WELLPOSEDNESS AND STABILITY RESULTS FOR THE NAVIER-STOKES EQUATIONS IN $\mathbf{R}^3$

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Abstract. In [8] a class of initial data to the three dimensional, periodic, incompressible Navier-Stokes equations was presented, generating a global smooth solution although the norm of the initial data may be chosen arbitrarily large. The aim of this article is twofold. First, we adapt the construction of [8] to the case of the whole space: we prove that if a certain nonlinear function of the initial data is small enough, in a Koch-Tataru [15] type space, then there is a global solution to the Navier-Stokes equations. We provide an example of initial data satisfying that nonlinear smallness condition, but whose norm is arbitrarily large in C^{-1}. Then we prove a stability result on the nonlinear smallness assumption. More precisely we show that the new smallness assumption also holds for linear superpositions of translated and dilated iterates of the initial data, in the spirit of a construction in [2], thus generating a large number of different examples.

1. Introduction

1.1. **On the global wellposedness of the Navier-Stokes system.** We consider the three dimensional, incompressible Navier-Stokes system in $\mathbf{R}^3$,

$$(NS) \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u = -\nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0. \end{cases}$$

Here u is a three-component vector field $u = (u_1, u_2, u_3)$ representing the velocity of the fluid, p is a scalar denoting the pressure, and both are unknown functions of the space variable $x \in \mathbf{R}^3$ and of the time variable $t \in \mathbf{R}^+$. We have chosen the kinematic viscosity of the fluid equal to one for simplicity – a comment on the dependence of our results on viscosity is given further down in this introduction.

It is well-known that (NS) has a global, smooth solution if the initial data is small enough in the scale invariant space $\dot{H}^{\frac{1}{2}}$, where we recall that $\dot{H}^s$ is the set of tempered distributions f with Fourier transform $\widehat{f}$ in $L^1_{loc}(\mathbf{R}^3)$ and such that

$$\|f\|_{\dot{H}^s} \stackrel{\text{def}}{=} \left(\int_{\mathbf{R}^3} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

is finite. We recall that the scaling of (NS) is the following: for any positive λ, the vector field u is a solution associated with the data u_0 if u_λ is a solution associated with $u_{0,\lambda}$, where

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) = \lambda u_0(\lambda x).$$

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The result in $\dot H^{\frac12}$ is due to H. Fujita and T. Kato in [9] (see also [17] for a similar result, where the smallness of u_0 is measured by $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$). Since then, a number of works have been devoted to proving similar wellposedness results for larger classes of initial data; one should mention the result of T. Kato [14] where the smallness is measured in L^3 (see also [13]) and the result of M. Cannone, Y. Meyer and F. Planchon (see [4]) where the smallness is measured in the Besov space $\dot B^{-1+\frac3p}_{p,\infty}$. Let us recall that, for positive σ,

$$\|u\|_{\dot B^{-\sigma}_{p,r}} \stackrel{\rm def}{=} \Big\| t^{\frac\sigma2}\|\mathrm{S}(t)u\|_{L^p}\Big\|_{L^r(\mathbf{R}^+,\frac{dt}{t})}$$

where $\mathrm{S}(t)=e^{t\Delta}$ denotes the heat flow. The importance of this result can be illustrated by the following example: if ϕ is a function in the Schwartz space $\mathcal{S}(\mathbf{R}^3)$, let us introduce the family of divergence free vector fields

$$\phi_\varepsilon(x)=\cos\Big(\frac{x_3}{\varepsilon}\Big)(\partial_2\phi,-\partial_1\phi,0).$$

Then, for small ε, the size of $\|\phi_\varepsilon\|_{\dot B^{-\sigma}_{p,r}}$ is ε^σ.

Let us also mention the result by H. Koch and D. Tataru in [15] where the smallness is measured in the space BMO^{-1}, defined by

$$\|u\|_{BMO^{-1}} \stackrel{\rm def}{=} \|u\|_{\dot B^{-1}_{\infty,\infty}} + \sup_{\substack{x\in\mathbf{R}^3\\ R>0}} R^{-\frac32}\Big(\int_{P(x,R)} |\mathrm{S}(t)u(y)|^2\,dydt\Big)^{\frac12},$$

where $P(x,R)=[0,R^2]\times B(x,R)$ and $B(x,R)$ denotes the ball of radius R centered at zero.

As observed by H. Koch and D. Tataru, this norm seems to be the ultimate norm for the initial data for which the classical Picard's iterative scheme can work. Indeed the first iterate, $\mathrm{S}(t)u_0$ must be in L^2 locally in $\mathbf{R}^+\times\mathbf{R}^3$. In particular, $\mathrm{S}(t)u_0$ must be in $L^2([0,1]\times B(0,1))$. Then considering the norm of the space must be invariant by translation as well as by the scaling of the equation, we get the norm $\|\cdot\|_{BMO^{-1}}$. Moreover, let us notice that we have

$$\sup_{\substack{x\in\mathbf{R}^3\\ R>0}} R^{-\frac32}\Big(\int_{P(x,R)} |\mathrm{S}(t)u(y)|^2dy\Big)^{\frac12} \leq \|\mathrm{S}(t)u\|_{L^2(L^\infty)}.$$

and thus $\|u\|_{\dot B^{-1}_{\infty,\infty}} \leq \|u\|_{BMO^{-1}} \leq \|u\|_{\dot B^{-1}_{\infty,2}}$.

Moreover the space $\dot C^{-1}=\dot B^{-1}_{\infty,\infty}$ seems to be optimal independently of the method of resolution, due to the following argument (see [1] for instance). Let B be a Banach space continuously included in the space $\mathcal{S}'$ of tempered distributions on $\mathbf{R}^3$. Let us assume that, for any $(\lambda,a)\in\mathbf{R}^+_\star\times\mathbf{R}^3$,

$$\|f(\lambda(\cdot-a))\|_B=\lambda^{-1}\|f\|_B.$$

Then we have that $|\langle f,e^{-|\cdot|^2}\rangle|\leq C\|f\|_B$. By dilation and translation, we deduce that

$$\|f\|_{\dot C^{-1}}=\sup_{t>0} t^{\frac12}\|\mathrm{S}(t)f\|_{L^\infty}\leq C\|f\|_B.$$

We have proved that any Banach space included in $\mathcal{S}'$, translation invariant and which has the right scaling is included in $\dot C^{-1}$.

Let us point out that none of the results mentioned so far are specific to the Navier-Stokes equations, as they do not use the special structure of the nonlinear term in (NS).

Our aim in this paper is to go beyond the smallness condition on the initial data and to exhibit arbitrarily large initial data in $\dot{C}^{-1}$ which generate a unique, global solution. This was performed in [8] in the periodic case, where we presented a new, nonlinear smallness assumption on the initial data, which may hold despite the fact that the data is large. That result uses the structure of the nonlinear term, as it is based on the fact that the two dimensional Navier-Stokes equation is globally well posed.

The first theorem of this paper consists in a result of global existence under a non linear smallness hypothesis (Theorem 1 below). The proof consists mainly in introducing an idea of [6] in the proof of the Koch and Tataru Theorem. The non linear smallness hypothesis is, roughly speaking, that the first iterate $\mathrm{S}(t)u_0 \cdot \nabla \mathrm{S}(t)u_0$ is exponentially small with respect to $\|u_0\|^4_{\dot{B}^{-1}_{\infty,2}}$.

Then we exhibit an example of a family of initial data with very large $\dot{C}^{-1}$ norm which satisfies the non linear smallness hypothesis. This example fits the structure of the non linear term $u \cdot \nabla u$.

Then, we study the stability of this nonlinear smallness condition, but not in the usual sense of a perturbation by a small vector field. This problem has been solved by I. Gallagher, D. Iftimie and F. Planchon in [11] and by P. Auscher, S. Dubois and P. Tchamitchian in [1]. These authors proved that, in any adapted scaling space (for instance $\dot{H}^{\frac{1}{2}}$, L^3, $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ or BMO^{-1}) the set of initial data giving rise to global solution is open.

Our purpose is different. Once constructed an initial data generating a global solution, we want to generate a large family of global solutions that may not be close to the one we start with, in the $\dot{C}^{-1}$ norm. This is done with a fractal type transform (see the forthcoming Definition 1.3). Roughly speaking, this is the linear superposition of an arbitrarily large number of dilated and translated iterates of the initial data, and we will see that the initial data so-transformed still satisfies the nonlinear smallness assumption. That of course enables one to construct a very large class of initial data satisfying that smallness assumption; the transformation is based on a construction of [2].

1.2. Definitions.

Before presenting more precisely the results of this paper, let us give some definitions and notation. We shall be using Besov spaces, which are defined equivalently using the Littlewood-Paley decomposition or the heat operator. As both definitions will be useful in the following, we present them both in the next definition.

Definition 1.1. *Let $\varphi \in \mathcal{S}(\mathbf{R}^3)$ be such that $\widehat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$ and $\widehat{\varphi}(\xi) = 0$ for $|\xi| > 2$. Define, for $j \in \mathbf{Z}$, the function $\varphi_j(x) \stackrel{\text{def}}{=} 2^{3j}\varphi(2^j x)$, and the Littlewood–Paley operators $S_j \stackrel{\text{def}}{=} \varphi_j * \cdot$ and $\Delta_j \stackrel{\text{def}}{=} S_{j+1} - S_j$. Let f be in $\mathcal{S}'(\mathbf{R}^3)$. Then f belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbf{R}^3)$ if and only if*

- *The partial sum $\sum_{-m}^{m} \Delta_j f$ converges towards f as a tempered distribution;*
- *The sequence $\varepsilon_j \stackrel{\text{def}}{=} 2^{js}\|\Delta_j f\|_{L^p}$ belongs to $\ell^q(\mathbf{Z})$.*

In that case

$$\|f\|_{\dot B^s_{p,q}} \stackrel{\mathrm{def}}{=} \biggl(\sum_{j\in\mathbf{Z}} 2^{jsq}\|\Delta_j f\|^q_{L^p}\biggr)^{\frac1q}$$

and if $s<0$, the one has the equivalent norm

$$\|f\|_{\dot B^s_{p,q}} \sim \Big\|t^{-\frac s2}\|\mathrm{S}(t)f\|_{L^p}\Big\|_{L^q(\mathbf{R}^+;\frac{dt}{t})}. \tag{1.1}$$

Let us notice that the above equivalence comes from the inequality, proved for instance in [6],

$$\|\mathrm{S}(t)\Delta_j a\|_{L^p} \leq Ce^{-C^{-1}2^{2j}t}\|\Delta_j a\|_{L^p}. \tag{1.2}$$

Note that the following Sobolev-type continuous embeddings hold:

$$\dot B^{s_1}_{p_1,q_1}\subset \dot B^{s_2}_{p_2,q_2}, \quad \text{as soon as} \quad s_1-\frac{d}{p_1}=s_2-\frac{d}{p_2} \quad \text{with} \quad p_1\leq p_2 \quad \text{and} \quad q_1\leq q_2.$$

We shall denote by $\mathbf{P}$ the Leray projector onto divergence free vector fields

$$\mathbf{P}=\mathrm{Id}-\nabla\Delta^{-1}\mathrm{div}.$$

Before stating the first result of this paper, let us introduce the following space.

Definition 1.2. *We shall denote by E the space of functions f in $L^1(\mathbf{R}^+;\dot B^{-1}_{\infty,1})$ such that*

$$\sum_{j\in\mathbf{Z}} 2^{-j}\Big\|\|\Delta_j f(t)\|_{L^\infty}\Big\|_{L^2(\mathbf{R}^+;tdt)}<\infty$$

equipped with the norm

$$\|f\|_E \stackrel{\mathrm{def}}{=} \|f\|_{L^1(\mathbf{R}^+;\dot B^{-1}_{\infty,1})}+\sum_{j\in\mathbf{Z}} 2^{-j}\Big\|\|\Delta_j f(t)\|_{L^\infty}\Big\|_{L^2(\mathbf{R}^+;tdt)}.$$

Let us remark that, for any homogeneous function σ of order 0 smooth outside 0, we have

$$\forall p\in[1,\infty]\,,\ \|\sigma(D)\Delta_j f\|_{L^p}\leq C\|\Delta_j f\|_{L^p}.$$

Thus the Leray projection $\mathbf{P}$ onto divergence free vectors fields maps continuously E into E.

1.3. Statement of the results.

1.3.1. *Global existence results.* The first result we shall prove is a new global wellposedness result, under a nonlinear smallness assumption on the initial data.

Theorem 1. *There is a constant C_0 such that the following result holds. Let $u_0\in\dot H^{\frac12}(\mathbf{R}^3)$ be a divergence free vector field. Suppose that*

$$\Big\|\mathbf{P}\Big(\mathrm{S}(t)u_0\cdot\nabla\mathrm{S}(t)u_0\Big)\Big\|_E \leq C_0^{-1}\exp\Big(-C_0\|u_0\|^4_{\dot B^{-1}_{\infty,2}}\Big). \tag{1.3}$$

Then there is a unique, global solution to (NS) associated with u_0, satisfying

$$u\in C_b(\mathbf{R}^+;\dot H^{\frac12})\cap L^2(\mathbf{R}^+;\dot H^{\frac32}).$$

Remark For the sake of simplicity, we state the theorem for initial data in $H^{\frac12}$, but it obviously works for initial data in $\dot B^{-1}_{\infty,2}$.

The proof of Theorem 1 is given in Section 2 below; it consists in writing the solution u (which exists for a short time at least), as $u = \mathrm{S}(t)u_0 + R$ and in proving global wellposedness result for the perturbed Navier-Stokes equation satisfied by R, under assumption (1.3). While the proof follows the lines of Koch and Tataru's one (see [15]). a small modification of classical Picard's argument is needed to control the linear term, which is not small.

In fact, the main point of the paper is to exhibit examples of applications of this theorem which go beyond the assumption of smallness of $\|u_0\|_{\dot{B}^{-1}_{\infty,2}}$. The problem we have to solve is the construction of large initial data u_0 such that a quadratic functionnal, namely $\mathbf{P}(S(t)u_0 \cdot \nabla S(t)u_0)$ is small. This demands a careful use of the structure of this quadratic functionnal.

The situation here is different from our previous work (see [8]), devoted to the periodic case, where the structure of the equation was used through the fact that the bidimensionnel Navier-Stokes equation is globally wellposed.

Now let us state the theorem that ensures that Theorem 1 is relevant.

Theorem 2. *Let $\phi \in \mathcal{S}(\mathbf{R}^3)$ be a given function, and consider two real numbers ε and α in $]0,1[$. Define*

$$\varphi_\varepsilon(x) = \frac{(-\log\varepsilon)^{\frac{1}{5}}}{\varepsilon^{1-\alpha}} \cos\Big(\frac{x_3}{\varepsilon}\Big)\, \phi\Big(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\Big).$$

There is a constant $C > 0$ such that for ε small enough, the smooth, divergence free vector field

$$u_{0,\varepsilon}(x) = (\partial_2\varphi_\varepsilon(x), -\partial_1\varphi_\varepsilon(x), 0)$$

satisfies

$$C^{-1}(-\log\varepsilon)^{\frac{1}{5}} \leq \|u_{0,\varepsilon}\|_{\dot{B}^{-1}_{\infty,\infty}} \leq C(-\log\varepsilon)^{\frac{1}{5}},$$

and

$$\|\mathrm{S}(t)u_{0,\varepsilon} \cdot \nabla \mathrm{S}(t)u_{0,\varepsilon}\|_E \leq C\varepsilon^{\frac{\alpha}{3}}(-\log\varepsilon)^{\frac{2}{5}}. \tag{1.4}$$

Thus for ε small enough, the vector field $u_{0,\varepsilon}$ generates a unique, global solution to (NS).

The proof of Theorem 2 is the purpose of Section 3.

Remark One can also write this example in terms of the Reynolds number of the fluid: let $\mathrm{Re} > 0$ be the Reynolds number, and define the rescaled velocity field $v(t,x) = \nu u(\nu t, x)$ where $\nu = 1/\mathrm{Re}$. Then v satisfies the Navier-Stokes equation

$$\partial_t v + \mathbf{P}(v\cdot\nabla v) - \nu\Delta v = 0$$

and Theorem 2 states the following: the vector field

$$v_{0,\nu}(x) = (-\log\nu)^{\frac{1}{5}} \cos\Big(\frac{x_3}{\nu}\Big)\Big((\partial_2\phi)(x_1, \frac{x_2}{\nu^\alpha}, x_3), \nu^\alpha(-\partial_1\phi)(x_1, \frac{x_2}{\nu^\alpha}, x_3)\Big)$$

satisfies

$$\|v_{0,\nu}\|_{\dot{B}^{-1}_{\infty,\infty}} \sim C\nu(-\log\nu)^{\frac{1}{5}}$$

and generates a global solution to the Navier-Stokes equations if ν is small enough. Compared with the usual theory of global existence for the Navier-Stokes equations, we have gained a (power of a) logarithm in the smallness assumption in terms of the viscosity, since classically one expects the initial data to be small with respect to ν.

1.3.2. *Stability results.* The second aim of this paper is to give some stability properties of global solutions. It is known since [11] that any initial data in $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ giving rise to a unique global solution is stable: a small perturbation of that data also generates a global solution (see [1] for the case of BMO^{-1}). Here we present a stability result where the perturbation is as large as the initial data but has a special form: it consists in the superposition of dilated and translated duplicates of the initial data, in the spirit of profile decompositions of P. Gérard (see [12]). This transform is a version of the fractal transform used in [2] in the study of refined Sobolev and Hardy inequalities. Let us be more precise and define the transformation. We shall only be considering compactly supported intial data for this study, and up to a rescaling we shall suppose to simplify that the support of the initial data is restricted to the unit cube Q of $\mathbf{R}^3$ centered at 0.

Definition 1.3. *Let $X = (x_1, \dots, x_K)$ be a set of K distinct points in $\mathbf{R}^3$. For $\Lambda \in 2^{\mathbf{N}}$, let us define*

$$T_{\Lambda,X} \quad \left\{ \begin{array}{lll} \mathcal{S}' & \rightarrow & \mathcal{S}' \\ f & \mapsto & T_{\Lambda,X} f \stackrel{\text{def}}{=} \displaystyle\sum_{J \in \{1,\dots,K\}} T^J_\Lambda f \qquad \textit{with} \quad T^J_\Lambda f(x) \stackrel{\text{def}}{=} \Lambda f(\Lambda(x - x_J)). \end{array} \right.$$

It can be noted that this is a generalization of the fractal transformation T^k studied in [2].

The next statement is quite easy to prove: it shows that this transformation on the initial data preserves global wellposedness, as soon as the scaling parameter Λ is large enough (the threshold Λ being unknown as a function of the initial data). The theorem following that statement gives a quantitative approach to that stability: if the initial data u_0 satisfies the smallness assumption (1.3) of Theorem 1, then so does $T_{\Lambda,X} u_0$ as soon as Λ is large enough (the threshold being an explicit function of norms of u_0).

More precisely we have the following results.

Proposition 1.1. *Let u_0 be a divergence free vector field in $\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)$ generating a unique, global solution to the Navier-Stokes equations and X be a finite sequence of distinct points. There is $\Lambda_0 > 0$ such that, for any $\Lambda \geq \Lambda_0$, the vector field $T_{\Lambda,X} u_0$ also generates a unique, global solution.*

Remarks

- Using the global stability of global solutions proved in [11], a global solution associated to an initial data in $\dot{H}^{\frac{1}{2}}$ is always in $L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+; \dot{H}^{\frac{3}{2}})$.
- As the proof of that result in Section 4.1 will show (see Proposition 4.1), Proposition 1.1 can be generalized to the case where the vector field u_0 is replaced by any finite sequence of vector fields in $\dot{H}^{\frac{1}{2}}$ generating a global solution.
- As we shall see in the proof of Theorem 3 stated below, the functions $T_{\Lambda,X} u_0$ and u_0 have essentially the same norm in $\dot{C}^{-1}$.

Now let us state the quantitative stability theorem, in particular in the case of an initial data satisfying the assumptions of Theorem 1. In order to avoid excessive heaviness, we shall assume from now on that the initial data is compactly supported, and after scaling, supported in the unit cube $Q =]-\frac{1}{2}, \frac{1}{2}[^d$. We shall consider sequences X such that

$$\inf_{\substack{(J,J')\in\{1,\dots,K\}^2\\ J\neq J'}} \{d(x_J, x_{J'}); d(x_J, {}^cQ)\} \geq \delta > 0. \tag{1.5}$$

We shall prove the following theorem.

Theorem 3. *Let u_0 be a smooth $\dot H^{\frac12}$ divergence free vector field, compactly supported in the cube Q. Suppose that u_0 satisfies (1.3) in the following slightly looser sense: there is $\eta \in]0,1[$ such that*

$$\|\mathbf{P}(\mathrm{S}(t)u_0 \cdot \nabla \mathrm{S}(t)u_0)\|_E \leq C_0^{-1} \exp\Big(-C_0\big(\|u_0\|_{\dot B^{-1}_{\infty,2}} + \eta\big)^4\Big) - \eta. \tag{1.6}$$

Then there is a positive Λ_0 (depending only on $\eta, K, \delta, \|u_0\|_{\dot H^{-1}}$ and $\|u_0\|_{\dot B^{-3}_{\infty,\infty}}$) such that for any $\Lambda \geq \Lambda_0$, the vector field $T_{\Lambda,X}u_0$ satisfies (1.3) and in particular generates a global solution to (NS). Moreover, for all r in $[1,\infty]$,

$$\|u_0\|_{\dot B^{-1}_{\infty,r}} - \eta \leq \|T_{\Lambda,X}u_0\|_{\dot B^{-1}_{\infty,r}} \leq \|u_0\|_{\dot B^{-1}_{\infty,r}} + \eta.$$

Remarks

- The factor η appearing in (1.6) means that u_0 must not saturate the nonlinear smallness assumption (1.3) of Theorem 1.
- The proof of this theorem is based on the fact that the Besov norm of index -1 as well as $\|\mathbf{P}(\mathrm{S}(t)u_0 \cdot \nabla \mathrm{S}(t)u_0)\|_E$ are invariant under the action of $T_{\Lambda,X}$, up to some small error terms.

As a conclusion of this introduction, let us state the following result, which describes the action of $T_{\Lambda,X}$ on the family $u_{0,\varepsilon}$ introduced in Theorem 2.

Theorem 4. *Let $u_{0,\varepsilon}$ be the family introduced in Theorem 2. For any K and δ, a constant Λ_0 exists, which is independent of ε, such that the following result holds. For any family X and any $\Lambda \geq \Lambda_0$, there is a global solution smooth solution of (NS) with initial data $T_{\Lambda,X}u_{0,\varepsilon}$.*

Remark Let us point out that as opposed to Proposition 1.1, Theorem 3 (or rather Lemmas 4.1 and 4.2 which are the key to its proof) provides precise bounds on Λ_0 so that the constant Λ_0 appearing in Theorem 4 may be chosen independently of ε.

2. Proof of Theorem 1

2.1. **Main steps of the proof.** Let us start by remarking that in the case when u_0 is small then there is nothing to be proved, so in the following we shall suppose that $\|u_0\|_{\dot B^{-1}_{\infty,2}}$ is not small, say $\|u_0\|_{\dot B^{-1}_{\infty,2}} \geq 1$.

We follow the method introduced by H. Koch and D. Tataru in [15] in order to look for the solution u under the form $u_F + R$, where $u_F(t) \stackrel{\mathrm{def}}{=} \mathrm{S}(t)u_0$. Let us denote by $\mathcal{Q}$ the bilinear operator defined by

$$\mathcal{Q}(a,b)(t) \stackrel{\mathrm{def}}{=} -\frac12 \int_0^t \mathrm{S}(t-t')\mathbf{P}\left(a(t')\cdot\nabla b(t') + b(t')\cdot\nabla a(t')\right) dt'$$

Then R is the solution of

$$(MNS) \qquad R = \mathcal{Q}(u_F, u_F) + 2\mathcal{Q}(u_F, R) + \mathcal{Q}(R, R).$$

To prove the global existence of u, we are reduced to proving the global wellposedness of (MNS); that relies on the following easy lemma, the proof of which is omitted.

Lemma 2.1. *Let X be a Banach space, let L be a continuous linear map from X to X, and let B be a bilinear map from $X \times X$ to X. Let us define*

$$\|L\|_{\mathcal{L}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=1} \|Lx\| \quad \text{and} \quad \|B\|_{\mathcal{B}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=\|y\|=1} \|B(x, y)\|.$$

If $\|L\|_{\mathcal{L}(X)} < 1$, then for any x_0 in X such that

$$\|x_0\|_X < \frac{(1 - \|L\|_{\mathcal{L}(X)})^2}{4\|B\|_{\mathcal{B}(X)}},$$

the equation

$$x = x_0 + Lx + B(x, x)$$

has a unique solution in the ball of center 0 and radius $\dfrac{1 - \|L\|_{\mathcal{L}(X)}}{2\|B\|_{\mathcal{B}(X)}}$.

Let us introduce the functional space for which we shall apply the above lemma. We define the quantity

$$U(t) \stackrel{\text{def}}{=} \|u_F(t)\|_{L^\infty}^2 + t\|u_F(t)\|_{L^\infty}^4,$$

which satisfies

$$\begin{aligned} \int_0^\infty U(t)dt &\leq C\|u_0\|^2_{\dot{B}^{-1}_{\infty,2}} + C\|u_0\|^4_{\dot{B}^{-1}_{\infty,4}} \\ &\leq C\|u_0\|^4_{\dot{B}^{-1}_{\infty,2}} \end{aligned} \tag{2.1}$$

recalling that we have supposed that $\|u_0\|_{\dot{B}^{-1}_{\infty,2}} \geq 1$ to simplify the notation.

For all $\lambda \geq 0$, let us denote by X_λ the set of functions on $\mathbf{R}^+ \times \mathbf{R}^3$ such that

$$\|v\|_\lambda \stackrel{\text{def}}{=} \sup_{t>0}\Big(t^{\frac{1}{2}}\|v_\lambda(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbf{R}^3\\ R>0}} R^{-\frac{3}{2}}\Big(\int_{P(x,R)} |v_\lambda(t,y)|^2 dy\Big)^{\frac{1}{2}}\Big) < \infty, \tag{2.2}$$

where

$$v_\lambda(t, x) \stackrel{\text{def}}{=} v(t, x) \exp\Big(-\lambda \int_0^t U(t')dt'\Big)$$

while $P(x, R) = [0, R^2] \times B(x, R)$ and $B(x, R)$ denotes the ball of $\mathbf{R}^3$ of center x and radius R. Let us point out that, in the case when $\lambda = 0$, this is exactly the space introduced by H. Koch and D. Tataru in [15], and that for any $\lambda \geq 0$ we have due to (2.1),

$$\|v\|_\lambda \leq \|v\|_0 \leq C\|v\|_\lambda \exp\Big(C\lambda\|u_0\|^4_{\dot{B}^{-1}_{\infty,2}}\Big). \tag{2.3}$$

From Lemmas 3.1 and 3.2 of [15] together with the above equivalence of norms, we infer that

$$\|\mathcal{Q}(v, w)\|_\lambda \leq C\|v\|_\lambda\|w\|_\lambda \exp\Big(C\lambda\|u_0\|^4_{\dot{B}^{-1}_{\infty,2}}\Big). \tag{2.4}$$

Theorem 1 follows from the following two lemmas.

Lemma 2.2. *There is a constant $C>0$ such that the following holds. For any non negative λ, for any $t \geq 0$ and any $f \in E$, we have*

$$\Big\| \int_0^t \mathrm{S}(t-t') f(t') dt' \Big\|_\lambda \leq C \|f\|_E.$$

Lemma 2.3. *Let $u_0 \in \dot{B}^{-1}_{\infty,2}$ be given, and define $u_F(t) = \mathrm{S}(t) u_0$. There is a constant $C>0$ such that the following holds. For any $\lambda \geq 1$, for any $t \geq 0$ and any $v \in X_\lambda$, we have*

$$\|\mathcal{Q}(u_F, v)(t)\|_\lambda \leq \frac{C}{\lambda^{\frac14}} \|v\|_\lambda.$$

End of the proof of Theorem 1 Let us apply Lemma 2.1 to Equation (MNS) satisfied by R, in a space X_λ. We choose λ so that according to Lemma 2.3,

$$\|\mathcal{Q}(u_F, \cdot)(t)\|_{\mathcal{L}(X_\lambda)} \leq \frac14 \cdot$$

Then according to Lemma 2.1, there is a unique solution R to (MNS) in X_λ as soon as $\mathcal{Q}(u_F, u_F)$ satisfies

$$\|\mathcal{Q}(u_F, u_F)\|_{X_\lambda} \leq \frac{1}{16 \|\mathcal{Q}\|_{\mathcal{B}(X_\lambda)}} \cdot$$

But (2.4) guarantees that

$$\|\mathcal{Q}\|_{\mathcal{B}(X_\lambda)} \leq C \exp\Big(C\lambda \|u_0\|^4_{\dot{B}^{-1}_{\infty,2}} \Big),$$

so it is enough to check that for some constant C,

$$\|\mathcal{Q}(u_F, u_F)\|_{X_\lambda} \leq C^{-1} \exp\Big(-C\lambda \|u_0\|^4_{\dot{B}^{-1}_{\infty,2}} \Big).$$

By Lemma 2.2, this is precisely condition (1.3) of Theorem 1, so under assumption (1.3), there is a unique, global solution R to (MNS), in the space X_λ. This implies immediately that there is a unique, global solution u to the Navier-Stokes system in X_λ. The fact that u belongs to $C_b(\mathbf{R}^+; \dot{H}^{\frac12}) \cap L^2(\mathbf{R}^+; \dot{H}^{\frac32})$ is then simply an argument of propagation of regularity (see for instance [16]).

2.2. **Proof of Lemma 2.2.** Thanks to (2.3), it is enough to prove Lemma 2.2 for $\lambda = 0$.

Let us start by proving that $\displaystyle\int_0^t \mathrm{S}(t-t') f(t') dt'$ belongs to $L^2(\mathbf{R}^+; L^\infty)$; that will give in particular the boundedness of the second norm entering in the definition of X_λ.

Using (1.2), we get

$$\Big\| \int_0^t \Delta_j \mathrm{S}(t-t') f(t') dt' \Big\|_{L^\infty} \leq C \int_0^t e^{-C^{-1} 2^{2j} (t-t')} \|\Delta_j f(t')\|_{L^\infty} dt'.$$

Young's inequality then gives

$$\Big\| \int_0^t \Delta_j \mathrm{S}(t-t') f(t') dt' \Big\|_{L^2(\mathbf{R}^+; L^\infty)} \leq C 2^{-j} \|\Delta_j f\|_{L^1(\mathbf{R}^+; L^\infty)},$$

thus the series $\Big(\Delta_j \int_0^t \mathrm{S}(t-t')f(t')dt'\Big)_{j\in\mathbf{Z}}$ converges in $L^2(\mathbf{R}^+;L^\infty)$, and

$$\Big\|\int_0^t \mathrm{S}(t-t')f(t')dt'\Big\|_{L^2(\mathbf{R}^+;L^\infty)} \leq C\|f\|_E.$$

This implies in particular that

$$\sup_{\substack{x\in\mathbf{R}^3\\ R>0}} R^{-\frac32}\bigg(\int_{P(x,R)}\Big|\int_0^t (\mathrm{S}(t-t')f(t'))(y)dt'\Big|^2 dy\bigg)^{\frac12} \leq C\|f\|_E. \tag{2.5}$$

The second part of the norm defining $\|\cdot\|_{X_\lambda}$ in (2.2) is therefore controlled by the norm of f in E.

To estimate the first part of that norm, let us write that for any $t\geq 0$ and any $j\in\mathbf{Z}$,

$$\begin{aligned}
t^{\frac12}\Delta_j \int_0^t \mathrm{S}(t-t')f(t')dt' &= G_j^{(1)}(t)+G_j^{(2)}(t) \quad \text{with}\\
G_j^{(1)}(t) &\stackrel{\text{def}}{=} t^{\frac12}\int_0^{\frac t2} \mathrm{S}(t-t')\Delta_j f(t')dt' \quad \text{and}\\
G_j^{(2)}(t) &\stackrel{\text{def}}{=} t^{\frac12}\int_{\frac t2}^{t} \mathrm{S}(t-t')\Delta_j f(t')dt'.
\end{aligned}$$

Using again (1.2) we have, since $t\leq 2(t-t')$,

$$\begin{aligned}
\|G_j^{(1)}(t)\|_{L^\infty} &\leq C\int_0^{\frac t2}(t-t')^{\frac12}2^j e^{-C^{-1}2^{2j}(t-t')}2^{-j}\|\Delta_j f(t')\|_{L^\infty}dt'\\
&\leq 2^{-j}\|\Delta_j f\|_{L^1(\mathbf{R}^+;L^\infty)}.
\end{aligned}$$

In order to estimate $\|G_j^{(2)}(t)\|_{L^\infty}$, let us write, since $t\leq 2t'$,

$$\|G_j^{(2)}(t)\|_{L^\infty}\leq C\int_0^t e^{-C^{-1}2^{2j}(t-t')}t'^{\frac12}\|\Delta_j f(t')\|_{L^\infty}dt'.$$

Using the Cauchy-Schwarz inequality, we get

$$\|G_j^{(2)}(t)\|_{L^\infty}\leq C2^{-j}\|t^{\frac12}\Delta_j f(t)\|_{L^2(\mathbf{R}^+;L^\infty)}.$$

Then using (2.5) and summing over $j\in\mathbf{Z}$ concludes the proof of Lemma 2.2. □

2.3. **Proof of Lemma 2.3.** We have (see for instance [15] or [7]) that

$$\begin{aligned}
\mathcal{Q}(v,w)(t,x) &= \int_0^t\int_{\mathbf{R}^3} k(t-t',y)v(t',x-y)w(t',x-y)dydt'\\
&= k\star(vw)(t,x) \quad \text{with} \quad |k(\tau,\zeta)|\leq \frac{C}{(\sqrt{\tau}+|\zeta|)^4}.
\end{aligned}$$

The proof relies now mainly on the following proposition.

Proposition 2.1. *Let $u_0 \in \dot B^{-1}_{\infty,2}$ be given, and define $u_F(t) = \mathrm{S}(t)u_0$. There is a constant C such that the following holds. Consider, for any positive R and for $(\tau,\zeta) \in \mathbf{R}^+ \times \mathbf{R}^3$, the following functions:*

$$K_R^{(1)}(\tau,\zeta) \stackrel{\mathrm{def}}{=} \mathbf{1}_{|\zeta|\geq R}\frac{1}{|\zeta|^4} \quad \text{and} \quad K_R^{(2)}(\tau,\zeta) \stackrel{\mathrm{def}}{=} \mathbf{1}_{|\zeta|\leq R}\frac{1}{(\sqrt{\tau}+|\zeta|)^4}.$$

Then for any $\lambda \geq 1$ and any $R > 0$,

$$\left\| e^{-\lambda\int_0^t U(t')dt'} K_R^{(1)} \star (u_F v)\right\|_{L^\infty([0,R^2]\times\mathbf{R}^3)} \leq \frac{C}{\lambda^{\frac12}R}\|v\|_\lambda. \tag{2.6}$$

Moreover, for any $\lambda \geq 1$ and any $R > 0$,

$$\left\| e^{-\lambda\int_0^t U(t')dt'} K_R^{(2)} \star (u_F v)\right\|_{L^\infty([R^2,2R^2]\times\mathbf{R}^3)} \leq \frac{C}{\lambda^{\frac14}R}\|v\|_\lambda. \tag{2.7}$$

Proof of Proposition 2.1 Let us write that

$$\begin{aligned} V_\lambda^{(1)}(t,x) &\stackrel{\mathrm{def}}{=} e^{-\lambda\int_0^t U(t')dt'}|K_R^{(1)} \star (u_F v)(t,x)| \\ &\leq \int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4} e^{-\lambda\int_{t'}^t U(t'')dt''}\|u_F(t',\cdot)\|_{L^\infty}|v_\lambda(t',x-y)|dt'dy. \end{aligned}$$

By the Cauchy-Schwarz inequality and by definition of U, we infer that

$$\begin{aligned} V_\lambda^{(1)}(t,x) &\leq \Big(\int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4} e^{-2\lambda\int_{t'}^t U(t'')dt''}\|u_F(t',\cdot)\|^2_{L^\infty}dt'dy\Big)^{\frac12} \\ &\qquad\qquad \times\Big(\int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy\Big)^{\frac12} \\ &\leq \Big(\frac{C}{\lambda R}\Big)^{\frac12}\Big(\int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy\Big)^{\frac12}. \qquad (2.8) \end{aligned}$$

Now let us decompose the integral on the right on rings; this gives

$$\begin{aligned} \int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy &= \sum_{p=0}^\infty \int_0^t\int_{B(0,2^{p+1}R)\setminus B(0,2^pR)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy \\ &\leq \frac1R\sum_{p=0}^\infty 2^{-p+3}(2^{p+1}R)^{-3} \\ &\qquad\times\int_0^t\int_{B(0,2^{p+1}R)}|v_\lambda(t,x-y)|^2dtdy. \end{aligned}$$

As $t \leq R^2$ and p is non negative, we have

$$\begin{aligned} \int_0^t\int_{^cB(0,R)} \frac{1}{|y|^4}|v_\lambda(t',x-y)|^2dt'dy &\leq \frac{C}{R}\sum_{p=0}^\infty 2^{-p}(2^{p+1}R)^{-3}\int_{P(x,2^{p+1}R)}|v_\lambda(t,z)|^2dtdz \\ &\leq \frac{C}{R}\sum_{p=0}^\infty 2^{-p}\sup_{R'>0}\frac{R'^3}{\int}{}_{P(x,R')}|v_\lambda(t,z)|^2dtdz. \end{aligned}$$

By definition of $\|\cdot\|_\lambda$, we infer that

$$\int_0^t \int_{^cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t',x-y)|^2 dt'dy \leq \frac{C}{R}\|v\|_\lambda^2,$$

Then, using (2.8), we conclude the proof of (2.6).

In order to prove the second inequality, let us observe that

$$\begin{aligned} & e^{-\lambda\int_0^t U(t')dt'} |(K_R^{(2)} \star (u_F v))(t,x)| \leq \mathcal{K}_R^{(21)}(t,x) + \mathcal{K}_R^{(21)}(t,x) \quad \text{with} \\ \mathcal{K}_R^{(22)}(t,x) & \stackrel{\text{def}}{=} \int_0^{\frac{t}{2}} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)4} e^{-\lambda\int_{t'}^t U(t'')dt''} \|u_F(t',\cdot)\|_{L^\infty} |v_\lambda(t',x-y)| dt'dy\,, \\ \mathcal{K}_R^{(22)}(t,x) & \stackrel{\text{def}}{=} \int_{\frac{t}{2}}^{t} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)4} e^{-\lambda\int_{t'}^t U(t'')dt''} \|u_F(t',\cdot)\|_{L^\infty} |v_\lambda(t',x-y)| dt'dy. \end{aligned}$$

Using the Cauchy-Schwarz inequality, as $t\in[R,R^2]$ and $t\leq 2(t-t')$, we infer that

$$\begin{aligned} \mathcal{K}_R^{(21)}(t,x) & \leq \biggl(\int_0^{\frac{t}{2}} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'}+|y|)^8} e^{-2\lambda\int_{t'}^t U(t'')dt''} \|u_F(t',\cdot)\|_{L^\infty}^2 dt'dy\biggr)^{\frac12} \\ & \qquad\qquad \times \biggl(\int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t',x-y)|^2 dt'dy\biggr)^{\frac12} \\ & \leq \frac{C}{\lambda^{\frac12}} \biggl(\int_{B(0,R)} \frac{dy}{(R+|y|)^8}\biggr)^{\frac12} \biggl(\int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t',x-y)|^2 dt'dy\biggr)^{\frac12} \\ & \leq \frac{C}{(t\lambda)^{\frac12}} R^{-\frac32} \biggl(\int_0^{R^2} \int_{B(0,R)} |v_\lambda(t',x-y)|^2 dt'dy\biggr)^{\frac12}, \end{aligned}$$

so that

$$K_R^{(21)}(t,x) \leq \frac{C}{(t\lambda)^{\frac12}} \|v\|_\lambda. \tag{2.9}$$

In order to estimate $\mathcal{K}_R^{(22)}$, let us write that

$$\begin{aligned} \mathcal{K}_R^{(22)}(t,x) & \leq \int_{\frac{t}{2}}^{t} \int_{\mathbf{R}^3} \frac{1}{(\sqrt{t-t'}+|y|)^4} e^{-\lambda\int_{t'}^t U(t'')dt''} \|u_F(t',\cdot)\|_{L^\infty} \|v_\lambda(t',\cdot)\|_{L^\infty} dt'dy \\ & \leq C\|v\|_\lambda \int_{\frac{t}{2}}^{t} \frac{1}{\sqrt{t-t'}} e^{-\lambda\int_{t'}^t U(t'')dt''} \frac{\|u_F(t',\cdot)\|_{L^\infty}}{t'^{\frac12}} dt'. \end{aligned}$$

By definition of U and using the fact that $t\leq 2t'$, Hölder's inequality implies that

$$\begin{aligned} \mathcal{K}_R^{(22)}(t,x) & \leq \frac{C}{t^{\frac12}} \|v\|_\lambda \Bigl(\int_0^t e^{-4\lambda\int_{t'}^t U(t'')dt''} t' \|u_F(t',\cdot)\|_{L^\infty}^4 dt'\Bigr)^{\frac14} \\ & \leq \frac{C}{\lambda^{\frac14} t^{\frac12}} \|v\|_\lambda. \end{aligned}$$

Together with (2.9), this concludes the proof of the proposition. □

From this proposition, we infer immediately the following corollary. This corollary proves directly one half of Lemma 2.3, as it gives a control of $\mathcal{Q}(u_F, v)$ in the first norm out of the two entering in the definition of X_λ.

Corollary 2.1. *Under the assumptions of Proposition 2.1, we have*

$$t^{\frac{1}{2}} e^{-\lambda \int_0^t U(t')dt'} \|\mathcal{Q}(u_F, v)(t, \cdot)\|_{L^\infty} \leq \frac{C}{\lambda^{\frac{1}{4}}} \|v\|_\lambda.$$

Proof of Corollary 2.1. Let us write that

$$k \star (u_F v)(t, x) = k \star (u_F \mathbf{1}_{{}^c B(x, 2\sqrt{t})} v)(t, x) + k \star (u_F \mathbf{1}_{B(x, 2\sqrt{t})} v)(t, x).$$

From Proposition 2.1, we infer that

$$\begin{aligned} e^{-\lambda \int_0^t U(t')dt'} |k \star (u_F \mathbf{1}_{{}^c B(x, 2\sqrt{t})} v)(t, x)| &\leq e^{-\lambda \int_0^t U(t')dt'} K^{(1)}_{2\sqrt{t}} \star (|u_F \mathbf{1}_{B(x, 2\sqrt{t})} v|)(t, x) \\ &\leq \frac{C}{(t\lambda)^{\frac{1}{2}}} \|v\|_\lambda. \end{aligned}$$

Moreover, thanks to Proposition 2.1, we have also

$$\begin{aligned} e^{-\lambda \int_0^t U(t')dt'} |k \star (u_F \mathbf{1}_{B(x, 2\sqrt{t})} v)(t, x)| &\leq e^{-\lambda \int_0^t U(t')dt'} K^{(2)}_{2\sqrt{t}} \star (|u_F| \mathbf{1}_{B(x, 2\sqrt{t})} |v|)(t, x) \\ &\leq \frac{C}{\lambda^{\frac{1}{4}} t^{\frac{1}{2}}} \|v\|_\lambda. \end{aligned}$$

This proves the corollary. □

In order to conclude the proof of Lemma 2.3, let us estimate $\|k \star (u_F v)\|_{L^2(P(x,R))}$, for an arbitrary $x \in \mathbf{R}^3$. Let us write that

$$k \star (u_F v) = k \star (u_F \mathbf{1}_{{}^c B(x, 2R)} v) + k \star (u_F \mathbf{1}_{B(x, 2R)} v).$$

Observing that, for any $y \in B(x, R)$, we have

$$|k \star (u_F \mathbf{1}_{{}^c B(x, 2R)} v)(t, y)| \leq C K^{(1)}_R \star (|u_F| \mathbf{1}_{{}^c B(x, 2R)} |v|)(t, y),$$

and using Inequality (2.6) of Proposition 2.1, we get

$$e^{-\lambda \int_0^t U(t')dt'} \|k \star (u_F \mathbf{1}_{{}^c B(x, 2R)} v\|_{L^\infty(P(x,R))} \leq \frac{C}{\lambda^{\frac{1}{2}} R} \|v\|_\lambda.$$

As the volume of $P(x, R)$ is proportional to R^5, we infer that

$$\|k \star (u_F v_\lambda)\|_{L^2(P(x,R))} \leq \frac{C}{\lambda^{\frac{1}{2}}} R^{\frac{3}{2}} \|v\|_\lambda. \tag{2.10}$$

The following inequality is easy and classical, so its proof is omitted.

$$\left\| e^{-\lambda \int_0^t U(t')dt'} \mathcal{Q}(u_F, v)(t) \right\|_{L^2([0,T] \times \mathbf{R}^3)} \leq \frac{C}{\lambda^{\frac{1}{2}}} \|v_\lambda\|_{L^2([0,T] \times \mathbf{R}^3)}. \tag{2.11}$$

We deduce that

$$
\begin{aligned}
\Big\|e^{-\lambda\int_0^t U(t')dt'}k\star(u_F\mathbf{1}_{B(x,2R)})v\Big\|_{L^2(P(x,R))} &\leq \Big\|e^{-\lambda\int_0^t U(t')dt'}k\star(u_F\mathbf{1}_{B(x,2R)}v)\Big\|_{L^2([0,R^2]\times\mathbf{R}^3)}\\
&\leq \frac{C}{\lambda^{\frac12}}\|\mathbf{1}_{B(x,2R)}v_\lambda\|_{L^2([0,R^2]\times\mathbf{R}^3)}\\
&\leq \frac{C}{\lambda^{\frac12}}\|v_\lambda\|_{L^2(P(x,2R))}.
\end{aligned}
$$

This concludes the proof of Lemma 2.3. □

3. Proof of Theorem 2

In this paragraph we shall check that the vector field $u_{0,\varepsilon}$ introduced in Theorem 2 satisfies the nonlinear smallness assumption of Theorem 1, and we shall also show that its $\dot B^{-1}_{\infty,\infty}$ norm is equivalent to $(-\log\varepsilon)^{\frac15}$. Let us start by proving the following lemma.

Lemma 3.1. *Let $f\in\mathcal{S}(\mathbf{R}^3)$ be given and $\sigma\in\Big]0,3\Big(1-\dfrac1p\Big)\Big[$. There is a constant $C>0$ such that for any $\varepsilon\in]0,1[$, the function*

$$
f_\varepsilon(x)\overset{\text{def}}{=}e^{i\frac{x_3}{\varepsilon}}f\Big(x_1,\frac{x_2}{\varepsilon^\alpha},x_3\Big)
$$

satisfies, for all $p\geq1$,

$$
\|f_\varepsilon\|_{\dot B^{-\sigma}_{p,1}}\leq C\varepsilon^{\sigma+\frac{\alpha}{p}}\quad\text{and}\quad\|f_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}}\geq C^{-1}\varepsilon^\sigma.
$$

Proof. Let us recall that

$$
\|f_\varepsilon\|_{\dot B^{-\sigma}_{p,1}}=\sum_{j\in\mathbf{Z}}2^{-j\sigma}\|\Delta_jf_\varepsilon\|_{L^p}.
$$

We shall start by estimating the high frequencies, defining a threshold $j_0\geq0$ to be determined later on. We have

$$
\begin{aligned}
\sum_{j\geq j_0}2^{-j\sigma}\|\Delta_jf_\varepsilon\|_{L^p} &\leq C2^{-j_0\sigma}\|f_\varepsilon\|_{L^p}\\
&\leq C2^{-j_0\sigma}\varepsilon^{\frac{\alpha}{p}}\|f\|_{L^p}.
\end{aligned}\tag{3.1}
$$

On the other hand, we have

$$
\begin{aligned}
\Delta_jf_\varepsilon(x) &= 2^{3j}\int_{\mathbf{R}^3}h(2^j(x-y))f_\varepsilon(y)\,dy\\
&= 2^{3j}\int_{\mathbf{R}^3}h(2^j(x-y))e^{i\frac{y_3}{\varepsilon}}f(y_1,\frac{y_2}{\varepsilon^\alpha},y_3)\,dy,
\end{aligned}
$$

so noticing that $e^{i\frac{y_3}{\varepsilon}}=(-i\varepsilon\partial_3)^N(e^{i\frac{y_3}{\varepsilon}})$, we get for any $N\in\mathbf{N}$,

$$
\Delta_jf_\varepsilon(x)=(i\varepsilon)^N2^{3j}\sum_{\ell=0}^NC_N^\ell\int e^{i\frac{y_3}{\varepsilon}}\partial_3^\ell\big(h(2^j(x-y))\big)\partial_3^{N-\ell}f(y_1,\frac{y_2}{\varepsilon^\alpha},y_3)\,dy.
$$

Young's inequality enables us to infer that

$$2^{-j\sigma}\|\Delta_j f_\varepsilon\|_{L^p} \leq C\varepsilon^N 2^{j(3-\sigma)} \min\Big(\sum_{\ell=0}^{N} 2^{j(\ell-3)}\varepsilon^{\frac{\alpha}{p}}, \sum_{\ell=0}^{N} 2^{j(\ell-\frac{3}{p})}\varepsilon^{\alpha}\Big).$$

So, choosing N large enough and since $\sigma < 3(1-\frac{1}{p})$, we get

$$\begin{aligned}\sum_{j\leq j_0} 2^{-j\sigma}\|\Delta_j f_\varepsilon\|_{L^p} &\leq \sum_{j\leq 0} 2^{-j\sigma}\|\Delta_j f_\varepsilon\|_{L^p} + \sum_{j\geq 0} 2^{-j\sigma}\|\Delta_j f_\varepsilon\|_{L^p}\\ &\leq C\sum_{j<0} 2^{-j(\sigma+3(1-\frac{1}{p}))}\varepsilon^{N+\alpha} + C\sum_{0\leq j\leq j_0} 2^{j(N-\sigma)}\varepsilon^{N+\frac{\alpha}{p}}\\ &\leq C\varepsilon^{N+\alpha} + C2^{j_0(N-\sigma)}\varepsilon^{N+\frac{\alpha}{p}}. \end{aligned} \tag{3.2}$$

Finally choosing $2^{-j_0} = \varepsilon$ in (3.1) and (3.2) ends the proof of the bound on $\|f_\varepsilon\|_{\dot B^{-\sigma}_{p,1}}$.

In order to go from below $\|f_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}}$, let us first observe that, as the space of smooth compactly supported functions is dense in $\mathcal{S}$ and the Fourier transform is continuous on $\mathcal{S}$, for any positive η, a function g exists, the Fourier transform of which is smooth and compactly supported such that, denoting as before $g_\varepsilon(x) = e^{i\frac{x_3}{\varepsilon}} g(x_1, \dfrac{x_2}{\varepsilon^\alpha}, x_3)$,

$$\|f_\varepsilon - g_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}} \leq \eta\varepsilon^\sigma \quad\text{and}\quad \|f-g\|_{L^\infty} \leq \eta. \tag{3.3}$$

As the support of the Fourier transform of g is included in the ball $B(0,R)$ for some positive R, that of $g(x_1, \varepsilon^{-\alpha}x_2, x_3)$ is included in the ball $B(0, R\varepsilon^{-\alpha})$. Then the support of $\mathcal{F}g_\varepsilon$ is included in the ball $B(\varepsilon^{-1}(0,0,1), \varepsilon^{-\alpha}R)$. This ball is included in $\varepsilon^{-1}\mathcal{C}$ for some ring $\mathcal{C}$. Thanks to (1.1) we shall use the heat flow. Let us write that

$$\begin{aligned}\|g_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}} &\sim \sup_{t>0} t^{\frac{\sigma}{2}}\|\mathrm{S}(t)g_\varepsilon\|_{L^\infty}\\ &\geq C\varepsilon^\sigma\|\mathrm{S}(\varepsilon^2)g_\varepsilon\|_{L^\infty}.\end{aligned}$$

For any function h such that the support of $\widehat{h}$ is included in $\varepsilon^{-1}\mathcal{C}$, we have

$$\|\mathcal{F}^{-1}(e^{\varepsilon^2|\xi|^2}h)\|_{L^\infty} \leq C\|h\|_{L^\infty}.$$

Applied with $h = \mathrm{S}(\varepsilon^2)g_\varepsilon$, this inequality gives

$$\|g_\varepsilon\|_{L^\infty} \leq C\|\mathrm{S}(\varepsilon^2)g_\varepsilon\|_{L^\infty} \quad\text{and thus}\quad \|g_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}} \geq C^{-1}\varepsilon^\sigma\|g_\varepsilon\|_{L^\infty} = C^{-1}\varepsilon^\sigma\|g\|_{L^\infty}.$$

Now let us write that

$$\begin{aligned}\|f_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}} &\geq \|g_\varepsilon\|_{\dot B^{-\sigma}_{\infty,\infty}} - \eta\varepsilon^\sigma\\ &\geq C^{-1}\varepsilon^\sigma(\|f\|_{L^\infty} - 2\eta).\end{aligned}$$

This ends the proof of the lemma. □

This enables us to infer immediately the following corollary.

Corollary 3.1. *A constant C exists such that, for any $p \geq 1$, we have*

$$\|u_{0,\varepsilon}\|_{\dot B^{-1}_{p,1}} \leq C\varepsilon^{\frac{\alpha}{p}}(-\log\varepsilon)^{\frac{1}{5}} \quad\textit{and}\quad \|u_{0,\varepsilon}\|_{\dot B^{-1}_{\infty,\infty}} \geq C^{-1}(-\log\varepsilon)^{\frac{1}{5}}.$$

The last verification to be made is the nonlinear assumption (1.4). It is based on the following lemma.

Lemma 3.2. *There is a constant C such that the following result holds. Let f and g be in $\dot B^{-1}_{\infty,2}\cap \dot H^{-1}$. Then we have*

$$\|\mathbf{P}(\mathrm{S}(t)f\mathrm{S}(t)g)\|_E \leq C\Big(\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}\Big)^{\frac23}\Big(\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}\Big)^{\frac13}$$

Proof. As the Leray projection $\mathbf{P}$ is continuous on E, it is enough to prove the lemma without $\mathbf{P}$. Using Bernstein's estimate, we get that

$$\|\Delta_j(\mathrm{S}(t)f\mathrm{S}(t)g)\|_{L^\infty}\leq C2^{3j}\|\mathrm{S}(t)f\mathrm{S}(t)g\|_{L^1}.$$

Then, using the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} E_j &\stackrel{\mathrm{def}}{=} \|\Delta_j(\mathrm{S}(t)f\mathrm{S}(t)g)\|_{L^1(\mathbf{R}^+;L^\infty)}+\|t^{\frac12}\Delta_j(\mathrm{S}(t)f\mathrm{S}(t)g)\|_{L^2(\mathbf{R}^+;L^\infty)}\\ &\leq C2^{3j}\Big(\|\mathrm{S}(t)f\|_{L^2(\mathbf{R}^+;L^2)}+\|t^{\frac12}\mathrm{S}(t)f\|_{L^\infty(\mathbf{R}^+;L^2)}\Big)\|\mathrm{S}(t)g\|_{L^2(\mathbf{R}^+;L^2)}.\end{aligned}$$

So using (1.1), we deduce that

$$E_j\leq C2^{3j}\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}. \tag{3.4}$$

Let us observe that we also have

$$\begin{aligned} E_j &\leq C\Big(\|\mathrm{S}(t)f\|_{L^2(\mathbf{R}^+;L^\infty)}+\|t^{\frac12}\mathrm{S}(t)f\|_{L^\infty(\mathbf{R}^+;L^\infty)}\Big)\|\mathrm{S}(t)g\|_{L^2(\mathbf{R}^+;L^\infty)}\\ &\leq C\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}.\end{aligned}$$

Using this estimate for high frequencies and (3.4) for low frequencies, we get, for any j_0 in $\mathbf{Z}$,

$$\begin{aligned} \|\mathrm{S}(t)f\mathrm{S}(t)g\|_E &= \sum_j 2^{-j}E_j\\ &\leq C\Big(\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}\sum_{j\leq j_0}2^{2j}+\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}\sum_{j\geq j_0}2^{-j}\Big)\\ &\leq C\Big(\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}2^{2j_0}+\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}2^{-j_0}\Big).\end{aligned}$$

Choosing j_0 such that

$$2^{3j_0}\sim\frac{\|f\|_{\dot B^{-1}_{\infty,2}}\|g\|_{\dot B^{-1}_{\infty,2}}}{\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}}$$

gives the result. □

Finally we are ready to prove estimate (1.4). Note that the proof relies heavily on the special structure of the nonlinear term in the system. We indeed start by remarking that there is no derivative in the third direction since $u_{0,\varepsilon}$ does not have a third component. Then denoting $u_F(t)=\mathrm{S}(t)u_{0,\varepsilon}$, we have by an easy computation and with the notation as in Lemma 3.1,

$$u_F^1\partial_1u_F^1+u_F^2\partial_2u_F^1=\frac{1}{\varepsilon^2}(-\log\varepsilon)^{\frac25}\mathrm{S}(t)f_\varepsilon\mathrm{S}(t)g_\varepsilon\quad\text{and}$$

$$u_F^1\partial_1u_F^2+u_F^2\partial_2u_F^2=\frac{1}{\varepsilon^{2-\alpha}}(-\log\varepsilon)^{\frac25}\mathrm{S}(t)\widetilde f_\varepsilon\mathrm{S}(t)\widetilde g_\varepsilon,$$

where f, $\widetilde{f}$, g and $\widetilde{g}$ are smooth functions. The result follows immediately using Lemmas 3.2 and Corollary 3.1 together with the fact that the Leray projection onto divergence free vector fields maps continuously E into E. □

4. Stability results

In this section we shall prove Proposition 1.1, as well as Theorems 3 and 4 stated in the introduction. The proof of Proposition 1.1 is rather easy and is given for the sake of completeness in the next section. The proof of Theorem 3 is the object of Section 4.2 below. Finally Theorem 4 is an easy consequence of the methods developped in the proof of Theorem 3 and is postponed to the end of Section 4.2.

4.1. **Proof of Proposition 1.1.**

Proposition 1.1 is an immediate consequence of the following more general result.

Proposition 4.1. *Let $X = (x_1, \ldots, x_K)$ be a family of K distinct points, and $(u_{0,1}, \ldots, u_{0,K})$ a family of divergence free vector fields in $\dot{H}^{\frac{1}{2}}$, each generating a unique, global solution to the Navier-Stokes equations. Then there is $\Lambda_0 > 0$ such that for any $\Lambda \geq \Lambda_0$, the vector field*

$$u_{0,\Lambda} \stackrel{\text{def}}{=} \sum_{J \in \{1,\ldots,K\}} T_\Lambda^J(u_{0,J})$$

also generates a unique, global solution to the Navier-Stokes equations.

Proof. The proof of that result is similar to methods of [3] concerning profile decompositions (see [10] for the case of the Navier-Stokes equations). Let us denote by u_J the solution of (NS) associated with $u_{0,J}$, and define

$$u_{\Lambda,J}(t,x) = \Lambda u_J\big(\Lambda^2 t, \Lambda(x - x_J)\big),$$

which solves (NS) with data $u_{0,\Lambda,J} = T_\Lambda^J(u_{0,J})$. Then we define the solution u_Λ of (NS) with data $u_{0,\Lambda}$, which a priori exists only for a short time. We can decompose

$$u_\Lambda = \sum_{J \in \{1,\ldots,K\}} u_{\Lambda,J} + R_\Lambda = u_\Lambda^{(1)} + R_\Lambda,$$

and R_Λ solves the following perturbed Navier-Stokes equation

$$\partial_t R_\Lambda - \Delta R_\Lambda + \mathbf{P}(R_\Lambda \cdot \nabla R_\Lambda) + \mathbf{P}(u_\Lambda^{(1)} \cdot \nabla R_\Lambda) + \mathbf{P}(R_\Lambda \cdot \nabla u_\Lambda^{(1)}) = F_\Lambda$$

with initial data zero, and where

$$F_\Lambda = -\mathbf{P} \sum_{J \neq J'} u_{\Lambda,J} \cdot \nabla u_{\Lambda,J'}.$$

It is not difficult to prove (see for instance [10], Proposition A.2) that R_Λ is globally defined and unique in $L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+; \dot{H}^{\frac{3}{2}})$ under the condition that

$$\|F_\Lambda\|_{L^2(\mathbf{R}^+; \dot{H}^{-\frac{1}{2}})} \leq C_0^{-1} \exp\left(-C_0 \|u_\Lambda^{(1)}\|^4_{L^4(\mathbf{R}^+; \dot{H}^1)}\right), \tag{4.1}$$

so let us compute $\|F_\Lambda\|_{L^2(\mathbf{R}^+; \dot{H}^{-\frac{1}{2}})}$ and $\|u_\Lambda^{(1)}\|_{L^4(\mathbf{R}^+; \dot{H}^1)}$.

As mentioned in the introduction, any global solution belongs to $L^4(\mathbf{R}^+;\dot H^1)$. Thus, by definition of $u_\Lambda^{(1)}$, we have

$$\|u_\Lambda^{(1)}\|_{L^4(\mathbf{R}^+;\dot H^1)} \leq \sum_{J\in\{1,\cdots,K\}} \|u_{\Lambda,J}\|_{L^4(\mathbf{R}^+;\dot H^1)}.$$

Using a scaling argument, we infer

$$\begin{aligned}\|u_\Lambda^{(1)}\|_{L^4(\mathbf{R}^+;\dot H^1)} &\leq \sum_{J\in\{1,\cdots,K\}} \|u_J\|_{L^4(\mathbf{R}^+;\dot H^1)}\\ &\leq K \sup_{J\in\{1,\cdots,K\}} C_J \quad\text{with}\\ C_J &\stackrel{\text{def}}{=} \|u_J\|_{L^\infty(\mathbf{R}^+;\dot H^{\frac12})} + \|u_J\|_{L^2(\mathbf{R}^+;\dot H^{\frac32})}.\end{aligned}\tag{4.2}$$

In order to estimate $\|F_\Lambda\|_{L^2(\mathbf{R}^+;\dot H^{-\frac12})}$, let us start by noticing that F_Λ is bounded uniformly in Λ in the space $L^{\frac43}(\mathbf{R}^+;L^2)$, by a constant depending on K and on the initial data. Indeed Hölder's inequality and Sobolev embeddings give

$$\begin{aligned}\|u_{\Lambda,J}\cdot\nabla u_{\Lambda,J'}\|_{L^{\frac43}(\mathbf{R}^+;L^2)} &\leq \|u_{\Lambda,J}\|_{L^4(\mathbf{R}^+;L^6)}\|\nabla u_{\Lambda,J'}\|_{L^2(\mathbf{R}^+;L^3)}\\ &\leq C\|u_{\Lambda,J}\|_{L^4(\mathbf{R}^+;\dot H^1)}\|\nabla u_{\Lambda,J'}\|_{L^2(\mathbf{R}^+;\dot H^{\frac12})},\end{aligned}$$

so that by scale invariance

$$\begin{aligned}\|F_\Lambda\|_{L^{\frac43}(\mathbf{R}^+;L^2)} &\leq C\|u_J\|_{L^4(\mathbf{R}^+;\dot H^1)}\|\nabla u_{J'}\|_{L^2(\mathbf{R}^+;\dot H^{\frac12})}\\ &\leq CK^2\sup_{J,J'}(C_JC_{J'}).\end{aligned}$$

So by interpolation it is enough to prove that

$$\lim_{\Lambda\to\infty}\|F_\Lambda\|_{L^4(\mathbf{R}^+;\dot H^{-1})} = 0.\tag{4.3}$$

Let $J\neq J'$ be two integers in $\{1,\ldots,K\}$, and let $\varepsilon>0$ be given. There exists a positive R and two vector fields ψ_ε and φ_ε in $\mathcal{D}(\mathbf{R}\times B(0,R))$ such that

$$\|\psi_\varepsilon - u_J\|_{L^4(\mathbf{R}^+;\dot H^1)} + \|\varphi_\varepsilon - u_{J'}\|_{L^4(\mathbf{R}^+;\dot H^1)} \leq \varepsilon.$$

The support of $T_{\Lambda,J}\psi_\varepsilon$ (resp. $T_{\Lambda,J'}\varphi_\varepsilon$) is included in the ball $B(x_J,R\Lambda^{-1})$ (resp. $B(x_J,R\Lambda^{-1})$). Thus we have

$$\Lambda\geq 4\delta^{-1}R \Longrightarrow T_{\Lambda,J}\psi_\varepsilon\, T_{\Lambda,J'}\varphi_\varepsilon = 0.\tag{4.4}$$

Then Sobolev embeddings as above give the estimate

$$\begin{aligned}&\|u_J\otimes(\varphi_\varepsilon - u_{J'})\|_{L^4(\mathbf{R}^+;L^2)} + \|(\psi_\varepsilon - u_J)\otimes\varphi_\varepsilon\|_{L^4(\mathbf{R}^+;L^2)}\\ &\quad\leq C\Big(\|u_J\|_{L^\infty(\mathbf{R}^+;\dot H^{\frac12})}\|\varphi_\varepsilon - u_{J'}\|_{L^4(\mathbf{R}^+;\dot H^1)} + \|\varphi_\varepsilon\|_{L^\infty(\mathbf{R}^+;\dot H^{\frac12})}\|\psi_\varepsilon - u_J\|_{L^4(\mathbf{R}^+;\dot H^1)}\Big),\end{aligned}$$

so that, using the scaling,

$$\|u_{\Lambda,J}\otimes(T_{\Lambda,J'}\varphi_\varepsilon - u_{J'})\|_{L^4(\mathbf{R}^+;L^2)} + \|(T_{\Lambda,J}\psi_\varepsilon - u_J)\otimes T_{\Lambda,J'}\varphi_\varepsilon\|_{L^4(\mathbf{R}^+;L^2)} \leq C(C_J+C_{J'})\varepsilon.\tag{4.5}$$

Using (4.4), it follows that for Λ large enough,

$$\|F_\Lambda\|_{L^4(\mathbf{R}^+;\dot H^{-1})} \leq CK^2\varepsilon \sup_{J\in\{1,\cdots,K\}} C_J,$$

and (4.3) is proved. Plugging together that estimate with (4.2) gives (4.1) for Λ large enough, and Proposition 4.1 is proved. □

4.2. **Proof of Theorems 3 and 4.** Before starting the proofs, let us make a few comments on the transformation $T_{\Lambda,X}$ and state its main properties. In all that follows, we shall consider only the action of $T_{\Lambda,X}$ on functions compactly supported in Q. First, one can notice that if the family X of points satisfies (1.5), then if $\Lambda \geq 4\delta^{-1}$,

$$\operatorname{supp} T^J_\Lambda f \subset Q^J_\Lambda \stackrel{\text{def}}{=} \Big\{ x \Big/ d(x,x_J) \leq \Lambda^{-1} \Big\} \subset Q^J_\delta \stackrel{\text{def}}{=} \Big\{ x \Big/ d(x,x_J) \leq \frac{1}{4}\delta \Big\}.$$

This implies immediately that

$$\|T_{\Lambda,X} f\|_{L^p} = \Lambda^{1-\frac{3}{p}} K^{\frac{1}{p}} \|f\|_{L^p}. \tag{4.6}$$

Then let us state the following two lemmas, which are crucial for the proof of Theorem 3 and will be proved in Section 4.2.2.

Lemma 4.1. *Let $K \geq 1$ be an integer and $\delta > 0$ a real number. There is a constant $C_{K,\delta}$ such that the following results hold. Let r be in $[1,\infty]$ and consider a family X as in Definition 1.3. Then for any real number Λ in $2^{\mathbf{N}}$ greater than $4\delta^{-1}$ and for any $f \in \mathcal{D}(Q)$, we have*

$$\|f\|_{\dot{B}^{-1}_{\infty,r}} - C_{K,\delta}\Lambda^{-2}\|f\|_{\dot{B}^{-3}_{\infty,\infty}} \leq \|T_{\Lambda,X} f\|_{\dot{B}^{-1}_{\infty,r}} \leq \|f\|_{\dot{B}^{-1}_{\infty,r}} + C_{K,\delta}\Lambda^{-2}\|f\|_{\dot{B}^{-3}_{\infty,\infty}}.$$

Moreover the following estimate holds, where the constant C is universal:

$$\|T_{\Lambda,X} f\|_{\dot{H}^{-1}} \leq C\sqrt{K}\Lambda^{-\frac{3}{2}}\|f\|_{\dot{H}^{-1}}. \tag{4.7}$$

Remark Let us point out that L^1 is continuously included in $\dot{B}^{-3}_{\infty,\infty}$.

Lemma 4.2. *Let $K \geq 1$ be an integer and $\delta > 0$ a real number. There is a constant $C_{K,\delta}$ such that the following results hold. Consider a family X as in Definition 1.3. Then for any real number Λ in $2^{\mathbf{N}}$ greater than $4\delta^{-1}$ and for all divergence free vector fields f and g in $\mathcal{D}(Q)$, we have*

$$\|\mathbf{P}(\mathrm{S}(t)T_{\Lambda,X} f \cdot \nabla \mathrm{S}(t)T_{\Lambda,X} g)\|_E \leq \|\mathbf{P}(\mathrm{S}(t) f \cdot \nabla \mathrm{S}(t) g)\|_E + C_{K,\delta}\Lambda^{-3}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}.$$

4.2.1. *End of the proof of Theorem 3.* Let us consider a vector field $u_0 \in \mathcal{D}(Q)$ satisfying (1.6) for some $\eta \in]0,1[$. We know from Lemma 4.1 that for any $r \in [1,\infty]$ and any $\eta \in]0,1[$, for any Λ greater than some Λ_0 (depending on K and δ only), we have

$$\|u_0\|_{\dot{B}^{-1}_{\infty,r}} - \eta \leq \|T_{\Lambda,X} u_0\|_{\dot{B}^{-1}_{\infty,r}} \leq \|u_0\|_{\dot{B}^{-1}_{\infty,r}} + \eta. \tag{4.8}$$

Next let us consider the smallness condition (1.3). By Lemma 4.2 we know that as soon as Λ_0 is large enough, then for any $\Lambda \geq \Lambda_0$,

$$\|\mathbf{P}(\mathrm{S}(t)T_{\Lambda,X} u_0 \cdot \nabla \mathrm{S}(t)T_{\Lambda,X} u_0)\|_E \leq \|\mathbf{P}(\mathrm{S}(t) u_0 \cdot \nabla \mathrm{S}(t) u_0)\|_E + \eta.$$

So we infer that

$$\begin{aligned} \|\mathbf{P}(\mathrm{S}(t)T_{\Lambda,X} u_0 \cdot \nabla \mathrm{S}(t)T_{\Lambda,X} u_0)\|_E &\leq C_0^{-1} \exp\left(-C_0\big(\|u_0\|_{\dot{B}^{-1}_{\infty,2}} + \eta\big)^4\right) \\ &\leq C_0^{-1} \exp\left(-C_0\|T_{\Lambda,X} u_0\|^4_{\dot{B}^{-1}_{\infty,2}}\right) \end{aligned}$$

due to (4.8). So Theorem 3 is proved, up to the proof of Lemmas 4.1 and 4.2 which is the object of the coming section. □

4.2.2. *The properties of $T_{\Lambda,X}$.* In this section, we are going to prove the properties of the transformation $T_{\Lambda,X}$ required in the proof of Theorem 3, namely Lemmas 4.1 and 4.2. Before starting the proofs, let us give some more notation and prove preliminary results which will be used many times in the rest of this section.

We define

$$\widetilde{Q}_\delta = \bigcup_{J\in\{1,\dots,K\}} \widetilde{Q}^J_\delta, \quad \text{where} \quad \widetilde{Q}^J_\delta \stackrel{\text{def}}{=} \Big\{ x \Big/ d(x, Q^J_\delta) \leq \frac{1}{32}\delta \Big\}, \tag{4.9}$$

and we notice that this is a disjoint reunion.

The proof of Lemmas 4.1 and 4.2 relies on the fact that the Littlewood-Paley theory is almost local. More precisely, let us recall Lemma 9.2.2 of [5].

Lemma 4.3. *For any positive integer N and any real number r, a constant C_N exists such that the following result holds. Let F be a closed subset of $\mathbf{R}^3$ and u a distribution in $\dot B^r_{\infty,\infty}$ supported in F; then for any couple (j,h) in $\mathbf{Z}\times\mathbf{R}^+$ such that 2^j and 2^jh are greater than 1, we have*

$$\|\Delta_j u\|_{L^\infty({}^cF_h)} \leq C_N 2^{-jr}(2^jh)^{-N}\|u\|_{\dot B^r_{\infty,\infty}},$$

where $F_h = \Big\{ x\in\mathbf{R}^3 \Big/ d(x,F)\leq h\Big\}$.

¿From this lemma, we deduce the following corollary.

Corollary 4.1. *Let K, δ and X be as in Definition 1.3 and let $M\in\mathbf{N}$ be given. There is a constant C_M (depending only on M) such that the following holds. For any $\Lambda\geq 4\delta^{-1}$, for any distribution f in $\dot B^{-3}_{\infty,\infty}$, compactly supported in Q and for any $J\in\{1,\dots,K\}$, one has the following estimates:*

$$\forall j\in\mathbf{Z}, \quad \|\Delta_j T^J_\Lambda f\|_{L^\infty({}^c\widetilde{Q}^J_\delta)} \leq C_M \delta^{-(M+3)}\Lambda^{-2}2^{-jM}\|f\|_{\dot B^{-3}_{\infty,\infty}}. \tag{4.10}$$

Moreover there is a universal constant C such that for any positive R,

$$\|f\|_{\dot B^{-1}_{\infty,r}} \leq \Big\| \Big(2^{-j}\|\Delta_j f\|_{L^\infty(Q_R)}\Big)_j\Big\|_{\ell^r} + CR^{-2}\|f\|_{\dot B^{-3}_{\infty,\infty}}, \tag{4.11}$$

where $Q_R = \Big\{x\in\mathbf{R}^3 \Big/ d(x,Q)\leq R\Big\}$.

Proof. The first inequality is obvious when j is negative or when $2^j\delta\leq 1$. Indeed we have the scaling property

$$\Delta_j\left(f\big(\Lambda(\cdot - x_J)\big)\right)(x) = (\Delta_{j-\log_2\Lambda} f)(\Lambda(x-x_J)), \tag{4.12}$$

so that for any $s\in\mathbf{R}$,

$$\|f(\Lambda(\cdot - x_J))\|_{\dot B^s_{\infty,\infty}} = \Lambda^s\|f\|_{\dot B^s_{\infty,\infty}}$$

Thus let us assume that 2^j and $2^j\delta$ are greater than 1. Using Lemma 4.3, we get

$$\begin{aligned}\|\Delta_j T^J_\Lambda f\|_{L^\infty({}^c\widetilde{Q}^J_\delta)} &\leq C_M 2^{3j}(2^j\delta)^{-M}\|T^J_\Lambda f\|_{\dot B^{-3}_{\infty,\infty}}\\ &\leq C_M 2^{-j(M-3)}\delta^{-M}\Lambda^{-2}\|f\|_{\dot B^{-3}_{\infty,\infty}}.\end{aligned}$$

In order to prove the second inequality, let us note that, thanks to the triangle inequality and to the fact that $\|\cdot\|_{\ell^r} \leq \|\cdot\|_{\ell^1}$, we have for any integer j_0,

$$\|f\|_{\dot{B}^{-1}_{\infty,r}} \leq \Big\|\Big(2^{-j}\|\Delta_j f\|_{L^\infty(Q_R)}\Big)_j\Big\|_{\ell^r} + \sum_{j<j_0} 2^{-j}\|\Delta_j f\|_{L^\infty(\mathbf{R}^3)} + \sum_{j\geq j_0} 2^{-j}\|\Delta_j f\|_{L^\infty({}^c Q_R)}.$$

Lemma 4.3 claims in particular that, if $2^j R \geq 1$,

$$\|\Delta_j f\|_{L^\infty({}^c Q_R)} \leq C R^{-3}\|f\|_{\dot{B}^{-3}_{\infty,\infty}}.$$

Thus, if j_0 is such that $2^{j_0}R \geq 1$, we have, by definition of the norm of $\dot{B}^{-1}_{\infty,\infty}$,

$$\begin{aligned}\|f\|_{\dot{B}^{-1}_{\infty,r}} &\leq \Big\|\Big(2^{-j}\|\Delta_j f\|_{L^\infty(Q_R)}\Big)_j\Big\|_{\ell^r} + \Big(\sum_{j<j_0} 2^{2j} + R^{-3}\sum_{j\geq j_0} 2^{-j}\Big)\|f\|_{\dot{B}^{-3}_{\infty,\infty}}\\ &\leq \Big\|\Big(2^{-j}\|\Delta_j f\|_{L^\infty(Q_R)}\Big)_j\Big\|_{\ell^r} + (2^{2j_0} + 2^{-j_0}R^{-3})\|f\|_{\dot{B}^{-3}_{\infty,\infty}}.\end{aligned}$$

Choosing $2^{j_0} \sim R^{-1}$ gives the result. □

4.2.3. *Proof of Lemma 4.1.* We shall start by proving the second inequality, namely that

$$\|T_{\Lambda,X} f\|_{\dot{B}^{-1}_{\infty,r}} \leq \|f\|_{\dot{B}^{-1}_{\infty,r}} + C_{K,\delta}\Lambda^{-2}\|f\|_{\dot{B}^{-3}_{\infty,\infty}}. \tag{4.13}$$

Let us start with low frequencies. We can write that

$$\begin{aligned}\sum_{j<0} 2^{-j}\|\Delta_j T_{\Lambda,X} f\|_{L^\infty} &\leq \sum_{j<0} 2^{-j}\sum_J \|\Delta_j T^J_\Lambda f\|_{L^\infty}\\ &\leq \sum_J\Big(\sum_{j<0} 2^{2j}\Big)\|T^J_\Lambda f\|_{\dot{B}^{-3}_{\infty,\infty}}.\end{aligned}$$

Using the scaling equality (4.12) we get that

$$\sum_{j<0} 2^{-j}\|\Delta_j T_{\Lambda,X} f\|_{L^\infty} \leq K\Lambda^{-2}\|f\|_{\dot{B}^{-3}_{\infty,\infty}}. \tag{4.14}$$

Now let us concentrate on the high frequencies. Recalling the definition given in (4.9), let us start by considering the case when $x \notin \widetilde{Q}_\delta$. Using Inequality (4.10) of Corollary 4.1, we can write (choosing $M = 0$)

$$\begin{aligned}\|\Delta_j T_{\Lambda,X} f\|_{L^\infty({}^c\widetilde{Q}_\delta)} &\leq \sum_J \|\Delta_j T^J_\Lambda f\|_{L^\infty({}^c\widetilde{Q}_\delta)}\\ &\leq CK\delta^{-3}\Lambda^{-2}\|f\|_{\dot{B}^{-3}_{\infty,\infty}}.\end{aligned}$$

Then we infer that

$$\sum_{j\geq 0} 2^{-j}\|\Delta_j T_{\Lambda,X} f\|_{L^\infty({}^c\widetilde{Q}_\delta)} \leq CK\delta^{-3}\Lambda^{-2}\|f\|_{\dot{B}^{-3}_{\infty,\infty}}. \tag{4.15}$$

Now let us consider the case when $x \in \widetilde{Q}_\delta$. We can write

$$\|\Delta_j T_{\Lambda,X} f\|_{L^\infty(\widetilde{Q}_\delta)} \leq \sup_J \|\Delta_j T_{\Lambda,X} f\|_{L^\infty(\widetilde{Q}^J_\delta)},$$

and let us fix some $J \in \{1,\ldots,K\}$. We recall that

$$T_{\Lambda,X} f = T^J_\Lambda f + \sum_{J'\neq J} T^{J'}_\Lambda f,$$

and let us start with the estimate of $T_{\Lambda,X}f - T^J_\Lambda f$. We have

$$\|\Delta_j(T_{\Lambda,X}f - T^J_\Lambda f)\|_{L^\infty(\widetilde Q^J_\delta)} \leq \sum_{J'\neq J} \|\Delta_j T^{J'}_\Lambda f\|_{L^\infty({}^c\widetilde Q^{J'}_\delta)}.$$

Using Inequality (4.10) of Corollary 4.1, we get that

$$\|\Delta_j(T_{\Lambda,X}f - T^J_\Lambda f)\|_{L^\infty({}^c\widetilde Q^J_\delta)} \leq CK\delta^{-3}\Lambda^{-2}\|f\|_{\dot B^{-3}_{\infty,\infty}}.$$

Thus we infer

$$\sum_{j\geq 0} 2^{-j}\|\Delta_j(T_{\Lambda,X}f - T^J_\Lambda f)\|_{L^\infty(\widetilde Q^J_\delta)} \leq CK\delta^{-3}\Lambda^{-2}\|f\|_{\dot B^{-3}_{\infty,\infty}}. \tag{4.16}$$

Now let us examine the term $\|T^J_\Lambda f\|_{L^\infty(\widetilde Q^J_\delta)}$. From (4.12) we get

$$\begin{aligned} \Big\|\Big(\mathbf{1}_{j\geq 0}2^{-j}\|\Delta_j T^J_\Lambda f\|_{L^\infty(\widetilde Q^J_\delta)}\Big)_j\Big\|_{\ell^r} &\leq \Lambda\Big\|\Big(\mathbf{1}_{j\geq 0}2^{-j}\|\Delta_{j-\log_2\Lambda} f\|_{L^\infty(\mathbf{R}^3)}\Big)_j\Big\|_{\ell^r}\\ &\leq \|f\|_{\dot B^{-1}_{\infty,r}}. \end{aligned}$$

Once noticed that $\|\cdot\|_{\ell^r} \leq \|\cdot\|_{\ell^1}$, we plug together that estimate with (4.14), (4.15) and (4.16) to conclude the proof of (4.13).

Let us bound from below $\|T_{\Lambda,X}f\|_{\dot B^{-1}_{\infty,r}}$. As $\|g\|_{L^\infty(\widetilde Q_\delta)} = \sup_J \|g\|_{L^\infty(\widetilde Q^J_\delta)}$, we have

$$\begin{aligned} \|T_{\Lambda,X}f\|_{\dot B^{-1}_{\infty,r}} &\geq \Big\|\Big(2^{-j}\|\Delta_j T_{\Lambda,X}f\|_{L^\infty(\widetilde Q_\delta)}\Big)_j\Big\|_{\ell^r}\\ &\geq \Big\|\Big(2^{-j}\sup_J \|\Delta_j T_{\Lambda,X}f\|_{L^\infty(\widetilde Q^J_{k\delta})}\Big)_j\Big\|_{\ell^r}\\ &\geq \Big\|\Big(2^{-j}\|\Delta_j T_{\Lambda,X}f\|_{L^\infty(\widetilde Q^{J_0}_\delta)}\Big)_j\Big\|_{\ell^r} \end{aligned}$$

for some J_0 in $\{1,\ldots,K\}$. Using the fact that $\|\cdot\|_{\ell^r} \leq \|\cdot\|_{\ell^1}$, we can write that

$$\begin{aligned} \|T_{\Lambda,X}f\|_{\dot B^{-1}_{\infty,r}} \geq{}& \Big\|\Big(2^{-j}\|\Delta_j T^{J_0}_\Lambda f\|_{L^\infty(\widetilde Q^{J_0}_\delta)}\Big)_j\Big\|_{\ell^r}\\ &- \sum_{j<0} 2^{-j}\Big\|\Delta_j T^{J_0}_\Lambda f\Big\|_{L^\infty(\widetilde Q^{J_0}_\delta)} - \sum_{j\geq 0} 2^{-j}\Big\|\Delta_j(T_{\Lambda,X}f - T^{J_0}_\Lambda f)\Big\|_{L^\infty(\widetilde Q^{J_0}_\delta)}. \end{aligned}$$

Using (4.14) and (4.16), we infer that

$$\|T_{\Lambda,X}f\|_{\dot B^{-1}_{\infty,r}} \geq \Big\|\Big(2^{-j}\|\Delta_j T^{J_0}_{\Lambda,X} f\|_{L^\infty(\widetilde Q^{J_0}_\delta)}\Big)_j\Big\|_{\ell^r} - C_{K,\delta}\Lambda^{-2}\|f\|_{\dot B^{-3}_{\infty,\infty}}. \tag{4.17}$$

By scaling and translation, we have

$$\Big\|\Big(2^{-j}\|\Delta_j T^{J_0}_{\Lambda,X} f\|_{L^\infty(\widetilde Q^{J_0}_k)}\Big)_j\Big\|_{\ell^r} = \Big\|\Big(2^{-j}\|\Delta_j f\|_{L^\infty(Q_{\Lambda,\delta})}\Big)_j\Big\|_{\ell^r}$$

where $Q_{\Lambda,\delta}$ is the cube of size $2\Lambda\delta$. Using (4.11) with $R = 2\delta\Lambda$ and (4.17), we infer that

$$\|T_{\Lambda,X}f\|_{\dot B^{-1}_{\infty,r}} \geq \|f\|_{\dot B^{-1}_{\infty,r}} - C_{K,\delta}\Lambda^{-2}\|f\|_{\dot B^{-3}_{\infty,\infty}}.$$

This concludes the proof of the first part of the lemma.

Now let us prove the second part of the lemma, namely Estimate (4.7) on the $\dot H^{-1}$ norm. Let $f\in\mathcal{D}(Q)$ be given. Stating $f_m \stackrel{\rm def}{=} -\mathcal{F}^{-1}(i\xi_m|\xi|^{-2}\widehat f)$, we can write

$$f=\sum_{m=1}^3 \partial_m f_m, \quad \text{with} \quad \|f\|_{\dot H^{-1}} \sim \sum_{m=1}^3 \|f_m\|_{L^2}.$$

Let us recall that

$$\|T_{\Lambda,X}f\|_{\dot H^{-1}} = \sup_{\substack{g\in\mathcal{D}(Q)\\ \|g\|_{\dot H^1}\leq 1}} \int_{\mathbf{R}^3} T_{\Lambda,X}f(x)g(x)\,dx.$$

Let $\chi\in\mathcal{D}(Q)$ be equal to one on the support of g. We have

$$\begin{aligned}\int_{\mathbf{R}^3} T_{\Lambda,X}f(x)g(x)\,dx &= \Lambda\sum_J\int_{\mathbf{R}^3} f(\Lambda(x-x_J))g(x)\,dx\\ &= \Lambda^{-2}\sum_J\sum_m\int_{\mathbf{R}^3}\partial_m f_m(x)g(\Lambda^{-1}x+x_J)\,dx,\end{aligned}$$

so after an integration by parts and a change of variables again, we infer that

$$\begin{aligned}\int_{\mathbf{R}^3} T_{\Lambda,X}f(x)g(x)\,dx &= -\Lambda^{-3}\sum_J\sum_m\int_{\mathbf{R}^3}\chi(x)f_m(x)(\partial_m g)(\Lambda^{-1}x+x_J)\,dx\\ &= -\Lambda^{-1}\sum_m\int_{\mathbf{R}^3} T_{\Lambda,X}(\chi f_m)(x)\partial_m g(x)\,dx.\end{aligned}$$

In particular we get that

$$\begin{aligned}\|T_{\Lambda,X}f\|_{\dot H^{-1}} &\leq C\Lambda^{-1}\sum_m\|T_{\Lambda,X}(\chi f_m)\|_{L^2}\\ &\leq C\Lambda^{-1}\sum_m\|f_m\|_{L^2}\Lambda^{-\frac12}\sqrt K\\ &\leq C\Lambda^{-\frac32}\sqrt K\|f\|_{\dot H^{-1}},\end{aligned}$$

and the result is proved. □

4.2.4. *Proof of Lemma 4.2.* First, we observe that

$$\mathrm{S}(t)T_{\Lambda,X}f\cdot\nabla\mathrm{S}(t)T_{\Lambda,X}g=\sum_{\ell=1}^3\partial_\ell\Big(\mathrm{S}(t)T_{\Lambda,X}f^\ell\,\mathrm{S}(t)T_{\Lambda,X}g\Big), \tag{4.18}$$

so using Bernstein's inequalities, we can write

$$\begin{aligned}\mathcal{E}_j &\stackrel{\rm def}{=} \Big\|\Delta_j\Big(\mathrm{S}(t)T_{\Lambda,X}f\cdot\nabla\mathrm{S}(t)T_{\Lambda,X}g\Big)\Big\|_{L^1(\mathbf{R}^+;L^\infty)}\\ &\qquad\qquad + \Big\|t^{\frac12}\Delta_j\Big(\mathrm{S}(t)T_{\Lambda,X}f\cdot\nabla\mathrm{S}(t)T_{\Lambda,X}g\Big)\Big\|_{L^2(\mathbf{R}^+;L^\infty)}\\ &\leq C2^{4j}\Big(\|\mathrm{S}(t)T_{\Lambda,X}f\|_{L^2(\mathbf{R}^+;L^2)}\|\mathrm{S}(t)T_{\Lambda,X}g\|_{L^2(\mathbf{R}^+;L^2)}\\ &\qquad\qquad + \|t^{\frac12}\mathrm{S}(t)T_{\Lambda,X}f\|_{L^\infty(\mathbf{R}^+;L^2)}\|\mathrm{S}(t)T_{\Lambda,X}g\|_{L^2(\mathbf{R}^+;L^2)}\Big)\\ &\leq C2^{4j}\|T_{\Lambda,X}f\|_{\dot H^{-1}}\|T_{\Lambda,X}g\|_{\dot H^{-1}}.\end{aligned}$$

Using Lemma 4.1, we get

$$\mathcal{E}_j \leq C 2^{4j} K \Lambda^{-3} \|f\|_{\dot H^{-1}} \|g\|_{\dot H^{-1}}.$$

We therefore infer a bound on the low frequencies:

$$\sum_{j\leq 0} 2^{-j} \mathcal{E}_j \leq C K \Lambda^{-3} \|f\|_{\dot H^{-1}} \|g\|_{\dot H^{-1}}. \tag{4.19}$$

The high frequencies are more delicate to estimate. Let us write that

$$\begin{aligned} \mathrm{S}(t) T_{\Lambda,X} f \cdot \nabla \mathrm{S}(t) T_{\Lambda,X} g &= H_{\Lambda,X}(t) + K_{\Lambda,X}(t) \quad \text{with} \\ H_{\Lambda,X}(t) &\stackrel{\text{def}}{=} \sum_{J\neq J'} \sum_{\ell=1}^{3} \partial_\ell \Big(\mathrm{S}(t) T_\Lambda^J f^\ell \mathrm{S}(t) T_\Lambda^{J'} g \Big) \quad \text{and} \\ K_{\Lambda,X}(t) &\stackrel{\text{def}}{=} \sum_{J} \sum_{\ell=1}^{3} \partial_\ell \Big(\mathrm{S}(t) T_\Lambda^J f^\ell \mathrm{S}(t) T_\Lambda^{J} g \Big). \end{aligned}$$

We observe that

$$\begin{aligned} B_{\Lambda,X}^{J,J'}(f,g) &\stackrel{\text{def}}{=} \partial_\ell \Big(\mathrm{S}(t) T_\Lambda^J f^\ell \mathrm{S}(t) T_\Lambda^{J'} g \Big) \\ &= \frac{1}{(4\pi t)^3} \int_{\mathbf{R}^6} \partial_{x_\ell} \exp\Big(-\frac{|x-y|^2 + |x-z|^2}{4t} \Big) T_\Lambda^J f(y) T_\Lambda^{J'} g(z) dy dz \\ &= \frac{1}{(4\pi t)^3} \int_{\mathbf{R}^6} \frac{2x_\ell - y_\ell - z_\ell}{2t} \exp\Big(-\frac{|x-y|^2 + |x-z|^2}{4t} \Big) T_\Lambda^J f(y) T_\Lambda^{J'} g(z) dy dz. \end{aligned}$$

Due to the distance between x_J and $x_{J'}$, one gets that a smooth bounded function (as well as all its derivatives) χ on $\mathbf{R}$ exists such that χ vanishes identically near 0 and such that

$$\begin{aligned} B_{\Lambda,X}^{J,J'}(f,g)(t,x) &= \frac{1}{(4\pi t)^3} \int_{\mathbf{R}^6} \Theta_\delta(t,x,y,z) T_\Lambda^J f(y) T_\Lambda^{J'} g(z) dy dz \quad \text{with} \\ \Theta_\delta(t,x,y,z) &\stackrel{\text{def}}{=} \frac{1}{t^{\frac12}} \chi\left(C\delta(|x-y|^2 + |x-z|^2) \right) \frac{2x_\ell - y_\ell - z_\ell}{2t} \exp\Big(-\frac{|x-y|^2 + |x-z|^2}{4t^{\frac12}} \Big). \end{aligned}$$

As we have

$$\|a\otimes b\|_{\dot H^{-2}(\mathbf{R}^6)} \leq \|a\|_{\dot H^{-1}(\mathbf{R}^3)} \|b\|_{\dot H^{-1}(\mathbf{R}^3)}, \tag{4.20}$$

we infer, using the scaling, that

$$\begin{aligned} \|B_{\Lambda,X}^{J,J'}(f,g)(t,\cdot)\|_{L^\infty} &\leq \frac{1}{(4\pi t)^3} \sup_{x\in\mathbf{R}^3} \|\Theta_\delta(t,x,\cdot)\|_{\dot H^2(\mathbf{R}^6)} \|T_\Lambda^J f\|_{\dot H^{-1}(\mathbf{R}^3)} \|T_\Lambda^{J'} g\|_{\dot H^{-1}(\mathbf{R}^3)} \\ &\leq \frac{1}{(4\pi t)^3} \sup_{x\in\mathbf{R}^3} \|\Theta_\delta(t,x,\cdot)\|_{\dot H^2(\mathbf{R}^6)} \Lambda^{-3} \|f\|_{\dot H^{-1}(\mathbf{R}^3)} \|g\|_{\dot H^{-1}(\mathbf{R}^3)}. \end{aligned}$$

It is obvious that

$$|\nabla^2 \Theta_\delta(t,x,y,z)| \leq \frac{C}{t^{\frac32}} e^{-\frac{\delta}{Ct}} \exp\Big(-\frac{|x-y|^2 + |x-z|^2}{8t} \Big)$$

and thus that

$$\|B_{\Lambda,X}^{J,J'}(f,g)(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^3} \Lambda^{-3} e^{-\frac{\delta}{Ct}} \|f\|_{\dot H^{-1}(\mathbf{R}^3)} \|g\|_{\dot H^{-1}(\mathbf{R}^3)}.$$

We immediately infer, since $\|\Delta_j \mathbf{P}\, a\|_{L^\infty} \leq C\|\Delta_j a\|_{L^\infty}$, that

$$\sum_{j\geq 0} 2^{-j}\Big(\|\Delta_j \mathbf{P} H_{\Lambda,X}\|_{L^1(L^\infty)} + \|t^{\frac12}\Delta_j \mathbf{P} H_{\Lambda,X}\|_{L^2(L^\infty)}\Big) \leq C_\delta \Lambda^{-3}\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}. \tag{4.21}$$

Now let us consider the term $K_{\Lambda,X}$. To start with, let us write

$$\begin{aligned}\|\Delta_j \mathbf{P} K_{\Lambda,X}(t,\cdot)\|_{L^\infty(\mathbf{R}^3)} &\leq \|\Delta_j \mathbf{P} K_{\Lambda,X}(t,\cdot)\|_{L^\infty(\widetilde Q_\delta)} + \|\Delta_j \mathbf{P} K_{\Lambda,X}(t,\cdot)\|_{L^\infty({}^c\widetilde Q_\delta)}\\ &\leq \sup_J \|\Delta_j \mathbf{P} K_{\Lambda,X}(t,\cdot)\|_{L^\infty(\widetilde Q_\delta^J)} + \|\Delta_j \mathbf{P} K_{\Lambda,X}(t,\cdot)\|_{L^\infty({}^c\widetilde Q_\delta)}.\end{aligned}$$

By definition of $K_{\Lambda,X}$, and denoting $\widetilde\Delta_j = \Delta_j\,\mathbf{P}$, we get

$$\begin{aligned}\|\Delta_j \mathbf{P} K_{\Lambda,X}(t,\cdot)\|_{L^\infty(\mathbf{R}^3)} &\leq \sup_J \|\Delta_j \mathbf{P}(\mathrm{S}(t)T_\Lambda^J f\nabla \mathrm{S}(t)T_\Lambda^J g)\|_{L^\infty(\mathbf{R}^3)}\\ &\qquad + \sup_J \Big\|\widetilde\Delta_j \sum_{J'\neq J}\partial_\ell\Big(\mathrm{S}(t)T_\Lambda^{J'}f^\ell \mathrm{S}(t)T_{\Lambda,X}^{J'}g\Big)\Big\|_{L^\infty(\widetilde Q_\delta^J)}\\ &\qquad + \sum_{J'}\Big\|\widetilde\Delta_j\partial_\ell\Big(\mathrm{S}(t)T_\Lambda^{J'}f^\ell \mathrm{S}(t)T_\Lambda^{J'}g\Big)\Big\|_{L^\infty({}^c\widetilde Q_\delta)}\\ &\leq \sup_J \|\Delta_j \mathbf{P}(\mathrm{S}(t)T_\Lambda^J f\nabla \mathrm{S}(t)T_\Lambda^J g)\|_{L^\infty(\mathbf{R}^3)}\\ &\qquad + \sup_J \sum_{J'\neq J}\Big\|\widetilde\Delta_j\partial_\ell\Big(\mathrm{S}(t)T_\Lambda^{J'}f^\ell \mathrm{S}(t)T_{\Lambda,X}^{J'}g\Big)\Big\|_{L^\infty(\widetilde Q_\delta^J)}\\ &\qquad + \sum_{J'}\Big\|\widetilde\Delta_j\partial_\ell\Big(\mathrm{S}(t)T_\Lambda^{J'}f^\ell \mathrm{S}(t)T_{\Lambda,X}^{J'}g\Big)\Big\|_{L^\infty({}^c\widetilde Q_\delta)}\\ &\leq \sup_J \|\Delta_j \mathbf{P}(\mathrm{S}(t)T_\Lambda^J f\nabla \mathrm{S}(t)T_\Lambda^J g)\|_{L^\infty(\mathbf{R}^3)}\\ &\qquad + \sum_{J'}\Big\|\widetilde\Delta_j\partial_\ell\Big(\mathrm{S}(t)T_\Lambda^{J'}f^\ell \mathrm{S}(t)T_{\Lambda,X}^{J'}g\Big)\Big\|_{L^\infty({}^c\widetilde Q_\delta^{J'})}.\end{aligned}$$

By translation and scaling we infer that

$$\begin{aligned}\|\Delta_j \mathbf{P} K_{\Lambda,X}(t,\cdot)\|_{L^\infty(\mathbf{R}^3)} \leq \Lambda\|\Delta_{j-\log_2\Lambda}\mathbf{P}(\mathrm{S}(t)f\nabla\mathrm{S}(t)g)\|_{L^\infty(\mathbf{R}^3)}\\ + \sum_{J'}\Big\|\widetilde\Delta_j\partial_\ell\Big(\mathrm{S}(t)T_\Lambda^{J'}f^\ell \mathrm{S}(t)T_{\Lambda,X}^{J'}g\Big)\Big\|_{L^\infty({}^c\widetilde Q_\delta^{J})}.\end{aligned}$$

By definition of $\mathrm{S}(t)$, we have, for some $\widetilde h \in \mathcal{S}(\mathbf{R}^3)$,

$$\begin{aligned}B_{\Lambda,j}^{J'}(f,g)(t,x) &\stackrel{\text{def}}{=} \widetilde\Delta_j\partial_\ell\Big(\mathrm{S}(t)T_\Lambda^{J'}f^\ell \mathrm{S}(t)T_{\Lambda,X}^{J'}g\Big)(t,x)\\ &= \frac{2^{3j}}{(4\pi t)^3}\int_{\mathbf{R}\,9}\widetilde h(2^j(x-x'))\partial_{x'_\ell}\exp\Big(-\frac{|x'-y|^2+|x'-z|^2}{4t}\Big)\\ &\qquad\qquad \times T_\Lambda^{J'}f(y)T_\Lambda^{J'}g(z)dx'dydz.\end{aligned}$$

Now if x is in ${}^c\widetilde Q_\delta^{J'}$ and y in $\widetilde Q_\delta^{J'}$, one gets that a smooth bounded function (as well as all its derivatives) χ on $\mathbf{R}$ exists such that χ vanishes identically near 0 and has value 1 outside a ball centered at the origin, and such that

$$\begin{aligned}B^{J'}_{\Lambda,j}(f,g) &= B^{J',1}_{\Lambda,j}(f,g)+B^{J',2}_{\Lambda,j}(f,g)\quad\text{with}\\ B^{J',1}_{\Lambda,j}(f,g)(t,x) &\overset{\text{def}}{=} \frac{2^{3j}}{(4\pi t)^3}\int_{\mathbf{R}^9}\widetilde h(2^j(x-x'))\chi(C\delta|x-x'|^2)\\ &\qquad\times\partial_{x'_\ell}\exp\Bigl(-\frac{|x'-y|^2+|x'-z|^2}{4t}\Bigr)T^{J'}_\Lambda f(y)T^{J'}_\Lambda g(z)dx'dydz\quad\text{and}\\ B^{J',2}_{\Lambda,j}(f,g)(t,x) &\overset{\text{def}}{=} \frac{2^{3j}}{(4\pi t)^3}\int_{\mathbf{R}^9}\widetilde h(2^j(x-x'))\chi(C\delta|x'-y|^2)\\ &\qquad\times\partial_{x'_\ell}\exp\Bigl(-\frac{|x'-y|^2+|x'-z|^2}{4t}\Bigr)T^{J'}_\Lambda f(y)T^{J'}_\Lambda g(z)dx'dydz.\end{aligned}$$

By integration by parts, we get that

$$B^{J',1}_{\Lambda,j}(f,g)(t,x)=2^{3j}\int_{\mathbf{R}^9}\partial_{x'_\ell}\Bigl(\widetilde h(2^j(x-x'))\chi(C\delta|x-x'|^2)\Bigr)(\mathrm{S}(t)T^{J'}_\Lambda f)(x')(\mathrm{S}(t)T^{J'}_\Lambda g)(x')dx'.$$

By the Leibnitz formula we have,

$$\begin{aligned}2^{3j}\partial_{x'_\ell}\Bigl(\widetilde h(2^j(x-x'))\chi(C\delta|x-x'|^2)\Bigr) &= 2^{4j}(\partial_{x_\ell}h)(2^j(x-x'))\chi(C\delta|x-x'|^2)\\ &\quad+2^{3j}\widetilde h(2^j(x-x'))2C\delta(x_\ell-x'_\ell)\chi'(C\delta|x-x'|^2)\end{aligned}$$

Using the properties of the function χ, we infer

$$\Bigl|2^{3j}\partial_{x'_\ell}\Bigl(\widetilde h(2^j(x-x'))\chi(C\delta|x-x'|^2)\Bigr)\Bigr|\leq C_\delta\underline{h}(2^j(x-x'))$$

for some bounded function $\underline{h}$. Thus, by integration, we infer that

$$\|B^{J',1}_{\Lambda,j}(f,g)(t,\cdot)\|_{L^\infty}\leq C_\delta\|\mathrm{S}(t)T^{J'}_\Lambda f\|_{L^2}\|\mathrm{S}(t)T^{J'}_\Lambda g\|_{L^2}.$$

By definition of Besov spaces, we deduce that

$$\|B^{J',1}_{\Lambda,j}(f,g)\|_{L^1(L^\infty)}+\|t^{\frac12}B^{J',1}_{\Lambda,j}(f,g)\|_{L^2(L^\infty)}\leq C_\delta\|T^{J'}_\Lambda f\|_{\dot H^{-1}}\|T^{J'}_\Lambda g\|_{\dot H^{-1}}.$$

By scaling, we infer that

$$\|B^{J',1}_{\Lambda,j}(f,g)\|_{L^1(L^\infty)}+\|t^{\frac12}B^{J',1}_{\Lambda,j}(f,g)\|_{L^2(L^\infty)}\leq C_\delta\Lambda^{-3}\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}.\tag{4.22}$$

In order to estimate $B^{J',2}_{\Lambda,j}(f,g)$, let us write

$$\begin{aligned}B^{J',2}_{\Lambda,j}(f,g)(t,x) &= \frac{1}{(4\pi t)^3}\int_{\mathbf{R}^6}\Theta_{\delta,j}(t,x,y,z)T^{J'}_\Lambda f(y)T^{J'}_\Lambda(z)dydz\quad\text{with}\\ \Theta_{\delta,j}(t,x,y,z) &\overset{\text{def}}{=} \frac{2^{3j}}{t^{\frac12}}\int_{\mathbf{R}^3}\widetilde h(2^j(x-x'))\chi(C_\delta|x'-y|^2)\\ &\qquad\times\frac{2x'_\ell-y_\ell-z_\ell}{2t^{\frac12}}\exp\Bigl(-\frac{|x'-y|^2+|x'-z|^2}{4t}\Bigr)dx'.\end{aligned}$$

Using (4.20), the definition of the Besov norm and the scaling property, we deduce that

$$\begin{aligned}\|B^{J',2}_{\Lambda,j}(f,g)(t,\cdot)\|_{L^\infty} &\leq \sup_{x\in\mathbf{R}^3}\|\nabla^2_{y,z}\Theta_{\delta,j}(t,x,\cdot,\cdot)\|_{L^2(\mathbf{R}^6)}\|T^{J'}_\Lambda f\|_{\dot H^{-1}}\|T^{J'}_\Lambda g\|_{\dot H^{-1}}\\ &\leq \sup_{x\in\mathbf{R}^3}\|\nabla^2_{y,z}\Theta_{\delta,j}(t,x,\cdot,\cdot)\|_{L^2(\mathbf{R}^6)}\Lambda^{-3}\|f\|_{\dot H^{-1}}\|g\|_{\dot H^{-1}}.\end{aligned}$$

A straightforward computation shows that

$$\sup_{x\in\mathbf{R}^3} \|\nabla^2_{y,z}\Theta_{\delta,j}(t,x\cdot,\cdot)\|_{L^2(\mathbf{R}^6)} \leq \frac{C_\delta}{t^{\frac{3}{2}}} e^{-\frac{\delta}{Ct}}.$$

Thus, we get that

$$\|B^{J',2}_{\Lambda,j}(f,g)\|_{L^1(L^\infty)} + \|t^{\frac{1}{2}} B^{J',2}_{\Lambda,j}(f,g)\|_{L^2(L^\infty)} \leq C_\delta \Lambda^{-3} \|f\|_{\dot{H}^{-1}} \|g\|_{\dot{H}^{-1}}$$

Using (4.19), (4.21) and (4.22) we infer that

$$\begin{aligned} \|\mathbf{P}(\mathrm{S}(t)T_{\Lambda,X} f \nabla \mathrm{S}(t) T_{\Lambda,X} g)\|_E \quad &\leq \quad \sum_j 2^{-j}\Lambda\Big(\|\Delta_{j-\log_2\Lambda} \mathbf{P}\mathrm{S}(t) f \nabla \mathrm{S}(t) g\|_{L^1(L^\infty)} \\ &\qquad + \|t^{\frac{1}{2}} \Delta_{j-\log_2\Lambda} \mathbf{P}\mathrm{S}(t) f \nabla \mathrm{S}(t) g\|_{L^2(L^\infty)}\Big) \\ &\qquad\qquad + C_{K,\delta}\Lambda^{-3}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}} \\ &\leq \quad \|\mathbf{P}(\mathrm{S}(t) f \nabla \mathrm{S}(t) g)\|_E + C_{K,\delta}\Lambda^{-3}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}. \end{aligned}$$

That ends the proof of Lemma 4.2. □

4.2.5. *Proof of Theorem 4.* The proof is straightforward: in order to apply Theorem 3, we define $\eta > 0$ and we need to find Λ_0 uniform in ε so that, according to Lemmas 4.1 and 4.2, the following two conditions are satisfied:

$$\Lambda_0^{-3} C_{\delta,K} \|u_{0,\varepsilon}\|^2_{\dot{H}^{-1}} = \eta \quad \text{and} \quad \Lambda_0^{-2} C_{\delta,K} \|u_{0,\varepsilon}\|_{\dot{B}^{-3}_{\infty,\infty}} = \eta.$$

Due to Corollary 3.1 this is trivially possible as soon as $\alpha > 0$. □

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