

## WELL-POSEDNESS OF SYSTEMS OF LINEAR ELASTICITY WITH DIRICHLET BOUNDARY CONTROL AND OBSERVATION\*

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**Abstract.** An open-loop system of linear elasticity with Dirichlet boundary control and collocated observation is considered. The main result we obtained states that this system is well-posed in the sense of Salamon. The result deduces the exponential stability of the closed-loop system under proportional output feedback. Hence it answers positively an open question proposed by Liu and Krstić [*IMA J. Appl. Math.*, 65 (2000), pp. 109–121]. Moreover, the well-posedness, together with the regularity of the open-loop system in the sense of Weiss that was obtained in our separate paper [S. G. Chai and B. Z. Guo, *Feedthrough Operator for Linear Elasticity System with Boundary Control and Observation*, Preprint, School of Computational and Applied Mathematics, University of the Witwatersrand, South Africa, 2008], makes this infinite-dimensional system parallel in many ways to a linear finite-dimensional system in the framework of well-posed and regular linear infinite-dimensional systems.

**Key words.** linear elasticity, transfer function, well-posed and regular system, boundary control and observation

**AMS subject classifications.** 35J50, 93C20, 93C25

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**1. Introduction and main results.** Verifying the well-posedness and regularity of the control systems described by partial differential equations (PDEs) is important, since, once these properties are established for an infinite-dimensional linear system, the system can be treated in many ways that parallel those for finite-dimensional ones. Extensive studies have been done in this direction over the last two decades. However, only a limited number of PDEs are proved to be well-posed and regular up to the present (see, e.g., [3, 5, 9, 11, 16, 17, 18, 19, 20, 21, 32, 33]). Those include the PDEs with control and observation possibly imposed on the subregions or boundaries.

In this paper, we study the well-posedness and regularity of a system of linear elasticity. Linear elasticity models the macroscopic mechanical properties of solids with “small” deformations. A key part of the elasticity theory consists of describing the deformation of solids under external forces. There are already many contributions to the controllability, observability, and stabilizability of the linear elastodynamic systems (see, e.g., [2, 7, 8, 24, 26, 27, 36, 37]). However, in contrast to Neumann boundary feedbacks, Dirichlet boundary feedbacks had not been considered in the literature until the strong stability was obtained in [39]. It was also indicated in Remark 3.2 of [39] that the exponential stability of systems of linear elasticity under Dirichlet boundary feedbacks is still an open problem, although this property has

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been known for the scalar wave equation [35]. The well-posedness confirmed in our paper gives an affirmative answer to the problem.

Let  $\Omega \subset \mathbb{R}^n (n \geq 2)$  be a bounded open region with a boundary  $\partial\Omega =: \Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$  of class  $C^2$ .  $\Gamma_0, \Gamma_1$  are disjoint parts of the boundary relatively open in  $\partial\Omega$  with  $\text{int}(\Gamma_0) \neq \emptyset$ . Let  $u(x, t) = (u_1(x, t), \dots, u_n(x, t))$  be the displacement vector at the position  $x \in \Omega$  and time  $t \in \mathbb{R}$ . The strain tensor  $\varepsilon(u) = (\varepsilon_{ij}(u))$  is defined by

$$\varepsilon_{ij}(u) := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad 1 \leq i, j \leq n.$$

The stress tensor  $\sigma(u) = (\sigma_{ij}(u))$  is given by

$$\sigma_{ij}(u) := \lambda \sum_{k=1}^n \varepsilon_{kk}(u) \delta_{ij} + 2\mu \varepsilon_{ij}(u) = \lambda \text{div}(u) \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad 1 \leq i, j \leq n,$$

where  $\delta_{ij}$  is the Kronecker delta, i.e.,  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise;  $\lambda$  and  $\mu$  are the Lamé constants satisfying

$$\mu > 0, \quad n\lambda + (n+1)\mu > 0.$$

We consider the following system of isotropic linear elasticity:

$$(1.1) \quad \begin{cases} u'' - \nabla \cdot \sigma(u) = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ u = g & \text{on } \Gamma_0 \times (0, \infty), \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ y = -\sigma(\mathcal{A}^{-1}u')\nu & \text{on } \Gamma_0 \times (0, \infty), \end{cases}$$

where  $u' := \frac{\partial u}{\partial t}$ ,  $u'' := \frac{\partial^2 u}{\partial t^2}$ ,  $\nu = (\nu_1, \dots, \nu_n)$  is the unit normal vector to  $\partial\Omega$  pointing towards the exterior of  $\Omega$ ,  $g = (g_1, \dots, g_n)$  is the vector-valued input (or control) function,  $y = (y_1, \dots, y_n)$  is the vector-valued output (or observation) function, and

$$(1.2) \quad \mathcal{A}u := -\nabla \cdot \sigma(u) \quad \forall u \in D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega))^n.$$

The system (1.1) can be written more explicitly as

$$(1.3) \quad \begin{cases} u'' - \mu \Delta u - (\lambda + \mu) \nabla \text{div}(u) = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ u = g & \text{on } \Gamma_0 \times (0, \infty), \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ y = -\mu \frac{\partial \mathcal{A}^{-1}u'}{\partial \nu} - (\lambda + \mu) \nu \text{div}(\mathcal{A}^{-1}u') & \text{on } \Gamma_0 \times (0, \infty), \end{cases}$$

where in the last line we used the equality

$$\sigma(w) \cdot \nu = \mu \frac{\partial w}{\partial \nu} + (\lambda + \mu) \text{div}(w) \quad \text{when } w = 0 \quad \text{on } \Gamma \times (0, \infty).$$

A rigorous explanation for the above equality, from a geometric point of view, is available in [10].

We consider the system (1.1) (resp., (1.3)) in the state space  $\mathcal{H} = (L^2(\Omega))^n \times (H^{-1}(\Omega))^n$  and the control and observation space  $U = (L^2(\Gamma_0))^n$ . Our main result in this paper is Theorem 1.1.

**THEOREM 1.1.** *Let  $T > 0$ ,  $(u^0, u^1) \in \mathcal{H}$ , and  $g \in L^2(0, T; U)$ . There exists a unique solution  $(u, u') \in C([0, T]; \mathcal{H})$  to the system (1.1). Moreover, there exists a constant  $C_T > 0$ , independent of  $(u^0, u^1, g)$ , such that*

$$\|(u(T), u'(T))\|_{\mathcal{H}}^2 + \|y\|_{L^2(0, T; U)}^2 \leq C_T \left[ \|(u^0, u^1)\|_{\mathcal{H}}^2 + \|g\|_{L^2(0, T; U)}^2 \right].$$

In the case of scalar wave equations, the corresponding interior regularity for  $\{(u(T), u'(T))\}$  in  $L^2(\Omega) \times H^{-1}(\Omega)$  was given in [28]; instead, the corresponding regularity of the boundary observation  $y$  in  $L^2(0, T; L^2(\Gamma_0))$  in the case of second order hyperbolic equations was asserted in [3]. A different and complete proof for  $y \in L^2(0, T; L^2(\Gamma_0))$  in the case of second order hyperbolic equations with constant coefficients was given in [33]. This same proof has been applied to the wave equation with variable coefficients in [20] for completeness. In the present paper dealing with the system of elasticity (1.1) or (1.3), our proof in section 4 for  $y \in L^2(0, T; U)$  will follow very closely the strategy of [33], complemented by the analysis of [24] and [25].

In the case of the system of elasticity, the interior regularity of  $\{(u(T), u'(T))\}$  in  $\mathcal{H}$  was given in Chapter IV of [37] in a dual form. This interior regularity was also contained in Theorem 1 of [7] on page 148 for a more general Lamé system.

Theorem 1.1 implies that the open-loop system (1.1) is well-posed in the state space  $\mathcal{H}$  and the input and output space  $U$  in the sense of Salamon (see our paper [19]). To obtain the exact controllability of system (1.1), we need to suppose that there exists a fixed point  $x_0 \in \mathbb{R}^n$  such that

$$(1.4) \quad (x - x_0) \cdot \nu \leq 0 \quad \forall x \in \Gamma_1.$$

This means that  $\Gamma_0$  contains all the points  $x \in \Gamma$  for which  $(x - x_0) \cdot \nu > 0$ . Set

$$R := \sup\{|x - x_0| \mid x \in \Omega\}.$$

The following result, which comes from Theorem 1.1 of [37] (see also Theorem 1.2 of [2]), shows that the system (1.1) is exactly controllable.

**THEOREM 1.2.** *Assume (1.4) and let  $T > T_0$ , where  $T_0$  is some positive constant depending only on  $\lambda, \mu$ , and  $R$ . Then for any given  $(u^0, u^1) \in \mathcal{H}$  and  $(\bar{u}^0, \bar{u}^1) \in \mathcal{H}$ , there exists  $g \in L^2_{loc}(\mathbb{R}; U)$  such that the solution to system (1.1) satisfies*

$$u(T) = \bar{u}^0 \text{ and } u'(T) = \bar{u}^1 \text{ in } \Omega.$$

*Remark 1.1.* It is trivial to see that (1.4) is satisfied when  $\Gamma_0 = \Gamma$  and  $\Gamma_1 = \emptyset$ .

By virtue of our Theorem 1.1 and Theorem 2.2 of [4] (see also Theorem 3 of [15]), the system (1.1) is exactly controllable in some time interval  $[0, T]$  if and only if its closed-loop system under the proportional output feedback  $g = -ky, k > 0$ , is exponentially stable. Set

$$E(t) = \frac{1}{2} \|(u(t), u'(t))\|_{\mathcal{H}}^2 = \frac{1}{2} \left[ \|u(t)\|_{(L^2(\Omega))^n}^2 + \|\mathcal{A}^{-1/2} u'(t)\|_{(L^2(\Omega))^n}^2 \right].$$

We then have the following corollary of Theorems 1.1 and 1.2.

COROLLARY 1.1. *The system (1.1) is exponentially stable under the proportional output feedback  $g = -ky, k > 0$ , that is to say, there exist constants  $M \geq 1$  and  $\delta > 0$  such that*

$$E(t) \leq Me^{-\delta t} E(0)$$

for all  $t > 0$ .

Corollary 1.1 answers affirmatively the question on the exponential stability proposed by Liu and Krstić in [39]. In the same paper, the strong stability for system (1.1) under the output feedback  $g = -y$  was proved by using the Nagy–Foiás–Foguel theory of decomposition of continuous semigroups of contractions.

Very recently, it was shown in [10] that the open-loop system (1.1) is regular in the state space  $\mathcal{H}$  and the input and output space  $U$  in the sense of Weiss (see our paper [19]). The analytic expression of the feedthrough operator  $D$  was also found in [10].

*Remark 1.2.* Set  $\mu = 1$  and  $\lambda = -1$ . Then the system (1.1) reduces to a system of  $n$  number of uncoupled scalar wave equations. Theorem 1.1 and regularity claimed in [10] ensure that this uncoupled system is well-posed and regular. We could thereby say that our results generalize the corresponding results of [3], [33], and [19] to the case of systems of linear elasticity.

The remainder of the paper is organized as follows. In section 2, we cast the system (1.1) into an abstract setting. In section 3, some preliminary results on the nonhomogeneous boundary value problems for systems of linear elasticity are developed. The proof of Theorem 1.1 is presented in section 4.

**2. Collocated formulation.** Let  $H = (H^{-1}(\Omega))^n$  be the dual space of the Sobolev space  $(H_0^1(\Omega))^n$  with the usual inner product. Let  $A$  be the positive self-adjoint operator in  $H$  induced by the bilinear form  $a(\cdot, \cdot)$  defined by

$$\begin{aligned} (2.1) \quad \langle Af_1, f_2 \rangle_{(H^{-1}(\Omega))^n \times (H_0^1(\Omega))^n} &= a(f_1, f_2) \\ &= \int_{\Omega} [\mu \nabla f_1 \cdot \nabla f_2 + (\lambda + \mu) \operatorname{div}(f_1) \cdot \operatorname{div}(f_2)] dx \\ &\quad \forall f_1, f_2 \in (H_0^1(\Omega))^n. \end{aligned}$$

By means of the Lax–Milgram theorem,  $A$  is a canonical isomorphism from  $D(A) = (H_0^1(\Omega))^n$  onto  $H$ . It is easy to show that  $Af = \mathcal{A}f$  whenever  $f \in (H^2(\Omega) \cap H_0^1(\Omega))^n$  and that  $A^{-1}f = \mathcal{A}^{-1}f$  for any  $f \in (L^2(\Omega))^n$ . Hence  $A$  is an extension of  $\mathcal{A}$  to the space  $(H_0^1(\Omega))^n$ .

As in [17], it can be easily shown that  $D(A^{1/2}) = (L^2(\Omega))^n$  and  $A^{1/2}$  is an isomorphism from  $(L^2(\Omega))^n$  onto  $H$ . Define the Dirichlet map  $\Upsilon$  by  $\Upsilon g = v$ , where  $v$  satisfies

$$\begin{cases} \mu \Delta v + (\lambda + \mu) \nabla \operatorname{div}(v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, v = g & \text{on } \Gamma_0. \end{cases}$$

It is well known from the elliptic theory (see, e.g., Theorems 10.1.1 and 10.1.2 in Chapter 10 of [41]) that  $\Upsilon \in \mathcal{L}((H^s(\Gamma_0))^n, (H^{s+1/2}(\Omega))^n)$  for all  $s \in \mathbb{R}$ . In particular,  $\Upsilon \in \mathcal{L}((L^2(\Gamma_0))^n, (H^{1/2}(\Omega))^n)$ . In terms of  $\Upsilon$ , one can write the system (1.1) as

$$(2.2) \quad u'' + A(u - \Upsilon g) = 0.$$

Since  $D(A)$  is dense in  $H$ , so is  $D(A^{1/2})$ . We now identify  $H$  with its dual  $H'$ . The following relations hold:

$$D(A^{1/2}) \hookrightarrow H = H' \hookrightarrow D(A^{1/2})'.$$

An extension  $\tilde{A} \in \mathcal{L}(D(A^{1/2}), D(A^{1/2})')$  of  $A$  is defined by

$$\langle \tilde{A}f_1, f_2 \rangle_{D(A^{1/2})' \times D(A^{1/2})} = \langle A^{1/2}f_1, A^{1/2}f_2 \rangle_H \quad \forall f_1, f_2 \in D(A^{1/2}).$$

As a result, (2.2) can be further written in  $D(A^{1/2})'$  as

$$u'' + \tilde{A}u + Bg = 0,$$

where  $B \in \mathcal{L}(U, D(A^{1/2})')$  is given by

$$(2.3) \quad Bg = -\tilde{A}\Upsilon g \quad \forall g \in U.$$

Define  $B^* \in \mathcal{L}(D(A^{1/2}), U)$  by

$$\langle B^*f, g \rangle_U = \langle f, Bg \rangle_{D(A^{1/2}) \times D(A^{1/2})'} \quad \forall f \in D(A^{1/2}), g \in U.$$

Then for any  $f \in D(A^{1/2})$  and  $g \in (C_0^\infty(\Gamma_0))^n$ , we have

$$\begin{aligned} \langle f, Bg \rangle_{D(A^{1/2}) \times D(A^{1/2})'} &= \langle f, \tilde{A}\tilde{A}^{-1}Bg \rangle_{D(A^{1/2}) \times D(A^{1/2})'} = \langle A^{1/2}f, A^{1/2}\tilde{A}^{-1}Bg \rangle_H \\ &= -\langle A^{1/2}f, A^{1/2}\Upsilon g \rangle_H = -\langle f, \Upsilon g \rangle_{(L^2(\Omega))^n} \\ &= -\langle \mathcal{A}\mathcal{A}^{-1}f, \Upsilon g \rangle_{(L^2(\Omega))^n} = -\langle \Upsilon^* \mathcal{A}\mathcal{A}^{-1}f, g \rangle_U \\ &= \left\langle \mu \frac{\partial \mathcal{A}^{-1}f}{\partial \nu} + (\lambda + \mu)\nu \operatorname{div}(\mathcal{A}^{-1}f), g \right\rangle_U. \end{aligned}$$

In the last step above, we used the fact (see formula (3.3) of [39])

$$\Upsilon^* \mathcal{A}\varphi = -\mu \frac{\partial \varphi}{\partial \nu} - (\lambda + \mu)\nu \operatorname{div}(\varphi) \text{ on } \Gamma \quad \forall \varphi \in D(A).$$

Since  $(C_0^\infty(\Gamma_0))^n$  is dense in  $U = (L^2(\Gamma_0))^n$ , we finally obtain that

$$(2.4) \quad B^* = \left( \mu \frac{\partial \mathcal{A}^{-1}}{\partial \nu} + (\lambda + \mu)\nu \operatorname{div}(\mathcal{A}^{-1}) \right) \Big|_{\Gamma_0}.$$

We thus have formulated the open-loop system (1.1) into an abstract form of a second order system in the state space  $\mathcal{H} = (L^2(\Omega))^n \times (H^{-1}(\Omega))^n$  and the control and observation space  $U = (L^2(\Gamma))^n$ :

$$(2.5) \quad \begin{cases} u''(t) + \tilde{A}u(t) + Bg(t) = 0, \\ y(t) = -B^*u'(t), \end{cases}$$

where  $B$  and  $B^*$  are defined by (2.3) and (2.4), respectively. The abstract system (2.5) has been studied in detail in [4] and [15] (see also [35] and Chapter 7 of [31]).

### 3. Preliminary results on nonhomogeneous boundary value problems.

In this section, we give some results on the nonhomogeneous boundary value problems for systems of linear elasticity. These results have appeared in the literature: see [7] and Chapter IV of [37]. These results parallel those for the scalar wave equation in [28]. We shall still give the proofs in detail for the sake of completeness and also because they are fundamental for the present paper. Our proofs follow those of [28].

In what follows, denote  $Q := \Omega \times (0, T)$ ,  $\Sigma := \Gamma \times (0, T)$ ,  $L^2(\Sigma) := H^0(\Sigma) = L^2(0, T; L^2(\Gamma))$ ,  $H^1(\Sigma) := L^2(0, T; H^1(\Gamma)) \cap H^1(0, T; L^2(\Gamma))$ ,  $H^{-1}(\Sigma) := (H_0^1(\Sigma))'$ , and  $(L^2(\Sigma))^n := L^2(0, T; (L^2(\Gamma))^n)$ ,  $(H^1(\Sigma))^n := L^2(0, T; (H^1(\Gamma))^n) \cap H^1(0, T; (L^2(\Gamma))^n)$ ,  $(H^{-1}(\Sigma))^n := ((H_0^1(\Sigma))^n)'$ .

Extending the value of  $g$  to be zero on  $\Gamma_1$ , we may assume, without loss of generality, that  $\Gamma_0 = \Gamma = \partial\Omega$  and  $g$  is defined on the whole boundary of  $\Omega$ . Consider the following system in the finite time interval  $[0, T]$ :

$$(3.1) \quad \begin{cases} u'' - \nabla \cdot \sigma(u) = \varphi & \text{in } Q, \\ u = g & \text{on } \Sigma, \\ u(0) = u^0, u'(0) = u^1 & \text{in } \Omega. \end{cases}$$

It is seen that when  $\varphi = 0$  and  $T = +\infty$ , the above system becomes system (1.1) without the last equation.

PROPOSITION 3.1. *For any given  $T > 0$ , suppose that*

$$\begin{cases} \varphi \in L^1(0, T; (H^{-1}(\Omega))^n), \\ g \in L^2(0, T; (L^2(\Gamma))^n), \\ u^0 \in (L^2(\Omega))^n, u^1 \in (H^{-1}(\Omega))^n. \end{cases}$$

*Then there exists a unique solution  $u$  to the system (3.1), which satisfies*

$$(3.2) \quad (u, u') \in C(0, T; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n)$$

*and*

$$(3.3) \quad \sigma(u)\nu \in (H^{-1}(\Sigma))^n.$$

*Remark 3.1.* With the same reason presented after Remark 2.5 of [28], it suffices to show  $(u, u') \in L^\infty(0, T; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n)$  for the proof of Proposition 3.1.

*Remark 3.2.* In Proposition 3.1, one can show that  $u$ ,  $u'$ , and  $\sigma(u)\nu$  depend continuously on the related given datum. Similar remarks apply to all regularity results in what follows.

*Remark 3.3.* Let  $\varphi = g = 0$  in Proposition 3.1. Then formula (3.2) shows that (3.1) associates with a  $C_0$ -semigroup on  $(L^2(\Omega))^n \times (H^{-1}(\Omega))^n$ , that is,

$$(u(t), u'(t)) = e^{Ct}(u(0), u'(0)),$$

where  $e^{Ct}$  is a  $C_0$ -semigroup on  $(L^2(\Omega))^n \times (H^{-1}(\Omega))^n$ .

In order to prove Proposition 3.1, we need several preliminary lemmas. First, we consider the following dual system of (3.1):

$$(3.4) \quad \begin{cases} w'' - \nabla \cdot \sigma(w) = \psi & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0) = 0, w'(0) = 0 & \text{in } \Omega. \end{cases}$$

The following result is in Chapter IV of [37] and Proposition 1 of [7] on page 149 for a more general Lamé system.

LEMMA 3.1. *Suppose that  $\psi \in L^1(0, T; (L^2(\Omega))^n)$  for some  $T > 0$ . Then there exists a unique solution  $w$  to (3.4), which satisfies*

$$(3.5) \quad (w, w') \in L^\infty(0, T; (H^1(\Omega))^n \times (L^2(\Omega))^n)$$

and

$$(3.6) \quad \sigma(w)\nu \in (L^2(\Sigma))^n.$$

*Proof.* From [12], we know that the system (3.4) admits a unique solution satisfying (3.5). Now we need to show only (3.6). First, since  $\Gamma$  is of class  $C^2$ , we can always take a vector field  $h : \bar{\Omega} \rightarrow \mathbb{R}^n$  of class  $C^1$  such that  $h = \nu$  on  $\Gamma$  (see, e.g., Lemma 4.1 of [20]).

For the sake of convenience, we rewrite, in the proof below, the system (3.4) as follows:

$$(3.7) \quad \begin{cases} w_i'' - \sum_j \sigma_{ij,j}(w) = \psi_i & \text{in } Q, \\ w_i = 0 & \text{on } \Sigma, \\ w_i(0) = 0, w_i'(0) = 0 & \text{in } \Omega, \end{cases} \quad i = 1, \dots, n,$$

where  $\sigma_{ij,j}(w) := \frac{\partial \sigma_{ij}(w)}{\partial x_j}$ . Here and in what follows for notational simplicity we omit the range of index of summation.

Denote  $w_{i,m} := \frac{\partial w_i}{\partial x_m}$ ,  $w_{i,jm} := \frac{\partial^2 w_i}{\partial x_j \partial x_m}$ ,  $(w'_i w'_i)_{,m} := \frac{\partial (w'_i w'_i)}{\partial x_m}$ . Multiply both sides of the  $i$ th equation of (3.7) by  $\sum_m h_m w_{i,m}$  and integrate over  $Q = \Omega \times (0, T)$  by parts to get

$$\begin{aligned} \int_0^T \int_\Omega \psi_i \sum_m h_m w_{i,m} dx dt &= \int_0^T \int_\Omega \left( w_i'' - \sum_j \sigma_{ij,j}(w) \right) \sum_m h_m w_{i,m} dx dt \\ &= \int_\Omega \sum_m h_m w_{i,m} w'_i dx \Big|_0^T - \int_0^T \int_\Gamma \sum_m h_m w_{i,m} \sum_j \sigma_{ij}(w) \nu_j d\Gamma dt \\ &\quad + \int_0^T \int_\Omega \left( \sum_{j,m} h_{m,j} \sigma_{ij}(w) w_{i,m} + \sum_{j,m} h_m \sigma_{ij}(w) w_{i,jm} - \frac{1}{2} \sum_m h_m (w'_i w'_i)_{,m} \right) dx dt. \end{aligned}$$

First, since

$$\sum_{i,j} \sigma_{ij}(w) w_{i,jm} = \sum_{i,j} \sigma_{ij}(w) \varepsilon_{ij,m}(w) = \frac{1}{2} \sum_{i,j} (\sigma_{ij}(w) \varepsilon_{ij}(w))_{,m},$$

it follows that

$$\begin{aligned}
(3.8) \quad & \int_0^T \int_{\Gamma} \left[ 2 \sum_{i,j,m} h_m w_{i,m} \sigma_{ij}(w) \nu_j + (h \cdot \nu) \sum_i \left( w'_i w'_i - \sum_j \sigma_{ij}(w) \varepsilon_{ij}(w) \right) \right] d\Gamma dt \\
&= \int_{\Omega} 2 \sum_{i,m} h_m w_{i,m} w'_i dx \Big|_0^T + \int_0^T \int_{\Omega} \left[ 2 \sum_{i,j,m} h_{m,j} \sigma_{ij}(w) w_{i,m} \right. \\
&\quad \left. - 2 \sum_{i,m} \psi_i h_m w_{i,m} + \operatorname{div}(h) \sum_i \left( w'_i w'_i - \sum_j \sigma_{ij}(w) \varepsilon_{ij}(w) \right) \right] dx dt.
\end{aligned}$$

Next, since  $w = 0$  on  $\Sigma$  and  $h = \nu$  on  $\Gamma$ , we have

$$w'_i = 0, \quad w_{i,m} \nu_j = w_{i,\nu} \nu_m \nu_j = w_{i,j} \nu_m \quad \text{on } \Gamma,$$

and hence

$$\sum_{i,j,m} h_m w_{i,m} \sigma_{ij}(w) \nu_j = (h \cdot \nu) \sum_{i,j} \sigma_{ij}(w) w_{i,j} = (h \cdot \nu) \sum_{i,j} \sigma_{ij}(w) \varepsilon_{ij}(w) = \sum_{i,j} \sigma_{ij}(w) \varepsilon_{ij}(w).$$

The left-hand side of (3.8) is then reduced to

$$\int_0^T \int_{\Gamma} \sum_{i,j} \sigma_{ij}(w) \varepsilon_{ij}(w) d\Gamma dt.$$

Now, by Lemma 2.1 of [2],

$$\frac{\alpha}{2} \sum_{i,j} \sigma_{ij}(w) \varepsilon_{ij}(w) \leq \sum_i \left( \sum_j \sigma_{ij}(w) \nu_j \right)^2 \leq \frac{\beta}{\alpha} \sum_{i,j} \sigma_{ij}(w) \varepsilon_{ij}(w),$$

where  $\alpha$  and  $\beta$  are two positive constants depending only on  $\lambda$  and  $\mu$ . Hence

$$\int_0^T \int_{\Gamma} \sum_i \left( \sum_j \sigma_{ij}(w) \nu_j \right)^2 d\Gamma dt \leq \frac{\beta}{\alpha} \int_0^T \int_{\Gamma} \sum_{i,j} \sigma_{ij}(w) \varepsilon_{ij}(w) d\Gamma dt = \frac{\beta}{\alpha} \text{RHS of (3.8)}.$$

Finally, from (3.5), we know that  $w$  and  $w'$  depend continuously on  $\psi$ , and we thus conclude from (3.8) that

$$\int_0^T \int_{\Gamma} \sum_i \left( \sum_j \sigma_{ij}(w) \nu_j \right)^2 d\Gamma dt \leq C \|\psi\|_{L^1(0,T;L^2(\Omega)^n)}^2$$

for some positive constant  $C$ . This shows the validity of (3.6).  $\square$

The first part of Proposition 3.1 is Lemma 3.2 following. When  $\varphi = 0$  in  $Q$  and  $u^0 = u^1 = 0$  in  $\Omega$ , Lemma 3.2 was already shown as Theorem 1 of [7] on page 148 for the more general system. Our proof is slightly different from that in [7] by using the “lifting theorem” in [29].



LEMMA 3.2. Equation (3.2) in Proposition 3.1 holds true. That is, under the conditions of Proposition 3.1, we have

$$(u, u') \in C(0, T; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

*Proof.* We first prove the result by taking  $\varphi = 0$ . This will be split into several steps.

*Step 1.* Let  $u$  be the solution to (3.1) and let  $w$  be the solution to the following system:

$$\begin{cases} w_i'' - \sum_j \sigma_{ij,j}(w) = \psi_i & \text{in } Q, \\ w_i = 0 & \text{on } \Sigma, \\ w_i(T) = 0, w_i'(T) = 0 & \text{in } \Omega. \end{cases} \quad i = 1, \dots, n,$$

Assume that all data are smooth. We claim that

$$(3.9) \quad \langle \varphi, w \rangle_Q + \langle \sigma(u)\nu, w \rangle_\Sigma = -\langle u^1, w(0) \rangle_\Omega + \langle u^0, w'(0) \rangle_\Omega + \langle u, \psi \rangle_Q + \langle u, \sigma(w)\nu \rangle_\Sigma.$$

In fact,

$$\begin{aligned} \langle \varphi, w \rangle_Q + \langle \sigma(u)\nu, w \rangle_\Sigma &= \sum_i \langle \varphi_i, w_i \rangle_Q + \sum_i \left\langle \sum_j \sigma_{ij}(u)\nu_j, w_i \right\rangle_\Sigma \\ &= \sum_i \left\langle u_i'' - \sum_j \sigma_{ij,j}(u), w_i \right\rangle_Q + \sum_i \left\langle \sum_j \sigma_{ij}(u)\nu_j, w_i \right\rangle_\Sigma \\ &= \int_0^T \int_\Omega \sum_i \left( u_i'' - \sum_j \sigma_{ij,j}(u) \right) w_i dx dt + \int_0^T \int_\Gamma \sum_{i,j} \sigma_{ij}(u)\nu_j w_i d\Gamma dt \\ &= \int_\Omega \sum_i u_i' w_i dx \Big|_0^T - \int_\Omega \sum_i u_i w_i' dx \Big|_0^T + \int_0^T \int_\Omega \sum_i u_i w_i'' dx dt \\ &\quad - \int_0^T \int_\Gamma \sum_{i,j} w_i \sigma_{ij}(u)\nu_j d\Gamma dt + \int_0^T \int_\Omega \sum_{i,j} \sigma_{ij}(u)\varepsilon_{ij}(w) dx dt \\ &\quad + \int_0^T \int_\Gamma \sum_{i,j} w_i \sigma_{ij}(u)\nu_j d\Gamma dt \\ &= - \int_\Omega \sum_i u_i'(0) w_i(0) dx + \int_\Omega \sum_i u_i(0) w_i'(0) dx + \int_0^T \int_\Omega \sum_i u_i w_i'' dx dt \\ &\quad + \int_0^T \int_\Omega \sum_{i,j} \sigma_{ij}(w)\varepsilon_{ij}(u) dx dt \\ &= - \int_\Omega \sum_i u_i'(0) w_i(0) dx + \int_\Omega \sum_i u_i(0) w_i'(0) dx + \int_0^T \int_\Omega \sum_i u_i w_i'' dx dt \end{aligned}$$

$$\begin{aligned}
 & + \int_0^T \int_{\Gamma} \sum_{i,j} u_i \sigma_{ij}(w) \nu_j d\Gamma dt - \int_0^T \int_{\Omega} \sum_{i,j} u_i \sigma_{ij,j}(w) dx dt \\
 & = - \int_{\Omega} \sum_i u'_i(0) w_i(0) dx + \int_{\Omega} \sum_i u_i(0) w'_i(0) dx \\
 & + \int_0^T \int_{\Omega} \sum_i u_i \left( w''_i - \sum_j \sigma_{ij,j}(w) \right) dx dt + \int_0^T \int_{\Gamma} \sum_{i,j} u_i \sigma_{ij}(w) \nu_j d\Gamma dt \\
 & = - \sum_i \langle u'_i(0), w_i(0) \rangle_{\Omega} + \sum_i \langle u_i(0), w'_i(0) \rangle_{\Omega} \\
 & + \sum_i \langle u_i, \psi_i \rangle_Q + \sum_i \left\langle u_i, \sum_j \sigma_{ij}(w) \nu_j \right\rangle_{\Sigma} \\
 & = - \langle u^1, w(0) \rangle_{\Omega} + \langle u^0, w'(0) \rangle_{\Omega} + \langle u, \psi \rangle_Q + \langle u, \sigma(w) \nu \rangle_{\Sigma}.
 \end{aligned}$$

The claim (3.9) is proved.

Since  $\varphi = 0$  in  $Q$  and  $w = 0$  on  $\Sigma$ , equality (3.9) yields

$$(3.10) \quad 0 = - \langle u^1, w(0) \rangle_{\Omega} + \langle u^0, w'(0) \rangle_{\Omega} + \langle u, \psi \rangle_Q + \langle u, \sigma(w) \nu \rangle_{\Sigma}.$$

*Step 2.* By virtue of Lemma 3.1 and its assumptions, the map

$$\psi \rightarrow \langle u^1, w(0) \rangle_{\Omega} - \langle u^0, w'(0) \rangle_{\Omega} - \langle u, \sigma(w) \nu \rangle_{\Sigma}$$

is continuous on  $L^1(0, T; (L^2(\Omega))^n)$ . From (3.10), the right-hand side of the above expression equals  $\langle u, \psi \rangle_Q$ . This implies that  $u$  belongs to the dual space of  $L^1(0, T; (L^2(\Omega))^n)$ :

$$u \in L^\infty(0, T; (L^2(\Omega))^n).$$

*Step 3.* From the final result in Step 2, we have

$$u'' = \nabla \cdot \sigma(u) \in L^\infty(0, T; (H^{-2}(\Omega))^n).$$

By Theorems 2.3 and 12.4 in Chapter 1 of [38], we get

$$u' \in L^2(0, T; [(L^2(\Omega))^n, (H^{-2}(\Omega))^n]_{1/2}) = L^2(0, T; (H^{-1}(\Omega))^n).$$

Furthermore, by appealing to the “lifting theorem” in [29] (see also Theorem 7.3.1 of [31]) we arrive at

$$u' \in C(0, T; (H^{-1}(\Omega))^n) \subset L^\infty(0, T; (H^{-1}(\Omega))^n).$$

Thus Lemma 3.2 is valid when  $\varphi = 0$ .

The proof will be complete if we can show the same conclusion for any  $\varphi \in L^1(0, T; (H^{-1}(\Omega))^n)$ . Let  $v$  be the solution to the following system:

$$\begin{cases} v'' - \nabla \cdot \sigma(v) = \varphi & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(0) = 0, v'(0) = 0 & \text{in } \Omega. \end{cases}$$

From [12] we know that there is a unique solution  $v$  to the above system, which satisfies

$$(v, v') \in L^\infty(0, T; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

Let  $u$  be the solution to the system (3.1) with  $\varphi = 0$ . Then  $u + v$  is the solution to the system (3.1) with  $\varphi \neq 0$ , and  $u + v$  satisfies (3.2) with  $u$  and  $u'$  being replaced by  $u + v$  and  $u' + v'$ , respectively. The proof is complete.  $\square$

LEMMA 3.3. *Consider the system (3.1) with  $\varphi = 0$  and suppose that*

$$\begin{cases} g, g' \in L^2(0, T; (L^2(\Gamma))^n), \\ u^0 \in (H^1(\Omega))^n, u^1 \in (L^2(\Omega))^n \end{cases}$$

with the compatibility condition

$$g|_{t=0} = u^0|_\Gamma.$$

Then the solution of (3.1) satisfies

$$(u', u'') \in L^\infty(0, T; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

*Proof.* We need to prove the result for smooth data only, since the general case can be accomplished by continuous extension and a density argument. Let  $\gamma := u'$ . The  $\gamma$  satisfies

$$\begin{cases} \gamma'' - \nabla \cdot \sigma(\gamma) = 0 & \text{in } Q, \\ \gamma = g' & \text{on } \Sigma, \\ \gamma(0) = u^1, \gamma'(0) = \nabla \cdot \sigma(u^0) & \text{in } \Omega. \end{cases}$$

Since  $\gamma'(0) = \nabla \cdot \sigma(u^0) \in (H^{-1}(\Omega))^n$ , applying Lemma 3.2 to the above system gives

$$(\gamma, \gamma') \in L^\infty(0, T; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

Lemma 3.3 then follows from the fact  $\gamma = u'$ .  $\square$

LEMMA 3.4. *Consider the system in Lemma 3.3. If in addition*

$$g \in L^\infty(0, T; (H^{1/2}(\Gamma))^n),$$

then

$$u \in L^\infty(0, T; (H^1(\Omega))^n).$$

*Proof.* According to Lemma 3.3, we have

$$\nabla \cdot \sigma(u) = u'' \in L^\infty(0, T; (H^{-1}(\Omega))^n).$$

Take  $t$  as a parameter and consider the Dirichlet problem

$$\begin{cases} \nabla \cdot \sigma(u) \in L^\infty(0, T; (H^{-1}(\Omega))^n), \\ u|_\Gamma = g \in L^\infty(0, T; (H^{1/2}(\Gamma))^n). \end{cases}$$

Since the Shapiro–Lopatinskij condition holds at each point of the boundary  $\partial\Omega$  (see [1, p. 56]), the above boundary value problem is elliptic in the sense of Agmon, Douglis, and Nirenberg (see [1, p. 53]). By Theorems 10.1.1 and 10.1.2 in Chapter 10 of [41] (take  $G = \Omega$ ,  $s = -1$ ,  $p = 2$ ,  $N = m = n$ ,  $\sigma_j = -2$ ,  $t_j = 2$ ,  $s_j = 0$ ,  $j = 1, \dots, n$ , then  $\kappa = 0$ ,  $\tau_1 = \dots = \tau_n = 2$ ,  $\prod_{j=1}^N \tilde{H}^{t_j+s, p, (\tau_j)}(G) = (\tilde{H}^{1,2,(2)}(\Omega))^n$ ,  $\prod_{j=1}^N \tilde{H}^{s-s_j, p, (\kappa-s_j)}(G) = (H^{-1,2}(\Omega))^n = (H^{-1}(\Omega))^n$ ,  $\prod_{h=1}^m B^{s-\sigma_h-1/p, p}(\partial G) = (H^{1/2,2}(\Gamma))^n = (H^{1/2}(\Gamma))^n$ ), we get

$$u \in L^\infty(0, T; (H^1(\Omega))^n). \quad \square$$

LEMMA 3.5. *Suppose that*

$$\begin{cases} \varphi \in L^1(0, T; (L^2(\Omega))^n), \\ g \in (H^1(\Sigma))^n = L^2(0, T; (H^1(\Gamma))^n) \cap H^1(0, T; (L^2(\Gamma))^n), \\ u^0 \in (H^1(\Omega))^n, u^1 \in (L^2(\Omega))^n \end{cases}$$

with the compatibility condition

$$g|_{t=0} = u^0|_\Gamma.$$

Then the solution  $u$  to the system (3.1) satisfies

$$(3.11) \quad (u, u') \in C([0, T]; (H^1(\Omega))^n \times (L^2(\Omega))^n)$$

and

$$(3.12) \quad \sigma(u)\nu \in (L^2(\Sigma))^n.$$

*Proof.* We first prove (3.11). From the proof of Lemma 3.2, it suffices to consider the case of  $\varphi = 0$ . Since  $g \in (H^1(\Sigma))^n = L^2(0, T; (H^1(\Gamma))^n) \cap H^1(0, T; (L^2(\Gamma))^n)$ , it follows that

$$g \in L^2(0, T; (H^1(\Gamma))^n), \quad g' \in L^2(0, T; (L^2(\Gamma))^n).$$

By Theorem 3.1 in Chapter 1 of [38] (take  $m = 1$ ,  $j = 0$ ,  $X = (H^1(\Gamma))^n$ , and  $Y = (L^2(\Gamma))^n$ ), we know that

$$g \in C(0, T; [(H^1(\Gamma))^n, (L^2(\Gamma))^n]_{1/2}) \subset L^\infty(0, T; (H^{1/2}(\Gamma))^n).$$

According to Lemmas 3.3 and 3.4, we have

$$u \in L^\infty(0, T; (H^1(\Omega))^n), \quad u' \in L^\infty(0, T; (L^2(\Omega))^n).$$

This is (3.11) by Remark 3.1.

Next, we prove (3.12). Assume that all data are smooth and take a vector field  $h : \bar{\Omega} \rightarrow \mathbb{R}^n$  of class  $C^1$  such that  $h = \nu$  on  $\Gamma$ . Using the identity (3.8), we have

(3.13)

$$\begin{aligned} & \int_0^T \int_{\Gamma} \left[ 2 \sum_{i,j,m} h_m u_{i,m} \sigma_{ij}(u) \nu_j + (h \cdot \nu) \sum_i \left( u'_i u'_i - \sum_j \sigma_{ij}(u) \varepsilon_{ij}(u) \right) \right] d\Gamma dt \\ &= \int_{\Omega} 2 \sum_{i,m} h_m u_{i,m} u'_i dx \Big|_0^T + \int_0^T \int_{\Omega} \left[ 2 \sum_{i,j,m} h_{m,j} \sigma_{ij}(u) u_{i,m} \right. \\ & \quad \left. - 2 \sum_{i,m} \varphi_i h_m u_{i,m} + \operatorname{div}(h) \sum_i \left( u'_i u'_i - \sum_j \sigma_{ij}(u) \varepsilon_{ij}(u) \right) \right] dx dt. \end{aligned}$$

Since  $u = g$  on  $\Gamma$ , we have (see formula (2.49) on page 161 of [28])

$$u_{i,m} \nu_j = (u_{i,\nu} \nu_m + \mathcal{T}_m g_i) \nu_j = u_{i,\nu} \nu_m \nu_j + \mathcal{T}_m g_i \cdot \nu_j = (u_{i,j} - \mathcal{T}_j g_i) \nu_m + \mathcal{T}_m g_i \cdot \nu_j,$$

where  $\mathcal{T}_k, k = 1, \dots, n$ , are first order differential operators on  $\Gamma$ . Hence

$$\sum_{i,j,m} h_m u_{i,m} \sigma_{i,j}(u) \nu_j = \sum_{i,j} (h \cdot \nu) \sigma_{ij}(u) u_{i,j} - \sum_{i,j} (h \cdot \nu) \sigma_{ij}(u) \mathcal{T}_j g_i + \sum_{i,j,m} h_m \nu_j \sigma_{ij}(u) \mathcal{T}_m g_i.$$

Since  $h = \nu$  on  $\Gamma$ , the left-hand side of (3.13) reduces to

$$\int_0^T \int_{\Gamma} \sum_{i,j} \left[ \sigma_{ij}(u) \varepsilon_{ij}(u) - 2 \sigma_{ij}(u) \left( \mathcal{T}_j g_i - \sum_m \nu_m \nu_j \mathcal{T}_m g_i \right) + (g'_i)^2 \right] d\Gamma dt.$$

Set

$$G := \|\varphi\|_{L^1(0,T;(L^2(\Omega))^n)}^2 + \|u^0\|_{(H^1(\Omega))^n}^2 + \|u^1\|_{(L^2(\Omega))^n}^2 + \|g\|_{(H^1(\Sigma))^n}^2.$$

Then the right-hand side of (3.13) can be majorized by  $CG$  for some positive constant  $C$  depending only on  $\lambda, \mu$ , and  $\Omega$ .

Again by Lemma 2.1 of [2], we know that

$$\sum_{i,j} (\sigma_{ij}(u))^2 \leq \frac{\beta}{\alpha} \sum_{i,j} \sigma_{ij}(u) \varepsilon_{ij}(u),$$

and hence

$$\begin{aligned} & \sigma_{ij}(u) \varepsilon_{ij}(u) - 2 \sigma_{ij}(u) \left( \mathcal{T}_j g_i - \sum_m \nu_m \nu_j \mathcal{T}_m g_i \right) + (g'_i)^2 \\ &= \sigma_{ij}(u) \varepsilon_{ij}(u) - 2 \sqrt{\frac{\alpha}{2\beta}} \sigma_{ij}(u) \cdot \sqrt{\frac{2\beta}{\alpha}} \left( \mathcal{T}_j g_i - \sum_m \nu_m \nu_j \mathcal{T}_m g_i \right) + (g'_i)^2 \\ &\geq \sigma_{ij}(u) \varepsilon_{ij}(u) - \frac{\alpha}{2\beta} (\sigma_{ij}(u))^2 - \frac{2\beta}{\alpha} \left( \mathcal{T}_j g_i - \sum_m \nu_m \nu_j \mathcal{T}_m g_i \right)^2 + (g'_i)^2 \\ &\geq \frac{1}{2} \sigma_{ij}(u) \varepsilon_{ij}(u) - \frac{2\beta}{\alpha} \left( \mathcal{T}_j g_i - \sum_m \nu_m \nu_j \mathcal{T}_m g_i \right)^2 + (g'_i)^2. \end{aligned}$$

Therefore, from (3.13), we have

$$\int_0^T \int_\Gamma \frac{1}{2} \sum_{i,j} \sigma_{ij}(u) \varepsilon_{ij}(u) d\Gamma dt \leq CG$$

$$+ \int_0^T \int_\Gamma \sum_{i,j} \left[ \frac{2\beta}{\alpha} \left( \mathcal{T}_j g_i - \sum_m \nu_m \nu_j \mathcal{T}_m g_i \right)^2 - (g'_i)^2 \right] d\Gamma dt \leq C'G$$

for some positive constant  $C'$ . Applying Lemma 2.1 of [2], we get

$$\sum_i \left( \sum_j \sigma_{ij}(u) \nu_j \right)^2 \leq \frac{\beta}{\alpha} \sum_{i,j} \sigma_{ij}(u) \varepsilon_{ij}(u).$$

Combine the last two inequalities to yield

$$\int_0^T \int_\Gamma \sum_i \left( \sum_j \sigma_{ij}(u) \nu_j \right)^2 d\Gamma dt \leq C''G$$

for some positive constant  $C''$ . This is (3.12). □

*Proof of Proposition 3.1.* In view of Lemma 3.2, only (3.3) needs to be proved. This is done by duality as in [28]. Let  $w$  be the solution to the following system:

$$(3.14) \quad \begin{cases} w''_i - \sigma_{ij,j}(w) = 0 & \text{in } Q, \\ w_i = p_i & \text{on } \Sigma, \\ w_i(T) = 0, w'_i(T) = 0 & \text{in } \Omega, \end{cases} \quad i = 1, \dots, n,$$

where  $p_i, i = 1, \dots, n$ , satisfy

$$p_i \in H^1(\Sigma), p_i(T) = 0 \text{ on } \Gamma.$$

Assuming all data to be smooth, using (3.9) and noticing  $\psi_i = 0$  in  $Q$  and  $w_i = p_i$  on  $\Sigma$ , we have

$$\langle \sigma(u)\nu, p \rangle_\Sigma = -\langle \varphi, w \rangle_Q + \langle g, \sigma(w)\nu \rangle_\Sigma - \langle u^1, w(0) \rangle_\Omega + \langle u^0, w'(0) \rangle_\Omega.$$

Applying Lemma 3.5 to the system (3.14) gives

$$|\langle \sigma(u)\nu, p \rangle_\Sigma| \leq C \|p\|_{(H^1(\Sigma))^n}$$

for some positive constant  $C$ . Equation (3.3) then follows by taking  $p \in (H^1_0(\Sigma))^n$ . □

To end this section, we introduce some preliminary notation and results in the theory of pseudodifferential operators. All are classical and standard (see, e.g., [14, 23, 40]).

Let  $\Omega \subset \mathbb{R}^n$  be an open set. Define the symbol class  $S^m(\Omega)$  ( $m \in \mathbb{R}$ ) as the set of  $p \in C^\infty(\Omega \times \mathbb{R}^n)$  satisfying that, for any compact  $K \subset \Omega$ , and any multi-index  $\alpha$ , there exists a constant  $C_K$  such that

$$|D^\alpha_\xi p(x, \xi)| \leq C_K (1 + |\xi|)^{m-|\alpha|}$$

for all  $x \in K$  and  $\xi \in \mathbb{R}^n$ .

The scalar pseudodifferential operator  $P: C_0^\infty(\Omega) \rightarrow C^\infty(\Omega)$  is a map given by

$$Pu(x) = (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} p(x, \xi) \widehat{u}(\xi) d\xi,$$

where  $\widehat{u}(\xi)$  denotes the Fourier transform of  $u(x)$ . If  $p(x, \xi) \in S^m(\Omega)$ ,  $P$  is said to belong to  $OP(S^m(\Omega))$ . We can extend  $P$  to be a continuous map from  $\mathcal{E}'(\Omega)$  to  $\mathcal{D}'(\Omega)$ , where  $\mathcal{D}'(\Omega)$  is the space of all distributions (generalized functions) in  $\Omega$  and  $\mathcal{E}'(\Omega)$  is the space of all distributions (generalized functions) with compact supports in  $\Omega$ .

We can consider a system of pseudodifferential operators in  $\Omega$  to be a matrix of scalar pseudodifferential operators. We still use  $P$  to denote, without confusion, an  $n \times n$  matrix formed pseudodifferential operator of order  $m$  from  $(C_0^\infty(\Omega))^n$  to  $(C^\infty(\Omega))^n$ :

$$P := \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix},$$

where  $P_{ij}$ ,  $1 \leq i, j \leq n$ , are scalar pseudodifferential operators of order  $m$ . Then for any  $u = (u_1, u_2, \dots, u_n) \in (C_0^\infty(\Omega))^n$ , the  $i$ th component of  $Pu$  is defined by

$$(Pu)_i = \sum_{j=1}^n P_{ij} u_j, \quad i = 1, \dots, n.$$

Again without confusion, we still use  $OP(S^m(\Omega))$  to denote the class of matrix formed pseudodifferential operators of order  $m$ . If the principal symbol of  $P_{ij}$  is denoted by  $\sigma_m(P_{ij})$ , then the principal symbol of  $P$  is just the matrix of  $\sigma_m(P_{ij})$ . The operator  $P$  is called elliptic if its principal symbol  $(\sigma_m(P_{ij}))$  is invertible everywhere in  $\Omega$ .

Many properties of scalar pseudodifferential operators can be generalized to the case of matrix formed pseudodifferential operators. For instance, the pseudolocal property of a matrix formed pseudodifferential operator  $P$  says that  $\text{singsupp}(Pu) \subset \text{supp}(u)$  for all  $u \in (\mathcal{E}'(\Omega))^n$  (here  $\text{singsupp}(Pu)$ , the singular support of  $Pu$ , is the set of points in  $\Omega$  having no open neighborhood to which the restriction of  $Pu$  is a  $(C^\infty)^n$  vector-valued function). The Sobolev space continuity of  $P \in OP(S^m(\Omega))$  means that  $P$  is a continuous map from  $(H_{\text{comp}}^s(\Omega))^n$  to  $(H_{\text{loc}}^{s-m}(\Omega))^n$ . The commutator of two matrix formed pseudodifferential operators has the following property: if  $P_1 \in OP(S^{m_1}(\Omega))$ ,  $P_2 \in OP(S^{m_2}(\Omega))$ , then the commutator  $[P_1, P_2] := P_1 P_2 - P_2 P_1 \in OP(S^{m_1+m_2-1}(\Omega))$ .

In the proof of Theorem 1.1, we need to consider the Dirichlet problem of a matrix formed elliptic pseudodifferential operator  $P$ :

$$\begin{cases} Pu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The well-developed theory of pseudodifferential boundary value problems can be found in [14, 40] and the references therein.

**4. Proof of Theorem 1.1.** As in section 3, we still assume that  $g$  is defined on  $\Gamma$  by the zero extension of  $g$  on  $\Gamma_1$ . We rewrite system (1.1) with zero initial data into the form

$$(4.1) \quad \begin{cases} u'' + \mathcal{A}u = 0 & \text{in } \Omega \times (0, \infty), \\ u = g & \text{on } \Gamma \times (0, \infty), \\ u(0) = 0, \ u'(0) = 0 & \text{in } \Omega, \\ y = -\sigma(\mathcal{A}^{-1}u')\nu & \text{on } \Gamma \times (0, \infty), \end{cases}$$

where  $\mathcal{A}$  is defined by (1.2) in section 1.

By virtue of the interior regularity of  $\{(u(T), u'(T))\}$  in  $\mathcal{H}$  we obtained in section 3, Theorem 1.1 is equivalent to saying that the solution to (4.1) satisfies

$$\|y\|_{L^2(0,T;U)} \leq C_T \|g\|_{L^2(0,T;U)} \quad \forall g \in L^2(0, T; U).$$

Since  $y = -B^*u' = -\sigma(\mathcal{A}^{-1}u')\nu$  from (2.5) and (1.1), the above formula is equivalent to

$$\|B^*u'\|_{L^2(0,T;U)} \leq C_T \|g\|_{L^2(0,T;U)} \quad \forall g \in L^2(0, T; U)$$

or

$$(4.2) \quad \|\sigma(\mathcal{A}^{-1}u')\nu\|_{L^2(0,T;U)} \leq C_T \|g\|_{L^2(0,T;U)} \quad \forall g \in L^2(0, T; U).$$

The system (4.1) can be transformed into (4.5) through a partition of unity and a change of variables [22, 24, 25]. The present setting yielding (4.5) is taken from [24, 25]. Let us say a few words about the process. Let  $(x, y)$  be the new variables such that  $x \in \mathbb{R}^1, y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ . Denote also by  $\Omega$  the new half-space  $\mathbb{R}_{x_+}^1 \times \mathbb{R}_y^{n-1}$  with the boundary  $\Gamma = \partial\Omega = \Omega|_{x=0} = \mathbb{R}_y^{n-1}$  without confusion, and also keep the notation  $u$  and  $g$  for the new solution and new control, respectively. Under such a change of coordinates, the operator  $\partial_t^2 + \mathcal{A}$  is changed locally to  $-\mathcal{T}(x, y)D_t^2 + P(x, y, D_x, D_y) - P_l(x, y, D_x, D_y)$ , where the differential operators  $D_t, D_x$ , and  $D_{y_j}, j = 1, \dots, n - 1$ , are defined by

$$D_t := \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t}, \quad D_x := \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x}, \quad D_{y_j} := \frac{1}{\sqrt{-1}} \frac{\partial}{\partial y_j}.$$

$\mathcal{T}(x, y)$  denotes a positive definite diagonal matrix  $(t_{ii}(x, y))_{i=1}^n$  and  $0 < t_{ii}(x, y) < M, i = 1, \dots, n$ , for all  $(x, y) \in \Omega$ ; here  $M$  is some positive constant depending on  $\lambda, \mu$ , and the original  $\Omega$  only.  $P(x, y, D_x, D_y)$  denotes the principal term, which is a matrix formed second order elliptic operator with symbol  $p(x, y, \xi, \eta)$ . The entries of the matrix  $p(x, y, \xi, \eta)$  can be assumed to have the following forms:

$$(4.3) \quad p_{ij}(x, y, \xi, \eta) := \begin{cases} \xi^2 + \tilde{p}_{ij}(x, y, \xi, \eta_k, \eta_k^2), & i = j, \ k = 1, \dots, n - 1, \\ \tilde{p}_{ij}(x, y, \xi, \eta_k, \eta_k^2), & i \neq j, \ k = 1, \dots, n - 1, \end{cases}$$

where the  $\tilde{p}_{ij}(x, y, \xi, \eta_k, \eta_k^2), 1 \leq i, j \leq n$ , are of second order in the variables  $\xi$  and  $\eta_k$  but do not contain  $\xi^2$  terms. In addition,  $\tilde{p}_{ij}(x, y, \xi, \eta_k, \eta_k^2)$  may depend on  $x$  and  $y$  but not on  $t$ .  $P_l(x, y, D_x, D_y)$  denotes the lower order terms. The boundary output (or observation) under the transformation can be expressed as

$$\mathcal{Y} = -\mathcal{B}((P - P_l)^{-1}\mathcal{T}u') \text{ on } \Gamma \times (0, \infty).$$



In the above expression, the symbol of  $\mathcal{B}$  has the following form (modulo zero order terms) on the boundary  $\partial\Omega = \Omega|_{x=0}$ :

$$(4.4) \quad b_{ij}(y, \xi, \eta) := \begin{cases} \xi + \tilde{b}_{ij}(y, \eta_k), & i = j, k = 1, \dots, n - 1, \\ \tilde{b}_{ij}(y, \eta_k), & i \neq j, k = 1, \dots, n - 1, \end{cases}$$

where  $\tilde{b}_{ij}(y, \eta_k), 1 \leq i, j \leq n$ , may be dependent on  $y$  but are no worse than the linear combinations of all variables  $\eta_k, k = 1, \dots, n - 1$ . Denote  $\mathcal{P} := -\mathcal{T}(x, y)D_t^2 + P(x, y, D_x, D_y)$ . Under the new coordinates, the system (4.1) takes a new form as shown in [24, 25]:

$$(4.5) \quad \begin{cases} \mathcal{P}u = -\mathcal{T}D_t^2u + Pu = P_lu & \text{in } \Omega \times (0, \infty), \\ u = g & \text{on } \Gamma \times (0, \infty), \\ u(0) = 0, u'(0) = 0 & \text{in } \Omega, \\ \mathcal{Y} = -\mathcal{B}((P - P_l)^{-1}\mathcal{T}u') & \text{on } \Gamma \times (0, \infty), \end{cases}$$

where  $u' := \frac{\partial u}{\partial t}$ .

We give several remarks concerning the system (4.5). They are essentially consequences of Proposition 3.1 and will be used in the proof of Theorem 1.1.

*Remark 4.1.* Since the system (4.5) is transformed from the system (4.1) through a locally smooth and invertible transformation, the new solution keeps many properties of the old one, including the well-posedness (in the sense of Hadamard) and the Sobolev regularity. In particular, as a consequence of Proposition 3.1, we assert that for any  $g \in L^2(0, T; (L^2(\Gamma))^n)$ , there exists a unique solution  $u$  to the system (4.5), which satisfies

$$(4.6) \quad (u, u') \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n), \quad \mathcal{B}u \in (H^{-1}(\Sigma))^n,$$

where  $\mathcal{B}u$  comes from  $\sigma(u)\nu$  through the locally smooth transformation, and we still use  $\Sigma = \Gamma \times (0, T)$ .

Since the boundary output (or observation)  $y = -\sigma(\mathcal{A}^{-1}u')\nu$  is transformed into  $\mathcal{Y} = -\mathcal{B}((P - P_l)^{-1}\mathcal{T}u')$  under the transformation, from the analysis above, we can assert that  $\sigma(\mathcal{A}^{-1}u')\nu \in L^2(0, T; U) = (L^2(\Sigma))^n$  if and only if  $\mathcal{B}((P - P_l)^{-1}\mathcal{T}u') \in (L^2(\Sigma))^n$ . Due to the equivalence of Theorem 1.1 and (4.2), Theorem 1.1 is valid if we can prove that

$$(4.7) \quad \mathcal{B}((P - P_l)^{-1}\mathcal{T}u') \in (L^2(\Sigma))^n.$$

*Remark 4.2.* Through the same transformation from (4.1) to (4.5), the system (3.1) with  $g = 0$  becomes the following system:

$$(4.8) \quad \begin{cases} \mathcal{P}u = \mathcal{T}u'' + Pu = P_lu + \varphi & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \Gamma \times (0, \infty), \\ u(0) = u^0, u'(0) = u^1 & \text{in } \Omega, \end{cases}$$

where we used the same notation  $u, \varphi, u^0$ , and  $u^1$  as in (3.1). By Remark 4.1, for any  $\varphi \in L^1(0, T; (H^{-1}(\Omega))^n)$  and  $(u^0, u^1) \in (L^2(\Omega))^n \times (H^{-1}(\Omega))^n$ , there exists a unique solution  $u$  to the system (4.8), which satisfies

$$(u, u') \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

Write (4.8) in the operator form

$$\frac{d}{dt}(u, u') = \mathcal{C}_1(u, u') + \mathcal{C}_2(u, u') + (0, \mathcal{T}^{-1}\varphi),$$

where

$$\mathcal{C}_1(f_1, f_2) = (f_2, -\mathcal{T}^{-1}P f_1), \quad \mathcal{C}_2(f_1, f_2) = (0, \mathcal{T}^{-1}P_l f_1).$$

We see that  $\mathcal{C}_2$  is a bounded operator on  $(L(\Omega))^n \times (H^{-1}(\Omega))^n$ . By Remarks 3.3 and 4.1,  $\mathcal{C}_1 + \mathcal{C}_2$  generates a  $C_0$ -semigroup on  $(L(\Omega))^n \times (H^{-1}(\Omega))^n$ , and so does  $\mathcal{C}_1$  due to the boundedness of  $\mathcal{C}_2$ .

Let  $w$  be the solution to (4.8) but with the low order terms removed. It is associated with the operator  $\mathcal{C}_1$  only, that is, it satisfies the following system:

$$\begin{cases} \mathcal{P}w = \mathcal{T}w'' + Pw = \varphi & \text{in } \Omega \times (0, \infty), \\ w = 0 & \text{on } \Gamma \times (0, \infty), \\ w(0) = w^0, w'(0) = w^1 & \text{in } \Omega. \end{cases}$$

Then for any  $\varphi \in L^1(0, T; (H^{-1}(\Omega))^n)$  and  $(w^0, w^1) \in (L^2(\Omega))^n \times (H^{-1}(\Omega))^n$ , we have

$$(w, w') \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

*Remark 4.3.* As it is stated in Remark 4.1, proving Theorem 1.1 is equivalent to showing (4.7). Moreover, we claim that (4.7) is further equivalent to

$$\mathcal{B}(P^{-1}u') \in (L^2(\Sigma))^n.$$

In fact, a straightforward computation shows that

$$\begin{aligned} (P - P_l)^{-1}\mathcal{T}u' &= (P - P_l)^{-1}PP^{-1}\mathcal{T}u' \\ &= (P - P_l)^{-1}((P - P_l) + P_l)P^{-1}\mathcal{T}u' \\ (4.9) \quad &= (I + (P - P_l)^{-1}P_l)P^{-1}\mathcal{T}u' = P^{-1}\mathcal{T}u' + (P - P_l)^{-1}P_lP^{-1}\mathcal{T}u' \\ &= ([P^{-1}, \mathcal{T}] + \mathcal{T}P^{-1})u' + (P - P_l)^{-1}P_lP^{-1}\mathcal{T}u' \\ &= \mathcal{T}P^{-1}u' + [P^{-1}, \mathcal{T}]u' + (P - P_l)^{-1}P_lP^{-1}\mathcal{T}u'. \end{aligned}$$

Here  $[P^{-1}, \mathcal{T}]$  is the commutator of  $P^{-1}$  and  $\mathcal{T}$  as defined at the end of section 3. Since  $\mathcal{T} \in OP(S^0(\Omega))$ ,  $P \in OP(S^2(\Omega))$ , and  $P_l \in OP(S^1(\Omega))$ , by the properties of pseudodifferential operators [23], we have  $[P^{-1}, \mathcal{T}] = P^{-1}\mathcal{T} - \mathcal{T}P^{-1} \in OP(S^{-3}(\Omega))$  and  $(P - P_l)^{-1}P_lP^{-1}\mathcal{T} \in OP(S^{-3}(\Omega))$ . Furthermore, from the a priori regularity of  $u'$  in (4.6), we have

$$[P^{-1}, \mathcal{T}]u' + (P - P_l)^{-1}P_lP^{-1}\mathcal{T}u' \in C([0, T]; (H^2(\Omega))^n).$$

Then the trace theorem for Sobolev spaces (see, e.g., Theorem 2.3 of [13]) can be applied to get

$$\mathcal{B}([P^{-1}, \mathcal{T}]u' + (P - P_l)^{-1}P_lP^{-1}\mathcal{T}u') \in C([0, T]; (H^{1/2}(\Gamma))^n) \subset (L^2(\Sigma))^n.$$

This together with (4.9) shows that

$$\mathcal{B}((P - P_l)^{-1}\mathcal{T}u') \in (L^2(\Sigma))^n \text{ if and only if } \mathcal{B}(\mathcal{T}P^{-1}u') \in (L^2(\Sigma))^n.$$

Finally, by the smoothness and invertibility of  $\mathcal{T}$ , we get

$$\mathcal{B}((P - P_l)^{-1}\mathcal{T}u') \in (L^2(\Sigma))^n \text{ if and only if } \mathcal{B}(P^{-1}u') \in (L^2(\Sigma))^n.$$

*Proof of Theorem 1.1.* The proof follows closely [33] for the scalar wave equation. The proof is done in five steps.

*Step 1.* In this step, we obtain the a priori estimate

$$(4.10) \quad \mathcal{B}z' \in (H^{-1}(\Sigma))^n,$$

where  $z := P^{-1}u'$  and  $u$  is the solution to (4.5) with  $g \in L^2(0, T; (L^2(\Gamma))^n)$ .

To do this, rewrite (4.5) as the following system:

$$(4.11) \quad \begin{cases} u'' + \mathcal{T}^{-1}(P - P_l)u = 0 & \text{in } \Omega \times (0, \infty), \\ u = g & \text{on } \Gamma \times (0, \infty), \\ u(0) = 0, u'(0) = 0 & \text{in } \Omega. \end{cases}$$

Define the map  $\tilde{\Upsilon} \in \mathcal{L}((L^2(\Gamma))^n, (H^{1/2}(\Omega))^n)$  by  $\tilde{\Upsilon}g = v$ , where  $v$  satisfies

$$\begin{cases} \mathcal{T}^{-1}(P - P_l)v = 0 & \text{in } \Omega, \\ v = g & \text{on } \Gamma. \end{cases}$$

From the classical elliptic theory (see Chapter 3 of [6] or Theorems 10.1.1 and 10.1.2 in Chapter 10 of [41]), we have

$$(4.12) \quad \tilde{\Upsilon}g \in L^2(0, T; (H^{1/2}(\Omega))^n), \quad \mathcal{B}(\tilde{\Upsilon}g) \in L^2(0, T; (H^{-1}(\Omega))^n) \subset (H^{-1}(\Sigma))^n.$$

In terms of  $\tilde{\Upsilon}$ , we can write (4.11) as

$$u'' + \mathcal{T}^{-1}(P - P_l)(u - \tilde{\Upsilon}g) = 0.$$

Since  $z = P^{-1}u'$ , the above equality implies that

$$(4.13) \quad \begin{aligned} \mathcal{B}z' &= \mathcal{B}P^{-1}u'' = -\mathcal{B}P^{-1}\mathcal{T}^{-1}(P - P_l)(u - \tilde{\Upsilon}g) \\ &= -\mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)u) + \mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)(\tilde{\Upsilon}g)). \end{aligned}$$

In order to prove  $\mathcal{B}z' \in (H^{-1}(\Sigma))^n$ , we first prove

$$(4.14) \quad \mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)u) \in (H^{-1}(\Sigma))^n.$$

In fact,

$$(4.15) \quad \begin{aligned} \mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)u) &= \mathcal{B}P^{-1}([\mathcal{T}^{-1}, P - P_l] + (P - P_l)\mathcal{T}^{-1})u \\ &= \mathcal{B}P^{-1}[\mathcal{T}^{-1}, P - P_l]u + \mathcal{B}P^{-1}(P - P_l)\mathcal{T}^{-1}u \\ &= \mathcal{B}P^{-1}[\mathcal{T}^{-1}, P - P_l]u + \mathcal{B}(I - P^{-1}P_l)\mathcal{T}^{-1}u \\ &= \mathcal{B}(\mathcal{T}^{-1}u) + \mathcal{B}(P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1})u \\ &= \mathcal{B}(\mathcal{T}^{-1}u) + [\mathcal{B}, P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1}]u \\ &\quad + (P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1})\mathcal{B}u. \end{aligned}$$

Since from (4.6),  $\mathcal{B}u \in (H^{-1}(\Sigma))^n$ , by the smoothness and invertibility of  $\mathcal{T}$ , it follows that

$$(4.16) \quad \mathcal{B}(\mathcal{T}^{-1}u) \in (H^{-1}(\Sigma))^n.$$

Furthermore, from  $P^{-1}[\mathcal{T}^{-1}, P - P_l] \in OP(S^{-1}(\Omega))$ ,  $P^{-1}P_l\mathcal{T}^{-1} \in OP(S^{-1}(\Omega))$ , and  $\mathcal{B} \in OP(S^1(\Omega))$ , we know that  $P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1} \in OP(S^{-1}(\Omega))$  and  $[\mathcal{B}, P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1}] \in OP(S^{-1}(\Omega))$ . This combines with the a priori regularity  $u \in C([0, T]; (L^2(\Omega))^n)$  to yield

$$[\mathcal{B}, P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1}]u \in C([0, T]; (H^1(\Omega))^n).$$

Then by the trace theorem of Sobolev spaces,

$$(4.17) \quad [\mathcal{B}, P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1}]u \in C([0, T]; (H^{1/2}(\Gamma))^n) \subset (H^{-1}(\Sigma))^n.$$

Again by  $\mathcal{B}u \in (H^{-1}(\Sigma))^n$  and  $P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1} \in OP(S^{-1}(\Omega))$ , we have

$$(4.18) \quad (P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1})\mathcal{B}u \in (H^{-1}(\Sigma))^n.$$

Combine (4.15), (4.16), (4.17), and (4.18) to obtain

$$\begin{aligned} \mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)u) &= \mathcal{B}(\mathcal{T}^{-1}u) + [\mathcal{B}, P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1}]u \\ &\quad + (P^{-1}[\mathcal{T}^{-1}, P - P_l] - P^{-1}P_l\mathcal{T}^{-1})\mathcal{B}u \in (H^{-1}(\Sigma))^n. \end{aligned}$$

This is (4.14).

Next, along the same line of proving (4.14) from the a priori regularity  $u \in C([0, T]; (L^2(\Omega))^n)$  and  $\mathcal{B}u \in (H^{-1}(\Sigma))^n$ , we can obtain

$$(4.19) \quad \mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)(\tilde{\Upsilon}g)) \in (H^{-1}(\Sigma))^n$$

from (4.12).

Finally, combine (4.13), (4.14), and (4.19) to yield

$$\mathcal{B}z' = -\mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)u) + \mathcal{B}(P^{-1}\mathcal{T}^{-1}(P - P_l)(\tilde{\Upsilon}g)) \in (H^{-1}(\Sigma))^n.$$

Equation (4.10) is thus concluded.

*Step 2.* As it was explained in the beginning of this section, through a partition of unity and a change of variables, the system (4.1) can be transformed into (4.5). As in [33], we perform a cutoff in time and divide the system (4.5) into two independent systems.

Since the solution  $u$  to (4.5) has zero initial data, one can extend  $u(t)$  to be zero for  $t < 0$ . Let  $\phi \in C_0^\infty(\mathbb{R})$ ,  $|\phi| \leq 1$ , be a smooth cutoff function in  $\mathbb{R}$  with  $\phi(t) = 0$  for  $t \geq (3/2)T$  and  $\phi(t) = 1$  for  $t \in [0, T]$ . Set

$$u_c := \phi u.$$

Then  $u_c$  satisfies

$$(4.20) \quad \begin{cases} \mathcal{P}u_c = [\mathcal{P}, \phi I]u + \phi P_l u & \text{in } \Omega \times (0, \infty), \\ u_c = \phi g & \text{on } \Gamma \times (0, \infty), \\ u_c(0) = 0, u'_c(0) = 0 & \text{in } \Omega, \\ \text{supp}(u_c) \subset [0, (3/2)T], \end{cases}$$

where  $I$  is the  $n \times n$  identity matrix and  $\phi I$  is thus an  $n \times n$  diagonal matrix with the same entries  $\phi$ .

Now, decompose  $u_c = v + w$ , where  $v$  and  $w$  satisfy (4.21) and (4.22) below, respectively:

$$(4.21) \quad \begin{cases} \mathcal{P}v = 0 & \text{in } \Omega \times (0, \infty), \\ v = \phi g & \text{on } \Gamma \times (0, \infty), \\ v(0) = 0, v'(0) = 0 & \text{in } \Omega, \end{cases}$$

$$(4.22) \quad \begin{cases} \mathcal{P}w = f := [\mathcal{P}, \phi I]u + \phi P_l u & \text{in } \Omega \times (0, \infty), \\ w = 0 & \text{on } \Gamma \times (0, \infty), \\ w(0) = 0, w'(0) = 0 & \text{in } \Omega. \end{cases}$$

By (4.6), the solution to (4.5) satisfies  $(u, u') \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n)$ . Hence

$$(4.23) \quad (u_c, u'_c) \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

Since  $(u, u') \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n)$ ,  $[\mathcal{P}, \phi I] \in OP(S^1(\Omega \times \mathbb{R}_t^1))$ , and  $P_l \in OP(S^1(\Omega))$ , we have  $[\mathcal{P}, \phi I]u \in C([0, T]; (H^{-1}(\Omega))^n)$  and  $\phi P_l u \in C([0, T]; (H^{-1}(\Omega))^n)$ . Thus

$$(4.24) \quad f = [\mathcal{P}, \phi I]u + \phi P_l u \in C([0, T]; (H^{-1}(\Omega))^n).$$

By Remark 4.2, the solution  $w$  to (4.22) satisfies

$$(4.25) \quad (w, w') \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

Noticing  $v = u_c - w$ , from (4.23) and (4.25), we obtain

$$(4.26) \quad (v, v') = (u_c, u'_c) - (w, w') \in C([0, T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n).$$

Next, we show that the a priori estimate obtained in Step 1 is still valid for  $u_c$ , that is,

$$(4.27) \quad \mathcal{B}\widehat{z}' \in (H^{-1}(\Sigma))^n,$$

where  $\widehat{z} := P^{-1}u'_c$ . In fact, since  $\phi(t) = 1$  for  $t \in [0, T]$ , we have  $\mathcal{B}\widehat{z}' = \mathcal{B}(P^{-1}(\phi u)') = \mathcal{B}(P^{-1}u') = \mathcal{B}z'$  on  $\Sigma = \Gamma \times (0, T)$ .

*Step 3.* We perform a region decomposition in this step, and this step is based on the work of [24, 25] in the elliptic sector. As shown in (4.5), let  $\xi \in \mathbb{R}$  and  $\eta \in \mathbb{R}^{n-1}$  be the Fourier variables corresponding to  $x$  and  $y$ , respectively, and let  $p(x, y, \xi, \eta)$  be the symbol of  $P(x, y, D_x, D_y)$ . Then  $P(x, y, D_x, D_y) = p(x, y, D_x, D_y)$ , where  $p(x, y, D_x, D_y)$  comes from the symbol  $p(x, y, \xi, \eta)$  with  $\xi$  and  $\eta$  being replaced by  $D_x$  and  $D_y$ , respectively.

Let  $\tau = \rho - \sqrt{-1}\gamma, \gamma > 0, \rho \in \mathbb{R}$ , be the Laplace variable corresponding to  $t$ . By the definition of  $\mathcal{P}$  in (4.5), the entries of the matrix formed symbol corresponding to  $\mathcal{P}$  become

$$(4.28) \quad \mathfrak{p}_{ij}(x, y, \xi, \eta, \tau) := \begin{cases} -t_{ii}(x, y)\tau^2 + p_{ij}(x, y, \xi, \eta), & i = j, \\ p_{ij}(x, y, \xi, \eta), & i \neq j, \end{cases}$$

where  $p_{ij}(x, y, \xi, \eta), 1 \leq i, j \leq n$ , are the entries of the matrix formed symbol  $p(x, y, \xi, \eta)$  as shown in (4.3). We may assume without loss of generality that  $\gamma = 0$ . Then (4.28) becomes

$$(4.29) \quad \mathbf{p}_{ij}(x, y, \xi, \eta, \rho) := \begin{cases} -t_{ii}(x, y)\rho^2 + p_{ij}(x, y, \xi, \eta), & i = j, \\ p_{ij}(x, y, \xi, \eta), & i \neq j. \end{cases}$$

Due to the symmetry of  $\mathbf{p}_{ij}(x, y, \xi, \eta, \rho)$  in the variables  $\eta$  and  $\rho$ , we can restrict ourselves to the region  $\mathbb{R}_+^{2n} = \{(x, y, \eta, \rho) \mid (x, y) \in \Omega, \rho > 0, \eta_j > 0, j = 1, \dots, n-1\}$ . As in [24, 25], divide the one quarter  $\eta/\rho$  space  $\mathbb{R}_+^n = \{(\eta, \rho) \mid \rho > 0, \eta_j > 0, j = 1, \dots, n-1\}$  into three regions:

$$\begin{aligned} \mathcal{R}_1 &:= \{(\eta, \rho) \in \mathbb{R}_+^n \mid \rho < c_0|\eta|\}, \\ \mathcal{R}_{tr} &:= \{(\eta, \rho) \in \mathbb{R}_+^n \mid c_0|\eta| \leq \rho \leq 2c_0|\eta|\}, \\ \mathcal{R}_2 &:= \{(\eta, \rho) \in \mathbb{R}_+^n \mid \rho > 2c_0|\eta|\}, \end{aligned}$$

where  $c_0$  is some positive constant such that  $\mathcal{R}_1 \cup \mathcal{R}_{tr}$  is contained in the elliptic region of the operator  $\mathcal{P}$ . The value of  $c_0$  will be determined by the formula (4.30).

Now, consider the region  $\mathcal{R}_1 \cup \mathcal{R}_{tr} = \{(\eta, \rho) \in \mathbb{R}_+^n \mid \rho \leq 2c_0|\eta|\}$ . Since  $P$  is a matrix formed second order elliptic operator, there exists a constant  $\alpha > 0$  such that the matrix formed symbol  $p(x, y, \xi, \eta)$  that corresponds to  $P$  satisfies

$$p(x, y, \xi, \eta)w \cdot w \geq \alpha(|\xi|^2 + |\eta|^2)|w|^2 \quad \forall w \in \mathbb{R}^n.$$

Thus by (4.29), the symbol  $\mathbf{p}(x, y, \xi, \eta, \rho)$  that corresponds to  $\mathcal{P}$  satisfies

$$\begin{aligned} \mathbf{p}(x, y, \xi, \eta, \rho)w \cdot w &= -\sum_{i=1}^n t_{ii}\rho^2 w_i^2 + p(x, y, \xi, \eta)w \cdot w \\ &\geq -\left(\max_{i=1, \dots, n} t_{ii}\right)\rho^2|w|^2 + \alpha(|\xi|^2 + |\eta|^2)|w|^2 \\ &\geq -4c_0^2\left(\max_{i=1, \dots, n} t_{ii}\right)|\eta|^2|w|^2 + \alpha(|\xi|^2 + |\eta|^2)|w|^2 \\ &\geq \left(\alpha - 4c_0^2 \max_{i=1, \dots, n} t_{ii}\right)(|\xi|^2 + |\eta|^2)|w|^2 \quad \text{in } \mathcal{R}_1 \cup \mathcal{R}_{tr}. \end{aligned}$$

Choosing  $c_0$  small enough such that

$$(4.30) \quad \beta := \alpha - 4c_0^2 \max_{i=1, \dots, n} t_{ii} > 0,$$

we then have

$$\begin{aligned} \mathbf{p}(x, y, \xi, \eta, \rho)w \cdot w &\geq \beta(|\xi|^2 + |\eta|^2)|w|^2 \geq \beta\left(|\xi|^2 + \frac{1}{2}|\eta|^2 + \frac{1}{8c_0^2}\rho^2\right)|w|^2 \\ &\geq \beta \min\left\{\frac{1}{2}, \frac{1}{8c_0^2}\right\}(|\xi|^2 + |\eta|^2 + \rho^2)|w|^2 \quad \text{in } \mathcal{R}_1 \cup \mathcal{R}_{tr}. \end{aligned}$$

This implies that  $\mathcal{P}$  is elliptic in the region  $\mathcal{R}_1 \cup \mathcal{R}_{tr}$ .

Let  $\chi(x, y, t, \eta, \rho) \in C^\infty$  be a homogeneous scalar symbol of order zero with respect to  $\eta$  and  $\rho$  such that  $0 \leq \chi(x, y, t, \eta, \rho) \leq 1$  and

$$\chi(x, y, t, \eta, \rho) = \begin{cases} 1 & \text{in } \mathcal{R}_1, \\ 0 & \text{in } \mathcal{R}_2, \end{cases} \quad \text{supp}(\chi) \subset \mathcal{R}_1 \cup \mathcal{R}_{tr}.$$

Let  $\mathcal{X} \in OP(S^0(\Omega \times \mathbb{R}_t^1))$  be the matrix formed pseudodifferential operator corresponding to  $\chi I$ . Then  $\mathcal{P}\mathcal{X}$  is a matrix formed elliptic pseudodifferential operator of order two. Similarly to [33], we can prove that

$$(4.31) \quad (I - \mathcal{X})\mathcal{B}\hat{z} \in (L^2(\Sigma))^n.$$

In fact, by the a priori estimate (4.27), we have

$$(1 + |\eta|^2 + \rho^2)^{-\frac{1}{2}} (\sqrt{-1}\rho)\mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \in (L^2(\mathbb{R}_{\eta,\rho}^n))^n,$$

where  $\mathcal{F}_{(y,t)}(\mathcal{B}\hat{z})$  is the partial Fourier transform of  $\mathcal{B}\hat{z}$  with respect to  $y$  and  $t$ . Hence

$$\int_{\mathbb{R}^n} \frac{\rho^2}{1 + |\eta|^2 + \rho^2} \left| \mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \right|^2 d\eta d\rho < +\infty.$$

Since

$$\frac{c_0^2}{2 + c_0^2} \leq \frac{c_0^2}{\frac{1}{|\eta|^2} + 1 + c_0^2} = \frac{c_0^2|\eta|^2}{1 + |\eta|^2 + c_0^2|\eta|^2} \leq \frac{\rho^2}{1 + |\eta|^2 + \rho^2} \quad \text{as } \rho \geq c_0|\eta|, |\eta| \geq 1,$$

we have

$$\begin{aligned} \int_{\rho \geq c_0|\eta|} \left| \mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \right|^2 d\eta d\rho &\leq \int_{\rho \geq c_0|\eta|} \frac{2 + c_0^2}{c_0^2} \frac{\rho^2}{1 + |\eta|^2 + \rho^2} \left| \mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \right|^2 d\eta d\rho \\ &\leq \frac{2 + c_0^2}{c_0^2} \int_{\mathbb{R}^n} \frac{\rho^2}{1 + |\eta|^2 + \rho^2} \left| \mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \right|^2 d\eta d\rho < +\infty. \end{aligned}$$

Hence from the fact  $\text{supp}(1 - \chi) = \overline{\{\chi \neq 1\}} \subset \{\rho \geq c_0|\eta|\}$ , we have

$$\begin{aligned} \int_{\mathbb{R}^n} |1 - \chi|^2 \left| \mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \right|^2 d\eta d\rho &= \int_{\text{supp}(1-\chi)} |1 - \chi|^2 \left| \mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \right|^2 d\eta d\rho \\ &\leq \int_{\rho \geq c_0|\eta|} \left| \mathcal{F}_{(y,t)}(\mathcal{B}\hat{z}) \right|^2 d\eta d\rho < +\infty. \end{aligned}$$

Equation (4.31) is thus verified.

*Step 4.* By Remark 4.3, proving Theorem 1.1 is equivalent to showing

$$\mathcal{B}(P^{-1}u') \in (L^2(\Sigma))^n.$$

Since  $\phi(t) = 1$  for  $t \in [0, T]$  and  $u_c = \phi u$ , the above assertion is equivalent to

$$\mathcal{B}\hat{z} \in (L^2(\Sigma))^n \quad \text{with } \hat{z} = P^{-1}u'_c.$$

Thus from (4.31), the proof of Theorem 1.1 will be accomplished if we can show that

$$(4.32) \quad \mathcal{X}\mathcal{B}\hat{z} \in (L^2(\Sigma))^n.$$

Recall that  $u_c = v + w$ ; we have

$$(4.33) \quad \mathcal{X}\mathcal{B}\widehat{z} = \mathcal{X}\mathcal{B}(P^{-1}u'_c) = \mathcal{X}\mathcal{B}(P^{-1}(v + w)') = \mathcal{X}\mathcal{B}(P^{-1}v') + \mathcal{X}\mathcal{B}(P^{-1}w').$$

In the rest of this step, we show that  $\mathcal{X}\mathcal{B}(P^{-1}v') \in (L^2(\Sigma))^n$ . The proof of  $\mathcal{X}\mathcal{B}(P^{-1}w') \in (L^2(\Sigma))^n$  will be done in the next step.

First, we prove

$$(4.34) \quad \mathcal{X}\mathcal{B}\Phi \in (L^2(\Sigma))^n,$$

where  $\Phi := P^{-1}v'$  and  $v$  is the solution to the system (4.21).

In fact, applying  $\mathcal{X}$  to both sides of (4.21), we see that  $\mathcal{X}v$  satisfies

$$(4.35) \quad \begin{cases} \mathcal{P}\mathcal{X}v = -[\mathcal{X}, \mathcal{P}]v \in (H^{-1}(\widetilde{Q}))^n, \\ \mathcal{X}v|_{\partial\widetilde{Q}} \in (L^2(\partial\widetilde{Q}))^n, \end{cases}$$

where henceforth we denote  $\widetilde{Q} := \Omega \times [-T, 2T]$  and  $\widetilde{\Sigma} := \Gamma \times [-T, 2T]$ . Let us say a few words about this fact. Indeed, by the smoothness of the variable  $\phi$ , we can extend the regularity of  $(v, v')$  in (4.26) to

$$(4.36) \quad (v, v') \in C([-T, 2T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n) \subset (L^2(\widetilde{Q}))^n \times (H^{-1}(\widetilde{Q}))^n.$$

This together with the fact  $[\mathcal{X}, \mathcal{P}] \in OP(S^1(\widetilde{Q}))$  yields the governing equation (4.35):  $[\mathcal{X}, \mathcal{P}]v \in (H^{-1}(\widetilde{Q}))^n$ . The boundary  $\partial\widetilde{Q}$  is composed of three parts:  $\Omega \times \{-T\}$ ,  $\Omega \times \{2T\}$ , and  $\widetilde{\Sigma}$ . Considering  $\widetilde{\Sigma}$  first, we have  $\mathcal{X}v|_{\widetilde{\Sigma}} = \mathcal{X}(\phi g) \in (L^2(\widetilde{\Sigma}))^n$ . Second, since  $\text{supp}(v) \subset [0, (3/2)T]$ , we know that  $v(\cdot, -T) = 0 \in C^\infty(\Omega)$  and  $v(\cdot, 2T) = 0 \in C^\infty(\Omega)$ . By the pseudolocal property of the pseudodifferential operator  $\mathcal{X}$ , we get  $(\mathcal{X}v)(\cdot, -T) \in (C^\infty(\Omega))^n$  and  $(\mathcal{X}v)(\cdot, 2T) \in (C^\infty(\Omega))^n$ . This implies  $\mathcal{X}v \in (L^2(\Omega \times \{-T\}))^n$  and  $\mathcal{X}v \in (L^2(\Omega \times \{2T\}))^n$ . We thus have  $\mathcal{X}v|_{\partial\widetilde{Q}} = \mathcal{X}v|_{\widetilde{\Sigma}} + \mathcal{X}v|_{\Omega \times \{-T\}} + \mathcal{X}v|_{\Omega \times \{2T\}} \in (L^2(\partial\widetilde{Q}))^n$ , which is the boundary condition of (4.35).

Now since  $\mathcal{P}\mathcal{X}$  is a matrix formed elliptic pseudodifferential operator of order two, we can apply the classical regularity theory of elliptic pseudodifferential boundary value problems (see Theorem 4.4 of [13] or [14, 40]) to (4.35) to obtain

$$(4.37) \quad \mathcal{X}v \in (H^{1/2}(\widetilde{Q}))^n + (H^1(\widetilde{Q}))^n = (H^{1/2}(\widetilde{Q}))^n,$$

where the first term in the middle of (4.37) comes from the boundary regularity of (4.35), and the second term comes from the interior regularity.

Next, recall that  $\Phi = P^{-1}v'$ . We consider the following elliptic boundary value problem in the space  $\widetilde{Q} = \Omega \times [-T, 2T]$ :

$$\begin{cases} P\Phi = v' \text{ in } \widetilde{Q}, \\ \Phi|_{\partial\widetilde{Q}} = 0. \end{cases}$$

Applying  $\mathcal{X}$  to both sides of the above system, we obtain

$$(4.38) \quad P\mathcal{X}\Phi = \mathcal{X}v' + [P, \mathcal{X}]\Phi = \frac{d}{dt}\mathcal{X}v - \left[\frac{d}{dt} \cdot I, \mathcal{X}\right]v + [P, \mathcal{X}]\Phi.$$

Since  $\mathcal{X} \in OP(S^0(\widetilde{Q}))$ ,  $P \in OP(S^2(\widetilde{Q}))$ , and  $\frac{d}{dt} \cdot I \in OP(S^1(\widetilde{Q}))$ , we have

$$(4.39) \quad [P, \mathcal{X}] \in OP(S^1(\widetilde{Q})), \quad \left[\frac{d}{dt} \cdot I, \mathcal{X}\right] \in OP(S^0(\widetilde{Q})).$$



From (4.36), we know that  $v' \in C([-T, 2T]; (H^{-1}(\Omega))^n)$ , and hence

$$(4.40) \quad \Phi = P^{-1}v' \in C([-T, 2T]; (H^1(\Omega))^n).$$

Again from (4.36), we have  $v \in (L^2(\tilde{Q}))^n$ . This together with (4.39) and (4.40) concludes that

$$(4.41) \quad - \left[ \frac{d}{dt} \cdot I, \mathcal{X} \right] v + [P, \mathcal{X}]\Phi \in (L^2(\tilde{Q}))^n.$$

From (4.37) and using the anisotropic Hörmander spaces on page 477 of [23], we have

$$\mathcal{X}v \in (H_{(\frac{1}{2}, \frac{1}{2})}(\tilde{Q}))^n \subset (H_{(0, \frac{1}{2})}(\tilde{Q}))^n.$$

In the space  $H_{(m,s)}(\tilde{Q})$ ,  $m$  is the order in the normal direction to the plane  $x = 0$  (which plays a significant role) and  $m + s$  is the order in the tangential direction in  $t$  and  $y$ . Since  $\frac{d}{dt} \cdot I$  is a first order differential operator along the tangential direction, we have

$$(4.42) \quad \frac{d}{dt} \mathcal{X}v \in (H_{(0, -\frac{1}{2})}(\tilde{Q}))^n \subset (H_{(-\frac{1}{2}, 0)}(\tilde{Q}))^n = (H^{-1/2}(\tilde{Q}))^n.$$

Finally, by (4.38), (4.41), and (4.42), we need to solve the following boundary value problem:

$$\begin{cases} P\mathcal{X}\Phi \in (H^{-1/2}(\tilde{Q}))^n + (L^2(\tilde{Q}))^n = (H^{-1/2}(\tilde{Q}))^n, \\ \mathcal{X}\Phi|_{\partial\tilde{Q}} = 0. \end{cases}$$

Since  $P\mathcal{X}$  is elliptic, again by the classical regularity theory of elliptic pseudodifferential boundary value problems (see Theorem 4.4 of [13] or [14, 40]), we have

$$\mathcal{X}\Phi \in (H^{3/2}(\tilde{Q}))^n, \quad \mathcal{B}(\mathcal{X}\Phi) \in (L^2(\tilde{\Sigma}))^n.$$

Since  $Q = \Omega \times (0, T) \subset \tilde{Q} = \Omega \times [-T, 2T]$  and  $\Sigma = \Gamma \times [0, T] \subset \tilde{\Sigma} = \Gamma \times [-T, 2T]$ , the above fact implies that

$$(4.43) \quad \mathcal{X}\Phi \in (H^{3/2}(Q))^n, \quad \mathcal{B}(\mathcal{X}\Phi) \in (L^2(\Sigma))^n.$$

Moreover, from  $\mathcal{X} \in OP(S^0(Q))$  and  $\mathcal{B} \in OP(S^1(Q))$ , we know that  $[\mathcal{X}, \mathcal{B}] \in OP(S^0(Q))$ . This together with  $\Phi = P^{-1}v' \in C([-T, 2T]; (H^1(\Omega))^n)$  that comes from (4.40) yields

$$[\mathcal{X}, \mathcal{B}]\Phi \in C([-T, 2T]; (H^1(\Omega))^n).$$

Then by the trace theorem for Sobolev spaces, we have

$$(4.44) \quad [\mathcal{X}, \mathcal{B}]\Phi \in C([0, T]; (H^{1/2}(\Gamma))^n) \subset (L^2(\Sigma))^n.$$

Thus from (4.43) and (4.44), we have

$$\mathcal{X}\mathcal{B}\Phi = [\mathcal{X}, \mathcal{B}]\Phi + \mathcal{B}(\mathcal{X}\Phi) \in (L^2(\Sigma))^n.$$

This is (4.34).

Step 5. This step is devoted to showing

$$(4.45) \quad \mathcal{X}\mathcal{B}\Psi \in (L^2(\Sigma))^n,$$

where  $\Psi := P^{-1}w'$  and  $w$  is the solution to the system (4.22). We obtain (4.45) along the same line of Step 4.

First, apply  $\mathcal{X}$  to both sides of (4.22) to obtain

$$(4.46) \quad \begin{cases} \mathcal{P}\mathcal{X}w = -[\mathcal{X}, \mathcal{P}]w + \mathcal{X}\mathcal{P}w = -[\mathcal{X}, \mathcal{P}]w + \mathcal{X}f \in (H^{-1}(\tilde{Q}))^n, \\ \mathcal{X}w|_{\partial\tilde{Q}} \in (C^\infty(\partial\tilde{Q}))^n. \end{cases}$$

By extending the result in (4.25) to the case of  $[-T, 2T]$ , we get

$$(4.47) \quad (w, w') \in C([-T, 2T]; (L^2(\Omega))^n \times (H^{-1}(\Omega))^n) \subset (L^2(\tilde{Q}))^n \times (H^{-1}(\tilde{Q}))^n.$$

Similarly, we get  $f \in C([-T, 2T]; (H^{-1}(\Omega))^n) \subset (H^{-1}(\tilde{Q}))^n$  from (4.24). From these results and the facts  $\mathcal{X} \in OP(S^0(\tilde{Q}))$  and  $[\mathcal{X}, \mathcal{P}] \in OP(S^1(\tilde{Q}))$ , we obtain  $\mathcal{P}\mathcal{X}w \in (H^{-1}(\tilde{Q}))^n$ . This is the interior regularity of the governing equation of (4.46). Moreover, as in Step 4, we can obtain  $\mathcal{X}w \in (C^\infty(\Omega \times \{-T\}))^n$  and  $\mathcal{X}w \in (C^\infty(\Omega \times \{2T\}))^n$  by the pseudolocal property of the pseudodifferential operator  $\mathcal{X}$ . This together with  $\mathcal{X}w|_{\tilde{\Sigma}} = 0 \in (C^\infty(\tilde{\Sigma}))^n$  yields the boundary condition of (4.46).

Once again, apply the classical regularity theory (see Theorem 4.4 of [13] or [14, 40]) to the elliptic system (4.46) to obtain  $\mathcal{X}w \in (H^1(\tilde{Q}))^n$ . Hence, from  $\frac{d}{dt} \cdot I \in OP(S^1(\tilde{Q}))$ , we have

$$(4.48) \quad \frac{d}{dt}\mathcal{X}w \in (L^2(\tilde{Q}))^n.$$

Next, notice that  $\Psi = P^{-1}w'$ . We consider the elliptic boundary value problem

$$\begin{cases} P\Psi = w' & \text{in } \tilde{Q}, \\ \Psi|_{\tilde{\Sigma}} = 0. \end{cases}$$

Apply  $\mathcal{X}$  to both sides of the above system to get

$$(4.49) \quad \mathcal{P}\mathcal{X}\Psi = \mathcal{X}w' + [P, \mathcal{X}]\Psi = \frac{d}{dt}\mathcal{X}w - \left[ \frac{d}{dt} \cdot I, \mathcal{X} \right] w + [P, \mathcal{X}]\Psi.$$

By virtue of (4.47), we know that  $w' \in C([-T, 2T]; (H^{-1}(\Omega))^n)$ , and hence

$$(4.50) \quad \Psi = P^{-1}w' \in C([-T, 2T]; (H^1(\Omega))^n).$$

Again by (4.47), we know  $w \in (L^2(\tilde{Q}))^n$ . This together with (4.39) and (4.50) yields

$$(4.51) \quad - \left[ \frac{d}{dt} \cdot I, \mathcal{X} \right] w + [P, \mathcal{X}]\Psi \in (L^2(\tilde{Q}))^n.$$

Finally, from (4.48), (4.49), and (4.51), we are led to solving the boundary value problem

$$(4.52) \quad \begin{cases} \mathcal{P}\mathcal{X}\Psi \in (L^2(\tilde{Q}))^n, \\ \mathcal{X}\Psi|_{\tilde{\Sigma}} = 0. \end{cases}$$

By the classical regularity theory of elliptic boundary value problems (see, e.g., Theorem 20.1.2 of [23]), we have

$$\mathcal{X}\Psi \in (H^2(\tilde{Q}))^n.$$

This implies that  $\mathcal{X}\Psi \in (H^2(Q))^n$  since  $Q \subset \tilde{Q}$ . By the trace theorem of Sobolev spaces, we obtain

$$\mathcal{B}(\mathcal{X}\Psi) \in (H^{1/2}(\Sigma))^n \subset (L^2(\Sigma))^n.$$

This together with the same argument at the end of Step 4 concludes (4.45).

Combine (4.33), (4.34), and (4.45) to yield

$$\mathcal{X}\mathcal{B}\hat{z} = \mathcal{X}\mathcal{B}\Phi + \mathcal{X}\mathcal{B}\Psi \in (L^2(\Sigma))^n.$$

This is (4.32). The proof is complete.  $\square$

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