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WELL-POSEDNESS OF THE BOUNDARY LAYER EQUATIONS*

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Abstract. We consider the mild solutions of the Prandtl equations on the half space. Requiring analyticity only with respect to the tangential variable, we prove the short time existence and the uniqueness of the solution in the proper function space. The proof is achieved applying the abstract Cauchy–Kowalewski theorem to the boundary layer equations once the convection-diffusion operator is explicitly inverted. This improves the result of [M. Sammartino and R. E. Caflisch, *Comm. Math. Phys.*, 192 (1998), pp. 433–461], as we do not require analyticity of the data with respect to the normal variable.

Key words. boundary layer, Prandtl equations

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1. Introduction.

In this paper we shall be concerned with the unsteady Prandtl equations on the half space. They describe the behavior of an incompressible fluid close to a physical boundary in the limit of small viscosity [19]. The system we shall deal with is the following:

$$(\partial_t - \partial_{YY})\, u^P + u^P \partial_x u^P + v^P \partial_Y u^P + \partial_x p^P = 0 \, , \tag{1.1}$$

$$\partial_Y p^P = 0 \, , \tag{1.2}$$

$$\partial_x u^P + \partial_Y v^P = 0 \, , \tag{1.3}$$

$$u^P(x, Y = 0, t) = v^P(x, Y = 0, t) = 0 \, , \tag{1.4}$$

$$u^P(x, Y \to \infty, t) \longrightarrow U(x, t) \, , \tag{1.5}$$

$$p^P(x, Y \to \infty, t) \longrightarrow p^E(x, y = 0, t) \, , \tag{1.6}$$

$$u^P(x, Y, t = 0) = u^P_{in} \, . \tag{1.7}$$

In the above equations (u^P, v^P) and p^P represent the components of the fluid velocity and the pressure inside the boundary layer. Equation (1.3) is the incompressibility condition and equations (1.4) are the boundary conditions: $u^P(x, Y = 0, t) = 0$ is the no-slip condition and $v^P(x, Y = 0, t) = 0$ is the no-influx condition. Equation (1.5) is the matching condition between the flow inside the boundary layer and the outer Euler flow; $U(x, t)$ is the tangential component of the Euler flow at the boundary; $x = (x_1, x_2)$ is the tangential variable, and Y the normal variable.

The Prandtl equations can be regarded as asymptotic equations of the Navier–Stokes equations in the limit of vanishing viscosity ($\nu \to 0$). In the limit case $\nu = 0$, the higher derivative term is dropped from the Navier–Stokes system and one gets

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the Euler equations, which rule the behavior of inviscid flows. Since the Euler system is first order, we have a reduction of the order of the equations, and a corresponding reduction must be done in the number of the boundary conditions: only the normal component of the velocity can be imposed at the boundary. Since the Navier–Stokes equations impose the value of both the velocity components at the boundary, one must allow a thin layer where there is a rapid variation of the fluid velocity from zero (imposed by the no-slip condition) to the value prescribed by the inviscid equations. Hence, in the boundary layer (whose size is $O(\sqrt{\nu})$), vorticity is generated so that the viscosity term $\nu\Delta\boldsymbol{u}$ is $O(1)$, even as the viscosity goes to zero. The fluid develops an internal length scale so that one is faced with a singular perturbation problem. Rescaling the normal variable with the square root of the viscosity, and writing the solution to the Navier–Stokes equations in the form of an asymptotic series, one gets the equations which rule the fluid inside the boundary layer, i.e., Prandtl equations.

The equations were first derived by Prandtl in 1904, and the practical success of the boundary layer theory was soon overwhelming. Nevertheless, the theoretical foundation of the boundary layer theory was rather unsatisfactory, and many fundamental questions are still debated. For instance, the problem of establishing a well-founded mathematical connection to the Navier–Stokes equation has been solved only recently, and neither existence, uniqueness, nor well-posedness of the boundary layer equation is proved in the general case.

Regarding the problem of the convergence of the Prandtl equations to the Navier–Stokes equations, a major complication is given by the fact that no uniqueness theorem with Sobolev-type initial data for the three-dimensional Navier–Stokes (nor Euler) equations has been proved, and the time of existence of a regular solution depends on the data and on the viscosity (see Marsden [13] and the monographs Constantin and Foias [7] and Temam [21]). In the absence of boundaries the convergence of viscous planar flow to ideal planar flow was shown by Swann [20] for a time which is independent of the viscosity and, lately, in the case of concentrated vorticity, by Constantin and Wu [8].

In the presence of boundaries the problem is harder. Kato [10] proved that a necessary and sufficient condition for the convergence of $\boldsymbol{u}^{NS}$ to the solution of Euler equations, $\boldsymbol{u}^{E}$, in $L^2(\Omega)$ uniformly in $t \in [0,T]$ is that the energy dissipation for $\boldsymbol{u}^{NS}$ in a small layer close to the boundary of size $O(\nu)$, during the interval $[0,T]$, tends to zero. However, such result gives no ultimate solution to the problem because of the unverified energy estimate on the Navier–Stokes solution. With a similar condition on the L^2-norm of the gradient of the pressure, Temam and Wang [22] proved the convergence of the Navier–Stokes solution to the solution of the Euler equation in a strip.

Analogously it is also hard to prove the convergence of the Navier–Stokes solution to the Prandtl solution under satisfactory hypotheses: the few existence and uniqueness theorems proved for the unsteady case hold in particular cases. For instance, Oleinik proved the existence and uniqueness of the Prandtl equations on the half space requiring prescribed horizontal velocities positive and strictly increasing. See [14] for a review.

The first results which do not require monotonicity of the initial data were proved by Sammartino and Caflisch, after the earlier work of Asano [2]. In [17], assuming analyticity of the initial data with respect to the spatial variables, they proved the existence and uniqueness of the Prandtl equations on the half space. They achieved the result using an abstract formulation of the Cauchy–Kowalewski theorem in the Banach spaces of analytic functions. In [18] they performed the asymptotic analysis of the Navier–Stokes equation in the limit of zero viscosity. They constructed the solution in the form of an asymptotic series in $\sqrt{\nu}$, whose zeroth order term is constituted by the sum of the Euler and the Prandtl solutions. The norm of the first order correction term is then proved to be bounded in the proper function space. They also proved an analogous result in the case of a curved boundary (see [5]).

In the linear case it has been possible to prove the convergence of the linearized Navier–Stokes equations to the corresponding inviscid equations for Sobolev-type initial data. The asymptotic analysis has been successfully performed for the Stokes equations on the half space (Sammartino [16]) and on the exterior of a disk (Lombardo, Caflisch, and Sammartino [11]). Similar results were achieved for the Oseen equations, i.e., the Navier–Stokes equations linearized around a nonzero flow, on a strip (see Lombardo and Sammartino [12] and Temam and Wang [23]).

Temam and Wang analyzed the linear case for a general $2-D$ exterior domain (see [24] and [25]), but they obtained weaker convergence results. In the nonlinear case, with blowing and suction boundary conditions [26], they were able to prove that these boundary conditions stabilize the boundary layer.

In the opposite direction Grenier [9] proved that a solution of the Prandtl equations is linearly and nonlinearly unstable, and, therefore, it does not converge in H^1 to the Navier–Stokes solutions.

A review about the mathematical aspects of the boundary layer theory can be found in [4].

In this paper we extend the result of [17] to a wider class of initial data, namely, the functions which are analytic only with respect to the tangential variable and L^2, together with their derivatives, with respect to the normal variable. Through the explicit expression of the Green's function, we invert the second order parabolic operator appearing in the Prandtl equation, including the first order Y-derivative. We are thus able to obtain a mild form of the system. The existence and the uniqueness of the solution are then proved using a slightly modified version of the abstract Cauchy–Kowalewski (ACK) theorem in the Banach spaces.

The results presented in this paper were previously announced in [6].

The paper is organized as follows. In section 2 we define the function spaces where existence and uniqueness will be proved. In section 3 we state the abstract Cauchy–Kowalewski theorem in the Banach spaces. In section 4 the parabolic initial-boundary value problem is explicitly solved and the norm of the corresponding operators bounded in the proper function spaces. The mild form of the Prandtl equation is given in section 5. In sections 6 and 7 the source term of the Prandtl equation is proved to satisfy the hypotheses of the ACK theorem. Finally the main theorem is stated in section 8. For convenience two appendices are inserted. In Appendix A a sketch of the proof of the ACK theorem is given. In Appendix B the estimates of the pseudodifferential operator defined in section 4 are proved.

2. Function spaces. In this section we introduce the function spaces used in the proof of the existence and uniqueness of the Prandtl equations. We first define the domain of analyticity with respect to the tangential variable:

$$D(\rho) = \{x \in \mathbb{C} \ : \Im x \in (-\rho, \rho)\} .$$

We now introduce the ambient spaces for the Prandtl equations.

DEFINITION 2.1. *The space $K^{l,\rho}$ is the space of the functions $f(x)$ such that*

- *f is analytic in $D(\rho)$;*
- *if $\Im x \in (-\rho,\rho)$ and $0 \le j \le l$, then $\partial_x^j f(\Re x + i\Im x)$ is square integrable in $\Re x$;*
- *$|f|_{l,\rho} \equiv \sum_{j=0}^{l} \sup_{\Im x\in(-\rho,\rho)} \|\partial_x^j f(\cdot + i\Im x)\|_{L^2(\Re x)} < \infty$.*

DEFINITION 2.2. *The space $K^{l,\rho,\mu}$, with $\mu > 0$, is the space of the functions $f(Y,x)$ such that*

$$e^{\mu Y}\partial_x^i\partial_Y^j f \in L^\infty(\mathbb{R}^+, K^{0,\rho}) \text{ when } i+j\le l \text{ and } j \le 2.$$

The norm in $K^{l,\rho,\mu}$ is defined as

$$|f|_{l,\rho,\mu} \equiv \sum_{j\le 2}\sum_{i\le l-j}\sup_{Y\in\mathbb{R}^+} e^{\mu Y}\,|\partial_Y^j\partial_x^i f(Y,\cdot)|_{0,\rho}.$$

DEFINITION 2.3. *The space $K^{l,\rho}_{\beta,T}$, with $\beta > 0$ and $\rho-\beta T>0$, is the space of the functions $f(x,t)$ such that*

$$\partial_t^i\partial_x^j f(x,t) \in K^{l,\rho-\beta t} \quad \forall 0\le t\le T, \text{ where } 0\le i+j\le l \text{ and } 0\le i\le 1.$$

Moreover,

$$|f|_{l,\rho,\beta,T} \equiv \sum_{0\le j\le 1}\sum_{i\le l-j}\sup_{0\le t\le T}|\partial_t^j\partial_x^i f(\cdot,t)|_{0,\rho-\beta t} < \infty\ .$$

DEFINITION 2.4. *The space $K^{l,\rho,\mu}_{\beta,T}$, with $\beta>0$, $\rho-\beta T>0$ and $\mu-\beta T>0$, is the space of the functions $f(x,Y,t)$ such that*

$$f\in K^{l,\rho-\beta t,\mu-\beta t} \text{ and } \partial_t\partial_x^i f\in K^{0,\rho-\beta t,\mu-\beta t} \quad \forall 0\le t\le T, \text{ where } 0\le i\le l-2.$$

Moreover,

$$\begin{aligned}|f|_{l,\rho,\mu,\beta,T} \equiv &\sum_{0\le j\le 2}\sum_{i\le l-j}\sup_{0\le t\le T}|\partial_Y^j\partial_x^i f(\cdot,\cdot,t)|_{0,\rho-\beta t,\mu-\beta t}\\ &+\sum_{i\le l-2}\sup_{0\le t\le T}|\partial_t\partial_x^i f(\cdot,\cdot,t)|_{0,\rho-\beta t,\mu-\beta t} < \infty\ .\end{aligned}$$

3. The abstract Cauchy–Kowalewski theorem.

To prove the existence and the uniqueness of the mild solution to the Prandtl equations, we shall give a slightly modified version of the abstract Cauchy–Kowalewski (ACK) theorem as given in [15] or [1] and [3].

For t in $[0,T]$, consider the equation

$$u + F(t,u) = 0. \tag{3.1}$$

Let $\{X_\rho : 0<\rho\le\rho_0\}$ be a Banach scale with norms $|\cdot|_\rho$ such that $X_{\rho'}\subset X_{\rho''}$ and $|\cdot|_{\rho''}\le|\cdot|_{\rho'}$ when $\rho''\le\rho'\le\rho_0$.

THEOREM 3.1 (ACK theorem). *Suppose that $\exists R>0$, $\rho_0>0$, and $\beta_0>0$ such that if $0<t\le\rho_0/\beta_0$, the following properties hold:*

(1) *$\forall\, 0 < \rho' < \rho \le \rho_0$ and $\forall u$ such that $\{u \in X_\rho : \sup_{0\le t\le T} |u(t)|_\rho \le R\}$ the map $F(t,u) : [0,T] \mapsto X_{\rho'}$ is continuous.*

(2) *$\forall\, 0 < \rho < \rho_0$ the function $F(t,0) : [0, \rho_0/\beta_0] \mapsto \{u \in X_\rho : \sup_{0\le t\le T} |u(t)|_\rho \le R\}$ is continuous and*

$$|F(t,0)|_\rho \le R_0 < R\,. \tag{3.2}$$

(3) *$\forall 0 < \rho' < \rho(s) < \rho_0$ and $\forall\ u^1$ and $u^2 \in \{u \in X_\rho : \sup_{0\le t\le T} |u(t)|_{\rho-\beta_0 t} \le R\}$,*

$$|F(t,u^1) - F(t,u^2)|_{\rho'} \le C \int_0^t ds \left(\frac{|u^1-u^2|_{\rho(s)}}{\rho(s)-\rho'} + \frac{|u^1-u^2|_{\rho'}}{\sqrt{t-s}} \right). \tag{3.3}$$

Then $\exists \beta > \beta_0$ such that $\forall 0 < \rho < \rho_0$, (3.1) has a unique solution $u(t) \in X_\rho$ with $t \in [0, (\rho_0-\rho)/\beta]$; moreover $\sup_{\rho<\rho_0-\beta t} |u(t)|_\rho \le R$.

The proof of the above theorem is given in Appendix A.

4. A parabolic equation.

The next section will be devoted to writing Prandtl equations in the form given by (3.1). The main difficulty in doing this is in the parabolic nature of the Prandtl equation. We shall solve this difficulty by inverting the parabolic operator $(\partial_t - \partial_{YY} + \alpha\, Y\, \partial_Y)$, giving the explicit expression of the Green's function.

We introduce the kernels

$$F_\alpha(x,Y,t) = \frac{1}{\sqrt{4\pi}} \frac{1}{\Psi(x,t)} \exp\left(- \frac{Y^2 e^{-2A(x,t)}}{4(\Psi(x,t))^2} \right), \tag{4.1}$$

$$E_\alpha(x,Y,t) = \int_0^\infty dY' \left[F_\alpha(x, Y-Y', t) - F_\alpha(x, Y+Y', t) \right], \tag{4.2}$$

$$H_\alpha(x,Y,t) = -\frac{\partial F_\alpha}{\partial Y}(x,Y,t) + \alpha(x,t) Y F_\alpha(x,Y,t) - \frac{1}{2}\alpha(x,t) E_\alpha(x,Y,t), \tag{4.3}$$

where α is a function of x and t, and $A(x,\tau)$ is defined as

$$A(x,\tau) = \int_0^\tau d\theta\ \alpha(x,\theta) \tag{4.4}$$

and

$$\Psi(x,t) = \left(\int_0^t d\tau\ e^{-2A(x,\tau)} \right)^{1/2}. \tag{4.5}$$

The operator M_0 is the convolution of the kernel F_α with the odd extension to $Y < 0$ of the function $u_0(x,Y)$:

$$M_0 u_0 = \int_0^\infty dY'\ \left[F_\alpha(Y-Y', t) - F_\alpha(Y+Y', t) \right]\, u_0(x,Y'). \tag{4.6}$$

It solves the following system:

$$(\partial_t - \partial_{YY} + \alpha\, Y\, \partial_Y) M_0 u_0 = 0, \tag{4.7}$$

$$M_0 u_0(x, Y=0, t) = 0, \tag{4.8}$$

$$M_0 u_0(x, Y, t=0) = u_0. \tag{4.9}$$

We now introduce the operator M_2:

$$M_2 f = \int_0^t ds \int_0^\infty dY' \, [F_\alpha(Y-Y', t-s) - F_\alpha(Y+Y', t-s)] \, f(x, Y', s). \tag{4.10}$$

It solves the parabolic equations with zero boundary and initial data:

$$(\partial_t - \partial_{YY} + \alpha Y \partial_Y) M_2 f = f, \tag{4.11}$$

$$M_2 f(x, Y=0, t) = 0, \tag{4.12}$$

$$M_2 f(x, Y, t=0) = 0. \tag{4.13}$$

The operator M_1 acts on functions defined on the boundary, namely,

$$M_1 g = 2 \int_0^t ds \; H_\alpha(Y, t-s) \, g(x, s), \tag{4.14}$$

and solves the following system:

$$(\partial_t - \partial_{YY} + \alpha Y \partial_Y) M_1 g = 0, \tag{4.15}$$

$$M_1 g(x, Y=0, t) = g, \tag{4.16}$$

$$M_1 g(x, Y, t=0) = 0. \tag{4.17}$$

Finally we define the operator $M_3 h$:

(4.18)

$$M_3 h = -\int_0^t ds \int_0^\infty dY' \, \partial_Y \, [F_\alpha(x, Y-Y', t-s) - F_\alpha(x, Y+Y', t-s)] \; h(x, Y', s).$$

Notice that if $h(x, Y=0, t) = 0$, then, integrating by parts, one gets $M_3 h \equiv M_2 \partial_Y h$.

We shall now give some estimates on the above operators. We begin with the estimates on the operator M_2.

PROPOSITION 4.1. *Let $\alpha \in K^{l,\rho}_{\beta,T}$, $f \in K^{l,\rho,\mu}_{\beta,T}$ with $f|_{Y=0} = 0$. If $\rho' < \rho - \beta t$ and $\mu' < \mu - \beta t$, then the following estimate holds:*

$$|M_2 f|_{l,\rho',\mu'} \le c \int_0^t ds \; |f(\cdot,\cdot,s)|_{l,\rho',\mu'} \le c \, |f|_{l,\rho,\mu,\beta,T},$$

where the constant c depends on $|\alpha|_{l,\rho,\beta,T}$.

PROPOSITION 4.2. *Let $\alpha \in K^{l,\rho}_{\beta,T}$, $f \in K^{l,\rho,\mu}_{\beta,T}$. Then $M_2 f \in K^{l,\rho,\mu}_{\beta,T}$ and the following estimate holds:*

$$|M_2 f|_{l,\rho,\mu,\beta,T} \le c \, |f|_{l,\rho,\mu,\beta,T} \; .$$

The following estimate of $M_3 h$ will be crucial in handling the nonlinear term containing the Y-derivative.

PROPOSITION 4.3. *Suppose $\alpha \in K^{l,\rho}_{\beta,T}$, $h \in K^{l,\rho,\mu}_{\beta,T}$ with $h|_{Y=0} = 0$, $\partial_Y h|_{Y=0} = 0$. If $0 < \mu' < \mu(s) < \mu - \beta s$, then $M_3 h \in K^{l,\rho,\mu'}$ for each $0 < t < T$ and the following estimate holds:*

$$|M_3 h|_{l,\rho,\mu'} \le c \int_0^t ds \; \left(\frac{|h(\cdot,\cdot,s)|_{l,\rho,\mu'}}{\sqrt{t-s}} + \frac{|h(\cdot,\cdot,s)|_{l,\rho,\mu(s)}}{\mu(s) - \mu'} \right).$$

The proofs of the above propositions are given in Appendix B.

We finally give some bounds on the operators M_0 and M_1.

PROPOSITION 4.4. *Let $\alpha \in K^{l,\rho}_{\beta,T}$ and $u_0(x,Y) \in K^{l,\rho,\mu}$. Moreover let the compatibility condition $u_0(x,Y=0)=0$. Then $M_0 u_0 \in K^{l,\rho,\mu}_{\beta,T}$ and the following estimate holds:*

$$|M_0 u_0|_{l,\rho,\mu,\beta,T} \le c\,|u_0|_{l,\rho,\mu}\ .$$

PROPOSITION 4.5. *Let $\alpha, g \in K^{l,\rho}_{\beta,T}$ and $g(x,t=0)=0$. Then $M_1 g \in K^{l,\rho,\mu}_{\beta,T}$ and the following estimate holds:*

$$|M_1 g|_{l,\rho,\mu,\beta,T} \le c\,|g|_{l,\rho,\beta,T}\ .$$

We will also need the following lemma.

LEMMA 4.6. *Let $\alpha \in K^{l,\rho}_{\beta,T}$, $w = u+g$ with $u \in K^{l,\rho,\mu}$, and $g \in K^{l,\rho}$, i.e., constant with respect to Y and t. Moreover, let $u(x,Y=0) = -g(x)$. Then $M_0(t)w - g \in K^{l,\rho,\mu}\,\forall t$ and the following estimate holds:*

$$\sup_{0\le t\le T} |M_0(t)w - g\ |_{\ l,\rho,\mu} \le c\,(|\alpha|_{l,\rho,\beta,T} + |u|_{l,\rho,\mu} + |g|_{l,\rho})\,.$$

5. The mild form of the Prandtl equations.

In this section, following the same procedure used in [17], we shall recast the Prandtl equations in a form suitable for the application of the ACK theorem.

First, one can get rid of the pressure gradient introducing the new variable u:

$$u = u^P - U\ . \tag{5.1}$$

In fact, written in terms of the variable u and using the Euler equation at the boundary,

$$\partial_t U + U\partial_x U + \partial_x p^E|_{y=0} = 0, \tag{5.2}$$

equations (1.1)–(1.7) become

$$(\partial_t - \partial_{YY} + Y\partial_x U\,\partial_Y)\,u + u\,\partial_x u - \left(\int_0^Y dY'\partial_x u\right)\partial_Y u + U\,\partial_x u + u\,\partial_x U = 0, \tag{5.3}$$

$$u(x,Y=0,t) = -U, \tag{5.4}$$

$$u(x,Y\to\infty,t) = 0, \tag{5.5}$$

$$u(t=0) = u^P_{in} - U(t=0) \equiv u_0, \tag{5.6}$$

where we have also used the incompressibility condition, written as

$$v^P = -\int_0^Y \partial_x u^P dY' = -\left(\int_0^Y \partial_x u\ dY' + Y\partial_x U\right). \tag{5.7}$$

We can now define the quantities

$$K_1(u,t) = -\left(2u\,\partial_x u + U\,\partial_x u + u\,\partial_x U\right), \tag{5.8}$$

$$K_2(u,t) = \partial_Y\left(u\int_0^Y dY'\ \partial_x u,\right) \tag{5.9}$$

and the operator $F(u,t)$ as

$$F(u,t) = M_2K_1(u,t) + M_2K_2(u,t) + \mathcal{C}, \tag{5.10}$$

where we have identified the $\alpha(x,t)$ appearing in the kernel F_α with $-\partial_x U(x,t)$, and where $\mathcal{C}$ is defined by

$$\mathcal{C} = M_0(t)\,(u_0 + U(t=0)) - M_1\,(U - U(t=0)) - U(t=0). \tag{5.11}$$

Given that $(u\int_0^Y dY'\ \partial_x u)\,|_{Y=0} = 0$, $F(u,t)$ can be written as

$$F(u,t) = M_2K_1(u,t) + M_3K_3(u,t) + \mathcal{C}, \tag{5.12}$$

where $K_3(u,t)$ is defined as

$$K_3(u,t) = u\int_0^Y dY'\ \partial_x u. \tag{5.13}$$

Therefore (5.3), together with the boundary and initial condition (5.4)–(5.6), can finally be written in the form

$$u = F(u,t). \tag{5.14}$$

We call (5.14) with $F(u,t)$ defined in (5.12), and with M_2, M_3, K_1, K_3 defined in (4.10), (4.18), (5.8), (5.13), respectively, the mild form of the Prandtl equations. We are now left to prove that the operator $F(u,t)$, given by (5.12), satisfies the hypotheses of the ACK theorem.

6. The forcing term. It is obvious that the operator $F(u,t)$ satisfies assumption 1 of the ACK theorem. In this section we shall show that it satisfies assumption 2, namely, that $F(0,t) \in K^{l,\rho,\mu}$ and that $\forall t \in [0,t]$

$$|F(0,t)|_{l,\rho,\mu} \leq R_0. \tag{6.1}$$

Since

$$F(0,t) = \mathcal{C}, \tag{6.2}$$

using Lemma 4.6 and Proposition 4.5, one gets the following.

Proposition 6.1. *Suppose that* $u_0 \in K^{l,\rho,\mu}$ *with* $u_0(\cdot, Y=0) = -U(t=0)$ *and* $U \in K^{l,\rho}_{\beta,T}$. *Then* $F(0,t) \in K^{l,\rho,\mu}_{\beta,T}$ *and the following estimate holds:*

$$|F(0,t)|_{l,\rho,\mu,\beta,T} \leq c\left(|U|_{l,\rho,\beta,T} + |u_0|_{l,\rho,\mu}\right).$$

This proves that the forcing term can be estimated in terms of the initial condition for Prandtl equations and the outer Euler flow. Notice that the compatibility condition $u_0(\cdot, Y=0) = -U(t=0)$ is necessary for the hypotheses of Lemma 4.6 to be verified.

7. The contractiveness property of the operator F. In this section we shall prove that the operator F, given by (5.10), satisfies assumption 3 of the ACK theorem. Namely, we shall prove the following.

THEOREM 7.1. *Suppose that* u^1 *and* u^2 *are in* $K^{l,\rho_0,\mu_0}_{\beta_0,T}$. *Suppose* $0 < \rho' < \rho(s) < \rho_0 s$ *and* $0 < \mu' < \mu(s) < \mu_0$. *Then the following estimate holds:*

$$\begin{aligned} &\left| \, F(u^1,t) - F(u^2,t) \, \right|_{l,\rho',\mu'} \\ &\le c \int_0^t ds \, \left(\frac{|u^1-u^2|_{l,\rho(s),\mu}}{\rho(s)-\rho'} + \frac{|u^1-u^2|_{l,\rho,\mu(s)}}{\mu(s)-\mu'} + \frac{|u^1-u^2|_{l,\rho',\mu'}}{\sqrt{t-s}} \right). \end{aligned} \tag{7.1}$$

To prove the above theorem we have to bound the operators M_2K_1 and M_3K_3. The first one contains two different kinds of terms: the nonlinear term, $u\partial_x u$, and two linear terms. They all will be estimated through the Cauchy estimate in the x-variable. The operator M_3K_3, which contains the nonlinear term involving the Y-derivative, will be estimated using the properties of the kernel of the operator M_3.

7.1. The operator M_2K_1. We start with the estimate of the nonlinear term involving the x-derivative. One has the following Cauchy estimate for the derivative of an analytic function.

PROPOSITION 7.2. *Let* $f \in K^{l,\rho''}$. *If* $\rho' < \rho''$, *then*

$$|\partial_x f|_{l,\rho'} \le \frac{|f|_{l,\rho''}}{\rho''-\rho'} \, . \tag{7.2}$$

Therefore the following proposition can be proved.

PROPOSITION 7.3. *Suppose that* u^1 *and* u^2 *are in* $K^{l,\rho_0,\mu_0}_{\beta_0,T}$. *Suppose* $0 < \rho' < \rho(s) < \rho_0$. *Then the following estimate holds:*

$$\left| \, u^1\partial_x u^1 - u^2\partial_x u^2 \, \right|_{l,\rho',\mu'} \le c \, \frac{|u^1-u^2|_{l,\rho,\mu}}{\rho-\rho'}, \tag{7.3}$$

where the constant c *depends only on* $|u^1|_{l,\rho_0,\mu_0,\beta,T}$ *and* $|u^2|_{l,\rho_0,\mu_0,\beta,T}$.

The proof of the above proposition can be found in [17].

The estimate of the linear terms is easily achieved using the following lemma.

LEMMA 7.4. *Let* $U \in K^{l,\rho}_{\beta,T}$ *and let* $\rho' < \rho$; *then* $\forall \; 0 < t \le T$

$$\sup_{x\in D(\rho')} |\partial^l_x U(\cdot,t)| \le c \; |U|_{l,\rho,\beta,T}.$$

The proof of the above lemma is a consequence of the Cauchy estimate for an analytic function and of the Sobolev inequality.

Finally, using Proposition 4.1 and the above lemmas, we get the following.

PROPOSITION 7.5. *Suppose that* u^1 *and* u^2 *are in* $K^{l,\rho,\mu}_{\beta,T}$. *Suppose* $0 < \rho' < \rho(s) < \rho$. *Then the following estimate holds:*

$$\left| \, M_2K_1(u^1,t) - M_2K_1(u^2,t) \, \right|_{l,\rho',\mu} \le c \, \int_0^t ds \, \frac{|u^1-u^2|_{l,\rho(s),\mu}}{\rho(s)-\rho'}, \tag{7.4}$$

where the constant c *depends only on* $|u^1|_{l,\rho,\mu,\beta,T}$ *and* $|u^2|_{l,\rho,\mu,\beta,T}$.

Notice that the difference $K_1(u^1,t) - K_1(u^2,t)$ has to be considered only for functions which satisfy the condition $u(x,Y=0,t) = -U$, so that $K_1(u^1,t) - K_1(u^2,t)\,|_{Y=0} = 0$. Therefore the requirement of Proposition 4.1 is fulfilled.

7.2. The operator M_3K_3. In this subsection we shall estimate the term containing the Y-derivative using Proposition 4.3. Since it involves also the x-derivative, one must pay attention to the way the derivatives are distributed. In the estimate of the term involving the $\partial_Y^2\partial_x^{l-2}$-derivatives, one has to invoke Proposition 4.3. On the other hand, in the estimate of the term involving the $\partial_Y\partial_x^{l-1}$-derivatives, one has to Cauchy estimate the x-derivative.

The following proposition then holds.

PROPOSITION 7.6. *Suppose that u^1 and u^2 are in $K^{l,\rho,\mu}_{\beta,T}$. Suppose $0<\rho'<\rho(s)<\rho$, $0<\mu'<\mu(s)<\mu$. Then the following estimate holds:*

$$\left|M_3K_3(u^1,t)-M_3K_3(u^2,t)\right|_{l,\rho',\mu'} \le c\int_0^t ds\left(\frac{|u^1-u^2|_{l,\rho(s),\mu'}}{\rho(s)-\rho'}+\frac{|u^1-u^2|_{l,\rho',\mu(s)}}{\mu(s)-\mu'}+\frac{|u^1-u^2|_{l,\rho',\mu'}}{\sqrt{t-s}}\right), \tag{7.5}$$

where the constant c depends only on $|u^1|_{l,\rho,\mu,\beta,T}$ and $|u^2|_{l,\rho,\mu,\beta,T}$.

We stress the fact that we are allowed to use Proposition 4.3, as both the hypotheses are satisfied. In fact the first hypothesis reads $[u^1\int_0^Y dY'\partial_x u^1 - u^2\int_0^Y dY'\partial_x u^2]_{Y=0} = 0$ and the second one

$$\begin{aligned}
&\left[\partial_Y\left(u^1\int_0^Y dY'\partial_x u^1-u^2\int_0^Y dY'\partial_x u^2\right)\right]_{Y=0}\\
&=\left[\partial_Y u^1\int_0^Y dY'\partial_x u^1-\partial_Y u^2\int_0^Y dY'\partial_x u^2\right]_{Y=0}+\left[u^1\partial_x u^1-u^2\partial_x u^2\right]_{Y=0}\\
&=\left[(u^1-u^2)\partial_x u^1+u^2\partial_x(u^1-u^2)\right]_{Y=0}=0,
\end{aligned}$$

where the last equality holds since both u^1 and u^2 have the same datum at the boundary.

This concludes the proof of Theorem 7.1.

8. The main result.

In the previous sections we have proved that the operator F satisfies all the hypotheses of the ACK theorem. Hence the following theorem, which is the main result of this paper, has been proved.

THEOREM 8.1. *Suppose $U\in K^{l,\rho_0}_{\beta_0,T}$ and $u^P_{in}-U\in K^{l,\rho_0,\mu_0}$. Moreover let the compatibility conditions*

$$u^P_{in}(x,Y=0) = 0, \tag{8.1}$$

$$u^P_{in}(x,Y\to\infty)-U\longrightarrow 0 \tag{8.2}$$

hold. Then there exist $0<\rho_1<\rho_0$, $0<\mu_1<\mu_0$, $\beta_1>\beta_0>0$, and $0<T_1<T$ such that (1.1)–(1.7) admit a unique mild solution u^P. This solution can be written as

$$u^P(x,Y,t)=u(x,Y,t)+U, \tag{8.3}$$

where $u\in K^{l,\rho_1,\mu_1}_{\beta_1,T_1}$.

9. Concluding remarks.

In this paper we have proved short time existence and uniqueness of the solution of the Prandtl equations. The main hypothesis we have imposed is the analyticity of the initial data and of the prescribed (Euler) flow with respect to the tangential variable. This improves the results of [17], where analyticity with respect to the normal variable was also imposed.

The main ideas in our proof are the following.

First, we inverted the convection-diffusion (in the normal variable) operator. This led us to introduce the mild form of the Prandtl equations and allowed us to put the Prandtl equations in a form (see (5.14)) suitable for the application of the ACK theorem.

Second, we introduced a modified form of the ACK theorem to deal with a term which has a mild singularity in time (see (3.3)). The origin of this mild singularity is in the fact that, due to the lack of analyticity with respect to the normal variable, we had to use the regularizing properties of the Green's function of the diffusion operator. The gain of regularity in the normal space variable was paid with a mild singularity in time.

Third, the analyticity in the tangential variable was used to deal with the nonlinear convection in the tangential direction. Application of our version of the ACK theorem gave the existence and uniqueness of the solution.

The result of this paper is more general than the results of [17]. Moreover it seems a necessary step toward a rigorous mathematical analysis of the boundary layer theory for curved boundaries. In fact, when the curvature is present, the requirement of analyticity with respect to the normal variable would not allow the asymptotic matching between the exterior and the interior solutions. Therefore the problem of proving the well-posedness of the boundary layer equations when geometries other than very special ones (e.g., the half space or the exterior of a circular domain) are involved does not seem to be out of reach. This would open the possibility of the analysis of the zero viscosity problem for a fluid confined in a general bounded domain.

Appendix A. Proof of the ACK theorem. The proof of Theorem 3.1 follows along the same lines as that of [15].

In fact we prove the ACK theorem by proving that $F(u,t)$ is contractive in an auxiliary Banach space $\mathbb{S}^\gamma$.

For $\gamma > 0$, we consider the weighted Banach space $\mathbb{S}^\gamma$ of continuous functions $u(t)$ with values in X_ρ, where $\rho + \beta t < \rho_0$. The norm in $\mathbb{S}^\gamma$ is defined as

$$\|u\|^{(\gamma)} = \sup_{\rho+\beta t<\rho_0} (\rho_0 - \rho - \beta_0 t)^\gamma \; |u(t)|_\rho. \tag{A.1}$$

The contractiveness of the $F(u)$ in $\mathbb{S}^\gamma$ can be proved as follows.

Let $0 < \rho' < \rho(s) < \rho_0$. We set

$$\rho(s) = \rho' + \frac{\lambda(s)}{2}, \tag{A.2}$$

where

$$\lambda(s) = \rho_0 - \rho' - \beta s. \tag{A.3}$$

Therefore

$$\rho_0 - \rho(s) - \beta s = \frac{\lambda(s)}{2} = \rho(s) - \rho'. \tag{A.4}$$

We can now make the estimate

$$
\begin{aligned}
|F(t,u^1)-F(t,u^2)|_{\rho'} &\le C\int_0^t ds\left(\frac{|u^1-u^2|_{\rho'}}{\sqrt{t-s}}+\frac{|u^1-u^2|_{\rho(s)}}{\rho(s)-\rho'}\right)\\
&\le C\int_0^t ds\left(\frac{|u^1-u^2|_{\rho'}}{\sqrt{t-s}}\frac{(\rho_0-\rho'-\beta s)^\gamma}{(\rho_0-\rho'-\beta t)^\gamma}+\frac{|u^1-u^2|_{\rho(s)}}{\rho(s)-\rho'}\frac{(\rho_0-\rho(s)-\beta s)^\gamma}{(\rho_0-\rho(s)-\beta s)^\gamma}\right)\\
&\le C\|u^1-u^2\|^{(\gamma)}\left[2\sqrt{t}(\rho_0-\rho'-\beta t)^{-\gamma}+\int_0^t ds\frac{2^{\gamma+1}}{(\rho_0-\rho'-\beta s)^{\gamma+1}}\right]\\
&\le C\frac{\|u^1-u^2\|^{(\gamma)}}{(\rho_0-\rho'-\beta t)^\gamma}\left[2\sqrt{\frac{\rho_0}{\beta}}+\frac{2^{\gamma+1}}{\gamma\beta}\right], \qquad \text{(A.5)}
\end{aligned}
$$

where C is the constant appearing in assumption 3. Passing from the second to the third line, we have used (A.3) and (A.4).

Taking the sup of (A.5) over $\rho'+\beta t<\rho_0$, we get

$$
\|F(t,u^1)-F(t,u^2)\|^{(\gamma)} \le 2\left(\sqrt{\frac{\rho_0}{\beta}}+\frac{2^\gamma}{\gamma\beta}\right)\|u^1-u^2\|^{(\gamma)}. \qquad \text{(A.6)}
$$

Therefore, to prove that the operator F is contractive in the (γ)-norm, it is enough to choose β big enough so that $\sqrt{\frac{\rho_0}{\beta}}+\frac{2^\gamma}{\gamma\beta}<\frac{1}{2}$. ∎

Appendix B. Proofs of Propositions 4.1, 4.2, 4.3, 4.4, and 4.5. We first prove some simple lemmas. Set

$$
\Psi(x,t)=\left(\int_0^t d\tau\; e^{-2A(x,\tau)}\right)^{1/2}. \qquad \text{(B.1)}
$$

LEMMA B.1.

$$
\sup_{x\in D(\rho)}\left|\frac{e^{-2A(x,t)}}{(\Psi(x,t))^2}\right| \le \frac{e^{4T\sup_{x,t}|\alpha|}}{t}.
$$

Proof.

$$
\begin{aligned}
\sup_{x\in D(\rho)}\left|\frac{e^{-2A(x,t)}}{(\Psi(x,t))^2}\right| &\le \frac{e^{2T\sup_{x,t}|\alpha|}}{\inf_{x\in D(\rho)}\left|\int_0^t d\tau\; e^{-2A(x,\tau)}\right|} \le \frac{e^{2T\sup_{x,t}|\alpha|}}{\left|\int_0^t d\tau\; e^{-2\sup_{x\in D(\rho)}A(x,\tau)}\right|}\\
&\le \frac{e^{2T\sup_{x,t}|\alpha|}}{\int_0^t d\tau\; e^{-2T\sup_{x\in D(\rho)}|\alpha(x,\tau)|}} \le \frac{e^{4T\sup_{x,t}|\alpha|}}{t}. \qquad ∎
\end{aligned}
$$

Using the above bound it is straightforward to prove the following lemmas.

LEMMA B.2.

$$
\sup_{x\in D(\rho)}\left|\partial_x^l F_\alpha(\cdot,Y,t)\right| \le c\;\frac{\exp\left(-\frac{Y^2\;e^{-4T\sup_{x,t}|\alpha|}}{4t}\right)}{\sqrt{t}}\sum_{i=0}^{l}\left(\frac{Y^2\;e^{4T\sup_{x,t}|\alpha|}}{2t}\right)^i.
$$

LEMMA B.3.

$$
\sup_{x\in D(\rho)}|\partial_Y F_\alpha(\cdot,Y,t)| \le c\;\frac{Y\;e^{4T\sup_{x,t}|\alpha|}}{t}\;\frac{\exp\left(-\frac{Y^2\;e^{-4T\sup_{x,t}|\alpha|}}{4t}\right)}{\sqrt{t}}.
$$

LEMMA B.4.

$$\sup_{x\in D(\rho)} \left|\partial_Y \partial_x^l F_\alpha(\cdot,Y,t)\right| \leq c\, \frac{\exp\left(-\frac{Y^2\ e^{-4T\sup_{x,t}|\alpha|}}{4t}\right)}{\sqrt{t}} \sum_{i=0}^{l} \left\{ \left(\frac{Y^2\ e^{4T\sup_{x,t}|\alpha|}}{2t}\right)^i \frac{Y\ e^{-4T\sup_{x,t}|\alpha|}}{2t} + \left(\frac{Y^2\ e^{4T\sup_{x,t}|\alpha|}}{2t}\right)^{i-1} \frac{Y\ e^{4T\sup_{x,t}|\alpha|}}{2t} \right\}.$$

In the proof of Proposition 4.5 we shall also need the following two lemmas.

LEMMA B.5.

$$\sup_{x\in D(\rho)} \left| \exp\left(-\frac{Y^2 e^{-2A(\cdot,Y^2/4\eta^2)}}{4\Psi^2(\cdot,Y^2/4\eta^2)}\right) \right| \leq\ c\ e^{-\eta^2}.$$

LEMMA B.6.

$$\sup_{x\in D(\rho)} \left|\ \Psi^n(\cdot,Y^2/4\eta^2)\ \right| \geq\ c\ \frac{Y^n}{2^n\eta^n}\ e^{-nT\sup|\alpha|}.$$

We now start with the proof of Proposition 4.3.

Proof of Proposition 4.3. In order to estimate $|M_3h|_{l,\rho,\mu'}$ we have to estimate $|\partial_x^i M_3 h|_{0,\rho,\mu'}$ with $i\leq l$, $|\partial_Y\partial_x^i M_3h|_{0,\rho,\mu'}$ with $i\leq l-1$, $|\partial_t\partial_x^i M_3h|_{0,\rho,\mu'}$ with $i\leq l-1$, and $|\partial_{YY}\partial_x^i M_3h|_{0,\rho,\mu'}$ with $i\leq l-2$.

We begin with $|\partial_x^i M_3h|_{0,\rho,\mu'}$ with $i\leq l$.

$$\begin{aligned}
&|\partial_x^i M_3 h|_{0,\rho,\mu'}\\
&= \sup_{Y\geq 0} e^{\mu' Y} \sup_{|\Im x|\leq\rho} \left\| \partial_x^i \int_0^t ds \int_0^\infty dY' \partial_Y \left[F_\alpha(x,Y-Y',t-s) - F_\alpha(x,Y+Y',t-s)\right] h(x,Y',s)\right\|_{L^2}\\
&\leq \sup_{Y\geq 0} e^{\mu' Y} \int_0^t ds \int_0^\infty dY' \sum_{k=0}^{i} \sup_x \left|\partial_x^k \partial_Y \left[F_\alpha(\cdot,Y-Y',t-s) - F_\alpha(\cdot,Y+Y',t-s)\right]\right|\\
&\qquad\qquad \times \sup_{|\Im x|\leq\rho} \|\partial_x^{i-k} h(\cdot,Y',s)\|_{L^2}\\
&\leq c \sup_{Y\geq 0} e^{\mu' Y} \int_0^t \frac{ds}{\sqrt{t-s}} \sum_{k=0}^{i} \left\{ \int_{\frac{-Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta\, e^{-\eta^2} \eta^{2k+1} \right.\\
&\qquad\qquad \times \sup_{|\Im x|\leq\rho} \left\|\partial_x^{i-k} h(x,Y+2\eta e^{2T\sup|\alpha|}\sqrt{t-s},s)\right\|_{L^2}\\
&\quad + \int_{\frac{-Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta\ e^{-\eta^2}\ k\ \eta^{2k-1} \sup_{|\Im x|\leq\rho} \left\|\partial_x^{i-k} h(x,Y+2\eta e^{2T\sup|\alpha|}\sqrt{t-s},s)\right\|_{L^2}\\
&\quad + \int_{\frac{Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta\ e^{-\eta^2} \eta^{2k+1} \sup_{|\Im x|\leq\rho} \left\|\partial_x^{i-k} h(x,-Y+2\eta e^{2T\sup|\alpha|}\sqrt{t-s},s)\right\|_{L^2}\\
&\quad \left. + \int_{\frac{Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta\ e^{-\eta^2}\ k\ \eta^{2k-1} \sup_{|\Im x|\leq\rho} \left\|\partial_x^{i-k} h(x,-Y+2\eta e^{2T\sup|\alpha|}\sqrt{t-s},s)\right\|_{L^2} \right\}\\
&\leq c\int_0^t ds\ \frac{1}{\sqrt{t-s}}\ |\partial_x^i h|_{0,\rho,\mu} \leq c\int_0^t ds\ \frac{1}{\sqrt{t-s}}\ |h|_{l,\rho,\mu},
\end{aligned}$$

where, in passing from the third to the fourth line, we have used Lemma B.4 and have posed $\eta = \frac{(Y'-Y)e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}$ in the first two integrals and $\eta = \frac{(Y'+Y)e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}$ in the third and fourth integrals.

We now pass to the estimates of $|\partial_Y \partial_x^i M_2 \partial_Y h|_{0,\rho,\mu'}$ with $i \leq l-1$.

$$
\begin{aligned}
&|\partial_Y \partial_x^i M_3 h|_{0,\rho,\mu'} \\
&= \sup_{Y\geq 0} e^{\mu' Y} \sup_{|\Im x|\leq\rho} \Bigg\| \partial_x^i \int_0^t ds \int_0^\infty dY' \, \partial_Y \left[F_\alpha(x, Y-Y', t-s) - F_\alpha(x, Y+Y', t-s) \right] \\
&\qquad\qquad \times \partial_{Y'} h(x, Y', s) \Bigg\|_{L^2} \\
&\leq \sup_{Y\geq 0} e^{\mu' Y} \int_0^t ds \int_0^\infty dY' \sum_{k=0}^{i} \sup_x \left| \partial_x^k \partial_Y \left[F_\alpha(\cdot, Y-Y', t-s) - F_\alpha(\cdot, Y+Y', t-s) \right] \right| \\
&\qquad\qquad \times \sup_{|\Im x|\leq\rho} \left\| \partial_x^{i-k} \partial_{Y'} h(\cdot, Y', s) \right\|_{L^2} \\
&\leq c \sup_{Y\geq 0} e^{\mu' Y} \int_0^t \frac{ds}{\sqrt{t-s}} \sum_{k=0}^{i} \Bigg\{ \int_{\frac{-Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta \, e^{-\eta^2} \eta^{2k+1} \\
&\qquad\qquad \times \sup_{|\Im x|\leq\rho} \left\| \partial_x^{i-k} \partial_Y h(x, Y + 2\eta e^{2T\sup|\alpha|}\sqrt{t-s}, s) \right\|_{L^2} \\
&\quad + \int_{\frac{-Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta \, e^{-\eta^2} \, k \, \eta^{2k-1} \sup_{|\Im x|\leq\rho} \left\| \partial_x^{i-k} \partial_Y h(x, Y + 2\eta e^{2T\sup|\alpha|}\sqrt{t-s}, s) \right\|_{L^2} \\
&\quad + \int_{\frac{Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta \, e^{-\eta^2} \eta^{2k+1} \sup_{|\Im x|\leq\rho} \left\| \partial_x^{i-k} \partial_Y h(x, -Y + 2\eta e^{2T\sup|\alpha|}\sqrt{t-s}, s) \right\|_{L^2} \\
&\quad + \int_{\frac{Y\ e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta \, e^{-\eta^2} \, k \, \eta^{2k-1} \sup_{|\Im x|\leq\rho} \left\| \partial_x^{i-k} \partial_Y h(x, -Y + 2\eta e^{2T\sup|\alpha|}\sqrt{t-s}, s) \right\|_{L^2} \Bigg\} \\
&\leq c \int_0^t ds \, \frac{1}{\sqrt{t-s}} \, |\partial_x^i \partial_Y h|_{0,\rho,\mu} \leq c \int_0^t ds \frac{1}{\sqrt{t-s}} \, |h|_{l,\rho,\mu}.
\end{aligned}
$$

The estimate of $|\partial_{YY} M_2 \partial_Y h|_{0,\rho,\mu'}$ is easily achieved by transforming the derivative ∂_{YY} acting on the kernel into $\partial_{Y'}\partial_Y$ and integrating by parts. It then proceeds analogously to the one given above, as the appearance of singular boundary terms is prevented by the condition $\partial_Y h(x, Y=0, t) = 0$.

Finally we have to bound the term $|\partial_t M_3 h|_{0,\rho,\mu'}$. We notice that $\partial_t M_3 h = \partial_{YY} M_3 h - \alpha Y \partial_Y M_3 h$; hence we need to estimate $|Y \partial_Y \partial_x^i M_3 h|_{0,\rho,\mu'}$ with $i \leq l-2$ and use the estimate given above.

$$
\begin{aligned}
&|Y \partial_Y \partial_x^i M_3 h|_{0,\rho,\mu'} \\
&= \sup_{Y\geq 0} e^{\mu' Y} Y \sup_{|\Im x|\leq\rho} \Bigg\| \partial_Y \partial_x^i \int_0^t ds \int_0^\infty dY' \left[F_\alpha(x, Y-Y', t-s) - F_\alpha(x, Y+Y', t-s) \right] \\
&\qquad\qquad \times \partial_{Y'} h(x, Y', s) \Bigg\|_{L^2} \\
&\leq \sup_{Y\geq 0} e^{\mu' Y} \sup_{|\Im x|\leq\rho} \Bigg\| \partial_x^i \int_0^t ds \int_0^\infty dY' \, Y \left[F_\alpha(x, Y-Y', t-s) + F_\alpha(x, Y+Y', t-s) \right] \\
&\qquad\qquad \times \partial_{Y'}^2 h(x, Y', s) \Bigg\|_{L^2}
\end{aligned}
$$

$$\begin{aligned}
&\le \sup_{Y\ge 0} e^{\mu' Y} \sup_{|\Im x|\le\rho} \left\| \partial_x^i \int_0^t ds \int_0^\infty dY' \; (Y-Y')\; F_\alpha(x, Y-Y', t-s)\; \partial_{Y'}^2 h(x,Y',s) \right\|_{L^2} \\
&+ \sup_{Y\ge 0} e^{\mu' Y} \sup_{|\Im x|\le\rho} \left\| \partial_x^i \int_0^t ds \int_0^\infty dY' \; (Y+Y')\; F_\alpha(x, Y+Y', t-s)\; \partial_{Y'}^2 h(x,Y',s) \right\|_{L^2} \\
&+ \sup_{Y\ge 0} e^{\mu' Y} \sup_{|\Im x|\le\rho} \Bigg\| \partial_x^i \int_0^t ds \int_0^\infty dY' \; Y' \left[F_\alpha(x, Y-Y', t-s) - F_\alpha(x, Y+Y', t-s) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \partial_{Y'}^2 h(x,Y',s) \Bigg\|_{L^2} \\
&\le c \sup_{Y\ge 0} e^{\mu' Y} \int_0^t \frac{ds}{\sqrt{t-s}} \Bigg\{ \sum_{k=0}^{i} \int_{\frac{-Y\; e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta\; e^{-\eta^2} \eta^{2k+1} \\
&\qquad\qquad \times \sup_{|\Im x|\le\rho} \left\| \partial_x^{i-k} \partial_Y^2 h(x, Y + 2\eta e^{2T\sup|\alpha|}\sqrt{t-s}, s) \right\|_{L^2} \\
&\quad + \sum_{k=0}^{i} \int_{\frac{Y\; e^{-2T\sup|\alpha|}}{2\sqrt{t-s}}}^\infty d\eta\; e^{-\eta^2} \eta^{2k+1} \sup_{|\Im x|\le\rho} \left\| \partial_x^{i-k} \partial_Y^2 h(x, -Y + 2\eta e^{2T\sup|\alpha|}\sqrt{t-s}, s) \right\|_{L^2} \Bigg\} \\
&+ c \sup_{Y\ge 0} \sup_{|\Im x|\le\rho} \Bigg\| \partial_x^i \int_0^t ds \int_0^\infty dY' \frac{e^{\mu'(Y-Y')}}{\mu-\mu'} \left[F_\alpha(x, Y-Y', t-s) - F_\alpha(x, Y+Y', t-s) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \sup_{Y'\ge 0} e^{\mu Y'}\; \partial_{Y'}^2 h(x,Y',s) \Bigg\|_{L^2} \\
&\le c \int_0^t ds \left(\frac{|\partial_x^i \partial_Y^2 h|_{0,\rho,\mu}}{\sqrt{t-s}} + \frac{|\partial_x^i \partial_Y^2 h|_{0,\rho,\mu}}{\mu-\mu'} \right),
\end{aligned}$$

where, in passing from the second to the third line, we have integrated by parts and used the condition $\partial_Y h(x, Y = 0, t) = 0$. In the last step, the third integral was estimated using Lemma B.2 and the boundedness of the integral with respect to Y'. ∎

Proofs of Propositions 4.1, 4.2, and 4.4. The proofs of Propositions 4.1, 4.2, and 4.4 are easily achieved by adopting the same techniques used to prove Proposition 4.3.

Proof of Proposition 4.5. To prove Proposition 4.5 it is useful to introduce the following change of variable into the expression (4.14) for the operator $M_1 g$:

$$\eta = \frac{Y}{2\Psi(x, t-s)}, \tag{B.2}$$

where the function $\Psi(x, t-s)$ has been defined by (B.1). Since $\Psi(x, t-s)$ is a monotone function of the time variable, one can express $t-s$ as a function of η. Namely, it exits the function Φ such that

$$s = t - \Phi(Y/2\eta).$$

Therefore the expression (4.14) becomes

$$
\begin{aligned}
\text{(B.3)}\quad M_1 g = 4 \int_{\frac{Y}{2\Psi(x,t)}}^{\infty} & d\eta \, \exp\left(-\eta^2 e^{-2A(x,\,\Phi(Y/2\eta))}\right) g(x,\ t - \Phi(Y/2\eta)) \\
& \times \left[1 + \frac{Y^2}{2\eta^2}\,\alpha(x,\ \Phi(Y/2\eta))\, e^{2A(x,\,\Phi(Y/2\eta))}\right] \\
& - \int_0^t dz\, g(x, t-z)\,\alpha(x,z) \left[\int_{-\frac{Ye^{-2A}}{2\Psi(x,z)}}^{\infty} d\eta\, e^{-\eta^2} - \int_{\frac{Ye^{-2A}}{2\Psi(x,z)}}^{\infty} d\eta\, e^{-\eta^2}\right],
\end{aligned}
$$

where, in the last integral, we have also posed $t - s = z$.

To estimate $|M_1 g|_{l,\rho,\mu}$ we have to estimate $|\partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l$, $|\partial_t \partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l-1$, $|\partial_Y \partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l-1$, and $|\partial_{YY} \partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l-2$.

The estimate of the term $|\partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l$ is easily achieved by letting the operator ∂_x^i act and by using the same techniques of Proposition 4.3.

Analogously, one can get the estimate of the term $|\partial_t \partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l-1$, noticing that, in the expression (B.3), the time derivative commutes with the integral because $g(x, t=0) = 0$.

We now estimate the term $|\partial_Y \partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l-1$. Recalling that if $f = f(\Phi(Y/2\eta))$, one has

$$
\partial_Y f = \frac{\partial f}{\partial \Phi} \frac{\partial \Phi}{\partial (Y/2\eta)} \frac{1}{2\eta} = - \frac{Y e^{2A(x,\,\Phi(Y/2\eta)\,)}}{2\eta^2} \frac{\partial f}{\partial \Phi},
$$

we obtain the expression for $\partial_Y M_1 g$:

$$
\begin{aligned}
\partial_Y M_1 g = 8Y \int_{\frac{Y}{2\Psi(x,t)}}^{\infty} & d\eta\, \exp\left(-\eta^2 e^{-2A(x,\,\Phi(Y/2\eta))}\right) g(x,\ t - \Phi(Y/2\eta)) \\
& \times \left[1 + \frac{Y^2}{2\eta^2}\,\alpha(x,\ \Phi(Y/2\eta))\, e^{2A(x,\,\Phi(Y/2\eta))}\right] \\
+ 2 \int_{\frac{Y}{2\Psi(x,t)}}^{\infty} & d\eta\, \exp\left(-\eta^2 e^{-2A(x,\,\Phi(Y/2\eta))}\right) \frac{Y e^{2A(x,\,\Phi(Y/2\eta))}}{\eta^2}\, \partial_t g(x,\ t - \Phi(Y/2\eta)) \\
& \times \left[1 + \frac{Y^2}{2\eta^2}\,\alpha(x,\ \Phi(Y/2\eta))\, e^{2A(x,\,\Phi(Y/2\eta))}\right] \\
+ 4 \int_{\frac{Y}{2\Psi(x,t)}}^{\infty} & d\eta\, \exp\left(-\eta^2 e^{-2A(x,\,\Phi(Y/2\eta))}\right) \frac{Y e^{2A(x,\,\Phi(Y/2\eta))}}{\eta^2}\, g(x,\ t - \Phi(Y/2\eta)) \\
& \times \left[\alpha - \frac{Y^2}{\eta}\, e^{2A(x,\,\Phi(Y/2\eta))} \left(\alpha - \frac{\partial_t \alpha}{2}\right)\right] \\
& - \int_0^t g(x, t-z)\,\alpha(x,\ z)\, \frac{\exp\left(-\frac{Y^2 e^{-2A(x,z)}}{4\Psi^2(x,z)}\right)}{\Psi(x,z)}.
\end{aligned}
$$

Using the above expression and Lemmas B.5 and B.6, the estimate of the terms $|\partial_Y \partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l-1$ and $|Y \partial_Y \partial_x^i M_1 g|_{0,\rho,\mu}$ with $i \le l-1$ is straightforward. The proof of Proposition 4.5 is thus achieved.

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