WELL-POSEDNESS OF THE FUNDAMENTAL BOUNDARY VALUE PROBLEMS FOR CONSTRAINED ANISOTROPIC ELASTIC MATERIALS

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Dedicated to Professor Joachim Nitsche on the occasion of his sixtieth birthday.

Abstract

We consider the equations of linear homogeneous anisotropic elasticity admitting the possibility that the material is internally constrained, and formulate a simple necessary and sufficient condition for the fundamental boundary value problems to be well-posed. For materials fulfilling the condition, we establish continuous dependence of the displacement and stress on the elastic moduli and ellipticity of the elasticity system. As an application we determine the orthotropic materials for which the fundamental problems are well-posed in terms of their Young's moduli, shear moduli, and Poisson ratios. Finally, we derive a reformulation of the elasticity system that is valid for both constrained and unconstrained materials and involves only one scalar unknown in addition to the displacements. For a two-dimensional constrained material a further reduction to a single scalar equation is outlined.

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1. Introduction

The equations of anisotropic elasticity are

$$\mathbf{A}\,\underset{\approx}{\sigma} = \underset{\approx}{\epsilon}(\underset{\sim}{u}) \quad \text{in } \Omega, \tag{1.1}$$

$$\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma} = \underset{\sim}{f} \quad \text{in } \Omega, \tag{1.2}$$

where $\underset{\approx}{\sigma} = (\sigma_{kl})$ is a 3×3 symmetric tensor of unknown stresses, $\underset{\sim}{u}$ is a 3-vector of unknown displacements, and $\underset{\sim}{f}$ is a given 3 vector of forces, all defined on a smoothly bounded domain $\Omega \subset \mathbb{R}^3$. The infinitesmal strain tensor $\underset{\approx}{\epsilon}(\underset{\sim}{u})$ is defined as the symmetric part of the gradient tensor $(\partial u_i/\partial x_j)$ and the vector-valued divergence $\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}$ is defined by applying the scalar-valued divergence operator to the rows of $\underset{\approx}{\sigma}$. The fourth order tensor $\mathbf{A}$, known as the compliance tensor, is a self-adjoint linear operator on the six-dimensional space $\underset{\approx}{\mathbb{R}}$ of symmetric 3×3 tensors, and characterizes the particular material. The compliance tensor may be determined by specifying 21 independent coefficients or elastic moduli.

We shall consider in this paper the fundamental displacement and traction boundary conditions:

$$\begin{aligned} \underset{\sim}{u} &= \underset{\sim}{g}_1 \quad \text{on } \Gamma_1, \\ \underset{\approx}{\sigma}\,\underset{\sim}{n} &= \underset{\sim}{g}_2 \quad \text{on } \Gamma_2. \end{aligned} \tag{1.3}$$

Here Γ_1 and Γ_2 are disjoint open subsets of $\partial\Omega$ with $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$. For now we assume that Γ_1 and Γ_2 are nonempty. The case of unmixed boundary conditions is considered in Section 5.

It is often assumed that the compliance tensor is positive definite. In this case, $\underset{\approx}{\sigma}$ can be eliminated and it can easily be shown that the resulting boundary value problem is well-posed. For many important materials, however, the compliance tensor is positive semidefinite but singular, or nearly so. If the compliance tensor is singular, admitting a nonzero tensor $\underset{\approx}{\sigma}_0$ in its nullspace, then the displacement fields which satisfy the constitutive equation (1.1) are not arbitrary, but automatically satisfy the linear relation

$$\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\sigma}_0 = 0.$$

This relation is called the material constraint and the material is said to be (internally) constrained. We term any nonzero tensor in the nullspace of the compliance tensor a *constraint tensor*. For example, an incompressible material is one for which the 3×3 identity matrix is a constraint tensor and the corresponding constraint is $\operatorname{div}\underset{\sim}{u} = 0$. A material which is inextensible in the direction $\underset{\sim}{s}$ has constraint tensor $\underset{\sim}{s}\underset{\sim}{s}^t$ and so satisfies the constraint $\underset{\sim}{s} \cdot \underset{\sim}{\operatorname{grad}}\,(\underset{\sim}{s} \cdot \underset{\sim}{u}) = 0$.

The boundary value problem (1.1)-(1.3) for a constrained material may or may not be well-posed. For an incompressible material, for example, well-posedness has long been known in the isotropic case and has been recently established in general [6]. For inextensible materials, in contrast, the boundary value problem is over-determined.

In this paper we assume only that the compliance tensor is semidefinite, and formulate a simple algebraic property of the compliance tensor which characterizes those materials for which the fundamental boundary value problem is well-posed. Moreover, for those materials we establish *a priori* bounds for the displacement and stress fields which are uniform with respect to the elastic moduli and establish continuous dependence of the solution on the moduli.

As an application of our analysis we consider the class of orthotropic materials. A material in this class is determined by nine independent physical constants and can be constrained in a variety of ways. We determine when the fundamental boundary value problems are well set in terms of these constants, and establish continuity of the solutions with respect to them.

The question of continuous dependence on the elastic moduli near an elastic constraint is of great importance. Without such continuous dependence results, the use of constrained models, which represent an idealization of nearly constrained materials, would be unjustified. Nonetheless this question remains largely unresolved. Our results apparently provide the first proof of convergence of unconstrained materials to a constrained material outside of the simplest case, that of an isotropic incompressible material. The isotropic case was examined by BRAMBLE and PAYNE [4], who proved continuous dependence results for the pure displacement and traction problems and, in particular, showed that as the Poisson ratio tends to 1/2 the displacement and each of its derivatives converge at interior points to the corresponding quantity for the incompressible problem. Results of the same sort have since been derived by MIKHLIN [17], KOBEL'KOV [13], LAZAREV [14], and ROSTAMIAN [19]. For nonlinear elastic materials asymptotic expansions have been devised which suggest the convergence of an almost constrained material to a constrained one, but of course these do not provide proofs of convergence. See SPENCER [21] for the constraint of incompressibility of an elastic solid and ANTMAN [2] for that of inextensibility of an elastica.

ROSTAMIAN [19] has derived abstract conditions on the compliance tensor of an anisotropic linearly elastic material which insure continuous dependence of the solution on the elastic moduli. His conditions, which are sufficient but not neccessary, are much more complex than the simple algebraic conditions that we give. He applied his theory only to the known case of isotropic elasticity, regaining the results of BRAMBLE and PAYNE [4] and also showing convergence of the stresses.

PIPKIN outlines the general theory of constraints in linear elasticity in [18]. He classifies constraints by their dimension, which he defines as the rank of the corresponding constraint tensor. For our purpose the crucial distinction is between constraint tensors of deficient rank and those of full rank. We term the corresponding constraints *singular* and *nonsingular* respectively. Our essential hypothesis on the material is that it admits only nonsingular constraints, that is, that no nonzero singular tensor $\underset{\approx}{\sigma}_0$ satisfies $\mathbf{A}\underset{\approx}{\sigma}_0 = 0$.

Let us comment on the physical significance of singular and nonsingular constraints. A material is constrained if and only if a smooth body composed of the material can be subject to a homogeneous state of stress without deforming. The constant stress tensor is then a constraint tensor. The constraint is singular if and only if the traction vanishes at some point on the boundary, since the normal at such a point is a nullvector of the constraint tensor. For example, an incompressible material supports a uniform pressure without deformation. In this state the traction never vanishes. Contrarily, an inextensible material under uniform tension does not deform, but the traction vanishes in any direction normal to the axis of tension.

To state uniform estimates we associate a quantitative measure with this hypothesis. Let $\mathcal{C}$ denote the space of positive semidefinite self-adjoint linear transformations of $\underset{\approx}{\mathbb{R}}$ into itself, and for $\mathbf{A} \in \mathcal{C}$ let $0 \le \lambda_1(\mathbf{A}) \le \lambda_2(\mathbf{A}) \le \cdots \le \lambda_6(\mathbf{A})$ denote its eigenvalues and $\underset{\approx}{\sigma}_1(\mathbf{A}), \underset{\approx}{\sigma}_2(\mathbf{A}), \cdots, \underset{\approx}{\sigma}_6(\mathbf{A}) \in \underset{\approx}{\mathbb{R}}$ a corresponding orthonormal basis of eigenvectors. The quantity that we use to measure the closeness of the material to having a singular constraint is denoted by $\chi(\mathbf{A})$ and defined by

$$\chi(\mathbf{A}) = \max[\lambda_1(\mathbf{A}), \lambda_2(\mathbf{A})/|\underset{\approx}{\sigma}_1(\mathbf{A})^{-1}|^2]. \tag{1.4}$$

In the next section we show that the definition of χ is independent of the choice of eigenbasis, and that $\chi : \mathcal{C} \to [0, \infty)$ is continuous and vanishes if and only if the material admits a singular constraint. We may now state our principal result.

Theorem 1.1: *Suppose that the compliance tensor* $\mathbf{A}$ *is positive semidefinite and admits no singular constraints. Then for any data* $(\underset{\sim}{f}, \underset{\sim}{g}_1, \underset{\sim}{g}_2) \in \underset{\sim}{L}^2(\Omega) \times \underset{\sim}{H}^{1/2}(\Gamma_1) \times \underset{\sim}{L}^2(\Gamma_2)$*, there exists a unique solution* $(\underset{\approx}{\sigma}, \underset{\sim}{u}) \in \underset{\approx}{L}^2(\Omega) \times \underset{\sim}{H}^1(\Omega)$ *to the mixed boundary value problem (1.1)–(1.3). Moreover, the a priori estimate*

$$\|\underset{\approx}{\sigma}\|_0 + \|\underset{\sim}{u}\|_1 \le C(\|\underset{\sim}{f}\|_{-1,D} + |\underset{\sim}{g}_1|_{1/2,\Gamma_1} + |\underset{\sim}{g}_2|_{-1/2,\Gamma_2}) \tag{1.5}$$

holds with C *a constant depending only on* Ω*, an upper bound for the compliances, and a lower bound for* $\chi(\mathbf{A})$*; and the solution* $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ *depends continuously on the compliance tensor* $\mathbf{A}$ *and the data* $\underset{\sim}{f}$*,* $\underset{\sim}{g}_1$*, and* $\underset{\sim}{g}_2$*.*

An outline of the paper is as follows. Section 2 contains additional notation used in the paper along with the statement of a theorem due to Brezzi [5] dealing with abstract saddle point problems. This theorem will play a major role in our subsequent analysis. The proof of Theorem 1.1 is given in Section 3. As an application of the theorem we consider the case of orthotropic materials in Section 4. In the next section we extend the results to the cases of pure traction and pure displacement boundary conditions. We then show in Section 6 that the hypothesis of nonsingularity of constraints is in some sense necessary. In Section 7 we prove ellipticity of the elastic system uniformly with respect to the elastic moduli and in Section 8 we use the ideas previously developed to derive two alternate formulations of the elasticity equations which may be more convenient for some computational and analytic purposes. In the first of these formulations the stress $\underset{\approx}{\sigma}$ is eliminated and a new scalar variable p is introduced. In the case of an isotropic incompressible material these equations are equivalent to the stationary Stokes equations. The second formulation is a further simplification possible in the two-dimensional constrained case and results in a single fourth order equation, analogous to reduction of the Stokes system to the biharmonic problem via the introduction of a stream function. Finally, in the last section, we remark on the case of plane elasticity.

2. Notation and Preliminary Results

We underscore 3×3 symmetric tensors by $\approx$ and 3-vectors by $\sim$. We endow $\underset{\approx}{\mathbb{R}}$, the space of real symmetric 3×3 tensors, with the Frobenius norm and use the notation $\underset{\approx}{\sigma} : \underset{\approx}{\tau} = \sum_{i,j=1}^{3} \sigma_{ij}\tau_{ij}$ for the associated inner product. The space $\underset{\sim}{\mathbb{R}}$ of three vectors carries the usual Euclidean norm and dot product. For vector-valued functions $\underset{\sim}{u} = (u_1, u_2, u_3)^t$, we write $\underset{\sim}{u} \in \underset{\sim}{H}^1(\Omega)$ if $u_i \in H^1(\Omega)$ for $i = 1, 2, 3$, and set $\|\underset{\sim}{u}\|_1 = (\sum_{i=1}^{3} \|u_i\|_1^2)^{1/2}$. For 3×3 symmetric tensors $\underset{\approx}{\sigma} = (\sigma_{ij})$, we write $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega)$ if $\sigma_{ij} \in L^2(\Omega)$ for $i, j = 1, 2, 3$ and set $\|\underset{\approx}{\sigma}\|_0 = (\sum_{i,j=1}^{3} \|\sigma_{ij}\|_0^2)^{1/2}$.

We shall require some spaces of functions defined on a smoothly bounded open subset Γ' of Γ. By $\underset{\sim}{H}^{1/2}(\Gamma')$ we denote the usual Sobolev space [16, Ch.1,Sec.7]. The subspace consisting of functions whose extension to Γ by zero lies in $\underset{\sim}{H}^{1/2}(\Gamma)$ is denoted by $\underset{\sim}{H}_{00}^{1/2}(\Gamma')$. The norm is taken as the graph norm of the extension by zero, which induces a finer topology than the $\underset{\sim}{H}^{1/2}(\Gamma')$ norm. By $\underset{\sim}{H}^{-1/2}(\Gamma')$ we mean the normed dual of $\underset{\sim}{H}_{00}^{1/2}(\Gamma')$. The norms in $\underset{\sim}{H}^{1/2}(\Gamma')$ and $\underset{\sim}{H}^{-1/2}(\Gamma')$ are denoted by $|\cdot|_{1/2,\Gamma'}$ and $|\cdot|_{-1/2,\Gamma'}$ respectively, with the subscript being dropped in case $\Gamma' = \Gamma$.

We further define

$$\underset{\sim}{H}_0^1(\Omega) = \{\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : \underset{\sim}{v}|_{\Gamma} = 0\},$$

and

$$\underset{\sim}{H}_D^1(\Omega) = \{\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : \underset{\sim}{v}|_{\Gamma_1} = 0\},$$

and denote by $\|\underset{\sim}{f}\|_{-1,0}$ and $\|\underset{\sim}{f}\|_{-1,D}$ the norms in the dual spaces of $\underset{\sim}{H}_0^1(\Omega)$ and $\underset{\sim}{H}_D^1(\Omega)$, respectively.

Many of the results in this paper will be derived using a theorem of F. Brezzi [5] dealing with saddle point problems of the following type:

Find $(\sigma, u) \in W \times V$ such that:

$$a(\sigma, \tau) + b(\tau, u) = \langle g, \tau \rangle \quad \text{for all } \tau \in W, \tag{2.1}$$

$$b(\sigma, v) = \langle f, v \rangle \quad \text{for all } v \in V, \tag{2.2}$$

where W and V are real Hilbert spaces, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms on $W \times W$ and $W \times V$ respectively, and g and f are given functions in W^* and V^* (the duals of W and V respectively).

Let $Z = \{\tau \in W : b(\tau, v) = 0 \ \text{ for all } v \in V\}$. One version of Brezzi's theorem is the following:

Theorem 2.1: *Suppose there is a constant $\gamma > 0$ such that*

$$a(\tau, \tau) \geq \gamma \|\tau\|_W^2 \quad \textit{for all } \tau \in Z$$

and

$$\inf_{0 \neq v \in V} \sup_{0 \neq \tau \in W} \frac{b(\tau, v)}{\|\tau\|_W \|v\|_V} \geq \gamma.$$

Then for all $(f,g) \in V^* \times W^*$, there is a unique solution $(\sigma, u) \in W \times V$ of (2.1), (2.2). Moreover,

$$\|\sigma\|_W + \|u\|_V \leq \; C(\|g\|_{W^*} + \|f\|_{V^*}),$$

where C depends only on γ and bounds for the bilinear forms a and b.

We will be applying Brezzi's theorem in the case

$$a(\underset{\approx}{\sigma}, \underset{\approx}{\tau}) = \int_\Omega \mathbf{A}\, \underset{\approx}{\sigma} : \underset{\approx}{\tau}\, d\underset{\sim}{x}, \qquad b(\underset{\approx}{\tau}, \underset{\sim}{v}) = -\int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{v}) : \underset{\approx}{\tau}\, d\underset{\sim}{x}. \tag{2.3}$$

Finally we establish some properties of the function $\chi(\mathbf{A})$ defined in (1.4). For any $\mathbf{A}$ with $\lambda_1(\mathbf{A}) \neq \lambda_2(\mathbf{A})$, $\underset{\approx}{\sigma}_1(\mathbf{A})$ is uniquely determined (up to sign), so the definition of $\chi(\mathbf{A})$ is independent of the choice of eigenbasis and, moreover, χ is certainly continuous in $\mathbf{A}$. On the other hand, if $\lambda_1(\mathbf{A}) = \lambda_2(\mathbf{A})$, then $\chi(\mathbf{A}) = \lambda_1(\mathbf{A})$, since regardless of the choice of basis

$$|\underset{\approx}{\sigma}_1(\mathbf{A})^{-1}| \in [1, \infty]. \tag{2.4}$$

Moreover, in view of (2.4),

$$0 \leq \lambda_1(\mathbf{A}) \leq \chi(\mathbf{A}) \leq \lambda_2(\mathbf{A}) < \infty,$$

from which it follows that χ is again continuous in $\mathbf{A}$. Thus in any case χ maps $\mathcal{C}$ continuously into $[0, \infty)$.

We next show that if $\lambda_2(\mathbf{A}) = 0$, then $\mathbf{A}$ admits a singular constraint. This is certainly so if $\underset{\approx}{\sigma}_1(\mathbf{A})$ is singular. If $\underset{\approx}{\sigma}_1(\mathbf{A})$ is nonsingular and $\lambda_2(\mathbf{A}) = 0$, then $\underset{\approx}{\sigma}^\mu = \underset{\approx}{\sigma}_2(\mathbf{A}) + \mu\, \underset{\approx}{\sigma}_1(\mathbf{A})$ is a (nonzero) constraint tensor for all real μ and its determinant is a polynomial in μ of degree exactly 3. When μ is a real root of this polynomial, $\underset{\approx}{\sigma}^\mu$ is a singular constraint tensor.

Now if $\mathbf{A}$ does not admit a singular constraint then $|\underset{\approx}{\sigma}_1(\mathbf{A})^{-1}| < \infty$ and as we have just seen $\lambda_2 > 0$, so $\chi(\mathbf{A}) > 0$. If, on the other hand, $\mathbf{A}$ admits a singular constraint tensor $\underset{\approx}{\sigma}_1(\mathbf{A})$, then $\lambda_1(\mathbf{A}) = 0$ and $|\underset{\approx}{\sigma}_1(\mathbf{A})^{-1}| = +\infty$, so $\chi(\mathbf{A}) = 0$. Thus $\chi(\mathbf{A})$ vanishes if and only if $\mathbf{A}$ admits a singular constraint.

3. Proof of Theorem 1.1

The crux of the argument is contained in the following lemma.

Lemma 3.1: *Let* $\mathbf{A}$ *be a semidefinite compliance tensor which does not admit a singular constraint. Let* $\underset{\approx}{G} \in \underset{\approx}{L}^2(\Omega)^*, \underset{\sim}{F} \in \underset{\sim}{H}^1_D(\Omega)^*$. *Then there exist unique functions* $\underset{\approx}{\rho} \in \underset{\approx}{L}^2(\Omega)$ *and* $\underset{\sim}{z} \in \underset{\sim}{H}^1_D(\Omega)$ *such that*

$$a(\underset{\approx}{\rho}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{z}) = \langle \underset{\approx}{G}, \underset{\approx}{\tau} \rangle \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega),$$

$$b(\underset{\approx}{\rho}, \underset{\sim}{v}) = \langle \underset{\sim}{F}, \underset{\sim}{v} \rangle \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\Omega).$$

Moreover

$$\|\underset{\approx}{\rho}\|_0 + \|\underset{\sim}{z}\|_1 \le C(\|\underset{\approx}{G}\|_0 + \|\underset{\sim}{F}\|_{-1,D})$$

where C *depends only on* Ω, *an upper bound for* $|\mathbf{A}|$, *and a lower bound for* $\chi(\mathbf{A})$.

The bilinear forms here are defined in (2.3). Before turning to the proof of this lemma, we deduce from it the proof of the theorem.

As is usual, we impose the Dirichlet boundary condition by setting $\underset{\sim}{u}^1 = \underset{\sim}{\mathcal{E}}(\underset{\sim}{g}_1)$ with $\underset{\sim}{\mathcal{E}} : \underset{\sim}{H}^{1/2}(\Gamma_1) \to \underset{\sim}{H}^1(\Omega)$ a continuous extension operator, and seek a pair $(\underset{\approx}{\sigma}, \underset{\sim}{u}^2)$ such that

$$\begin{aligned} \mathbf{A}\,\underset{\approx}{\sigma} - \underset{\approx}{\epsilon}(\underset{\sim}{u}^2) &= \underset{\approx}{\epsilon}(\underset{\sim}{u}^1), \\ \underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma} &= \underset{\sim}{f}, \\ \underset{\sim}{u}^2 &= 0 \quad \text{on } \Gamma_1, \\ \underset{\approx}{\sigma}\,\underset{\sim}{n} &= \underset{\sim}{g}_2 \quad \text{on } \Gamma_2. \end{aligned} \tag{3.1}$$

We then take $\underset{\sim}{u} = \underset{\sim}{u}^1 + \underset{\sim}{u}^2$, so that the problem (1.1)–(1.3) is satisfied. In terms of the bilinear forms (2.3), a weak form of (3.1) is:

Find $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega)$, $\underset{\sim}{u}^2 \in \underset{\sim}{H}^1_D(\Omega)$ such that

$$a(\underset{\approx}{\sigma}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{u}^2) = -b(\underset{\approx}{\tau}, \underset{\sim}{u}^1) \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega), \tag{3.2}$$

$$b(\underset{\approx}{\sigma}, \underset{\sim}{v}) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\; d\underset{\sim}{x} - \int_{\Gamma_2} \underset{\sim}{g}_2 \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\Omega). \tag{3.3}$$

By Lemma 3.1 this problem admits a unique solution and the estimate

$$\|\underset{\approx}{\sigma}\|_0 + \|\underset{\sim}{u}^2\|_1 \le C(\|\underset{\approx}{\epsilon}(\underset{\sim}{u}^1)\|_0 + \|\underset{\sim}{f}\|_{-1,D} + |\underset{\sim}{g}_2|_{-1/2,\Gamma_2})$$

holds with C a constant depending only on Ω, an upper bound for $|\mathbf{A}|$, and a lower bound for $\chi(\mathbf{A})$. Existence and uniqueness for the original problem and the *a priori* estimate (1.5) follow readily. The continuous dependence result follows by a standard argument which we sketch. Letting $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}})$ denote the solution to the elliptic system with compliance tensor $\bar{\mathbf{A}}$ and data $\bar{\underset{\sim}{f}}, \bar{\underset{\sim}{g}}_1, \bar{\underset{\sim}{g}}_2$, and writing $\bar{\underset{\sim}{u}} = \bar{\underset{\sim}{u}}^1 + \bar{\underset{\sim}{u}}^2$ as above, the pair $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}}^2)$ solves

$$a(\bar{\underset{\approx}{\sigma}}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \bar{\underset{\sim}{u}}^2) = -b(\underset{\approx}{\tau}, \bar{\underset{\sim}{u}}^1) + \int_\Omega (\mathbf{A} - \bar{\mathbf{A}})\bar{\underset{\approx}{\sigma}} : \underset{\approx}{\tau}\, d\underset{\sim}{x} \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega), \tag{3.4}$$

$$b(\bar{\underset{\approx}{\sigma}}, \underset{\sim}{v}) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, d\underset{\sim}{x} - \int_{\Gamma_2} \underset{\sim}{g}_2 \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\Omega). \tag{3.5}$$

We wish to show that if $|\mathbf{A} - \bar{\mathbf{A}}| + \|\underset{\sim}{f} - \bar{\underset{\sim}{f}}\|_{-1,D} + |\underset{\sim}{g}_1 - \bar{\underset{\sim}{g}}_1|_{1/2,\Gamma_1} + |\underset{\sim}{g}_2 - \bar{\underset{\sim}{g}}_2|_{-1/2,\Gamma_2} \to 0$, then $\|\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}\|_0 + \|\underset{\sim}{u} - \bar{\underset{\sim}{u}}\|_1 \to 0$. First $\|\underset{\sim}{u}^1 - \bar{\underset{\sim}{u}}^1\|_1 \to 0$ by the continuity of the extension operator $\underset{\sim}{\mathcal{E}}$. Subtracting (3.2), (3.3) from (3.4), (3.5), noting that $|\mathbf{A}| \to |\bar{\mathbf{A}}|$ and $\chi(\mathbf{A}) \to \chi(\bar{\mathbf{A}})$, and applying Lemma 3.1, we deduce that also $\|\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}\|_0 + \|\underset{\sim}{u}^2 - \bar{\underset{\sim}{u}}^2\|_1 \to 0$.

It remains to prove Lemma 3.1. We apply Brezzi's theorem (Theorem 2.1) to reduce Lemma 3.1 to the verification of the following two lemmas.

Lemma 3.2: *There exists a constant $\gamma > 0$ depending only on Ω and a lower bound for $\chi(\mathbf{A})$ such that*

$$\int_\Omega \mathbf{A}\,\underset{\approx}{\tau} : \underset{\approx}{\tau}\, d\underset{\sim}{x} \geq \gamma \|\underset{\approx}{\tau}\|_0^2 \quad \textit{for all } \underset{\approx}{\tau} \in \underset{\approx}{Z},$$

where

$$\underset{\approx}{Z} = \{\underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega) \;:\; \int_\Omega \underset{\approx}{\tau} : \underset{\approx}{\epsilon}(\underset{\sim}{v})\, d\underset{\sim}{x} = 0 \quad \textit{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\Omega)\}.$$

Lemma 3.3: *There exists a constant $\gamma > 0$ depending only on Ω such that*

$$\inf_{0 \neq \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\Omega)} \sup_{0 \neq \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega)} \frac{\int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{v}) : \underset{\approx}{\tau}\, d\underset{\sim}{x}}{\|\underset{\sim}{v}\|_1 \, \|\underset{\approx}{\tau}\|_0} \geq \gamma.$$

The proof of Lemma 3.3 is immediate: given $\underset{\sim}{v}$, we take $\underset{\approx}{\tau} = \underset{\approx}{\epsilon}(\underset{\sim}{v})$ and apply Korn's inequality. To prove Lemma 3.2 we decompose an arbitrary element $\underset{\approx}{\tau}$ of $\underset{\approx}{Z}$ as $\underset{\approx}{\tau} = \underset{\approx}{\tau}_T + \underset{\approx}{\tau}_D$ with $\underset{\approx}{\tau}_T = (\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1)\underset{\approx}{\sigma}_1$. Then clearly

$$\int_\Omega \mathbf{A}\,\underset{\approx}{\tau} : \underset{\approx}{\tau}\, d\underset{\sim}{x} \geq \max(\lambda_1 \|\underset{\approx}{\tau}\|_0^2, \lambda_2 \|\underset{\approx}{\tau}_D\|_0^2). \tag{3.6}$$

Now there exists $\underset{\sim}{p} \in \underset{\sim}{H}^1_D(\Omega)$ such that

$$\operatorname{div} \underset{\sim}{p} = \underset{\approx}{\tau} : \underset{\approx}{\sigma}_1 \quad \text{and} \quad \|\underset{\sim}{p}\|_1 \leq C_1 \|\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1\|_0,$$

where C_1 depends only on Ω. Let $\underset{\sim}{q} = \underset{\approx}{\sigma}_1^{-1}\underset{\sim}{p}$. Then

$$\|\underset{\sim}{q}\|_1 \leq C_1 |\underset{\approx}{\sigma}_1^{-1}| \|\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1\|_0$$

and

$$\underset{\approx}{\sigma}_1 : \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{q} = \operatorname{div} \underset{\sim}{p} = \underset{\approx}{\tau} : \underset{\approx}{\sigma}_1.$$

Consequently

$$\|\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1\|_0^2 = \int_\Omega (\underset{\approx}{\sigma}_1 : \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{q})(\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1)\, d\underset{\sim}{x}$$

$$= \int_\Omega \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{q} : \underset{\approx}{\tau}_T\, d\underset{\sim}{x} = \int_\Omega \underset{\approx}{\operatorname{grad}}\, \underset{\sim}{q} : (\underset{\approx}{\tau} - \underset{\approx}{\tau}_D)\, d\underset{\sim}{x}$$

$$= \int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{q}) : (\underset{\approx}{\tau} - \underset{\approx}{\tau}_D)\, d\underset{\sim}{x} = -\int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{q}) : \underset{\approx}{\tau}_D\, d\underset{\sim}{x},$$

since $\underset{\sim}{q} \in \underset{\sim}{H}_D^1(\Omega)$ and $\underset{\approx}{\tau} \in \underset{\approx}{Z}$. Thus

$$\|\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1\|_0^2 \le \|\underset{\sim}{q}\|_1 \|\underset{\approx}{\tau}_D\|_0 \le C_1 |\underset{\approx}{\sigma}_1^{-1}| \|\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1\|_0 \|\underset{\approx}{\tau}_D\|_0$$

and it follows easily that

$$\|\underset{\approx}{\tau}_D\|_0^2 \ge \frac{C_2}{|\underset{\approx}{\sigma}_1^{-1}|^2} \|\underset{\approx}{\tau}\|_0^2 \tag{3.7}$$

where C_2 depends only on Ω. The lemma is an immediate consequence of (3.6) and (3.7).

4. Orthotropic Materials

An elastic material which admits three orthogonal planes of symmetry is termed orthotropic. Included in this case are hexagonal and cubic crystalline structures [15, page 31]. Orthotropic materials are also used to model woods, plywood and other composites [15, pages 58-60], and some biological substances, such as the basilar membrane of the inner ear [11]. Constrained orthotropic materials, in particular incompressible ones, are studied frequently in the engineering literature [8],[20]. To state the constitutive equation for an orthotropic material concisely it is convenient to introduce the notations

$$\underset{\sim}{\operatorname{diag}}\, \underset{\approx}{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33})^t \quad \text{and} \quad \underset{\sim}{\operatorname{offd}}\, \underset{\approx}{\sigma} = (\sigma_{23}, \sigma_{13}, \sigma_{12})^t$$

for the diagonal and offdiagonal of a symmetric 3×3 tensor. The constitutive equation may then be written

$$B\, \underset{\sim}{\operatorname{diag}}\, \underset{\approx}{\sigma} = \underset{\sim}{\operatorname{diag}}\, \underset{\approx}{\epsilon}(\underset{\sim}{u}),$$

$$G\, \underset{\sim}{\operatorname{offd}}\, \underset{\approx}{\sigma} = \underset{\sim}{\operatorname{offd}}\, \underset{\approx}{\epsilon}(\underset{\sim}{u}),$$

where

$$B = \begin{pmatrix} 1/E_1 & -\nu_{12}/E_2 & -\nu_{13}/E_3 \\ -\nu_{21}/E_1 & 1/E_2 & -\nu_{23}/E_3 \\ -\nu_{31}/E_1 & -\nu_{32}/E_2 & 1/E_3 \end{pmatrix}$$

and

$$G = \begin{pmatrix} 1/G_1 & 0 & 0 \\ 0 & 1/G_2 & 0 \\ 0 & 0 & 1/G_3 \end{pmatrix}.$$

Here the E_i are the Young's moduli of the material, the G_i are the shear moduli, and the ν_{ij} are the Poisson ratios. The relations

$$\nu_{ij} E_i = \nu_{ji} E_j, \; 1 \le i < j \le 3,$$

are satisfied, so an orthotropic material is defined by nine independent constants and the matrix B is symmetric.

The Young's modulus E_i is the ratio of tension to extension when the body is in a state of pure tension in the *ith* coordinate direction. The shear modulus G_i is the ratio of shear stress to shear strain when the body is in a state of pure shear orthogonal to the *ith* coordinate direction. The Poisson ratio ν_{ij} is the ratio of compression in the *ith* direction to extension in the *jth* direction for a material in a state of pure tension in the *jth* direction.

The condition that the compliance tensor be positive semidefinite implies that $E_i > 0$ and $G_i > 0$. It is *a priori* possible that one of these quantities is infinite, but in that case it is easy to see that the material admits a singular constraint. Henceforth we assume that the Young's moduli and shear moduli are positive finite real numbers. It is rare, though apparently possible, for some of the Poisson ratios to be negative [9].

Noting that $\operatorname{sign}(\nu_{jk}) = \operatorname{sign}(\nu_{kj})$, we introduce the symmetrized Poisson ratios

$$\nu_i = \operatorname{sign}(\nu_{jk}) \sqrt{\nu_{jk} \nu_{kj}}$$

where $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. Setting

$$D = \begin{pmatrix} E_1^{-1/2} & 0 & 0 \\ 0 & E_2^{-1/2} & 0 \\ 0 & 0 & E_3^{-1/2} \end{pmatrix}$$

and

$$M = \begin{pmatrix} 1 & -\nu_3 & -\nu_2 \\ -\nu_3 & 1 & -\nu_1 \\ -\nu_2 & -\nu_1 & 1 \end{pmatrix}$$

we have $B = DMD$. Thus the compliance tensor is positive semidefinite if and only if M is. Since the diagonal elements of M are positive, this holds if and only if the principle minors and the determinant of M are nonnegative, i.e., if and only if

$$1 - \nu_i^2 \ge 0, \; i = 1, 2, 3, \tag{4.1}$$

and

$$1 - \nu_1^2 - \nu_2^2 - \nu_3^2 - 2\nu_1\nu_2\nu_3 \ge 0. \tag{4.2}$$

Figure 4.1: The solid P of admissible Poisson ratios.

The region $P \subset \underset{\sim}{\mathbb{R}}$ described by these inequalities is a compact convex set which may be described as a solid curvilinear tetrahedron. Its vertices are the points $(-1,1,1)^t$, $(1,-1,1)^t$, $(1,1,-1)^t$, and $(-1,-1,-1)^t$, which are the only singular points of ∂P. The six line segments connecting these points form the 1-skeleton of a 3-simplex. This skeleton, which we denote by K, is entirely contained in ∂P and decomposes it into four curvilinear triangles in $\underset{\sim}{\mathbb{R}}$ with straight edges. These triangles are joined along their edges in a manner yielding a surface which is smooth except at the vertices. (See Figure 4.1.) To verify these assertions we note that for

$$|\nu_1| \leq 1, \quad |\nu_2| \leq 1 \tag{4.3}$$

(4.2) may be solved for ν_3 to give

$$-\nu_1\nu_2 - \sqrt{(1-\nu_1^2)(1-\nu_2^2)} \leq \nu_3 \leq -\nu_1\nu_2 + \sqrt{(1-\nu_1^2)(1-\nu_2^2)}. \tag{4.4}$$

Moreover (4.3) and (4.4) together imply that $|\nu_3| \leq 1$. Thus the systems (4.1-4.2) and (4.3-4.4) are equivalent. It is then easy to give parametric representations of the four curvilinear triangles forming ∂P. They are

$$\begin{aligned}
\nu_3 &= -\nu_1\nu_2 + \sqrt{(1-\nu_1^2)(1-\nu_2^2)}, \quad \nu_1 \geq -1,\, \nu_2 \geq -1,\, \nu_1 + \nu_2 \leq 0, \\
\nu_3 &= -\nu_1\nu_2 + \sqrt{(1-\nu_1^2)(1-\nu_2^2)}, \quad \nu_1 \leq 1,\, \nu_2 \leq 1,\, \nu_1 + \nu_2 \geq 0, \\
\nu_3 &= -\nu_1\nu_2 - \sqrt{(1-\nu_1^2)(1-\nu_2^2)}, \quad \nu_1 \leq 1,\, \nu_2 \geq -1,\, \nu_1 - \nu_2 \geq 0, \\
\nu_3 &= -\nu_1\nu_2 - \sqrt{(1-\nu_1^2)(1-\nu_2^2)}, \quad \nu_1 \geq -1,\, \nu_2 \leq 1,\, \nu_1 - \nu_2 \leq 0.
\end{aligned}$$

One easily checks that a point $\underset{\sim}{\nu} = (\nu_1, \nu_2, \nu_3)^t \in P$ lies on ∂P if and only if $\det M = 1 - \nu_1^2 - \nu_2^2 - \nu_3^2 - 2\nu_1\nu_2\nu_3 = 0$. Consequently if the vector of Poisson ratios of an orthotropic material lies in the interior of P, the material is unconstrained, while if it lies on the boundary the material is constrained. We now show that is $\underset{\sim}{\nu} \in \partial P \setminus K$ then the constraint is nonsingular, but if $\underset{\sim}{\nu} \in K$ the material admits a singular constraint. First suppose that $\underset{\sim}{\nu} \in K$. Without loss of generality we may suppose that $\underset{\sim}{\nu}$ lies on the line segment joining $(-1, 1, 1)^t$ and $(1, -1, 1)^t$. Then $\nu_3 = 1$ and $\nu_2 = -\nu_1$ so

$$M = \begin{pmatrix} 1 & -1 & \nu_1 \\ -1 & 1 & -\nu_1 \\ \nu_1 & -\nu_1 & 1 \end{pmatrix}.$$

In view of the form of the compliance tensor, we conclude that

$$\begin{pmatrix} E_1^{1/2} & 0 & 0 \\ 0 & E_2^{1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a constraint tensor, which is manifestly singular.

Next suppose that $\underset{\sim}{\nu} \in \partial P \setminus K$. We show that any nontrivial nullvector of M must then have all nonvanishing components. Indeed if $\underset{\sim}{z}$ denotes such a nullvector and z_3, for example, were to vanish, then $(z_1, z_2)^t$ would be a nontrivial nullvector of the matrix

$$\begin{pmatrix} 1 & -\nu_3 \\ -\nu_3 & 1 \end{pmatrix},$$

and consequently $\nu_3 = \pm 1$. From (4.2) it then follows that $\nu_1 \pm \nu_2 = 0$, whence $\underset{\sim}{\nu} \in K$, a contradiction. Again recalling the form of the compliance tensor for an orthotropic material, we deduce that when $\underset{\sim}{\nu} \in \partial P \setminus K$ the only constraint tensors are diagonal tensors with nonzero diagonal elements, which are nonsingular.

We are now in a position to invoke Theorem 1.1, with the following conclusion.

Theorem 4.1: *Let the elastic moduli of an orthotropic material satisfy*

$$(E_1, E_2, E_3, G_1, G_2, G_3, \nu_1, \nu_2, \nu_3) \in (0, \infty)^6 \times (P \setminus K).$$

Then the boundary value problem (1.1)–(1.3) is well-posed (in the sense of Theorem 1.1), the solution depending continuously on the load, boundary data, and elastic moduli. The a priori *estimate (1.5) holds with constant C uniform for elastic moduli in any compact subset of* $(0, \infty)^6 \times (P \setminus K)$.

We first presented this result in [3] under the additional assumption that the Poisson ratios are nonnegative.

5. Pure Traction and Pure Displacement Boundary Conditions

In this section we briefly indicate the changes necessary to analyze the elasticity system (1.1), (1.2) when the mixed boundary conditions (1.3) are replaced by either the displacement boundary condition

$$\underset{\sim}{u} = \underset{\sim}{g} \quad \text{on } \Gamma = \partial\Omega, \tag{5.1}$$

or the traction boundary condition

$$\underset{\approx}{\sigma}\,\underset{\sim}{n} = \underset{\sim}{g} \quad \text{on } \Gamma. \tag{5.2}$$

The latter case is entirely straightforward and we dispose of it immediately. A necessary and sufficient condition for the existence of a solution is the compatibility condition

$$\int_\Gamma \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, d\underset{\sim}{x} \quad \text{for all } \underset{\sim}{v} \in \mathrm{R}\underset{\sim}{\mathrm{M}}, \tag{5.3}$$

where

$$\mathrm{R}\underset{\sim}{\mathrm{M}} = \{\underset{\sim}{v} \in \underset{\sim}{L}^2(\Omega) : \underset{\sim}{v} = \underset{\sim}{c} + \underset{\approx}{Q}\underset{\sim}{x},\ \underset{\sim}{c} \in \mathbb{R},\ \underset{\approx}{Q} \in \mathbb{R}^{3\times 3},\ \underset{\approx}{Q} = -\underset{\approx}{Q}^t\}$$

is the space of rigid motions. When (5.3) holds, the solution is determined up to the addition of a rigid motion, and uniqueness may be obtained by requiring that $\underset{\sim}{u}$ belong to $\underset{\sim}{H}^1_\perp(\Omega)$, the orthogonal complement of $\mathrm{R}\underset{\sim}{\mathrm{M}}$ in $\underset{\sim}{H}^1(\Omega)$. The analogue of Theorem 1.1 for the traction problem thus applies to data $(\underset{\sim}{f}, \underset{\sim}{g}) \in \underset{\sim}{L}^2(\Omega) \times \underset{\sim}{L}^2(\Gamma)$ satisfying (5.3) and asserts existence and uniqueness of a solution in $\underset{\approx}{L}^2(\Omega) \times \underset{\sim}{H}^1_\perp(\Omega)$.

To prove the theorem, we consider a weak formulation of the traction problem which seeks $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega),\ \underset{\sim}{u} \in \underset{\sim}{H}^1_\perp(\Omega)$ such that

$$a(\underset{\approx}{\sigma}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{u}) = 0 \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega),$$

$$b(\underset{\approx}{\sigma}, \underset{\sim}{v}) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, d\underset{\sim}{x} - \int_\Gamma \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_\perp(\Omega).$$

Note that the latter equation actually holds for all $\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega)$ when the compatibility condition (5.3) is satisfied, so this weak formulation is justified. Proceeding as in Section 3, we may apply Brezzi's theorem to the analysis of this formulation to obtain the theorem.

The case of displacement boundary conditions is considerably more complicated, due to the existence of a compatibility condition only for constrained materials, the condition depending, moreover, on the compliance tensor. From (1.1), (5.1) and the fact that the material is homogeneous (specifically that $\underset{\approx}{\sigma}_1 = \underset{\approx}{\sigma}_1(\mathbf{A})$ is independent of $\underset{\sim}{x} \in \Omega$), we see that

$$\begin{aligned} \lambda_1 \int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_1\, d\underset{\sim}{x} &= \int_\Omega \underset{\approx}{\sigma} : \mathbf{A}\,\underset{\approx}{\sigma}_1\, d\underset{\sim}{x} \\ &= \int_\Omega \mathbf{A}\,\underset{\approx}{\sigma} : \underset{\approx}{\sigma}_1\, d\underset{\sim}{x} = \int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\sigma}_1\, d\underset{\sim}{x} \end{aligned} \tag{5.4}$$

$$= -\int_{\Omega} \underset{\sim}{u} \cdot \underset{\sim}{\operatorname{div}} \underset{\approx}{\sigma}_1 \, d\underset{\sim}{x} + \int_{\Gamma} \underset{\sim}{u} \cdot \underset{\approx}{\sigma}_1 \underset{\sim}{n} \, ds = \int_{\Gamma} \underset{\sim}{g} \cdot \underset{\approx}{\sigma}_1 \underset{\sim}{n} \, ds.$$

When $\mathbf{A}$ is singular, $\lambda_1 = 0$, implying the necessary condition

$$\int_{\Gamma} \underset{\sim}{g} \cdot \underset{\approx}{\sigma}_1(\mathbf{A}) \underset{\sim}{n} \, ds = 0. \tag{5.5}$$

When (5.5) does hold, uniqueness fails in that $(0, \underset{\approx}{\sigma}_1(\mathbf{A}))$ satisfies the homogeneous system. Uniqueness is restored by adding the side condition

$$\int_{\Omega} \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_1(\mathbf{A}) \, d\underset{\sim}{x} = 0. \tag{5.6}$$

Note that for $\lambda_1 \neq 0$, (5.6) follows from (5.5) by (5.4).

We remark that for the constraint of incompressibility, $\underset{\approx}{\sigma}_1$ is the identity tensor. In this case the compatibility condition (5.5) reduces to

$$\int_{\Gamma} \underset{\sim}{g} \cdot \underset{\sim}{n} \, ds = 0$$

and the side condition (5.6) to

$$\int_{\Omega} \operatorname{tr}(\underset{\approx}{\sigma}) \, d\underset{\sim}{x} = 0.$$

We now establish existence, uniqueness, and an *a priori* estimate for the displacement boundary value problem (assuming that the compliance tensor does not admit any singular constraints). For a weak formulation of the problem, we define the space

$$\underset{\approx}{W}_{\mathbf{A}} = \{\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega) : \int_{\Omega} \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_1(\mathbf{A}) \, d\underset{\sim}{x} = 0\}.$$

The proof of the following lemma, which differs only slightly from that of Lemma 3.1, will be discussed at the end of the section.

Lemma 5.1: *Let $\underset{\approx}{G} \in \underset{\approx}{W}_{\mathbf{A}}^*$, $\underset{\sim}{F} \in \underset{\sim}{H}_0^1(\Omega)^*$. Then there is a unique pair $(\underset{\approx}{\rho}, \underset{\sim}{z}) \in \underset{\approx}{W}_{\mathbf{A}} \times \underset{\sim}{H}_0^1(\Omega)$ such that*

$$a(\underset{\approx}{\rho}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{z}) = \langle \underset{\approx}{G}, \underset{\approx}{\tau} \rangle \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{W}_{\mathbf{A}}, \tag{5.7}$$

$$b(\underset{\approx}{\rho}, \underset{\sim}{v}) = \langle \underset{\sim}{F}, \underset{\sim}{v} \rangle \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega).$$

Moreover,

$$\|\underset{\approx}{\rho}\|_0 + \|\underset{\sim}{z}\|_1 \leq C(\|\underset{\approx}{G}\|_{\underset{\approx}{W}_{\mathbf{A}}^*} + \|\underset{\sim}{F}\|_{-1,0}),$$

where C depends only on Ω, an upper bound for $|\mathbf{A}|$, and a lower bound for $\chi(\mathbf{A})$.

Note that if

$$\langle \underset{\approx}{G}, \underset{\approx}{\sigma}_1(\mathbf{A}) \rangle = 0, \tag{5.8}$$

which will be the case for a Dirichlet problem with compatible data, then the solution of (5.7) satisfies the first equation also for $\underset{\approx}{\tau} = \underset{\approx}{\sigma}_1(\mathbf{A})$ and hence for all $\underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega)$, not just $\underset{\approx}{\tau} \in \underset{\approx}{W}_{\mathbf{A}}$. Therefore (5.7) is a valid weak formulation of the Dirichlet problem.

Now suppose that the displacement boundary data $\underset{\sim}{g}$ satisfies (5.5). Then the solution to the boundary value problem (1.1), (1.2), (5.1) may be written as $(\underset{\approx}{\sigma}, \underset{\sim}{u}^1 + \underset{\sim}{u}^2)$, where $\underset{\sim}{u}^1 = \underset{\sim}{\mathcal{E}}(\underset{\sim}{g})$ with $\underset{\sim}{\mathcal{E}} : \underset{\sim}{H}^{1/2}(\Gamma) \to \underset{\sim}{H}^1(\Omega)$ a bounded extension operator, and the pair $(\underset{\approx}{\sigma}, \underset{\sim}{u}^2)$ satisfies (5.7) with $\langle \underset{\sim}{F}, \underset{\sim}{v} \rangle = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, d\underset{\sim}{x}$, $\langle \underset{\approx}{G}, \underset{\approx}{\tau} \rangle = -b(\underset{\approx}{\tau}, \underset{\sim}{u}^1)$. The compatibility condition (5.5) insures (5.8), and so Lemma 5.1 implies first, that the displacement problem admits a unique solution $(\underset{\approx}{\sigma}, \underset{\sim}{u})$, and second, that

$$\|\underset{\approx}{\sigma}\|_0 + \|\underset{\sim}{u}\|_1 \le C(\|\underset{\sim}{f}\|_{-1,0} + |\underset{\sim}{g}|_{1/2}) \tag{5.9}$$

with C depending only on $\Omega, |\mathbf{A}|$, and $\chi(\mathbf{A})$.

If the displacement boundary data violates (5.5), then both these conclusions are false. Existence and uniqueness do not hold for a constrained material. Even for an unconstrained material the *a priori* estimate (5.9) does not hold uniformly. More precisely, $\int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_1(\mathbf{A})\, d\underset{\sim}{x}$ cannot be bounded independently of the material constants. However we can derive a uniform *a priori* bound on $\underset{\sim}{u}$ and on the orthogonal projection $\hat{\underset{\approx}{\sigma}}$ of $\underset{\approx}{\sigma}$ on the complement of the one-dimensional space spanned by $\underset{\approx}{\sigma}_1 = \underset{\approx}{\sigma}_1(\mathbf{A})$. To this end we decompose the solution as

$$(\underset{\approx}{\sigma}, \underset{\sim}{u}) = (\check{\underset{\approx}{\sigma}}, \check{\underset{\sim}{u}}) + (\hat{\underset{\approx}{\sigma}}, \hat{\underset{\sim}{u}}),$$

where

$$\check{\underset{\approx}{\sigma}} = \theta \underset{\approx}{\sigma}_1/\lambda_1, \quad \check{\underset{\sim}{u}} = \theta \underset{\approx}{\sigma}_1 \underset{\sim}{x}, \quad \theta = \int_\Gamma \underset{\sim}{g} \cdot \underset{\approx}{\sigma}_1 \underset{\sim}{n}\, ds/\text{measure}(\Omega).$$

Then $\hat{\underset{\approx}{\sigma}}$ is indeed the projection of $\underset{\approx}{\sigma}$ orthogonal to $\underset{\approx}{\sigma}_1$, as follows from (5.4), and the pair $(\hat{\underset{\approx}{\sigma}}, \hat{\underset{\sim}{u}})$ solves the boundary value problem

$$\mathbf{A}\, \hat{\underset{\approx}{\sigma}} = \underset{\approx}{\epsilon}(\hat{\underset{\sim}{u}}) \quad \text{in } \Omega,$$

$$\underset{\sim}{\operatorname{div}}\, \hat{\underset{\approx}{\sigma}} = \underset{\sim}{f} \quad \text{in } \Omega,$$

$$\hat{\underset{\sim}{u}} = \underset{\sim}{g} - \theta \underset{\approx}{\sigma}_1 \underset{\sim}{x} \quad \text{on } \partial\Omega.$$

The boundary data for this problem is compatible since

$$\int_\Gamma \underset{\approx}{\sigma}_1 \underset{\sim}{x} \cdot \underset{\approx}{\sigma}_1 \underset{\sim}{n}\, ds = \int_\Omega \underset{\approx}{\epsilon}(\underset{\approx}{\sigma}_1 \underset{\sim}{x}) : \underset{\approx}{\sigma}_1\, d\underset{\sim}{x}$$

$$= \int_\Omega |\underset{\approx}{\sigma}_1|^2\, d\underset{\sim}{x} = \text{measure}(\Omega).$$

Thus Lemma 5.1 implies

$$\|\hat{\underset{\approx}{\sigma}}\|_0 + \|\hat{\underset{\sim}{u}}\|_1 \le C(\|\underset{\sim}{f}\|_{-1,0} + |\underset{\sim}{g} - \theta \underset{\approx}{\sigma}_1 \underset{\sim}{x}|_{1/2}) \le C(\|\underset{\sim}{f}\|_{-1,0} + |\underset{\sim}{g}|_{1/2}).$$

Clearly also $\|\check{\underset{\sim}{u}}\|_1 \le C|\underset{\sim}{g}|_{1/2}$, so

$$\|\hat{\underset{\approx}{\sigma}}\|_0 + \|\underset{\sim}{u}\|_1 \le C(\|\underset{\sim}{f}\|_{-1,0} + |\underset{\sim}{g}|_{1/2}),$$

which gives the desired *a priori* bound.

Finally we consider the continuous dependence of the solution on the elastic moduli. Thus we fix a value $\bar{\mathbf{A}}$ of the compliance tensor and data $\bar{\underset{\sim}{f}}$ and $\bar{\underset{\sim}{g}}$, and denote by $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}})$ the corresponding solution. One might hope to show that if $(\mathbf{A}, \underset{\sim}{f}, \underset{\sim}{g})$ is sufficiently close to $(\bar{\mathbf{A}}, \bar{\underset{\sim}{f}}, \bar{\underset{\sim}{g}})$, then the solution $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ determined by $(\mathbf{A}, \underset{\sim}{f}, \underset{\sim}{g})$ is arbitrarily near $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}})$, i.e., that

$$\lim_{(\mathbf{A}, \underset{\sim}{f}, \underset{\sim}{g}) \to (\bar{\mathbf{A}}, \bar{\underset{\sim}{f}}, \bar{\underset{\sim}{g}})} (\underset{\approx}{\sigma}, \underset{\sim}{u}) = (\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}}) \quad \text{in } \underset{\approx}{L}^2(\Omega) \times \underset{\sim}{H}^1(\Omega). \tag{5.10}$$

Of course we assume that neither compliance tensor $\bar{\mathbf{A}}$ nor $\mathbf{A}$ admits a singular constraint. Moreover we may assume that the limiting material is constrained, i.e., that $\bar{\mathbf{A}}$ is singular, since otherwise the result is obvious. Now for $\bar{\mathbf{A}}$ singular we must suppose that

$$\int_\Gamma \bar{\underset{\sim}{g}} \cdot \bar{\underset{\approx}{\sigma}}_1 \underset{\sim}{n}\, ds = 0, \tag{5.11}$$

(where $\bar{\underset{\approx}{\sigma}}_1 = \underset{\approx}{\sigma}_1(\bar{\mathbf{A}})$) in order that the solution $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}})$ exist and (5.10) make sense. This condition is not, however, sufficient to make sense of (5.10), since even if (5.11) holds there may exist singular tensors $\mathbf{A}$ arbitrarily near $\bar{\mathbf{A}}$ for which $\underset{\sim}{g}$ is *not* compatible and hence for which $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ is undefined. We may circumvent this difficulty in two ways. First, we may consider the special case $\underset{\sim}{g} = 0$. In this case there is no problem of incompatibility and (5.10) follows from (5.9) by a straightforward argument. Second, to derive a result valid for nonzero $\bar{\underset{\sim}{g}}$ satisfying (5.11), we consider the singular compliance tensor $\bar{\mathbf{A}}$ as the limit of positive definite tensors $\mathbf{A}$, i.e., we restrict $\mathbf{A}$ in (5.10) to be nonsingular. Even with this restriction, however, it is not hard to see that (5.10) is not valid, as $\underset{\approx}{\sigma}$ may have a component in the direction of $\underset{\approx}{\sigma}_1(\mathbf{A})$ which becomes unbounded as $\mathbf{A}$ tends to $\bar{\mathbf{A}}$. However we shall show that

$$\lim(\|\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}\|_{\underset{\approx}{L}^2(\Omega)/\underset{\approx}{\sigma}_1(\mathbf{A})} + \|\underset{\sim}{u} - \bar{\underset{\sim}{u}}\|_1) = 0 \tag{5.12}$$

with the quotient seminorm in (5.12) defined by

$$\|\underset{\approx}{\rho}\|_{\underset{\approx}{L}^2(\Omega)/\underset{\approx}{\sigma}_1(\mathbf{A})} = \inf_{c \in \mathbb{R}} \|\underset{\approx}{\rho} + c\underset{\approx}{\sigma}_1(\mathbf{A})\|_{\underset{\approx}{L}^2(\Omega)},$$

and the limit taken as $(\mathbf{A}, \underset{\sim}{f}, \underset{\sim}{g})$ tends to $(\bar{\mathbf{A}}, \bar{\underset{\sim}{f}}, \bar{\underset{\sim}{g}})$ with $\mathbf{A}$ nonsingular. (This seminorm depends on $\mathbf{A}$, but for all $\mathbf{A}$ exceeds the quotient seminorm on $\underset{\approx}{L}^2(\Omega)$ induced by the six-dimensional subspace of constant tensors.)

To prove (5.12) we note that

$$a(\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{u} - \bar{\underset{\sim}{u}}) = \int_\Omega (\bar{\mathbf{A}} - \mathbf{A})\bar{\underset{\approx}{\sigma}} : \underset{\approx}{\tau}\, d\underset{\sim}{x} \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{W}_{\mathbf{A}},$$

$$b(\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}, \underset{\sim}{v}) = \int_\Omega (\underset{\sim}{f} - \bar{\underset{\sim}{f}}) \cdot \underset{\sim}{v} \, d\underset{\sim}{x} \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega).$$

Now let $\underset{\approx}{\rho}$ denote the projection of $\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}$ on the orthogonal complement of $\underset{\approx}{\sigma}_1(\mathbf{A})$ in $\underset{\approx}{L}^2(\Omega)$, and let $\underset{\sim}{z} = \underset{\sim}{u} - \bar{\underset{\sim}{u}} - \underset{\sim}{\mathcal{E}}(\underset{\sim}{g} - \bar{\underset{\sim}{g}})$. Then $(\underset{\approx}{\rho}, \underset{\sim}{z}) \in \underset{\approx}{W}_{\mathbf{A}} \times \underset{\sim}{H}_0^1(\Omega)$ and

$$a(\underset{\approx}{\rho}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{z}) = \int_\Omega (\bar{\mathbf{A}} - \mathbf{A}) \, \bar{\underset{\approx}{\sigma}} : \underset{\approx}{\tau} \, d\underset{\sim}{x} - b(\underset{\approx}{\tau}, \underset{\sim}{\mathcal{E}}(\underset{\sim}{g} - \bar{\underset{\sim}{g}})) \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{W}_{\mathbf{A}},$$

$$b(\underset{\approx}{\rho}, \underset{\sim}{v}) = \int_\Omega (\underset{\sim}{f} - \bar{\underset{\sim}{f}}) \cdot \underset{\sim}{v} \, d\underset{\sim}{x} \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega).$$

By Lemma 5.1

$$\|\underset{\approx}{\rho}\|_0 + \|\underset{\sim}{z}\|_1 \le C(|\bar{\mathbf{A}} - \mathbf{A}| \|\bar{\underset{\approx}{\sigma}}\|_0 + \|\underset{\sim}{\mathcal{E}}(\underset{\sim}{g} - \bar{\underset{\sim}{g}})\|_1 + \|\underset{\sim}{f} - \bar{\underset{\sim}{f}}\|_{-1,0})$$

$$\le C(|\bar{\mathbf{A}} - \mathbf{A}| + |\underset{\sim}{g} - \bar{\underset{\sim}{g}}|_{1/2,\Gamma} + \|\underset{\sim}{f} - \bar{\underset{\sim}{f}}\|_{-1,0}).$$

Further,

$$\|\underset{\sim}{u} - \bar{\underset{\sim}{u}}\|_1 \le \|\underset{\sim}{z}\|_1 + C|\underset{\sim}{g} - \bar{\underset{\sim}{g}}|_{1/2,\Gamma}$$

and

$$\|\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}\|_{\underset{\approx}{L}^2(\Omega)/\underset{\approx}{\sigma}_1(\mathbf{A})} = \|\underset{\approx}{\rho}\|_0,$$

and so (5.12) is established.

We close this section with a brief discussion of the proof of Lemma 5.1. It follows the proof of Lemma 3.1 very closely and differs significantly in only one point. In the statement of Lemma 3.2, which was used in the proof of Lemma 3.1, we must of course replace the space $\underset{\sim}{H}_D^1(\Omega)$ with $\underset{\sim}{H}_0^1(\Omega)$. We must also replace the space $\underset{\approx}{L}^2(\Omega)$ with $\{\underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega) : \int_\Omega \underset{\approx}{\tau} : \underset{\approx}{\sigma}_1 \, d\underset{\sim}{x} = 0\}$. Only when $\underset{\approx}{\tau}$ lies in this space does the differential equation

$$\operatorname{div} \underset{\sim}{p} = \underset{\approx}{\tau} : \underset{\approx}{\sigma}_1$$

have a solution in $\underset{\sim}{H}_0^1(\Omega)$, which enables the proof of Lemma 3.2 to be carried out as before. The additional hypothesis that $\underset{\approx}{\tau}$ be orthogonal to $\underset{\approx}{\sigma}_1$ causes no problem, since in the application to the proof of Lemma 5.1 this hypothesis follows from the membership of $\underset{\approx}{\rho}$ in $\underset{\approx}{W}_{\mathbf{A}}$.

6. Necessity of the Nonsingularity Condition

In this section, we show that if the compliance tensor *does* admit a singular constraint, then the elasticity problem is very ill-posed for a large class of boundary value problems. Indeed, for these problems, no solution exists unless the displacement boundary data satisfies an infinite number of linearly independent constraints. Further, the homogeneous problem admits an infinite dimensional space of solutions.

If the compliance tensor admits a singular constraint, then there exists $0 \neq \underset{\approx}{\sigma}_1 \in \underset{\approx}{\mathbb{R}}$, $0 \neq \underset{\sim}{m} \in \underset{\sim}{\mathbb{R}}$ such that $\mathbf{A}\,\underset{\approx}{\sigma}_1 = 0$, $\underset{\approx}{\sigma}_1 \underset{\sim}{m} = 0$. We suppose that there exists a nonempty interval I such that the cross-section $\Gamma^{(q)} = \{\underset{\sim}{x} \in \Gamma | \underset{\sim}{x} \cdot \underset{\sim}{m} = q\}$ is contained in Γ_1 for all $q \in I$. This hypothesis excludes the pure traction problem, but permits the displacement problem and a wide variety of mixed boundary value problems as well. First we note that the problem (1.1)–(1.3) does not admit a solution unless the Dirichlet data $\underset{\sim}{g}_1$ satisfies the linear constraint

$$\int_{\bigcup_{q\in J} \Gamma^{(q)}} \underset{\sim}{g}_1 \cdot \underset{\approx}{\sigma}_1 \underset{\sim}{n}\, ds = 0$$

for all subintervals J of I. This follows from the equation $\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\sigma}_1 = 0$ (itself a consequence of (1.1)), by integrating over $\{\underset{\sim}{x} \in \Omega | \underset{\sim}{x} \cdot \underset{\sim}{m} \in J\}$. Also, associated to any J, there is a solution to the homogeneous boundary value problem given by $\underset{\sim}{u} \equiv 0$ and $\underset{\approx}{\sigma}(\underset{\sim}{x}) = \underset{\approx}{\sigma}_1$ if $\underset{\sim}{x} \cdot \underset{\sim}{m} \in J$, $\underset{\approx}{\sigma}(\underset{\sim}{x}) = 0$ otherwise. (Although $\underset{\approx}{\sigma}$ is discontinuous, it is easy to check that $\underset{\sim}{\operatorname{div}}\, \underset{\approx}{\sigma} = 0$ in the sense of distributions, since $\underset{\approx}{\sigma}\underset{\sim}{m} = 0$.)

7. Ellipticity

The system (1.1),(1.2) of anisotropic elasticity is elliptic in the sense of Agmon, Douglis, and Nirenberg [1] when the compliance tensor is positive definite. In this section we show that ellipticity of the system holds for precisely those materials admitting no nonsingular constraints, and, more importantly, that the ellipticity is uniform with respect to the compliances in the sense that the symbolic determinant whose nonvanishing defines ellipticity may be bounded above and below by positive constants depending only on an upper bound for the compliances and a lower bound for $\chi(\mathbf{A})$. This implies (among other things) uniform interior regularity estimates on the solution of the equations [7].

For the verification of ellipticity we define, for any 3-vector $\underset{\sim}{\theta}$, the operator $E(\underset{\sim}{\theta}) : \underset{\sim}{\mathbb{R}} \to \underset{\approx}{\mathbb{R}}$ by

$$E(\underset{\sim}{\theta})\underset{\sim}{v} = \frac{1}{2}(\underset{\sim}{\theta}\,\underset{\sim}{v}^t + \underset{\sim}{v}\,\underset{\sim}{\theta}^t), \quad \underset{\sim}{v} \in \underset{\sim}{\mathbb{R}}.$$

The adjoint operator $E(\underset{\sim}{\theta})^t : \underset{\approx}{\mathbb{R}} \to \underset{\sim}{\mathbb{R}}$ is given by

$$E(\underset{\sim}{\theta})^t \underset{\approx}{\tau} = \underset{\approx}{\tau}\,\underset{\sim}{\theta}, \quad \underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}}.$$

Then $E(\underset{\sim}{\nabla})\underset{\sim}{u} = \underset{\approx}{\epsilon}(\underset{\sim}{u})$ and $E(\underset{\sim}{\nabla})^t \underset{\approx}{\sigma} = \underset{\sim}{\operatorname{div}}\, \underset{\approx}{\sigma}$, so we may write the system (1.1), (1.2) as

$$\mathcal{L}(\underset{\sim}{\nabla})(\underset{\approx}{\sigma}, \underset{\sim}{u}) = (0, \underset{\sim}{f})$$

where

$$\mathcal{L}(\underset{\sim}{\theta})(\underset{\approx}{\tau}, \underset{\sim}{v}) = (\mathbf{A}\,\underset{\approx}{\tau} - E(\underset{\sim}{\theta})\underset{\sim}{v}, E(\underset{\sim}{\theta})^t\underset{\approx}{\tau}) \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}},\ \underset{\sim}{v}, \underset{\sim}{\theta} \in \underset{\sim}{\mathbb{R}}.$$

To speak of ellipticity we must identify the principal part of the differential operator $\mathcal{L}(\underset{\sim}{\nabla})$. This is done in [1] and [7] by the assignment of weights s_i and t_j to the *ith* equation and *jth* unknown respectively. Without introducing an arbitrary numbering of the equations and unknowns, we associate weight 0 with the six scalar stress unknowns and the six scalar equations given by (1.1), and weight 1 with the three scalar displacement unknowns and the three equations given by (1.2). It is then easily seen that $\mathcal{L}(\underset{\sim}{\nabla})$ coincides with its principal part. Therefore the system of anisotropic elasticity is elliptic if and only if $\det[\mathcal{L}(\underset{\sim}{\theta})]$ is nonzero for all nonzero $\underset{\sim}{\theta}$. ($\mathcal{L}$ is a linear operator on the nine-dimensional space $\underset{\approx}{\mathbb{R}} \times \underset{\sim}{\mathbb{R}}$, so we may speak of its determinant.) Now if the material admits a singular constraint, then we have $\underset{\approx}{\sigma}_1 \neq 0, \underset{\sim}{s} \neq 0$ with $\mathbf{A}\,\underset{\approx}{\sigma}_1 = 0,\ \underset{\approx}{\sigma}_1\underset{\sim}{s} = 0$. It follows that $\mathcal{L}(\underset{\sim}{s})(\underset{\approx}{\sigma}_1, 0) = 0$ so $\det[\mathcal{L}(\underset{\sim}{s})] = 0$, and the elasticity system is not elliptic. The following theorem establishes the uniform ellipticity of the system if the material admits no singular constraints.

Theorem 7.1: *Suppose that the material admits no singular constraints. Then there exists a positive constant β depending only on an upper bound for $\mathbf{A}$ and a lower bound for $\chi(\mathbf{A})$ such that*

$$\beta|\underset{\sim}{\theta}|^2 \leq \det[\mathcal{L}(\underset{\sim}{\theta})] \leq \beta^{-1}|\underset{\sim}{\theta}|^2 \quad \textit{for all } \underset{\sim}{\theta} \in \underset{\sim}{\mathbb{R}}. \tag{7.1}$$

Proof: Since $\det[\mathcal{L}(\underset{\sim}{\theta})]$ is a homogeneous polynomial of degree 2 in $\underset{\sim}{\theta}$, (7.1) is equivalent to the condition

$$\beta \leq \det[\mathcal{L}(\underset{\sim}{\theta})] \leq \beta^{-1} \quad \text{for all unit vectors } \underset{\sim}{\theta} \in \underset{\sim}{\mathbb{R}}.$$

The asserted upper bound is obvious, and we discuss only the lower bound. We shall show that $\mathcal{L}(\underset{\sim}{\theta})$ is invertible and bound the spectral norm $\|\mathcal{L}(\underset{\sim}{\theta})^{-1}\|$ by a constant C depending only on $|\mathbf{A}|$ and $\chi(\mathbf{A})$. This will imply that the eigenvalues of $\mathcal{L}(\underset{\sim}{\theta})$ are all bounded below by $1/C$, so that $\det[\mathcal{L}(\underset{\sim}{\theta})] \geq 1/C^9$ as desired.

To prove the invertibility of $\mathcal{L}(\underset{\sim}{\theta})$ and establish the uniform bound on $\mathcal{L}(\underset{\sim}{\theta})^{-1}$, we apply Brezzi's theorem (Theorem 2.1) to the finite dimensional problem:

Given $(\underset{\approx}{G}, \underset{\sim}{F}) \in \underset{\approx}{\mathbb{R}} \times \underset{\sim}{\mathbb{R}}$, find $(\underset{\approx}{\sigma}, \underset{\sim}{u}) \in \underset{\approx}{\mathbb{R}} \times \underset{\sim}{\mathbb{R}}$ such that

$$\mathbf{A}\,\underset{\approx}{\sigma} : \underset{\approx}{\tau} - \underset{\sim}{u} \cdot \underset{\approx}{\tau}\,\underset{\sim}{\theta} = \underset{\approx}{G} : \underset{\approx}{\tau} \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}}$$

and

$$\underset{\approx}{\sigma}\,\underset{\sim}{\theta} \cdot \underset{\sim}{v} = \underset{\sim}{F} \cdot \underset{\sim}{v} \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\mathbb{R}}.$$

It is easily checked that $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ solves this problem if and only if

$$\mathcal{L}(\underset{\sim}{\theta})(\underset{\approx}{\sigma}, \underset{\sim}{u}) = (\underset{\approx}{G}, \underset{\sim}{F}).$$

Thus we must prove that this problem has a unique solution $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ and that

$$|\underset{\approx}{\sigma}| + |\underset{\sim}{u}| \leq C(|\underset{\approx}{G}| + |\underset{\sim}{F}|).$$

By BREZZI's theorem, it suffices to show that there exists $\gamma > 0$ such that

$$\mathbf{A}\,\underset{\approx}{\tau} : \underset{\approx}{\tau} \geq \gamma\,|\underset{\approx}{\tau}|^2 \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}} \text{ satisfying } \underset{\approx}{\tau}\,\underset{\sim}{\theta} = 0, \tag{7.2}$$

and

$$\inf_{0 \neq \underset{\sim}{v} \in \underset{\sim}{\mathbb{R}}} \; \sup_{0 \neq \underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}}} \frac{\underset{\approx}{\tau}\,\underset{\sim}{\theta} \cdot \underset{\sim}{v}}{|\underset{\approx}{\tau}|\;|\underset{\sim}{v}|} \geq \gamma. \tag{7.3}$$

The proof of (7.3) is direct. If $\underset{\approx}{\tau} = \sqrt{3}U^t R(U\underset{\sim}{v})U$, where U is an orthogonal matrix chosen so that $\sqrt{3}U\underset{\sim}{\theta} = (1,1,1)^t$ and $R(\underset{\sim}{z})$ is a diagonal matrix with $\underset{\sim}{\operatorname{diag}}\, R(\underset{\sim}{z}) = \underset{\sim}{z}$, then $|\underset{\approx}{\tau}| \leq C|\underset{\sim}{v}|$ and $\underset{\approx}{\tau}\,\underset{\sim}{\theta} = \underset{\sim}{v}$.

To prove (7.2) we decompose $\underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}}$ as $\underset{\approx}{\tau}_D + \underset{\approx}{\tau}_T$ with $\underset{\approx}{\tau}_T = (\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1)\underset{\approx}{\sigma}_1$. Then (cf. (3.6))

$$\mathbf{A}\,\underset{\approx}{\tau} : \underset{\approx}{\tau} \geq \max(\lambda_1|\underset{\approx}{\tau}|^2, \lambda_2|\underset{\approx}{\tau}_D|^2). \tag{7.4}$$

Now

$$\underset{\approx}{\tau}_T\underset{\sim}{\theta} \cdot \underset{\approx}{\sigma}_1^{-1}\underset{\sim}{\theta} = (\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1)\underset{\approx}{\sigma}_1\underset{\sim}{\theta} \cdot \underset{\approx}{\sigma}_1^{-1}\underset{\sim}{\theta} = \underset{\approx}{\tau} : \underset{\approx}{\sigma}_1$$

and by hypothesis,

$$\underset{\approx}{\tau}_D\underset{\sim}{\theta} = (\underset{\approx}{\tau} - \underset{\approx}{\tau}_T)\underset{\sim}{\theta} = -\underset{\approx}{\tau}_T\underset{\sim}{\theta}.$$

Therefore,

$$|\underset{\approx}{\tau}_T| = |\underset{\approx}{\tau} : \underset{\approx}{\sigma}_1| = |\underset{\approx}{\tau}_D\underset{\sim}{\theta} \cdot \underset{\approx}{\sigma}_1^{-1}\underset{\sim}{\theta}| \leq |\underset{\approx}{\sigma}_1^{-1}|\;|\underset{\approx}{\tau}_D|. \tag{7.5}$$

Combining (7.4) and (7.5) with the identity $|\underset{\approx}{\tau}|^2 = |\underset{\approx}{\tau}_T|^2 + |\underset{\approx}{\tau}_D|^2$ gives (7.2).

8. The Displacement - Pressure Formulation of Anisotropic Elasticity

The system (1.1), (1.2) of three-dimensional elasticity involves nine independent scalar unknowns. This is often considered too many for computational purposes, and other formulations are preferred. When the compliance tensor is invertible, the simplest possibility is to solve (1.1) for $\underset{\approx}{\sigma}$ and substitute in (1.2) to obtain the displacement equations of elasticity, which involve only the three displacements as unknowns. However, when the compliance tensor is singular this procedure is not possible, and when it is nearly singular it is often not advisable. For isotropic materials, incompressible or not, another formulation is widely used. This formulation involves only the displacement and one stress quantity (a pressure) as unknowns, and in the incompressible limit reduces to the Stokes equations. Here we introduce an analogous formulation valid for any anisotropic material, constrained or not, as long as the nullspace of the compliance tensor has dimension less than two (in particular if the material admits no singular constraints).

In the case of orthotropic elasticity, Key [12] and Taylor, Pfister, and Herrmann [22] have derived related formulations, extending work of Herrmann [10] for isotropic elasticity. Debognie [6] used a similar formulation to study incompressible anisotropic materials.

Our derivation is based on the decomposition of $\underset{\approx}{\mathbb{R}}$ into the one-dimensional subspace spanned by $\underset{\approx}{\sigma}_1 = \underset{\approx}{\sigma}_1(\mathbf{A})$ and its orthogonal complement

$$\underset{\approx}{Y} = \{\underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}} : \underset{\approx}{\tau} : \underset{\approx}{\sigma}_1 = 0\}.$$

Clearly, $\mathbf{A}$ maps $\underset{\approx}{Y}$ into itself and, since $\lambda_2 > 0$, the restriction $\mathbf{A}|_{\underset{\approx}{Y}}$ is positive definite. Define $\mathbf{A}^+ : \underset{\approx}{\mathbb{R}} \to \underset{\approx}{\mathbb{R}}$ by

$$\mathbf{A}^+ \underset{\approx}{\tau} = (\mathbf{A}|_{\underset{\approx}{Y}})^{-1} \underset{\approx}{\tau} \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{Y},$$

$$\mathbf{A}^+ \underset{\approx}{\sigma}_1 = 0.$$

Again decomposing

$$\underset{\approx}{\sigma} = p\, \underset{\approx}{\sigma}_1 + \underset{\approx}{\sigma}_D, \tag{8.1}$$

with $p = \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_1$ and $\underset{\approx}{\sigma}_D \in \underset{\approx}{Y}$, we deduce from (1.1) that

$$\underset{\approx}{\epsilon}(\underset{\sim}{u}) = \mathbf{A}\, \underset{\approx}{\sigma} = \lambda_1 p\, \underset{\approx}{\sigma}_1 + \mathbf{A}\, \underset{\approx}{\sigma}_D. \tag{8.2}$$

Applying $\mathbf{A}^+$ to this equation we get

$$\mathbf{A}^+ \underset{\approx}{\epsilon}(\underset{\sim}{u}) = \underset{\approx}{\sigma}_D$$

so (8.1) becomes

$$\underset{\approx}{\sigma} = \mathbf{A}^+ \underset{\approx}{\epsilon}(\underset{\sim}{u}) + p\, \underset{\approx}{\sigma}_1.$$

Inserting into (1.2) and noting that $\underset{\sim}{\operatorname{div}}\,(p\, \underset{\approx}{\sigma}_1) = \underset{\approx}{\sigma}_1 \underset{\sim}{\operatorname{grad}}\, p$ yields

$$\underset{\sim}{\operatorname{div}}\, \mathbf{A}^+ \underset{\approx}{\epsilon}(\underset{\sim}{u}) + \underset{\approx}{\sigma}_1 \underset{\sim}{\operatorname{grad}}\, p = \underset{\sim}{f}. \tag{8.3}$$

Next, taking the inner product of (8.2) with $\underset{\approx}{\sigma}_1$ and noting that $\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\sigma}_1 = \operatorname{div}(\underset{\approx}{\sigma}_1 \underset{\sim}{u})$ we get

$$\operatorname{div}(\underset{\approx}{\sigma}_1 \underset{\sim}{u}) - \lambda_1 p = 0. \tag{8.4}$$

Equations (8.3) and (8.4) give the desired formulation.

For a two-dimensional constrained anisotropic material it is possible to reduce the elastic system further, to a fourth order elliptic equation for a single scalar unknown. In the incompressible isotropic case this is the biharmonic equation. Define

$$\underset{\sim}{\operatorname{rot}}\,\phi = \begin{pmatrix} \partial\phi/\partial x_2 \\ -\partial\phi/\partial x_1 \end{pmatrix}, \quad \operatorname{curl}\underset{\sim}{\psi} = -\partial\psi_1/\partial x_2 + \partial\psi_2/\partial x_1,$$

(where now $\underset{\sim}{\psi} = (\psi_1, \psi_2)^t$ is a 2 vector). Multiply the analogue of (8.3) for two-dimensional elasticity by $\underset{\approx}{\sigma}_1^{-1}$ and take the curl to get

$$\operatorname{curl}[\underset{\approx}{\sigma}_1^{-1} \underset{\sim}{\operatorname{div}}\,\mathbf{A}^+ \underset{\approx}{\epsilon}(\underset{\sim}{u})] = \operatorname{curl}\underset{\approx}{\sigma}_1^{-1} \underset{\sim}{f}. \tag{8.5}$$

Now in the two-dimensional constrained case $\operatorname{div}(\underset{\approx}{\sigma}_1 \underset{\sim}{u}) = 0$ (cf. (8.4)), so $\underset{\sim}{u} = \underset{\approx}{\sigma}_1^{-1} \underset{\sim}{\operatorname{rot}}\,\phi$ for some scalar function ϕ. Substituting in (8.5) gives the self-adjoint fourth order differential equation

$$\operatorname{curl}[\underset{\approx}{\sigma}_1^{-1} \underset{\sim}{\operatorname{div}}\,\mathbf{A}^+ \underset{\approx}{\epsilon}(\underset{\approx}{\sigma}_1^{-1} \underset{\sim}{\operatorname{rot}}\,\phi)] = \operatorname{curl}\underset{\approx}{\sigma}_1^{-1} \underset{\sim}{f}.$$

Using the identity

$$\underset{\approx}{\sigma}_1 : \underset{\approx}{\epsilon}(\underset{\approx}{\sigma}_1^{-1} \underset{\sim}{\operatorname{rot}}\,\phi) = \operatorname{div} \underset{\sim}{\operatorname{rot}}\,\phi = 0,$$

it is easy to check that this defines a coercive variational problem on $H_0^2(\Omega)$.

9. Plane Elasticity

The results of the previous sections adapt to elasticity in $\mathbb{R}^2$ with one difference. By the method of proof of Section 3, it can be shown that if 0 is not a double eigenvalue of the compliance tensor (now a semidefinite operator on the space of 2×2 symmetric matrices) and if there is no nonzero singular nulltensor, then the fundamental boundary value problems are well-posed and the constants in the *a priori* estimates depend on

$$\chi(\mathbf{A}) = \max[\lambda_1(\mathbf{A}), \lambda_2(\mathbf{A})/|\underset{\approx}{\sigma}_1(\mathbf{A})^{-1}|^2]$$

as before. However, in the two-dimensional case it is possible for the compliance tensor to admit zero as a double eigenvalue without admitting a singular constraint. (This was ruled out in the three-dimensional case in Section 2.) That is, we may have $\chi(\mathbf{A}) = 0$ even though the material does not admit a singular constraint. We regard this as a pathological case. An example (which is essentially canonical) is given by the compliance tensor

$$\mathbf{A}\,\underset{\approx}{\tau} = \operatorname{tr}(\underset{\approx}{\tau})\underset{\approx}{\delta}.$$

It is easily verified that the homogeneous Dirichlet problem

$$\mathbf{A}\,\underset{\approx}{\tau} = \underset{\approx}{\epsilon}(\underset{\sim}{u}) \quad \text{in } \Omega, \tag{9.1}$$

$$\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau} = 0 \quad \text{in } \Omega, \tag{9.2}$$

$$\underset{\sim}{u} = 0 \quad \text{on } \Gamma = \partial\Omega,$$

($\Omega \subset \mathbb{R}^2$) admits an infinite dimensional solution space, namely

$$\underset{\sim}{u} = 0, \quad \underset{\approx}{\tau} = \begin{pmatrix} \partial^2\phi/\partial y^2 & -\partial^2\phi/\partial x\partial y \\ -\partial^2\phi/\partial x\partial y & \partial^2\phi/\partial x^2 \end{pmatrix},$$

where ϕ is any harmonic function on Ω. It is interesting to note that the differential equations (9.1),(9.2) form an elliptic system in the sense of Agmon, Douglis, and Nirenberg, even though the Dirichlet problem is not Fredholm. In fact, the result of Section 7 that the system of elasticity is elliptic if and only if the compliance tensor admits no singular constraint holds also in two dimensions, although the proof must be modified.

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