligned}\|\Psi(v)\|_{L^p(I,L^p)}+\|\Psi(v)\|_{L^a(I,\dot H_a^{\gamma_{\rm w}-\frac{\sigma}{2}})}&\le\varepsilon+CN^\nu,\\ \|[\Psi(v)]\|_{L^\infty(I,\dot H^{\gamma_{\rm w}})}&\le C\|[v](0)\|_{\dot H^{\gamma_{\rm w}}}+CN^\nu,\\ d(\Psi(v),\Psi(w))&\le CN^{\nu-1}d(v,w).\end{aligned}$$

Now by setting $N=2\varepsilon$ and $M=2C\|[v](0)\|_{\dot H^{\gamma_{\rm w}}}$ and choosing $\varepsilon>0$ small enough (provided either T is small or $\|[v](0)\|_{\dot H^{\gamma_{\rm w}}}$ is small) such that

$$CN^\nu\le\min\Big\{\varepsilon,C\|[v](0)\|_{\dot H^{\gamma_{\rm w}}}\Big\},\quad CN^{\nu-1}\le\frac{1}{2},$$

we see that Y is stable by Ψ and Ψ is a contraction on Y. By the fixed point theorem, there exists a unique solution $v\in Y$ to (NLFW). Note that when $\|[v](0)\|_{\dot H^{\gamma_{\rm w}}}$ is small enough, we can take $T=\infty$. The uniqueness in $C(I,\dot H^{\gamma_{\rm w}})\cap C^1(I,\dot H^{\gamma_{\rm w}-\sigma})\cap L^p(I,L^p)\cap L^a(I,\dot H_a^{\gamma_{\rm w}-\frac{\sigma}{2}})$ follows as in Theorem 1 by using (5.5). Here $\|v\|_{L^p(I,L^p)}$ and $\|v\|_{L^a(I,\dot H_a^{\gamma_{\rm w}-\frac{\sigma}{2}})}$ can be small as T is small.

We now prove the scattering property of the global solution. Let us denote

$$V(t):=\left[\begin{array}{c}v(t)\\ \partial_t v(t)\end{array}\right],\quad A:=\left(\begin{array}{cc}0&1\\ -\Lambda^{2\sigma}&0\end{array}\right),\quad G(V(t)):=\left[\begin{array}{c}0\\ F(v(t))\end{array}\right].$$

The (NLFW) can be written as

$$\partial_t V(t)-AV(t)=G(V(t)),$$

or

$$V(t)=e^{tA}V(0)+\int_0^t e^{(t-s)A}G(V(s))ds,$$

where

$$e^{tA} := \begin{pmatrix} \cos t\Lambda^\sigma & \frac{\sin t\Lambda^\sigma}{\Lambda^\sigma} \\ -\Lambda^\sigma \sin t\Lambda^\sigma & \cos t\Lambda^\sigma \end{pmatrix}.$$

The adjoint estimates of $e^{\pm it\Lambda^\sigma} : L^a([t_1,t_2],L^a) \to \dot{H}^{\gamma_{a,a}}$ with $\gamma_{a,a} = \sigma/2$ imply

$$\begin{aligned}\Big\| \int_{t_1}^{t_2} e^{\pm is\Lambda^\sigma} F(v(s))ds \Big\|_{\dot{H}^{\gamma_{\rm w}-\sigma}} &= \Big\| \int_{t_1}^{t_2} \Lambda^{-\frac{\sigma}{2}} e^{\pm is\Lambda^\sigma} \Lambda^{\gamma_{\rm w}-\frac{\sigma}{2}} F(v(s))ds \Big\|_{L^2} \\ &\lesssim \|F(v)\|_{L^{a'}([t_1,t_2],\dot{H}^{\gamma_{\rm w}-\frac{\sigma}{2}}_{a'})} \\ &\lesssim \|v\|^{\nu-1}_{L^p([t_1,t_2],L^p)} \|v\|_{L^a([t_1,t_2],\dot{H}^{\gamma_{\rm w}-\frac{\sigma}{2}}_{a})} \to 0\end{aligned}$$

as $t_1, t_2 \to +\infty$. This implies that

$$\|[e^{-t_2A}V(t_2) - e^{-t_1A}V(t_1)]\|_{\dot{H}^{\gamma_{\rm w}}} = \Big\| \Big[\int_{t_1}^{t_2} e^{-sA} G(V(s))ds \Big] \Big\|_{\dot{H}^{\gamma_{\rm w}}} \to 0 \quad (5.6)$$

as $t_1, t_2 \to +\infty$. Therefore, the limit

$$V^+(0) := \lim_{t\to+\infty} e^{-tA}V(t)$$

exists in $\dot{H}^{\gamma_{\rm w}} \times \dot{H}^{\gamma_{\rm w}-\sigma}$. We also have

$$V(t) - e^{tA}V^+(0) = -\int_t^{+\infty} e^{(t-s)A} G(V(s))ds.$$

Using the unitary property of $e^{\pm it\Lambda^\sigma}$ in L^2 and (5.6), we have $\|[V(t) - e^{tA}V^+(0)]\|_{\dot{H}^{\gamma_{\rm w}}} \to 0$ as $t \to +\infty$. This completes the proof of Item 1.

2. We next consider the case (1.21). The proof is similar as above, thus we only give the main steps. We will solve (NLFW) in

$$Y := \Big\{ v \in C(I, \dot{H}^{\gamma_{\rm w}}) \cap C^1(I, \dot{H}^{\gamma_{\rm w}-\sigma}) \cap L^p(I, L^p) \ : \ \|[v]\|_{L^\infty(I,\dot{H}^{\gamma_{\rm w}})} \leq M, \|v\|_{L^p(I,L^p)} \leq N \Big\},$$

equipped with the distance

$$d(v,w) := \|[v-w]\|_{L^\infty(I,\dot{H}^{\gamma_{\rm w}})} + \|v-w\|_{L^p(I,L^p)},$$

where p is as in Item 1. It is easy to check that under the assumption (1.21), (p,p) and (b,b) are admissible and

$$\gamma_{p,p} = \gamma_{\rm w} = \gamma_{b',b'} + 2\sigma,$$

where $b' = p/\nu$. By (2.20), we have $\|v_{\text{hom}}\|_{L^p(I,L^p)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma_{\text{w}}}}$. Therefore, $\|v_{\text{hom}}\|_{L^p(I,L^p)} \leq \varepsilon$ for some $\varepsilon > 0$ small enough provided T is small or $\|[v](0)\|_{\dot{H}^{\gamma_{\text{w}}}}$ is small. Similarly,

$$\|v_{\text{inh}}\|_{L^p(I,L^p)} \lesssim \|F(v)\|_{L^{b'}(I,L^{b'})} \lesssim \|v\|^{\nu}_{L^p(I,L^p)},$$

where the last inequality follows from the Hölder inequality with the fact that

$$\frac{1}{b'} = \frac{1}{p} + \frac{\nu-1}{p}.$$

We also have from (2.20) that

$$\|F(v) - F(w)\|_{L^{b'}(I,L^{b'})} \lesssim \Big(\|v\|^{\nu-1}_{L^p(I,L^p)} + \|w\|^{\nu-1}_{L^p(I,L^p)}\Big)\|v-w\|_{L^p(I,L^p)}. \quad (5.7)$$

This implies for all $v, w \in Y$, there exists $C > 0$ independent of $(\varphi, \phi) \in \dot{H}^{\gamma_{\text{w}}} \times \dot{H}^{\gamma_{\text{w}}-\sigma}$ such that

$$\begin{aligned} \|\Psi(v)\|_{L^p(I,L^p)} &\leq \varepsilon + CN^{\nu}, \\ \|[\Psi(v)]\|_{L^\infty(I,\dot{H}^{\gamma_{\text{w}}})} &\leq C\|[v](0)\|_{\dot{H}^{\gamma_{\text{w}}}} + CN^{\nu}, \\ d(\Psi(v), \Phi(w)) &\leq CN^{\nu-1}d(v,w). \end{aligned}$$

Now by setting $N = 2\varepsilon$ and $M = 2C\|[v](0)\|_{\dot{H}^{\gamma_{\text{w}}}}$ and choosing $\varepsilon > 0$ small enough, we have the existence of solution $v \in Y$ to (NLFW). The uniqueness in $C(I, \dot{H}^{\gamma_{\text{w}}}) \cap C^1(I, \dot{H}^{\gamma_{\text{w}}-\sigma}) \cap L^p(I, L^p)$ follows as in Theorem 1 by using (5.7). Here $\|v\|_{L^p(I,L^p)}$ can be small as T is small.

Using the adjoint Strichartz estimates with the fact that $\gamma_{b,b} = -\gamma_{b',b'} - \sigma = -\gamma_{\text{w}} + \sigma$, we have

$$\begin{aligned} \Big\| \int_{t_1}^{t_2} e^{\pm is\Lambda^\sigma} F(v(s))ds \Big\|_{\dot{H}^{\gamma_{\text{w}}-\sigma}} &= \Big\| \int_{t_1}^{t_2} \Lambda^{\gamma_{\text{w}}-\sigma} e^{\pm is\Lambda^\sigma} F(v(s))ds \Big\|_{L^2} \\ &\lesssim \|F(v)\|_{L^{b'}([t_1,t_2],L^{b'})} \\ &\lesssim \|v\|^{\nu}_{L^p([t_1,t_2],L^p)} \to 0 \end{aligned}$$

as $t_1, t_2 \to +\infty$. This implies

$$\|[e^{-t_2A}V(t_2) - e^{-t_1A}V(t_1)]\|_{\dot{H}^{\gamma_{\text{w}}}} = \Big\|\Big[\int_{t_1}^{t_2} e^{-sA}G(V(s))ds\Big]\Big\|_{\dot{H}^{\gamma_{\text{w}}}} \to 0$$

as $t_1, t_2 \to +\infty$. The same argument as in Item 1 proves the scattering property for the global solution. The proof of Theorem 11 is complete. □

Proof of Theorem 12. The proof is similar to the one of Theorem 11. We thus give a sketch of the proof. We emphasize that here $\nu = 1 + 4\sigma/(d-2\sigma)$ with σ as in (1.22). We will solve (NLFW) in

$$Y := \Big\{ v \in C(I, \dot{H}^\sigma) \cap C^1(I, L^2) \cap L^\nu(I, L^{2\nu}) \ : \\ \|[v]\|_{L^\infty(I,\dot{H}^\sigma)} \leq M, \|v\|_{L^\nu(I,L^{2\nu})} \leq N \Big\}$$

equipped with the distance

$$d(v,w) := \|[v-w]\|_{L^\infty(I,\dot{H}^\sigma)} + \|v-w\|_{L^\nu(I,L^{2\nu})},$$

where $I = [0,T]$ and $M, N > 0$ will be chosen later. It is easy to check that under the assumption (1.22), $(\nu, 2\nu)$ is admissible with $\gamma_{\nu,2\nu} = \sigma = \gamma_{1,2} + 2\sigma$. The Strichartz estimate (2.20) then implies $\|v_{\rm hom}\|_{L^\nu(I,L^{2\nu})} \lesssim \|[v](0)\|_{\dot{H}^\sigma}$. Thus $\|v_{\rm hom}\|_{L^\nu(I,L^{2\nu})} \leq \varepsilon$ for some $\varepsilon > 0$ small enough provided T is small or $\|[v](0)\|_{\dot{H}^\sigma}$ is small. The Strichartz estimate (2.20) also gives

$$\|v_{\rm inh}\|_{L^\nu(I,L^{2\nu})} \lesssim \|F(v)\|_{L^1(I,L^2)} = \|v\|^\nu_{L^\nu(I,L^{2\nu})}.$$

Similarly,

$$\|F(v)-F(w)\|_{L^1(I,L^2)} \\ \lesssim \Big(\|v\|^{\nu-1}_{L^\nu(I,L^{2\nu})} + \|w\|^{\nu-1}_{L^\nu(I,L^{2\nu})} \Big) \|v-w\|_{L^\nu(I,L^{2\nu})}. \quad (5.8)$$

Thus for all $v, w \in Y$, there exists $C > 0$ independent of $(\varphi, \phi) \in \dot{H}^\sigma \times L^2$ such that

$$\begin{aligned} \|\Psi(v)\|_{L^\nu(I,L^{2\nu})} &\leq \varepsilon + CN^\nu, \\ \|[\Psi(v)]\|_{L^\infty(I,\dot{H}^\sigma)} &\leq C\|[v](0)\|_{\dot{H}^\sigma} + CN^\nu, \\ d(\Psi(v), \Phi(w)) &\leq CN^{\nu-1} d(v,w). \end{aligned}$$

Now by setting $N = 2\varepsilon$ and $M = 2C\|[v](0)\|_{\dot{H}^\sigma}$ and choosing $\varepsilon > 0$ small enough, we have the existence of solution $v \in Y$ to (NLFW). The uniqueness in $C(I, \dot{H}^\sigma) \cap C^1(I, L^2) \cap L^\nu(I, L^{2\nu})$ follows as in Theorem 1 by using (5.8). Here $\|v\|_{L^\nu(I,L^{2\nu})}$ can be small as T is small.

The scattering property is very similar as in the proof of Theorem 11. We have

$$\Big\| \int_{t_1}^{t_2} e^{\pm is\Lambda^\sigma} F(v(s))ds \Big\|_{L^2} \leq \|F(v)\|_{L^1([t_1,t_2],L^2)} = \|v\|^\nu_{L^\nu([t_1,t_2],L^{2\nu})} \to 0$$

as $t_1, t_2 \to +\infty$. This implies

$$\|[e^{-t_2 A} V(t_2) - e^{-t_1 A} V(t_1)]\|_{\dot{H}^\sigma} \to 0$$

as $t_1, t_2 \to +\infty$. This completes the proof. □

Acknowledgments

The author would like to express his deep thanks to his wife Uyen Cong for her encouragement and support. He would like to thank his supervisor Prof. Jean-Marc Bouclet for the kind guidance and constant encouragement. He also would like to thank the reviewer for his/her helpful comments and suggestions.

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