

WELL-POSEDNESS, REGULARITY AND EXACT CONTROLLABILITY FOR THE PROBLEM OF TRANSMISSION OF THE SCHRÖDINGER EQUATION

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Abstract. In this paper, we shall study the system of transmission of the Schrödinger equation with Dirichlet control and colocated observation. Using the multiplier method, we show that the system is well-posed with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$. The regularity of the system is also established, and the feedthrough operator is found to be zero. Finally, the exact controllability of the open-loop system is obtained by proving the observability inequality of the dual system.

1. Introduction. In [19], Salamon introduced the class of well-posed linear systems. The aim was to provide a unifying abstract framework to formulate and solve control problems for systems described by functional and partial differential equations. Roughly speaking, a well-posed linear system is a linear time invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. This means that every well-posed system has a well-defined transfer function $G(s)$. An important subclass of well-posed linear systems is formed by the regular systems. A regular system ([22]) is a well-posed system satisfying the extra requirement that $\lim_{s \in \mathbb{R}, s \rightarrow +\infty} G(s) = D$ exists.

There is now a rich literature on the abstract theory for regular well-posed linear systems and from a practical point of view, the construction of specific examples of distributed parameter systems which belong to this class is of considerable importance. In recent years, a limited number of PDEs with boundary control and observation are proved to be well-posed and regular (see [2], [4], [7], [5], [6], [8], [9], [1], [10], [3]).

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In this paper, we shall study the system of transmission of the Schrödinger equation with Dirichlet control and colocated observation. Using the multiplier method, we show that the system is well-posed with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$. The regularity of the system is also established and the feedthrough operator is found to be zero. Finally, the exact controllability of the open-loop system is obtained by proving the observability inequality of the dual system.

2. System description and main results. Let Ω be an open bounded domain of \mathbb{R}^n ($n \geq 2$) with smooth boundary Γ , and let Ω_1 be a bounded domain contained inside Ω ; $\overline{\Omega_1} \subset \Omega$ with smooth boundary Γ_1 , Ω_2 is the domain $\Omega \setminus \Omega_1$ and ν is the unit normal of Γ or Γ_1 pointing toward the exterior of Ω_2 .

Let a time $T > 0$ and two distinct constants $a_1, a_2 > 0$ be given.

In this paper, we shall be concerned with the following system of transmission of the Schrödinger equation with Dirichlet control and colocated observation.

$$y'(x, t) = \mathbf{i} \operatorname{div}(a(x) \nabla y(x, t)), \quad (x, t) \in \Omega \times (0, T), \tag{2.1}$$

$$y(x, 0) = y^0(x), \quad x \in \Omega, \tag{2.2}$$

$$y_2(x, t) = u(x, t), \quad (x, t) \in \Gamma \times (0, T), \tag{2.3}$$

$$y_1(x, t) = y_2(x, t), \quad (x, t) \in \Gamma_1 \times (0, T), \tag{2.4}$$

$$a_1 \frac{\partial y_1(x, t)}{\partial \nu} = a_2 \frac{\partial y_2(x, t)}{\partial \nu}, \quad (x, t) \in \Gamma_1 \times (0, T), \tag{2.5}$$

$$z(x, t) = \mathbf{i} \frac{\partial}{\partial \nu} (A^{-1} y_2(x, t)), \quad (x, t) \in \Gamma \times (0, T), \tag{2.6}$$

where

$$y'(x, t) = \frac{\partial y(x, t)}{\partial t},$$

$$a(x) = \begin{cases} a_1, & x \in \Omega_1 \\ a_2, & x \in \Omega_2 \end{cases},$$

$$y(x, t) = \begin{cases} y_1(x, t), & (x, t) \in \Omega_1 \times (0, T) \\ y_2(x, t), & (x, t) \in \Omega_2 \times (0, T) \end{cases},$$

$A : H^{-1}(\Omega) \rightarrow H^{-1}(\Omega)$ is a positive selfadjoint operator defined by

$$Af = -\Delta f, \quad D(A) = H_0^1(\Omega),$$

$u(., .)$ is the input function, and

$z(., .)$ is the output function.

Equation (2.1), known as the position-dependent-mass (effective mass) Schrödinger equation, has important applications in the field of material science and condensed matter physics such as semiconductors, quantum dots, He clusters, quantum liquids, and semiconductor heterostructure (see [16] and [20] and the references therein).

When $a_1 = a_2$, Guo and Shao [4] have shown that the system (2.1)–(2.6) is well-posed with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$ and regular with zero as the feedthrough operator. One of the aims of this paper is to investigate the well-posedness and the regularity of the system (2.1)–(2.6) in the case where $a_1 \neq a_2$. Indeed, we shall prove the following

THEOREM 2.1. The equations (2.1)–(2.6) determine a well-posed linear system with input and output space $U = L^2(\Gamma)$ and state space $X = H^{-1}(\Omega)$.

THEOREM 2.2. The system (2.1)–(2.6) is regular with zero feedthrough operator. This means that if the initial state $y(\cdot, 0) = 0$ and $u(\cdot, t) = u(t) \in U$ is a step input, then the corresponding output satisfies

$$\lim_{\sigma \rightarrow 0} \int_{\Gamma} \left| \frac{1}{\sigma} \int_0^{\sigma} z(x, t) dt \right| d\Gamma = 0. \tag{2.7}$$

The second aim is to study the exact controllability problem for the open-loop system (2.1)–(2.5). Exact controllability of the Schrödinger equation with smooth coefficients in the elliptic principal part and subject to boundary control was treated in [11], [17] and [21]. To state our exact controllability result, we need the following assumptions:

(A1) $\Gamma = \Gamma^0 \cup \Gamma^1$; Γ^0 is possibly empty while Γ^1 is nonempty and relatively open.

(A2) $a_2 < a_1$.

(A3) There exists a real vector field $h(\cdot) \in (C^1(\overline{\Omega}))^n$ such that

(A3a)

$$\operatorname{Re} \left(\int_{\Omega} H(x) v(x) \cdot \overline{v(x)} dx \right) \geq \rho \int_{\Omega} \|v(x)\|^2 dx$$

for all $v(\cdot) \in (L^2(\Omega))^n$ for some $\rho > 0$, where

$$H(x) = \left(\frac{\partial h_i(x)}{\partial x_j} \right), i = 1, \dots, n \text{ and } j = 1, \dots, n.$$

(A3b)

$$h(x) \cdot \nu(x) \leq 0, \quad x \in \Gamma_1.$$

(A3c)

$$h(x) \cdot \nu(x) \leq 0, \quad x \in \Gamma^0.$$

THEOREM 2.3. Let $T > 0$ be arbitrary. Assume hypotheses (A1) and (A2). Then for any initial data $y^0 \in H^{-1}(\Omega)$, there exists a control $u \in L^2(0, T; L^2(\Gamma))$ with $u = 0$ on Γ^0 such that the corresponding solution of the system (2.1)–(2.5) satisfies $y(x, T) = 0$.

As a consequence of Theorem 2.1, Theorem 2.3 and Proposition 3.1 of [13], we have the following uniform stabilization result for the system (2.1)–(2.5) on the space $H^{-1}(\Omega)$.

COROLLARY 2.4. Let the hypotheses of Theorem 2.3 hold true. Then there exist positive constants M, ω such that the solution of (2.1)–(2.5) with $u = -\alpha z$ ($\alpha > 0$) satisfies

$$\|y(t)\|_X \leq M e^{-\omega t} \|y^0\|_X.$$

3. Abstract formulation. We define the space

$$H^2(\Omega, \Gamma_1) = \{y \in H_0^1(\Omega) : y_i = y|_{\Omega_i} \in H^2(\Omega_i); i = 1, 2; \\ a_1 \frac{\partial y_1}{\partial \nu} = a_2 \frac{\partial y_2}{\partial \nu} \text{ on } \Gamma_1\}$$

with the norm

$$\|y\|_{H^2(\Omega, \Gamma_1)}^2 = \|y_1\|_{H^2(\Omega_1)}^2 + \|y_2\|_{H^2(\Omega_2)}^2.$$

It can be shown that $H^2(\Omega, \Gamma_1)$ is dense in $H_0^1(\Omega)$.

Let $A_1 : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ be the extension of $-div(a(x)\nabla \cdot)$ to $H_0^1(\Omega)$. This means that $A_1 f = -div(a(x)\nabla f)$ whenever $f \in H^2(\Omega, \Gamma_1)$ and that $A_1^{-1} g = -(div(a(x)\nabla))^{-1} g$ for any $g \in L^2(\Omega)$.

Let $A_{-1} : H^{-1}(\Omega) \rightarrow (D(A))'$ be the extension of A_1 to $H^{-1}(\Omega)$. Notice that $(D(A))'$ is the dual of $D(A)$ with respect to the pivot space $H^{-1}(\Omega)$.

Define the Dirichlet map γ by

$$\gamma u = v$$

if and only if

$$\begin{aligned} div(a(x)\nabla v) &= 0 \text{ in } \Omega, \\ v &= u \text{ on } \Gamma, \\ v_1 &= v_2 \text{ on } \Gamma_1, \\ a_1 \frac{\partial v_1}{\partial \nu} &= a_2 \frac{\partial v_2}{\partial \nu} \text{ on } \Gamma_1. \end{aligned}$$

Then $\gamma \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$ ([12]).

Using the operators introduced above, we can rewrite (2.1), (2.3)–(2.5) on $(D(A))'$ as

$$y'(t) = -\mathbf{i}A_{-1}y(t) + Bu(t)$$

where $B \in \mathcal{L}(U, (D(A))')$ is given by

$$Bu = \mathbf{i}A_{-1}\gamma u.$$

We have, via Green's second theorem,

$$\gamma^* A \psi = -\frac{\partial \psi}{\partial \nu}, \quad \psi \in D(A).$$

Hence the adjoint C of B is given by

$$C \psi = \mathbf{i} \frac{\partial}{\partial \nu} (A^{-1} \psi).$$

Now, we can reformulate the system (2.1)–(2.6) into an abstract form in the state space $H^{-1}(\Omega)$ as follows:

$$y'(t) = -\mathbf{i}A_1 y(t) + Bu(t), \tag{3.1}$$

$$y(0) = y^0, \tag{3.2}$$

$$z(t) = Cu(t). \tag{3.3}$$

4. Proof of Theorem 2.1. The fact that the operator $-\mathbf{i}A_1$ generates a C_0 -group of unitary operators $S(t)$ on X is a consequence of Stone's Theorem (see [18]). In order to establish the admissibility of B and C for the group $S(t)$, we need the following identity, which is a particular case of the identity (7.16) in the appendix.

LEMMA 4.1. Let $m(\cdot)$ be a real vector field on $\overline{\Omega}$ of class C^1 such that

$$m = \nu \text{ on } \Gamma \text{ and } m = 0 \text{ in } \Omega_0,$$

where Ω_0 is an open domain in \mathbb{R}^n that satisfies

$$\overline{\Omega}_1 \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Omega.$$

Let $\{\xi_i^0, f_i\} \in H^1(\Omega_i) \times L^1(0, T, L^2(\Omega_i)), i = 1, 2$, such that

$$\begin{aligned} \xi_1^0 &= \xi_2^0 \text{ on } \Gamma_1, \\ \xi_2^0 &= 0 \text{ on } \Gamma. \end{aligned}$$

Then for every weak solution of

$$\xi'(x, t) = \mathbf{idiv}(a(x)\nabla\xi(x, t)) + f(x, t), (x, t) \in \Omega \times (0, T), \tag{4.1}$$

$$\xi(x, 0) = \xi_0(x), \quad x \in \Omega, \tag{4.2}$$

$$\xi_2(x, t) = 0, (x, t) \in \Gamma \times (0, T), \tag{4.3}$$

$$\xi_1(x, t) = \xi_2(x, t), (x, t) \in \Gamma_1 \times (0, T), \tag{4.4}$$

$$a_1 \frac{\partial \xi_1(x, t)}{\partial \nu} = a_2 \frac{\partial \xi_2(x, t)}{\partial \nu}, (x, t) \in \Gamma_1 \times (0, T), \tag{4.5}$$

the following identity holds true:

$$\begin{aligned} a_2 \int_0^T \int_{\Gamma} \left| \frac{\partial \xi}{\partial \nu} \right|^2 d\Gamma dt &= \text{Im} \left(\int_{\Omega_2} \xi m \cdot \nabla \bar{\xi} d\Omega \right)_0^T + a_2 \text{Re} \int_0^T \int_{\Omega_2} \xi \nabla \bar{\xi} \cdot \nabla (\text{div} m) d\Omega dt \\ &- 2a_2 \text{Re} \int_0^T \int_{\Omega_2} \nabla \xi \cdot m \nabla \bar{\xi} d\Omega dt + \text{Re} \int_0^T \int_{\Omega_2} \bar{f} \xi \text{div} m d\Omega dt - 2\text{Im} \int_0^T \int_{\Omega_2} f m \cdot \nabla \bar{\xi} d\Omega dt. \end{aligned} \tag{4.6}$$

REMARK 4.2. Liu and Williams [8] made use of the vector field m to establish a boundary regularity result for the problem of transmission of the plate equation.

4.1. *Admissibility of B and C for the group S(t).* Since the system (3.1)–(3.3) is colocated, the admissibility of B for the group $S(t)$ is equivalent to the admissibility of C for the group $S(t)$. But the latter means that

$$\int_0^T \int_{\Gamma} |CS(t)\psi|^2 d\Gamma dt \leq k \|\psi\|_X^2 \tag{4.7}$$

for all $\psi \in D(A)$ and for some $T > 0$.

Here and throughout the rest of the paper, k is a positive constant that takes different values at different occurrences.

An equivalent partial differential equation characterization of the estimate (4.7) is given by

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt \leq k \|\varphi_0\|_{H_0^1(\Omega)}^2, \tag{4.8}$$

where $\varphi^0 = A^{-1}\psi$ and φ is the solution of

$$\varphi'(x, t) = \mathbf{idiv}(a(x)\nabla\varphi(x, t)), (x, t) \in \Omega \times (0, T), \tag{4.9}$$

$$\varphi(x, 0) = \varphi^0(x), \quad x \in \Omega, \tag{4.10}$$

$$\varphi_2(x, t) = 0, (x, t) \in \Gamma \times (0, T), \tag{4.11}$$

$$\varphi_1(x, t) = \varphi_2(x, t), (x, t) \in \Gamma_1 \times (0, T), \tag{4.12}$$

$$a_1 \frac{\partial \varphi_1(x, t)}{\partial \nu} = a_2 \frac{\partial \varphi_2(x, t)}{\partial \nu}, (x, t) \in \Gamma_1 \times (0, T). \tag{4.13}$$

Specialization of the identity (4.6) to the φ -problem (4.9)–(4.13) yields

$$\begin{aligned} \int_0^T \int_{\Gamma} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt &= \text{Im} \left(\int_{\Omega_2} \varphi m \cdot \nabla \bar{\varphi} d\Omega \right)_0^T + a_2 \text{Re} \int_0^T \int_{\Omega_2} \varphi \nabla \bar{\varphi} \cdot \nabla (\text{div} m) d\Omega dt \\ &\quad - 2a_2 \text{Re} \int_0^T \int_{\Omega_2} \nabla \varphi \cdot M \nabla \bar{\varphi} d\Omega dt. \end{aligned} \tag{4.14}$$

Using Schwarz and Poincaré inequalities, we obtain from (4.14)

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt \leq k \int_0^T \int_{\Omega} |\nabla \varphi|^2 d\Omega dt + k \int_{\Omega} |\nabla \varphi(x, 0)|^2 d\Omega + k \int_{\Omega} |\nabla \varphi(x, T)|^2 d\Omega.$$

But

$$\int_{\Omega} |\nabla \varphi(x, t)|^2 d\Omega = \int_{\Omega} |\nabla \varphi^0|^2 d\Omega.$$

Thus

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt \leq k \|\varphi^0\|_{H_0^1(\Omega)}^2.$$

4.2. *Boundedness of the input-output map.* It suffices to show that the solution of (2.1)–(2.5) with $y(x, 0) = 0$ satisfies

$$\int_0^T \int_{\Gamma} \left| \frac{\partial A^{-1}y(x, t)}{\partial \nu} \right|^2 d\Gamma dt \leq k \int_0^T \int_{\Gamma} |u(x, t)|^2 d\Gamma dt \tag{4.15}$$

for all $u \in L^2(0, T; U)$.

From the admissibility of B , we have $y \in C(0, T; H^{-1}(\Omega))$ for every $y^0 \in H^{-1}(\Omega)$.

Let us introduce a new variable by setting

$$w(t) = A^{-1}y(t) \in C(0, T; H_0^1(\Omega)).$$

Thus by (3.1), we obtain the abstract equation

$$w'(t) = -\mathbf{i}A_1y(t) + \mathbf{i}\gamma u(t)$$

whose corresponding partial differential problem is

$$w'(x, t) = \mathbf{i} \text{div}(a(x)\nabla w(x, t)) + \mathbf{i}\gamma u(x, t), \quad (x, t) \in \Omega \times (0, T), \tag{4.16}$$

$$w(x, 0) = 0, \quad x \in \Omega, \tag{4.17}$$

$$w_2(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T), \tag{4.18}$$

$$w_1(x, t) = w_2(x, t), \quad (x, t) \in \Gamma_1 \times (0, T), \tag{4.19}$$

$$a_1 \frac{\partial w_1(x, t)}{\partial \nu} = a_2 \frac{\partial w_2(x, t)}{\partial \nu}, \quad (x, t) \in \Gamma_1 \times (0, T). \tag{4.20}$$

The estimate (4.15) becomes

$$\int_0^T \int_{\Gamma} \left| \frac{\partial w(x, t)}{\partial \nu} \right|^2 d\Gamma dt \leq k \int_0^T \int_{\Gamma} |u(x, t)|^2 d\Gamma dt. \tag{4.21}$$

As for (4.8), the estimate can also be deduced from the identity (4.6). Indeed, setting $f = \mathbf{i}\gamma u$ in (4.6) and using the fact that $\gamma \in L(L^2(\Gamma), L^2(\Omega))$, we obtain

$$\begin{aligned} \int_0^T \int_{\Gamma} \left| \frac{\partial w(x, t)}{\partial \nu} \right|^2 d\Gamma dt &\leq k \int_0^T \int_{\Omega} |\nabla w|^2 d\Omega dt + k \int_{\Omega} |\nabla w(x, T)|^2 d\Omega + k \int_0^T \int_{\Gamma} |u|^2 d\Gamma dt \\ &\leq k(\|w\|_{C(0, T; H_0^1(\Omega))}^2 + \|u\|_{L^2(0, T; L^2(\Gamma))}^2). \end{aligned}$$

This together with the admissibility of B for the C_0 -group $S(t)$ yields (4.21).

5. Proof of Theorem 2.2. Since the system (2.1)–(2.6) is well-posed, its transfer function $G(s)$ is bounded on some right half-plane (see [23]). To continue, we need the following results.

The assertion of Theorem 2 holds true if for any $u \in C_0^\infty(\Gamma)$ the solution y of

$$sy(x) - \mathbf{i} \operatorname{div}(a(x)\nabla y(x)) = 0, \quad x \in \Omega, \tag{5.1}$$

$$y_2(x) = u(x), \quad x \in \Gamma, \tag{5.2}$$

$$y_1(x) = y_2(x), \quad x \in \Gamma_1, \tag{5.3}$$

$$a_1 \frac{\partial y_1(x)}{\partial \nu} = a_2 \frac{\partial y_2(x)}{\partial \nu}, \quad x \in \Gamma_1, \tag{5.4}$$

satisfies

$$\lim_{s \in \mathbb{R}, s \rightarrow +\infty} \int_{\Gamma} \left| \frac{1}{s} \frac{\partial y}{\partial \nu} \right|^2 d\Gamma = 0. \tag{5.5}$$

Proof. We know from [23] that in the frequency domain, (2.7) is equivalent to

$$\lim_{s \in \mathbb{R}, s \rightarrow +\infty} G(s)u = 0 \tag{5.6}$$

in the strong topology of U for any $u \in U$. Due to the boundedness of $G(s)$ and the density of $L^2(\Gamma)$ in $C_0^\infty(\Gamma)$, it suffices to establish (5.6) for all $u \in C_0^\infty(\Gamma)$. Now for $u \in C_0^\infty(\Gamma)$ and $s > 0$, let

$$y = (sI + \mathbf{i}A_1)^{-1}Bu.$$

Then y satisfies (5.1)–(5.4) and

$$(G(s)y)(x) = \mathbf{i} \frac{\partial(A^{-1}y)}{\partial \nu}(x), \quad x \in \Gamma.$$

It follows from Lemma 7.1 in the appendix that there exists a function $v \in H^2(\Omega, \Gamma_1)$ satisfying the following boundary value problem:

$$\begin{aligned} \operatorname{div}(a(x)\nabla v(x)) &= 0, \quad x \in \Omega, \\ v_2(x) &= u(x), \quad x \in \Gamma, \\ v_1(x) &= v_2(x), \quad x \in \Gamma_1, \\ a_1 \frac{\partial v_1(x)}{\partial \nu} &= a_2 \frac{\partial v_2(x)}{\partial \nu}, \quad x \in \Gamma_1. \end{aligned}$$

Consequently, (5.1)–(5.4) can be written as

$$\begin{aligned} sy(x) - \mathbf{idiv}(a(x)\nabla(y(x) - v(x))) &= 0, \quad x \in \Omega, \\ y_2(x) - v_2(x) &= 0, \quad x \in \Gamma, \\ y_1(x) - v_1(x) &= y_2(x) - v_2(x), \quad x \in \Gamma_1, \\ a_1 \frac{\partial(y_1(x) - v_1(x))}{\partial\nu} &= a_2 \frac{\partial(y_2(x) - v_2(x))}{\partial\nu}, \quad x \in \Gamma_1. \end{aligned}$$

Hence

$$(G(s)y)(x) = \frac{a_2}{s} \frac{\partial y(x)}{\partial\nu} - \frac{a_2}{s} \frac{\partial v(x)}{\partial\nu}.$$

This gives (5.5). □

LEMMA 5.1. Let m be the vector field introduced in Section 4.1. Let $u \in C_0^\infty(\Gamma)$. Then the solution of (5.1) satisfies

$$\begin{aligned} a_2 \int_\Gamma \left| \frac{\partial y_2}{\partial\nu} \right|^2 d\Gamma &= -\frac{s}{a_2} \operatorname{Im} \int_\Omega ym \cdot \nabla \bar{y} + 2\operatorname{Re} \int_{\Omega_2} \nabla y_2 \cdot M \nabla \bar{y}_2 d\Omega \\ &\quad - \int_{\Omega_2} |\nabla y_2|^2 \operatorname{div} m d\Omega + \int_\Gamma |\nabla_\sigma y_2|^2 d\Gamma. \end{aligned} \tag{5.7}$$

Proof. We multiply both sides of (5.1) by $m \cdot \nabla \bar{y}$ and integrate over Ω . Using Green’s first theorem, we find

$$\begin{aligned} s \int_\Gamma |y|^2 m \cdot \nu d\Gamma - s \int_\Omega \bar{y} m \cdot \nabla y d\Omega - s \int_\Omega |y|^2 \operatorname{div} m d\Omega + \mathbf{ia}_1 \int_{\Gamma_1} \frac{\partial y_1}{\partial\nu} m \cdot \nabla \bar{y}_1 d\Gamma \\ + \mathbf{ia}_1 \int_{\Omega_1} \nabla y_1 \cdot \nabla(m \cdot \nabla \bar{y}_1) d\Omega - \mathbf{ia}_2 \int_{\Gamma_1} \frac{\partial y_2}{\partial\nu} m \cdot \nabla \bar{y}_2 d\Gamma - \mathbf{ia}_2 \int_\Gamma \frac{\partial y_2}{\partial\nu} m \cdot \nabla \bar{y}_2 d\Gamma \\ + \mathbf{ia}_2 \int_{\Omega_2} \nabla y_2 \cdot \nabla(m \cdot \nabla \bar{y}_2) d\Omega = 0. \end{aligned} \tag{5.8}$$

Recalling the assumptions made on the vector field m , we simplify (5.8) to

$$\begin{aligned} s \int_\Gamma |y|^2 m \cdot \nu d\Gamma - s \int_\Omega \bar{y} m \cdot \nabla y d\Omega - s \int_\Omega |y|^2 \operatorname{div} m d\Omega - \mathbf{ia}_2 \int_\Gamma \left| \frac{\partial y_2}{\partial\nu} \right|^2 d\Gamma \\ + \mathbf{ia}_2 \int_{\Omega_2} \nabla y_2 \cdot \nabla(m \cdot \nabla \bar{y}_2) d\Omega = 0 \end{aligned}$$

from which we obtain

$$a_2 \int_\Gamma \left| \frac{\partial y_2}{\partial\nu} \right|^2 d\Gamma = -s \operatorname{Im} \int_\Omega \bar{y} m \cdot \nabla y d\Omega + a_2 \operatorname{Re} \int_{\Omega_2} \nabla y_2 \cdot \nabla(m \cdot \nabla \bar{y}_2) d\Omega. \tag{5.9}$$

On the other hand, we have

$$\operatorname{Re} \int_{\Omega_2} \nabla y_2 \cdot \nabla(m \cdot \nabla \bar{y}_2) d\Omega = \operatorname{Re} \int_{\Omega_2} \nabla y_2 \cdot M \nabla \bar{y}_2 d\Omega + \frac{1}{2} \int_\Gamma |\nabla y_2|^2 d\Gamma - \frac{1}{2} \int_{\Omega_2} |\nabla y_2|^2 \operatorname{div} m d\Omega \tag{5.10}$$

where

$$M = \left(\frac{\partial m_i}{\partial x_j} \right)_{i=1,n; j=1,n}.$$

Using the fact that

$$|\nabla y_2|^2 = |\nabla_\sigma y_2|^2 + \left| \frac{\partial y_2}{\partial \nu} \right|^2 \quad \text{on } \Gamma,$$

(5.10) becomes

$$\begin{aligned} \operatorname{Re} \int_{\Omega_2} \nabla y_2 \cdot \nabla (m \cdot \nabla \bar{y}_2) d\Omega &= \operatorname{Re} \int_{\Omega_2} \nabla y_2 \cdot M \nabla \bar{y}_2 d\Omega - \frac{1}{2} \int_{\Omega_2} |\nabla y_2|^2 \operatorname{div} m d\Omega \\ &+ \frac{1}{2} \int_{\Gamma} |\nabla y_\sigma|^2 d\Gamma + \frac{1}{2} \int_{\Gamma} \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma. \end{aligned} \tag{5.11}$$

Insertion of (5.11) into (5.9) yields (5.7). □

LEMMA 5.2. Let y be a solution of (5.1)–(5.4). Then

$$s \int_{\Omega} |y|^2 d\Omega + \mathbf{i} \int_{\Omega} a(x) |\nabla y|^2 d\Omega = \mathbf{i} a_2 \int_{\Gamma} \frac{\partial y_2}{\partial \nu} \bar{y}_2 d\Gamma. \tag{5.12}$$

Proof. We multiply both sides of (5.1) by \bar{y} and integrate over Ω . From Green’s first theorem, we have

$$\begin{aligned} s \int_{\Omega} |y|^2 d\Omega - \mathbf{i} \{ a_2 \int_{\Gamma} \frac{\partial y_2}{\partial \nu} \bar{y}_2 d\Gamma + a_2 \int_{\Gamma_1} \frac{\partial y_2}{\partial \nu} \bar{y}_2 d\Gamma - a_2 \int_{\Omega_2} |\nabla y_2|^2 d\Omega \\ - a_1 \int_{\Gamma_1} \frac{\partial y_1}{\partial \nu} \bar{y}_1 d\Gamma - a_1 \int_{\Omega_2} |\nabla y_2|^2 d\Omega \}. \end{aligned} \tag{5.13}$$

Inserting the boundary condition (5.4) into (5.12), we find that this simplifies to (5.12). □

5.1. *Completion of the proof of Theorem 2.2.* We first introduce some constants:

$$a = \min(a_1, a_2), \quad \mu_1 = \sup_{\bar{\Omega}} |m(x)|, \quad \mu_2 = \sup_{\bar{\Omega}} \|M(x)\|, \quad \mu_3 = \sup_{\bar{\Omega}} |\operatorname{div} v(x)|.$$

From (5.7), we have the estimate

$$\begin{aligned} \frac{1}{s^2} \int_{\Gamma} \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma &\leq \frac{\mu_1}{2a_2 s^{1/2}} \int_{\Omega} |y|^2 d\Omega + \frac{\mu_1}{2a_2 s^{3/2}} \int_{\Omega} |\nabla y|^2 d\Omega \\ &+ \frac{2\mu_2 + \mu_3}{s^2} \int_{\Omega_2} |\nabla y_2|^2 d\Omega + \frac{1}{s^2} \int_{\Gamma} |\nabla_\sigma y_2|^2 d\Gamma. \end{aligned} \tag{5.14}$$

On the other hand, (5.12) implies

$$\frac{1}{s^{1/2}} \int_{\Omega} |y|^2 d\Omega \leq \frac{a_2}{2s^{1/2}} \int_{\Gamma} |y_2|^2 d\Gamma + \frac{a_2}{2s^{5/2}} \int_{\Gamma} \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma, \tag{5.15}$$

$$\frac{1}{s^{3/2}} \int_{\Omega} |\nabla y|^2 d\Omega \leq \frac{a_2}{2as^{1/2}} \int_{\Gamma} |y_2|^2 d\Gamma + \frac{a_2}{2as^{5/2}} \int_{\Gamma} \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma. \tag{5.16}$$

Substituting (5.15), (5.16) into (5.14), we get

$$\begin{aligned} \frac{1}{s^2} \int_{\Gamma} \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma &\leq \left(\frac{\mu_1}{4s^{1/2}} + \frac{\mu_1}{4as^{1/2}} + \frac{a_2(2\mu_2 + \mu_3)}{2as} \right) \int_{\Gamma} |y_2|^2 d\Gamma \\ &+ \left(\frac{\mu_1}{4s^{1/2}} + \frac{\mu_1}{4as^{1/2}} + \frac{a_2(2\mu_2 + \mu_3)}{2as} \right) \frac{1}{s^2} \int_{\Gamma} \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma + \frac{1}{s^2} \int_{\Gamma} |\nabla_\sigma y_2|^2 d\Gamma. \end{aligned} \tag{5.17}$$

Since

$$y_2 = u \text{ on } \Gamma \times (0, T)$$

and

$$\|y\|_{H^1(\Gamma)}^2 = \|y\|_{L^2(\Gamma)}^2 + \|\nabla_\sigma y\|_{L^2(\Gamma)}^2,$$

we rewrite (5.17) as follows:

$$\begin{aligned} \frac{1}{s^2} \int_\Gamma \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma &\leq \left(\frac{\mu_1}{4s^{1/2}} + \frac{\mu_1}{4as^{1/2}} + \frac{a_2(2\mu_2 + \mu_3)}{2as} + \frac{1}{s^2} \right) \|u\|_{H^1(\Gamma)}^2 \\ &+ \left(\frac{\mu_1}{4s^{1/2}} + \frac{\mu_1}{4as^{1/2}} + \frac{a_2(2\mu_2 + \mu_3)}{2as} \right) \frac{1}{s^2} \int_\Gamma \left| \frac{\partial y_2}{\partial \nu} \right|^2 d\Gamma. \end{aligned}$$

This last estimate shows that

$$\lim_{s \in \mathbb{R}, s \rightarrow +\infty} \int_\Gamma \left| \frac{1}{s} \frac{\partial y}{\partial \nu} \right|^2 d\Gamma = 0.$$

6. Proof of Theorem 2.3. Let

$$E(t) = \int_\Omega a(x) |\nabla \varphi|^2 d\Omega$$

be the energy corresponding to the solution of the system (4.9)–(4.13). Then

$$E(t) = E(0) \text{ for all } t > 0.$$

By classical duality theory, to prove Theorem 2.3 it is enough to establish the associated observability inequality

$$\int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt \geq k \|\varphi_0\|_{H_0^1(\Omega)}^2, \tag{6.1}$$

where φ is the solution of the homogeneous system (4.9)–(4.13).

To this end, we apply the identity (7.16) to the φ -problem (4.9)–(4.13) to obtain

$$\begin{aligned} &a_2 \int_0^T \int_\Gamma \left| \frac{\partial \varphi_2}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt \\ &= 2a_1 \left(1 - \frac{a_1}{a_2}\right) \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi_1}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt + a_2 \int_0^T \int_{\Gamma_1} |\nabla \varphi_2|^2 h \cdot \nu d\Gamma dt \\ &\quad - a_1 \int_0^T \int_{\Gamma_1} |\nabla \varphi_1|^2 h \cdot \nu d\Gamma dt + \text{Im} \left[\int_\Omega \varphi \cdot \nabla \bar{\varphi} d\Omega \right]_0^T + 2\text{Re} \int_0^T \int_\Omega a(x) \nabla \varphi \cdot H \nabla \bar{\varphi} d\Omega dt \\ &\quad + \text{Re} \int_0^T \int_\Omega a(x) \varphi \nabla \bar{\varphi} \cdot \nabla (\text{div} h) d\Omega dt. \end{aligned} \tag{6.2}$$

But

$$|\nabla \varphi_i|^2 = \left| \frac{\partial \varphi_i}{\partial \nu} \right|^2 + |\nabla_\sigma \varphi_i|^2 \text{ on } \Gamma_1 \times (0, T), i = 1, 2$$

and

$$|\nabla_\sigma \varphi_1|^2 = |\nabla_\sigma \varphi_2|^2 \text{ on } \Gamma_1 \times (0, T).$$

Then (A2) and (A3b) imply that

$$\begin{aligned}
 & 2a_1\left(1 - \frac{a_1}{a_2}\right) \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi_1}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt + a_2 \int_0^T \int_{\Gamma_1} |\nabla \varphi_2|^2 h \cdot \nu d\Gamma dt - a_1 \int_0^T \int_{\Gamma_1} |\nabla \varphi_1|^2 h \cdot \nu d\Gamma dt \\
 & = a_1\left(1 - \frac{a_1}{a_2}\right) \int_0^T \int_{\Gamma_1} \left| \frac{\partial \varphi_1}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt - (a_1 - a_2) \int_0^T \int_{\Gamma_1} |\nabla_{\sigma} \varphi_1|^2 h \cdot \nu d\Gamma dt \geq 0. \tag{6.3}
 \end{aligned}$$

From (6.2) and (6.3), we deduce that

$$\begin{aligned}
 & 2\rho T \int_{\Omega} a(x) |\nabla \varphi^0|^2 d\Omega \leq a_2 \int_0^T \int_{\Gamma^1} \left| \frac{\partial \varphi_2}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt - \text{Im} \left[\int_{\Omega} \varphi h \cdot \nabla \bar{\varphi} d\Omega \right]_0^T \\
 & + \text{Re} \int_0^T \int_{\Omega} a(x) \varphi \nabla \bar{\varphi} \cdot \nabla (\text{div} h) d\Omega dt. \tag{6.4}
 \end{aligned}$$

Application of Schwarz and Poincaré inequalities to the \int_{Ω} -terms on the right-hand side of (6.4) yields

$$\begin{aligned}
 (2\rho T - \frac{c_1 \varepsilon}{a_2}) \int_{\Omega} a(x) |\nabla \varphi^0|^2 d\Omega & \leq a_2 c_1 \int_0^T \int_{\Gamma^1} \left| \frac{\partial \varphi_2}{\partial \nu} \right|^2 d\Gamma dt \\
 & + \frac{1}{2} ((c_2 c_p + c_2) T + \frac{2c_1 c_p}{a_2 \varepsilon}) \|\varphi\|_{C(0,T;H_0^1(\Omega))}^2 \tag{6.5}
 \end{aligned}$$

where

$$c_1 = \sup_{\Omega} |h(x)|, \quad c_2 = \sup_{\Omega} |\nabla(\text{div} h)|,$$

$$c_p \text{ is the Poincaré constant: } \int_{\Omega} |\varphi|^2 d\Omega \leq c_p \int_{\Omega} |\nabla \varphi|^2 d\Omega, \text{ and}$$

ε is an arbitrary positive small constant.

The sought-after estimate follows now from (6.5) by a compactness/uniqueness argument.

7. Appendix.

LEMMA 7.1. Let f be a solution to the following elliptic problem:

$$\text{div} (a(x) \nabla f(x)) = g(x), \quad x \in \Omega, \tag{7.1}$$

$$f_2(x) = u(x), \quad x \in \Gamma, \tag{7.2}$$

$$f_1(x) = f_2(x), \quad x \in \Gamma_1, \tag{7.3}$$

$$a_1 \frac{\partial f_1(x)}{\partial \nu} = a_2 \frac{\partial f_2(x)}{\partial \nu}, \quad x \in \Gamma_1, \tag{7.4}$$

for $g \in L^2(\Omega)$ and $u \in H^{3/2}(\Omega)$. Then there exists a constant k independent of f, g and u such that

$$\|f\|_{H^2(\Omega, \Gamma_1)} \leq k \{ \|g\|_{L^2(\Omega)} + \|u\|_{H^{3/2}(\Gamma)} \}.$$

Proof. Let f be a solution to (7.1)–(7.4). Then f can be written as

$$f(x) = \begin{cases} f_1(x), & x \in \Omega_1 \\ f_2(x), & x \in \Omega_2 \end{cases},$$

where f_2 and f_1 are respectively the solutions of

$$\begin{aligned} a_2 \Delta f_2(x) &= g(x), \quad x \in \Omega, \\ f_2(x) &= u(x), \quad x \in \Gamma, \end{aligned}$$

and

$$\begin{aligned} a_1 \Delta f_1(x) &= g(x), \quad x \in \Omega_1, \\ f_1(x) &= f_2(x), \quad x \in \Gamma_1, \\ a_1 \frac{\partial f_1(x)}{\partial \nu} &= a_2 \frac{\partial f_2(x)}{\partial \nu}, \quad x \in \Gamma_1. \end{aligned} \tag{7.5}$$

From elliptic regularity theory (see [14]), we have

$$\|f_2\|_{H^2(\Omega)} \leq k\{\|g\|_{L^2(\Omega)} + \|u\|_{H^{3/2}(\Gamma)}\}. \tag{7.6}$$

It follows from the trace theorem that $f_2|_{\Gamma_1} \in H^{3/2}(\Gamma_1)$ and

$$\|f_2\|_{H^{3/2}(\Gamma_1)} \leq k \|f_2\|_{H^2(\Omega)}. \tag{7.7}$$

(7.5) together with (7.7) implies again via the elliptic regularity that $f_1 \in H^2(\Omega_1)$ and

$$\|f_1\|_{H^2(\Omega_1)} \leq k\{\|g\|_{L^2(\Omega)} + \|f_2\|_{H^{3/2}(\Gamma_1)}\}. \tag{7.8}$$

Combining (7.6), (7.8) and (7.7), we obtain

$$\|f_1\|_{H^2(\Omega_1)} + \|f_2\|_{H^2(\Omega)} \leq k\{\|g\|_{L^2(\Omega)} + \|u\|_{H^{3/2}(\Gamma)}\}$$

from which follows the desired estimate, since

$$\|f\|_{H^2(\Omega, \Gamma_1)}^2 = \|f_1\|_{H^2(\Omega_1)}^2 + \|f_2\|_{H^2(\Omega)}^2.$$

□

LEMMA 7.2. Let h be a real vector field of class C^1 on $\bar{\Omega}$. Then for every solution of the problem

$$\xi'(x, t) = \mathbf{div}(a(x)\nabla\xi(x, t)) + g(x, t), \quad (x, t) \in \Omega \times (0, T), \tag{7.11}$$

$$\xi(x, 0) = \xi^0(x), \quad x \in \Omega, \tag{7.12}$$

$$\xi_2(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T), \tag{7.13}$$

$$\xi_1(x, t) = \xi_2(x, t), \quad (x, t) \in \Gamma_1 \times (0, T), \tag{7.14}$$

$$a_1 \frac{\partial \xi_1(x, t)}{\partial \nu} = a_2 \frac{\partial \xi_2(x, t)}{\partial \nu}, \quad (x, t) \in \Gamma_1 \times (0, T), \tag{7.15}$$

we have

$$\begin{aligned}
 & a_2 \int_0^T \int_{\Gamma} \left| \frac{\partial \xi_2}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt + 2a_1 \left(\frac{a_1}{a_2} - 1 \right) \int_0^T \int_{\Gamma_1} \left| \frac{\partial \xi_1}{\partial \nu} \right|^2 h \cdot \nu d\Gamma dt \\
 & - a_2 \int_0^T \int_{\Gamma_1} |\nabla \xi_2|^2 h \cdot \nu d\Gamma dt + a_1 \int_0^T \int_{\Gamma_1} |\nabla \xi_1|^2 h \cdot \nu d\Gamma dt \\
 & = \operatorname{Im} \left[\int_{\Omega} \xi h \cdot \nabla \bar{\xi} d\Omega \right]_0^T + 2\operatorname{Re} \int_0^T \int_{\Omega} a(x) \nabla \xi \cdot H \nabla \bar{\xi} d\Omega dt \\
 & \quad + \operatorname{Re} \int_0^T \int_{\Omega} a(x) \xi \nabla \bar{\xi} \cdot \nabla (\operatorname{div} h) d\Omega dt \\
 & \quad + \operatorname{Im} \int_0^T \int_{\Omega} \bar{g} \xi \operatorname{div} h d\Omega dt - 2\operatorname{Im} \int_0^T \int_{\Omega} g h \cdot \nabla \bar{\xi} d\Omega dt.
 \end{aligned} \tag{7.16}$$

Proof. The identity (7.16) will be established for strong solutions and the general case will follow then by a standard density argument. To this end, let $\{\xi_i^0, f_i\} \in H^2(\Omega_i) \times H^1(\Omega_i) \times L^1(0, T; H^1(\Omega_i))$ such that

$$\begin{aligned}
 \xi_1^0 &= \xi_2^0 \text{ on } \Gamma_1, \\
 g_1 &= g_2 \text{ on } \Gamma_1 \times (0, T), \\
 \xi_2^0 &= 0 \text{ on } \Gamma, \\
 g_2 &= 0 \text{ on } \Gamma \times (0, T), \\
 a_1 \frac{\partial \xi_1^0}{\partial \nu} &= a_2 \frac{\partial \xi_2^0}{\partial \nu} \text{ on } \Gamma_1.
 \end{aligned}$$

We multiply both sides of (7.11) by $h \cdot \nabla \bar{\xi}$ and integrate over $\Omega \times (0, T)$ to obtain

$$\int_0^T \int_{\Omega} \xi' h \cdot \nabla \bar{\xi} d\Omega dt = \mathbf{i} \int_0^T \int_{\Omega} \operatorname{div}(a(x) \nabla \xi) h \cdot \nabla \bar{\xi} d\Omega dt + \int_0^T \int_{\Omega} g h \cdot \nabla \bar{\xi} d\Omega dt. \tag{7.17}$$

We have

$$\begin{aligned}
 \int_0^T \int_{\Omega} \xi' h \cdot \nabla \bar{\xi} d\Omega dt &= \left[\int_{\Omega} \xi h \cdot \nabla \bar{\xi} d\Omega \right]_0^T - \int_0^T \int_{\Gamma} \xi \bar{\xi}' h \nu d\Gamma dt \\
 &\quad + \int_0^T \int_{\Omega} (-\mathbf{i} \operatorname{div}(a(x) \nabla \bar{\xi}) + \bar{g}) h \cdot \nabla \xi d\Omega dt + \int_0^T \int_{\Omega} \xi \bar{\xi}' \operatorname{div} h d\Omega dt.
 \end{aligned} \tag{7.18}$$

Substituting (7.18) into (7.17), we get

$$\begin{aligned}
 & \left[\int_{\Omega} \xi h \cdot \nabla \bar{\xi} d\Omega \right]_0^T - \int_0^T \int_{\Gamma} \xi \bar{\xi}' h \nu d\Gamma dt + \int_0^T \int_{\Omega} (-\mathbf{i} \operatorname{div}(a(x) \nabla \bar{\xi}) + \bar{g}) h \cdot \nabla \xi d\Omega dt \\
 & + \int_0^T \int_{\Omega} \xi \bar{\xi}' \operatorname{div} h d\Omega dt = \mathbf{i} \int_0^T \int_{\Omega} \operatorname{div}(a(x) \nabla \bar{\xi}) h \cdot \nabla \bar{\xi} d\Omega dt + \int_0^T \int_{\Omega} g h \cdot \nabla \bar{\xi} d\Omega dt.
 \end{aligned}$$

Hence

$$\begin{aligned}
 2\operatorname{Re} \int_0^T \int_{\Omega} \operatorname{div}(a(x)\nabla\bar{\xi})h.\nabla\bar{\xi}d\Omega dt &= \operatorname{Im} \left[\int_{\Omega} \xi h.\nabla\bar{\xi}d\Omega \right]_0^T - \operatorname{Im} \int_0^T \int_{\Gamma} \xi\bar{\xi}' h\nu d\Gamma dt \\
 &+ \operatorname{Im} \int_0^T \int_{\Omega} \xi\bar{\xi}' \operatorname{div}h d\Omega dt - 2\operatorname{Im} \int_0^T \int_{\Omega} gh.\nabla\bar{\xi}d\Omega dt.
 \end{aligned}
 \tag{7.19}$$

Using Green’s first theorem along with the identity

$$2\operatorname{Re} \int_{\Omega} \nabla w.\nabla(h.\nabla\bar{w})d\Omega = 2\operatorname{Re} \int_{\Omega} \nabla w.H\nabla\bar{w}d\Omega + \int_{\Omega} h.\nabla(|\nabla w|^2)d\Omega,$$

we rewrite the left-hand side of (7.19) as

$$\begin{aligned}
 2\operatorname{Re} \int_0^T \int_{\Omega} \operatorname{div}(a(x)\nabla\xi)h.\nabla\bar{\xi}d\Omega dt &= 2a_2\operatorname{Re} \int_0^T \int_{\Gamma_1} \frac{\partial\xi_2}{\partial\nu} h.\nabla\bar{\xi}_2 d\Gamma dt + 2a_2\operatorname{Re} \int_0^T \int_{\Gamma} \frac{\partial\xi_2}{\partial\nu} h.\nabla\bar{\xi}_2 d\Gamma dt \\
 &- 2a_1\operatorname{Re} \int_0^T \int_{\Gamma_1} \frac{\partial\xi_1}{\partial\nu} h.\nabla\bar{\xi}_1 d\Gamma dt - a_2 \int_0^T \int_{\Gamma} |\nabla\xi_2|^2 h.\nu d\Gamma dt - a_2 \int_0^T \int_{\Gamma_1} |\nabla\xi_2|^2 h.\nu d\Gamma dt \\
 &+ a_1 \int_0^T \int_{\Gamma_1} |\nabla\xi_1|^2 h.\nu d\Gamma dt - 2a_2\operatorname{Re} \int_0^T \int_{\Omega_2} \nabla\xi_2.H\nabla\bar{\xi}_2 d\Omega dt + a_2 \int_0^T \int_{\Omega_2} |\nabla\xi_2|^2 \operatorname{div}h d\Omega dt \\
 &- 2a_1\operatorname{Re} \int_0^T \int_{\Omega_1} \nabla\xi_1.H\nabla\bar{\xi}_1 d\Omega dt + a_1 \int_0^T \int_{\Omega_2} |\nabla\xi_1|^2 \operatorname{div}h d\Omega dt.
 \end{aligned}
 \tag{7.20}$$

Recalling the boundary conditions (7.13)–(7.15), we have

$$h.\nabla\xi_2 = \frac{\partial\xi_2}{\partial\nu} h.\nu \text{ on } \Gamma \times (0, T),
 \tag{7.21}$$

$$\begin{aligned}
 h.\nabla(\xi_1 - \xi_2) &= \frac{\partial(\xi_1 - \xi_2)}{\partial\nu} h.\nu \\
 &= \left(1 - \frac{a_1}{a_2}\right) \frac{\partial\xi_1}{\partial\nu} \text{ on } \Gamma_1 \times (0, T).
 \end{aligned}
 \tag{7.22}$$

Inserting (7.21) and (7.22) into (7.20), we find that this simplifies to

$$\begin{aligned}
 2\operatorname{Re} \int_0^T \int_{\Omega} \operatorname{div}(a(x)\nabla\bar{\xi})h.\nabla\bar{\xi}d\Omega dt &= -2a_1\left(1 - \frac{a_1}{a_2}\right) \int_0^T \int_{\Gamma_1} \left| \frac{\partial\xi_1}{\partial\nu} \right|^2 h.\nu d\Gamma dt \\
 &+ a_2 \int_0^T \int_{\Gamma} \left| \frac{\partial\xi_2}{\partial\nu} \right|^2 h.\nu d\Gamma dt - a_2 \int_0^T \int_{\Gamma_1} |\nabla\xi_2|^2 h.\nu d\Gamma dt + a_1 \int_0^T \int_{\Gamma_1} |\nabla\xi_1|^2 h.\nu d\Gamma dt \\
 &- 2\operatorname{Re} \int_0^T \int_{\Omega} a(x)\nabla\xi.H\nabla\bar{\xi}d\Omega dt + \int_0^T \int_{\Omega} a(x)|\nabla\xi|^2 \operatorname{div}h d\Omega dt.
 \end{aligned}
 \tag{7.23}$$

Now, we consider the third integral on the right-hand side of (7.19). Applying Green’s first theorem and taking into consideration the boundary condition (7.13), we obtain

$$\begin{aligned}
 \operatorname{Im} \int_0^T \int_{\Omega} \xi\bar{\xi}' \operatorname{div}h d\Omega dt &= \operatorname{Re} \int_0^T \int_{\Omega} a(x)|\nabla\xi|^2 \operatorname{div}h + \operatorname{Re} \int_0^T \int_{\Omega} a(x)\xi\nabla\bar{\xi}.\nabla(\operatorname{div}h)d\Omega dt \\
 &+ \operatorname{Im} \int_0^T \int_{\Omega} \bar{g}\xi \operatorname{div}h d\Omega dt.
 \end{aligned}
 \tag{7.24}$$

Substituting (7.23) and (7.24) into (7.19) and using the boundary condition (7.13), we obtain (7.16). \square

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