Nonlinear Differ. Equ. Appl. 22 (2015), 1247–1258

1021-9722/15/051247-12
published online April 10, 2015
DOI 10.1007/s00030-015-0321-6

Nonlinear Differential Equations and Applications NoDEA

Wellposedness for stochastic continuity equations with Ladyzhenskaya–Prodi–Serrin condition

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Abstract. We consider the stochastic divergence-free continuity equations with Ladyzhenskaya–Prodi–Serrin condition. Wellposedness is proved meanwhile uniqueness may fail for the deterministic PDE. The main issue of strong uniqueness, in the probabilistic sense, relies on stochastic characteristic method and the generalized Itô–Wentzell–Kunita formula. The stability property for the unique solution is proved with respect to the initial data. Moreover, a persistence result is established by a representation formula.

Mathematics Subject Classification. 60H15 · 35R60 · 35F10 · 60H30.

Keywords. Stochastic partial differential equation, Continuity equation, Well-posedness, Stochastic characteristic method, Cauchy problem.

1. Introduction

In this paper we establish wellposedness for stochastic divergence-free continuity equations. Namely, we consider the following Cauchy problem: Given an initial-data u_0, find $u(t,x;\omega) \in \mathbb{R}$, satisfying

$$\begin{cases} \partial_t u\,(t,x;\omega) + \operatorname{div}\left(u\,(t,x;\omega)\ \left(b\,(t,x) + \dfrac{dB_t}{dt}(\omega)\right)\right) = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

$((t,x) \in U_T, \omega \in \Omega)$, where $U_T = [0,T] \times \mathbb{R}^d$, for $T > 0$ be any fixed real number, $(d \in \mathbb{N})$, $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field, with $\operatorname{div} b(t,x) = 0$, $B_t = (B_t^1, \dots, B_t^d)$ is a standard Brownian motion in $\mathbb{R}^d$ and the stochastic integration is taken (unless otherwise mentioned) in the Stratonovich sense. In fact, through of this paper, we fix a stochastic basis with a d-dimensional Brownian motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P}, (B_t))$.

The Cauchy problem for the stochastic transport equation

$$\partial_t u(t,x;\omega) + \left(b\,(t,x) + \frac{dB_t}{dt}(\omega)\right) \cdot \nabla u(t,x;\omega) = 0$$

has taken great attention recently, see for instance [3,4,14,18–20], and more recently the initial-boundary value problem in [23]. Concerning the deterministic case of the problem (1.1), also in a non-regular framework, the reader is mostly addressed to [9] and [2]. Those papers deal respectively with the Sobolev and the BV spatial regularity case, where the uniqueness proof relies on commutators, see DeLellis [8] for a nice review on that. The reader is directed towards the following references at the cited papers.

The main issue in this paper is to prove uniqueness of weak L^∞−solution (see Definition 1.1) of the Cauchy problem (1.1) for vector fields

$$\begin{aligned} &b \in L^q([0,T],(L^p(\mathbb{R}^d))^d), \quad p,q<\infty,\\ &p \geq 2, \quad q>2, \quad \text{and} \quad \frac{d}{p}+\frac{2}{q}<1. \end{aligned} \tag{1.2}$$

The last condition (1.2) is known in the fluid dynamic's literature as the Ladyzhenskaya–Prodi–Serrin condition, with $\leq$ in place of $<$. Here, we do not assume any differentiability (one of the main assumptions in [3]), nor boundedness (also important in [14]) of the vector field b. The uniqueness result, see Theorem 2.1, is established using the transportation property of the continuity equation for divergence free vector fields. Indeed, under the Ladyzhenskaya–Prodi–Serrin condition for b, we are allowed to compose the solution u to the transport equation with the stochastic flow, in fact its inverse (see (1.4)), then we bring on it with all its space derivatives on the test function. Thus, avoiding the commutator and the problems there in. Consequntly, we have sharpened the answer of the following question: Why noise improves the deterministic theory for transport/continuity equations?

In fact, that noise could improve the theory of transport equations was first discovered by [14]. More precisely, the condition assumed in [14] is Hölder continuity and boundedness of b, and an integrability condition on the divergence. Others results appear in [3] where no L^∞−control on the divergence is required, however, weak differentiability is assumed. Our result is more advanced in the sense that we work with integrable and not differentiability coefficients. Further, no boundedness of b is assumed.

We recall that the Ladyzhenskaya–Prodi–Serrin condition (1.2) (with local integrability, and $p,q \leq \infty$) was first considered by Krylov and Röckner [16]. In that paper, they proved the existence and uniqueness of strong solutions for SDE

$$X_{s,t}(x) = x + \int_s^t b\left(r, X_{s,r}(x)\right)\, dr + B_t - B_s, \tag{1.3}$$

where given $t \in [0,T]$ and $x \in \mathbb{R}^d$, it was shown that

$$\mathbb{P}\left\{ \int_0^T |b(t,X_t)|^2 \, dt < \infty \right\} = 1.$$

More recently, Fedrizzi and Flandoli see [10,11] proved the α-Hölder continuity of the stochastic flow $x \to X_{s,t}$ for any $\alpha \in (0,1)$. Moreover, they prove that it is a stochastic flow of homeomorphism.

Similarly, we may consider for convenience the inverse $Y_{s,t} := X_{s,t}^{-1}$, which satisfies the following backward stochastic differential equations,

$$Y_{s,t}(y) = y - \int_s^t b\left(r, Y_{r,t}(y)\right)\ dr - (B_t - B_s), \tag{1.4}$$

for $0 \leq s \leq t$. Usually Y is called the time reversed process of X.

One of the main motivations to consider problem (1.1) comes from the study of stochastic partial differential equations in fluid dynamics. In this direction, our ansatz is based on rational Continuum Mechanics. Let ψ be a physical quantity, which is a tensor field of order m. Also, we consider the supply and the flux of ψ, denoted respectively σ_ψ, ϕ_ψ, which are tensor fields of order m and $m+1$. Then, the general stochastic balance equation has (at least formally) the form

$$\partial_t \psi + \operatorname{div}\left(\psi \otimes \frac{dX_t}{dt} - \phi_\psi\right) = \sigma_\psi,$$

with X_t given for instance by

$$X_t(x) = x + \int_0^t \mathbf{v}(s, X_s(x))\ ds + V_t,$$

where $\mathbf{v}$ is the velocity field, and V_t is a stochastic process, which is not due necessarily to a Brownian motion, but posses for instance a Markov property. Therefore, our main assumption is to randomly perturb the motion of the physical quantity ψ. In particular, taking $\psi = \rho$, and $\psi = \rho\,\mathbf{v}$, the Cauchy problem for the incompressible non-homogeneous stochastic Navier–Stokes equations may be written as

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho\ \left(\mathbf{v} + \frac{dV_t}{dt}\right)\right) = 0, \\ \operatorname{div}\mathbf{v} = 0, \\ \partial_t(\rho\,\mathbf{v}) + \operatorname{div}\left(\rho\,\mathbf{v} \otimes \left(\mathbf{v} + \frac{dV_t}{dt}\right) - T\right) = \rho\, f, \\ \rho(0) = \rho_0, \quad \mathbf{v}(0) = \mathbf{v}_0, \end{cases} \tag{1.5}$$

where ρ is the density, and T is the stress tensor field given by

$$T = 2\,\mu(\rho)\,D(\mathbf{v}) - p\,I_d$$

with the scalar function p called pressure. Moreover, $D(\mathbf{v})$ is the symmetric part of the gradient of the velocity field, μ is the dynamic viscosity, and f is an external body force. The above problem (1.5) seems to us an onset of turbulence, which is a challenging phenomenon to understand in the (incompressible) fluid dynamics theory. The reader is further addressed to Flandoli [12], Mikulevicius and Rozovskii [22], and references therein.

Moreover, the uniqueness result obtained by the authors for the stochastic continuity equation here in this paper, have to open new directions to establish existence of solutions to stochastic scalar conservation laws, for non-homogenous flux functions $f(t, x, u)$ with low regularity in the time-space variables (t, x). In this direction, we recall the stochastic averaging lemmas for kinetic equations studied recently by Lions et al. [21].

The plan of exposition is as follows: In the rest of this section, we shall prove existence of weak $L^\infty-$solutions via the Itô–Wentzell–Kunita formula.

In Sect. 2, we prove the uniqueness of weak $L^\infty-$solutions. Moreover, we show that the unique solution is given by a representation formula, in terms of the initial data and the stochastic flow associated to equation (1.1). In Sect. 3, we present stability results for the solution with respect to the initial datum. Finally, we discuss in Section 4 some extensions, regularity results and interesting open problems are pointed out.

1.1. Existence of weak solutions

Hereupon, we assume

$$b \in L^1_{\mathrm{loc}}(U_T), \quad \operatorname{div} b \in L^1\left((0,T)\,; L^\infty\left(\mathbb{R}^d\right)\right). \tag{1.6}$$

Actually, it was point out by an anonymous referee that, when the drift b satisfies condition (1.2) the derivative and thus the Jacobian of the flow are regularized by the noise, and in particular almost bounded, without any hypothesis on the divergence. Therefore, it could be that, one does not need an L^∞ spatial bound on the divergence of b, but just one weaker. We leave this interesting question open. Also, we consider that $u_0 \in L^\infty(\mathbb{R}^d)$.

The next definition tell us in which sense a stochastic process is a weak solution of (1.1). Hereafter the usual summation convention is used.

Definition 1.1. *A stochastic process $u \in L^\infty(U_T \times \Omega)$ is called a weak $L^\infty-$solution of the Cauchy problem (1.1), when for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, the real value process $\int u(t,x)\varphi(x)dx$ has a continuous modification which is a $\mathcal{F}_t$-semimartingale, and for all $t \in [0,T]$, we have $\mathbb{P}$-almost sure*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\ dx + \int_0^t\!\!\int_{\mathbb{R}^d} u(s,x)\ b^i(s,x)\partial_i\varphi(x)\ dxds \\ + \int_0^t\!\!\int_{\mathbb{R}^d} u(s,x)\ \partial_i\varphi(x)\ dx \circ dB_s^i.\end{aligned} \tag{1.7}$$

Lemma 1.2. *Under condition (1.6), there exists a weak $L^\infty-$solution u of the Cauchy problem (1.1).*

Proof. 1. First, let us consider the following auxiliary Cauchy problem for the continuity equation, that is to say

$$\begin{cases} \partial_t v(t,x) + \operatorname{div}\left(v(t,x)\ b(t,x+B_t)\right) = 0, \\ v(0,x) = u_0(x). \end{cases} \tag{1.8}$$

According to a minor modification of the arguments in DiPerna and Lions [9], see Proposition II.1 (taking only test functions defined on $\mathbb{R}^d$), it follows that, there exists a function $v \in L^\infty(U_T \times \Omega)$, which is a solution of the auxiliary problem (1.8) in the sense that, it satisfies for each test function $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} v(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\ dx \\ + \int_0^t\!\!\int_{\mathbb{R}^d} v(s,x)\,b(s,x+B_s)\cdot\nabla\varphi(x)\ dxds.\end{aligned} \tag{1.9}$$

One observes that, the process $\int v(t,x)\varphi(x)dx$ is adapted, since it is the weak limit in $L^2([0,T]\times\Omega)$ of adapted processes, see [24] Chapter III for details.

2. Now, let us define for each $y\in\mathbb{R}^d$,

$$F(y) := \int_{R^d} v(t,x)\,\varphi(x+y)\ dx.$$

Then, applying the Itô–Wentzell–Kunita Formula, see Theorem 8.3 of [17], to $F(B_t)$, it follows from (1.9)

$$\begin{aligned}\int_{\mathbb{R}^d} v(t,x)\,\varphi(x+B_t)\ dx &= \int_{\mathbb{R}^d} u_0(x)\,\varphi(x)\ dx\\ &\quad+\int_0^t\int_{\mathbb{R}^d} b(s,x+B_s)\cdot\nabla\varphi(x+B_s)v(s,x)\ dxds\\ &\quad+\int_0^t\int_{\mathbb{R}^d} v(s,x)\ \partial_i\varphi(x+B_s)dx\circ dB_s^i, \qquad (1.10)\end{aligned}$$

where we have used that

$$\frac{\partial}{\partial y_i}\varphi(x+y)=\frac{\partial}{\partial x_i}\varphi(x+y).$$

3. Finally, defining $u(t,x) := v(t,x-B_t)$ we obtain from equation (1.10) that, $u(t,x)$ is a weak $L^\infty-$solution of the stochastic Cauchy problem (1.1). □

2. Uniqueness

We prove the uniqueness result in this section, where it will be considered the divergence-free condition, that is

$$\operatorname{div} b = 0 \qquad (2.11)$$

(understood in the sense of distributions), and also the Ladyzhenskaya–Prodi–Serrin condition (1.2).

As mentioned in the introduction, under condition (1.2) we have suitable regularity of the stochastic characteristics. Indeed, under the divergence-free condition the continuity equation turns to transport equation. Therefore, the main feature of the transport equation, which is the transportation property, is used by the authors to show uniqueness in a completely different way from the renormalization idea (which exploits commutators) used in [3,4,14,20]. Then, we have the following

Theorem 2.1. *Assume conditions* (1.2)*, and* (2.11)*. If* $u,v\in L^\infty(U_T\times\Omega)$ *are two weak* $L^\infty-$*solutions for the Cauchy problem* (1.1)*, with the same initial data* $u_0\in L^\infty(\mathbb{R}^d)$*, then for each* $t\in[0,T]$*,* $u(t)=v(t)$ *almost everywhere in* $\mathbb{R}^d\times\Omega$*.*

Proof. By linearity, it is enough to show that a weak $L^\infty-$solution u with initial condition $u_0(x)=0$ vanishes identically. Let $\phi_\varepsilon,\phi_\delta$ be standard symmetric mollifiers. Thus $u_\varepsilon(t,\cdot)=u(t,\cdot)*\phi_\varepsilon$ verifies

$$
\begin{aligned}
\int_{\mathbb{R}^d} u(t,z)\phi_\varepsilon(y-z)dz = \int_0^t\int_{\mathbb{R}^d} u(s,z)\ b^i(s,z)\partial_i\phi_\varepsilon(y-z)\ dzds \\
+ \int_0^t\int_{\mathbb{R}^d} u(s,z)\ \partial_i\phi_\varepsilon(y-z)\ dz \circ dB_s^i.
\end{aligned} \tag{2.12}
$$

One remarks that, for each $\varepsilon > 0$ the equation for u_ε is strong in the analytic sense (which is to say, it does not need test functions).

Now, we denote by b^δ the standard mollification of b by ϕ_δ, and let X_t^δ be the associated flow given by the SDE (1.3) replacing b by b^δ. Similarly, we consider Y_t^δ, which satisfies the backward SDE (1.4).

Since $\operatorname{div} b^\delta = 0$ (in other words, the Jacobian of the stochastic flow is identically one), for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, it follows that

$$
\int_{\mathbb{R}^d} (u*\phi_\varepsilon)(X_t^\delta)\ \varphi(y)\ dy = \int_{\mathbb{R}^d} (u*\phi_\varepsilon)(y)\ \varphi(Y_t^\delta)\ dy, \tag{2.13}
$$

for each $t \in [0,T]$.

Now, we observe that $v^\delta(t,x) = \varphi(Y_t^\delta)$, satisfies the transport equation in the classical sense, that is, it satisfies

$$
\begin{cases} dv^\delta(t,x;\omega) + b(t,x)\nabla v^\delta(t,x;\omega)(t,x;\omega)dt + \nabla v^\delta(t,x;\omega)(t,x;\omega)\circ dB_t = 0, \\ v^\delta|_{t=0} = \varphi(x), \end{cases} \tag{2.14}
$$

On the other hand, recall that u_ε is strong in analytic sense. Then we may apply Itô's formula to the product

$$
(u*\phi_\varepsilon)(y)\ \varphi(Y_t^\delta),
$$

and obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^d} (u*\phi_\varepsilon)(y)\ \varphi(Y_t^\delta)\ dy = &- \int_0^t\int_{\mathbb{R}^d} u_\varepsilon(s,y)\ b^\delta(s,y)\cdot\nabla[\varphi(Y_s^\delta)]\ dyds \\
&- \int_0^t\int_{\mathbb{R}^d} u_\varepsilon(s,y)\ \partial_i[\varphi(Y_s^\delta)]dy\ \circ dB_s^i \\
&+ \int_0^t\int_{\mathbb{R}^d} \varphi(Y_s^\delta)\int_{\mathbb{R}^d} u(s,z)b(s,z)\cdot\nabla\phi_\varepsilon(y-z)\ dzdyds \\
&+ \int_0^t\int_{\mathbb{R}^d} \varphi(Y_s^\delta)\int_{\mathbb{R}^d} u(s,z)\ \partial_i\phi_\varepsilon(y-z)\ dzdy\ \circ dB_s^i.
\end{aligned} \tag{2.15}
$$

Then, by integration by parts, we may bring all the derivatives on $\varphi(Y^\delta)$, hence it will be easy to pass to the limit when $\varepsilon \to 0$ (avoiding commutators). Therefore, from (2.13), (2.15), we may write

$$
\begin{aligned}
\int_{\mathbb{R}^d} (u*\phi_\varepsilon)(X_t^\delta)\ \varphi(x)\ dx = &- \int_0^t\int_{\mathbb{R}^d} u_\varepsilon(s,y)\ b^\delta(s,y)\cdot\nabla[\varphi(Y_s^\delta)]\ dyds \\
&- \int_0^t\int_{\mathbb{R}^d} u_\varepsilon(s,y)\ \partial_i[\varphi(Y_s^\delta)]dy\ \circ dB_s^i
\end{aligned}
$$

$$+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(s,z)\,\phi_\varepsilon(y-z)\;b(s,z)\cdot\nabla[\varphi(Y_s^\delta)]\;dzdyds$$
$$+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(s,z)\,\phi_\varepsilon(y-z)\;\partial_i[\varphi(Y_s^\delta)]\;dzdy\;\circ dB_s^i,$$

where we have used that, ϕ_ε is symmetric.

Now for $\delta>0$ fixed, passing to the limit as ε goes to 0^+, we obtain from the above equation

$$\int_{\mathbb{R}^d} u(X_t^\delta)\;\varphi(x)\;dx = -\int_0^t\int_{\mathbb{R}^d} u(s,y)\;b^\delta(s,y)\cdot\nabla[\varphi(Y_s^\delta)]\;dyds$$
$$+\int_0^t\int_{\mathbb{R}^d} u(s,y)\;b(s,z)\cdot\nabla[\varphi(Y_s^\delta)]\;dyds. \tag{2.16}$$

At this point, we use important and recent results obtained by Fedrizzi and Flandoli [10], more precisely, Lemma 3 and Lemma 5 in that paper. And for safeness of the reader, let us recall here the last one:

Lemma 5 (Fedrizzi and Flandoli) For every $p\geq 1$, there exists $C_{d,p,T}>0$ such that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\mathbb{E}[|\nabla Y_t^\delta(x)|^p]\leq C_{d,p,T},\quad \text{uniformly in } \delta>0.$$

Then, applying the Dominated Convergence Theorem we can pass to the limit in (2.16) as δ goes to 0^+, to conclude that

$$\int_{\mathbb{R}^d} u(X_t)\varphi(x)=0 \tag{2.17}$$

for each $\varphi\in C_c^\infty(\mathbb{R}^d)$, and $t\in[0,T]$.

Finally, let K be any compact set in $\mathbb{R}^d$. Then, we have

$$\begin{aligned}\int_K \mathbb{E}|u(t,x)|\;dx &= \lim_{\delta\to 0}\int_K \mathbb{E}|u(t,X_t^\delta(Y_t^\delta))|\;dx\\ &= \lim_{\delta\to 0}\;\mathbb{E}\int_{Y_t^\delta(K)}|u(t,X_t^\delta)|\;dx\\ &= \mathbb{E}\int_{Y_t(K)}|u(t,X_t)|\;dx=0,\end{aligned}$$

where we have used (2.17) and the regularity of the stochastic flow. Consequently, the thesis of our theorem is proved. □

We also have a representation formula in terms of the initial condition u_0 and the (inverse) stochastic flow associated to SDE (1.3). Then, we have the following

Proposition 2.2. *Assume conditions (1.2), and (2.11). Given $u_0\in L^\infty(\mathbb{R}^d)$, the stochastic process $u(t,x):=u_0(X_t^{-1}(x))$ is the unique weak $L^\infty-$solution of the Cauchy problem (1.1).*

Proof. 1. First, let us assume that b is regular, and we denote by u_0^δ the standard mollification of u_0. It is well known, see for instance [19], that $u^\delta(t,x):=$

$u_0^\delta(X_t^{-1}(x))$ is the unique classical solution of the associated transport equation, thus a weak $L^\infty-$solution o of (1.1) with u^δ and u_0^δ in place of u and u_0. For each test function $\varphi \in C_c^\infty(\mathbb{R}^d)$, it follows that

$$\int_{\mathbb{R}^d} u^\delta(t,x)\,\varphi(x)\;dx = \int_{\mathbb{R}^d} u_0^\delta(X_t^{-1})\,\varphi(x)\;dx$$

converges strongly in $L^2([0,T]\times\Omega)$ to

$$\int_{\mathbb{R}^d} u_0(X_t^{-1})\,\varphi(x)\;dx = \int_{\mathbb{R}^d} u(t,x)\,\varphi(x)\;dx.$$

Then, $u_0(X_t^{-1})$ is a weak $L^\infty-$solution of the Cauchy problem (1.1). By Theorem 2.1, uniqueness theorem, it is the only one.

2. Now, we denote by b^δ the standard mollification of b, and let X_t^δ be the associated flow given by the SDE (1.3), i.e. replacing b by b^δ. From item 1, we have that $u^\delta(t,x) = u_0(X_t^{\delta,-1})$ is the unique weak L^∞-solution of (1.1) with u^δ and b^δ in place of u and b. Then, applying Lemma 3 of [10], we have for each test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ that

$$\int_{\mathbb{R}^d} u^\delta(t,x)\,\varphi(x)\;dx = \int_{\mathbb{R}^d} u_0(X_t^{\delta,-1})\,\varphi(x)\;dx$$

converges strongly in $\in L^2([0,T]\times\Omega)$ to

$$\int_{\mathbb{R}^d} u_0(X_t^{-1})\,\varphi(x)\;dx = \int_{\mathbb{R}^d} u(t,x)\,\varphi(x)\;dx.$$

Therefore, $u_0(X_t^{-1})$ is the representative formula, which is the unique weak $L^\infty-$solution of the Cauchy problem (1.1). □

3. Stability

To end up the well-posedness for the Cauchy problem (1.1), it remains to show the stability property for the solution with respect to the initial datum. First, we establish a weak-stability result, then we show a strong one, assuming the strong convergence of the initial data.

Theorem 3.1. *Assume conditions (1.2), and (2.11). Let $\{u_0^n\}$ be any sequence, with $u_0^n \in L^\infty(\mathbb{R}^d)$ $(n \geq 1)$, converging weakly-star to $u_0 \in L^\infty(\mathbb{R}^d)$. Let $u(t,x)$, $u^n(t,x)$ be the unique weak $L^\infty-$solution of the Cauchy problem (1.1), for respectively the initial data u_0 and u_0^n. Then, for all $t \in [0,T]$, and for each function $\varphi \in C_c^0(\mathbb{R}^d)$ $\mathbb{P}-$ a.s.*

$$\int_{\mathbb{R}^d} u^n(t,x)\,\varphi(x)\;dx \quad \textit{converges to} \quad \int_{\mathbb{R}^d} u(t,x)\,\varphi(x)\;dx \quad \mathbb{P}-\ a.s..$$

Moreover, if u_0^n converges to u_0 in $L^\infty(\mathbb{R}^d)$, then

$$u^n(t,x) \textit{ converge to } u(t,x) \quad \textit{in } L^\infty(U_T\times\Omega).$$

Proof. 1. From Proposition 2.2, we may write

$$u^n(t,x) = u_0^n(X_t^{-1}), \quad \text{and} \quad u(t,x) = u_0(X_t^{-1}).$$

Since div $b = 0$ (in other words, the Jacobian of the stochastic flow is identically one), for each $\varphi \in C_c^\infty(\mathbb{R}^d)$, it follows that

$$\int_{\mathbb{R}^d} u^n(t,x)\varphi(x)\ dx = \int_{\mathbb{R}^d} u_0^n(X_t^{-1})\varphi(x)\ dx = \int_{\mathbb{R}^d} u_0^n(x)\varphi(X_t)\ dx \qquad (3.18)$$

and analogously

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\ dx = \int_{\mathbb{R}^d} u_0(X_t^{-1})\varphi(x)\ dx = \int_{\mathbb{R}^d} u_0(x)\varphi(X_t)\ dx. \qquad (3.19)$$

Now, by hypothesis u_0^n converges weak$*$ $L^\infty(\mathbb{R}^d)$ to u_0, thus

$$\int u_0^n(x)\varphi(X_t)\ dx \text{ converge to } \int u_0(x)\varphi(X_t)\ dx,$$

for all $t \in [0,T]$ and $\mathbb{P}-$ a.s. From equations (3.18) and (3.19) we deduce that $u^n(t,x)$ converge to $u(t,x)$ weak$*$ $L^\infty(\mathbb{R}^d)$ for all $t \in [0,T]$ and $\mathbb{P}-$ a.s., which finish the proof of weak-stability.

2. Now, let us consider the strong-stability. Again, from Proposition 2.2 we have

$$u^n(t,x) = u_0^n(X_t^{-1}) \quad \text{and} \quad u(t,x) = u_0(X_t^{-1}).$$

Therefore, we have

$$\begin{aligned} \sup_{U_T\times\Omega} |u^n(t,x) - u(t,x)| &= \sup_{U_T\times\Omega} |u_0^n(X_t^{-1}) - u_0(X_t^{-1})| \\ &\le \sup_{\mathbb{R}^d} |u_0^n(x) - u_0(x)|. \end{aligned}$$

Then, the thesis follows by the hypothesis that u_0^n converges to u_0 in $L^\infty(\mathbb{R}^d)$. □

Remark 3.2. *It remains open the stability result with respect to b under the Ladyzhenskaya–Prodi–Serrin condition (1.2). If additionally, we assume that $b(t)$ belongs to $W^{1,1}_{loc}$ or BV_{loc}, we can show the stability property for the solution with respect to b. In fact, the notion of renormalized solutions is valid and can be extended quite easily to a stochastic framework. For interesting remarks on renormalized solutions in the stochastic case see [3].*

4. Final comments

1. Following the same arguments in the proof of Theorem 2.1, we may also treat the case in which b is Hölder not necessarily bounded, with div$b = 0$. Indeed, from Theorem 7 in [13], if b is Hölder not necessarily bounded, then X_t is a Hölder continuous stochastic flow of diffeomorphisms. Then, a similar result of Theorem 2.1 and Proposition 2.2 hold.

2. As pointed out by Colombini et al. [7], there exists an important example of $b \in L^\infty \cap W^{1,p}$, $(\forall p < \infty)$, such that the propagation of the continuity in the deterministic transport equation is missing. That is to say, even if the

uniqueness is established in this case, the persistence condition is not, one may start with a continuous initial data, but the deterministic solution of the transport equation is not continuous. However, in the stochastic case we have the persistence property. In fact, let $u(t,x)$ be the unique weak $L^\infty-$ solution of the Cauchy problem (1.1), with $u_0 \in C_b(\mathbb{R}^d)$ (i.e. a continuous bounded function). By Proposition 2.2, we have

$$u(t,x) = u_0(X_t^{-1}),$$

and recall that under condition (1.2), X_t is a Hölder continuous stochastic flow of homeomorphisms (see Section 5 of [11]). Therefore, the continuity of u follows. Analogously, we have the persistence property when b is Hölder, not necessarily bounded, with $\mathrm{div} b = 0$. Indeed, see item 1 above.

Also concerning the persistence property, we recall from [10] that a certain Sobolev regularity is maintained under the Ladyzhenskaya–Prodi–Serrin condition, that is,

$$u_0 \in \bigcap_{r\geq 1} W^{1,r} \Rightarrow u(t,.) \in \bigcap_{r\geq 1} W^{1,r}_{\mathrm{loc}}.$$

3. It seems to us a very interesting question if (1.2) is sharp, which is to say if we can consider

$$\frac{d}{p} + \frac{2}{q} \leq 1 \quad \text{instead of} \quad \frac{d}{p} + \frac{2}{q} < 1. \tag{4.20}$$

This question is posed in particular for SDEs, and personal communications from Krylov and Röckner tell us that it remains open.

If we assume that b does not depend on t, $\mathrm{div} b = 0$ (i.e. autonomous divergence free vector fields), consider dimension $d = 2$, and suppose that (4.20) is true, then we may have a uniqueness result for the stochastic transport equation without any geometrical condition as required in the deterministic case, see in particular Hauray [15], and also Alberti et ak. [1]. Moreover, these results could open new ideas to solve the Muskat Problem (at least in dimension 2), see Chemetov and Neves [5,6].

Acknowledgements

We would like to thank the anonymous referee for providing very constructive comments and help us improve this paper. Wladimir Neves is partially supported by CNPq through the following Grants 484529/2013-7, 308652/2013-4 and also by FAPESP through the Grant 2013/15795-9. Christian Olivera is partially supported by FAEPEX 1324/12, FAPESP 2012/18739-0, FAPESP 2012/18780-0, and CNPq through the Grant 460713/2014-0.

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Received: 22 July 2014.
Accepted: 9 March 2015.