



*-Weyl Curvature Tensor within the Framework of Sasakian and (κ, μ) -Contact Manifolds

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Abstract. The object of the present paper is to study *-Weyl curvature tensor within the framework of Sasakian and (κ, μ) -contact manifolds.

1 Introduction

Let M be a $(2n + 1)$ -dimensional Riemannian manifold with metric g and let TM be the Lie algebra of differentiable vector fields in M . The Ricci operator Q of (M, g) is defined by $g(QX, Y) = Ric(X, Y)$, where Ric denotes the Ricci tensor of type $(0,2)$ on M and $X, Y \in TM$. Weyl ([20, 21]) constructed a generalized curvature tensor on a Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric. The Weyl conformal curvature tensor is defined by

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{2n - 1} \{g(QY, Z)X - g(QX, Z)Y + g(Y, Z)QX - g(X, Z)QY\} + \frac{r}{2n(2n - 1)} \{g(Y, Z)X - g(X, Z)Y\},$$

for $X, Y, Z \in TM$, where R and r denotes the Riemannian curvature tensor and the scalar curvature of M respectively. If $n = 1$ then $C(X, Y)Z = 0$ and if $n \geq 1$ then M is locally conformal flat if and only if $C(X, Y)Z = 0$. The condition of locally conformal flat holds for 3-dimensional Riemannian manifolds if and only if the Cotton tensor of M , which is given by

$$K(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4} \{(Xr)Y - (Yr)X\},$$

vanishes identically.

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In [16] Okumura showed that a conformally flat Sasakian manifold of dimension >3 is of constant curvature 1 and in [19] Tanno extended this result to the K -contact case and for dimensions ≥ 3 . Noting that the Ricci operator Q commutes with the fundamental collineation φ for a Sasakian manifold, but this commutativity need not hold for a contact metric manifold, Blair and Koufogiorgos [3] proved that a conformally flat contact metric manifold on which Q commutes with φ , is of constant curvature 1. This generalized the above mentioned result of Okumura. In [9], the authors classified a classes of conformally flat contact metric manifolds and characterized a conformally flat contact manifolds as a hypersurfaces of 4-dimensional Kaehler Einstein manifolds. Miyazawa and Yamaguchi [15] proved that a conformally symmetric Sasakian manifold is also locally isometric to the unit sphere. Chaki and Tarafdar [6] obtained the same result for a Sasakian manifold satisfying the condition $R(X, Y) \cdot C = 0$, for $X, Y \in TM$.

In 1959 Tachibana [17] defined $*$ -Ricci tensor Ric^* on almost Hermitian manifold. In [11] Hamada gave the definition of $*$ -Ricci tensor Ric^* in the following way

$$Ric^*(X, Y) = \frac{1}{2} \text{trace}(Z \rightarrow R(X, \varphi Y) \varphi Z), \quad (1.1)$$

for all $X, Y \in TM$. He also presented $*$ -Einstein, i.e., $g(Q^*X, Y) = \lambda g(X, Y)$, where λ is a constant multiple of $g(X, Y)$ and provided classification of $*$ -Einstein hypersurfaces. Ivey and Ryan in [13] extended the Hamada's work and studied the equivalence of $*$ -Einstein condition with other geometric conditions such as the pseudo-Einstein and the pseudo-Ryan condition. By using the concept of $*$ -Ricci tensor, the present authors with Naik [12] studied the some curvature properies on contact metric generalized (κ, μ) -space form.

Recently, Kaimakamis and Panagiotidou [14] introduced the notion of $*$ -Weyl curvature tensor on real hypersurfaces in non-flat complex space forms and it is defined in the following way

$$\begin{aligned} C^*(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1} \{g(Q^*Y, Z)X - g(Q^*X, Z)Y + g(Y, Z)Q^*X \\ &\quad - g(X, Z)Q^*Y\} + \frac{r^*}{2n(2n-1)} \{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \quad (1.2)$$

for all $X, Y, Z \in TM$, where Q^* is the $*$ -Ricci operator and r^* is the $*$ -scalar curvature corresponding to Q^* of M .

Motivated by the above studies, in this paper we study certain curvature conditions on the $*$ -Weyl curvature tensor in Sasakian and (κ, μ) -contact manifolds. The paper is organized as follows: In section 2, we give brief introduction of contact metric manifolds. In section 3, we study $*$ -Weyl curvature tensor within the background of Sasakian manifolds. Here, we show that Sasakin manifold with vanishing $*$ -Weyl curvature tensor is $\psi(F)_{2n+1}$, and also proved that if Sasakian manifold satisfies $R \cdot C^* = 0$, then it is η -Einstein. In section 4, we study $*$ -Weyl curvature tensor in (κ, μ) -contact manifolds. In this section, we show that a non-Sasakian (κ, μ) -contact manifold with vanishing $*$ -Weyl curvature tensor is a flat for $n = 1$ and locally isometric

to a Riemannian product $E^{n+1} \times S^n(4)$. Finally, it is proved that if non-Sasakian (κ, μ) -contact manifold satisfies $\nabla C^* = \pi \otimes C^*$, then it is locally isometric to a Riemannian product $E^{n+1} \times S^n(4)$.

2 Preliminaries

First, we shall review the basic definitions and formulas of contact metric manifolds. A $(2n+1)$ -dimensional smooth manifold M is said to be contact if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M . This 1-form is called a contact 1-form. For a contact 1-form η , there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact sub-bundle D (defined by $\eta = 0$), we obtain a Riemannian metric g and a $(1,1)$ -tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 X = -X + \eta(X)\xi, \quad (2.1)$$

for all $X, Y \in TM$. From these equations one can also deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2.2)$$

The structure (φ, ξ, η, g) on M is known as a contact metric structure and the metric g is called an associated metric. A Riemannian manifold M together with the structure (φ, ξ, η, g) is said to be a contact metric manifold and we denote it by $(M, \varphi, \xi, \eta, g)$. On a contact metric manifold (see [1])

$$\nabla_X \xi = -\varphi X - \varphi hX, \quad h\varphi + \varphi h = 0, \quad (2.3)$$

for any vector field X, Y on M and ∇ denotes the operator of covariant differentiation of g . If the vector field ξ is Killing (equivalently, $h = 0$) with respect to g , then the contact metric manifold M is said to be K -contact. On a Sasakian manifold, the following formulas are known [1]

$$\nabla_X \xi = -\varphi X, \quad (2.4)$$

$$Q\xi = 2n\xi, \quad (2.5)$$

where Q denote the Ricci operator of M . A contact metric manifold is said to be Sasakian if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X. \quad (2.6)$$

On a Sasakian manifold, the curvature tensor satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \quad (2.7)$$

Also, the contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the metric cone $(M \times R^+, r^2 g + dr^2)$ over M , is Kaehler [1]. A Sasakian manifold is K -contact but the converse is true only in dimension 3. For more details see [1] and [5].

3 *-Weyl Curvature Tensor and Sasakian Manifolds

We are now in a position to find the expression of *-Weyl curvature tensor in the background of Sasakian manifolds. In [10], Ghosh and Patra derive the expression of *-Ricci tensor on Sasakian manifold, which is of the form

$$Ric^*(X, Y) = Ric(X, Y) - (2n - 1)g(X, Y) - \eta(X)\eta(Y). \quad (3.1)$$

Contracting this over X yields

$$r^* = r - 4n^2. \quad (3.2)$$

Using (3.1) and (3.2) in (1.2), we obtain

$$\begin{aligned} C^*(X, Y)Z &= R(X, Y)Z - \frac{1}{2n-1}\{g(QY, Z)X - g(QX, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY\} + \frac{1}{2n-1}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi \\ &\quad - g(X, Z)\eta(Y)\xi\} + \left(\frac{r-4n^2}{2n(2n-1)} + 2\right)\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \quad (3.3)$$

Let M be a Sasakian manifold with vanishing *-Weyl curvature tensor, that is, $C^*(X, Y)Z = 0$. Relation for $C^*(X, Y)Z = 0$ implies that

$$\begin{aligned} R(X, Y)Z &= \frac{1}{2n-1}\{g(QY, Z)X - g(QX, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \\ &\quad - \frac{1}{2n-1}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi \\ &\quad - g(X, Z)\eta(Y)\xi\} - \left(\frac{r-4n^2}{2n(2n-1)} + 2\right)\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \quad (3.4)$$

Covariant differentiation of above relation along W and then contracting the resultant equation over W yields

$$\begin{aligned} &\frac{2(n-1)}{2n-1}\{g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z)\} \\ &= \frac{2(n-1)}{4n(2n-1)}\{(Xr)g(Y, Z) - (Yr)g(X, Z)\} \\ &\quad - \frac{1}{2n-1}\{2g(X, \varphi Y)\eta(Z) + g(X, \varphi Z)\eta(Y) - g(Y, \varphi Z)\eta(X)\}. \end{aligned} \quad (3.5)$$

In a Sasakian manifold we have the following relation:

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \quad (3.6)$$

For a Sasakian manifold we know that ξ is Killing and hence $L_\xi Ric = 0$. Therefore, by (2.4) we easily get $\nabla_\xi Q = \varphi Q - Q\varphi$. For a Sasakian manifold Q and φ commute (see [1]) and hence $\nabla_\xi Q = 0$. Now replacing Y by ξ in (3.5), recalling the last relation and (3.6) we find

$$2(n-1)g(Q\varphi X, Z) - (4n(n-1) + 1)g(\varphi X, Z) - \frac{2(n-1)}{4n}(Xr)\eta(Z) = 0.$$

Replacing Z by φZ in the foregoing equation and making use of (2.2) we obtain

$$Ric(X, Z) = ag(X, Z) - b\eta(X)\eta(Z), \quad (3.7)$$

where $a = \frac{4n(n-1)+1}{2(n-1)}$ and $b = \frac{1}{2(n-1)}$. Substituting (3.7) in (3.4), we obtain

$$\begin{aligned} R(X, Y, Z, W) = & p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + +q[g(X, W)\eta(Y)\eta(Z) \\ & - g(X, Z)\eta(Y)\eta(W) + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)], \end{aligned} \quad (3.8)$$

where $p = \left\{ \frac{2a}{2n-1} - \frac{r-4n^2}{2n(2n-1)} - 2 \right\}$ and $q = -\frac{b+1}{2n-1}$. The relation (3.8) leads to

$$R(X, Y, Z, W) = F(Y, Z)F(X, W) - F(X, Z)F(Y, W), \quad (3.9)$$

where

$$F(X, Y) = \sqrt{p}g(X, Y) + \frac{q}{\sqrt{p}}\eta(X)\eta(Y). \quad (3.10)$$

An n -dimensional Riemannian manifold whose curvature tensor R of type (0,4) satisfies the condition (3.9), where F is a symmetric tensor of type (0, 2) is called a special manifold with the associated symmetric tensor F , and is denoted by $\psi(F)_n$. Such type of manifolds have been studied by Chern [7] in 1956.

By virtue of (3.9), we have the following;

Theorem 3.1. *A Sasakian manifold with vanishing *-Weyl curvature tensor is $\psi(F)_{2n+1}$ with associated symmetric tensor F given by (3.10).*

In view of (3.8), we have following result;

Theorem 3.2. *A Sasakian manifold with vanishing *-Weyl curvature tensor is a manifold of quasi constant curvature.*

A Riemannian manifold (M, g) is said to be semisymmetric if $R \cdot R = 0$. Now we study, Sasakian manifold with *-Weyl curvature tensor satisfying relation $R \cdot C^* = 0$ and prove that

Theorem 3.3. *If a $(2n+1)$ -dimensional Sasakian manifold satisfies $R \cdot C^* = 0$, then it is η -Einstein manifold.*

Proof. Let us consider a $(2n+1)$ -dimensional Sasakian manifold satisfying $(R(X, Y) \cdot C^*)(U, V)W = 0$. Then, by definition we have

$$\begin{aligned} R(X, Y)C^*(U, V)W - C^*(R(X, Y)U, V)W \\ - C^*(U, R(X, Y)V)W - C^*(U, V)R(X, Y)W = 0. \end{aligned}$$

Replacing X by ξ in the above equation and then taking inner product of resultant equation, we obtain

$$\begin{aligned} \eta(R(\xi, Y)C^*(U, V)W) - \eta(C^*(R(\xi, Y)U, V)W) \\ - \eta(C^*(U, R(\xi, Y)V)W) - \eta(C^*(U, V)R(\xi, Y)W) = 0. \end{aligned} \quad (3.11)$$

In view of (2.7), it follows from (3.11) that

$$\begin{aligned} C^*(U, V, W, Y) - \eta(C^*(U, V)W)\eta(Y) - g(Y, U)\eta(C^*(\xi, V)W) \\ + \eta(U)\eta(C^*(Y, V)W) - g(Y, V)\eta(C^*(U, \xi)W) + \eta(V)\eta(C^*(U, Y)W) \\ - g(Y, W)\eta(C^*(U, V)\xi) + \eta(W)\eta(C^*(U, V)Y) = 0. \end{aligned}$$

Replacing Y by U in the above equation, we have

$$\begin{aligned} C^*(U, V, W, U) - g(U, U)\eta(C^*(\xi, V)W) \\ - g(U, V)\eta(C^*(U, \xi)W) + \eta(W)\eta(C^*(U, V)U) = 0. \end{aligned} \quad (3.12)$$

By virtue of (3.3), one can easily see that

$$\begin{aligned} \eta(C^*(X, Y)Z) = -\frac{1}{2n-1}\{Ric(Y, Z)\eta(X) - Ric(X, Z)\eta(Y)\} \\ + \left(\frac{r}{2n(2n-1)} - 4\right)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \end{aligned} \quad (3.13)$$

$$\eta(C^*(X, Y)\xi) = 0, \quad (3.14)$$

$$\begin{aligned} \eta(C^*(X, \xi)Z) = \frac{1}{2n-1}Ric(X, Z) - \frac{2n}{2n-1}\eta(X)\eta(Z) \\ - \left(\frac{r}{2n(2n-1)} - 4\right)\{g(X, Z) - \eta(X)\eta(Z)\}, \end{aligned} \quad (3.15)$$

$$\sum_{i=1}^{2n+1} C^*(e_i, Y, Z, e_i) = \frac{2n(2n-2)+1}{2n-1}g(Y, Z) + \eta(Y)\eta(Z), \quad (3.16)$$

where $\{e_i\}_{i=1}^{2n+1}$ is an orthonormal basis of the tangent space at any point of the manifold. Taking $U = e_i$ in (3.12) and summing over i and making use of (3.13)-(3.16), we get

$$Ric(V, W) = \alpha g(V, W) - \beta \eta(V)\eta(W),$$

where $\alpha = \frac{20n^2-12n+1-r}{2n}$ and $\beta = \frac{20n^2-8n+1-r}{2n}$. The above relation shows that the manifold is Sasakian. This completes the proof. \square

4 *-Weyl Curvature Tensor and (κ, μ) -contact Manifolds

In [2], Blair et al. introduced and studied a new type of contact metric manifold known as a (κ, μ) -contact manifold. Later on, Boeckx [4] classified these manifolds completely. A contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be (κ, μ) -space if the curvature tensor satisfies

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \quad (4.1)$$

for all vector fields X, Y on M and for some real numbers (κ, μ) . This type of space arises through a D -homothetic deformation ([18]) to a contact metric manifold which satisfies $R(X, Y)\xi = 0$. The class of (κ, μ) -spaces covers Sasakian manifolds (for $\kappa = 1$) and the trivial sphere bundle $E^{n+1} \times S^n(4)$ (for $\kappa = \mu = 0$). There exist examples of non-Sasakian (κ, μ) -contact metric manifolds. For instance, the unit tangent bundles of Riemannian manifolds of constant curvature $\kappa \neq 1$. Since a D -homothetic deformation preserves (κ, μ) -contact structures, one can construct a lot of examples of (κ, μ) -contact structures (see [2]). The following formulas are also valid for a non-Sasakian (κ, μ) -contact manifolds [2]:

$$\begin{aligned} QX &= [2(n-1) - n\mu]X + [2(n-1) + \mu]hX \\ &+ [2(1-n) + n(2\kappa + \mu)]\eta(X)\xi, \end{aligned} \quad (4.2)$$

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X &= (1 - \kappa)[2g(X, \varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X] \\ &+ (1 - \mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX], \end{aligned} \quad (4.3)$$

$$Q\xi = 2n\kappa\xi, \quad (4.4)$$

$$h^2 = (\kappa - 1)\varphi^2, \quad \kappa < 1, \quad (4.5)$$

equality holds when $\kappa = 1$ (equivalently, $h = 0$), i.e., M is Sasakian. For the non-Sasakian case, i.e., $\kappa < 1$, the (κ, μ) -nullity condition determines the curvature of M completely. In view of this, Boeckx [4] proved that a non-Sasakian (κ, μ) -contact manifold is locally homogeneous and hence analytic. Moreover, the constant scalar curvature r of such structures is given by

$$r = 2n(2(n-1) + \kappa - n\mu),$$

which is constant. On a (κ, μ) -contact manifold we have

$$(\nabla_\xi Q)X = \mu(2(n-1) + \mu)h\varphi X, \quad (4.6)$$

for any vector field X on M . In [10], Ghosh and Patra gave a expression of *-Ricci tensor on non-Sasakian (κ, μ) -contact manifolds, which is of the form

$$Ric^*(X, Y) = (n\mu + \kappa)\{-g(X, Y) + \eta(X)\eta(Y)\}. \quad (4.7)$$

Contracting this over X provides

$$r^* = -2n(n\mu + \kappa). \quad (4.8)$$

Making use of (4.7) and (4.8) in (1.2), we ultimately have

$$\begin{aligned} C^*(X, Y)Z = & R(X, Y)Z - \frac{n\mu + \kappa}{2n - 1} \{ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(Y, Z)X \\ & + g(Y, Z)\eta(X)\xi + g(X, Z)Y - g(X, Z)\eta(Y)\xi \}. \end{aligned} \quad (4.9)$$

Let M be a (κ, μ) -contact manifold with vanishing $*$ -Weyl curvature tensor, i.e., $C^*(X, Y)Z = 0$. Relation (4.9) for $C^*(X, Y)Z = 0$ implies

$$\begin{aligned} R(X, Y)Z = & \frac{n\mu + \kappa}{2n - 1} \{ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(Y, Z)X \\ & + g(Y, Z)\eta(X)\xi + g(X, Z)Y - g(X, Z)\eta(Y)\xi \}. \end{aligned}$$

Substituting Z by ξ in the above equation, we obtain

$$R(X, Y)\xi = 0.$$

Hence by Blair's theorem (see [1], p.122) M is locally flat in dimension 3, and in higher dimension it is locally isometric to the trivial bundle $E^{n+1} \times S^n(4)$. Thus we state the following theorem;

Theorem 4.1. *Let M be a non-Sasakian (κ, μ) -contact manifold with vanishing $*$ -Weyl curvature tensor. Then M is flat for $n = 1$ and for $n > 1$, M is locally isometric to a Riemannian product $E^{n+1} \times S^n(4)$.*

A Riemannian manifold (M, g) is said to be recurrent if there exists a 1-form ω such that Riemannian curvature tensor R satisfies $\nabla R = \pi \otimes R$, where ∇ is Levi-Civita connection of g . This type of manifold appears as a generalization of symmetric manifold. In [8] Ghosh studied conformally recurrent (κ, μ) -contact manifold of dimension > 3 and show that it is locally isometric to either (i) unit sphere $S^{2n+1}(1)$ or (ii) $E^{n+1} \times S^n(4)$. Now we study, non-Sasakian (κ, μ) -contact manifold with $*$ -Weyl curvature tensor satisfying recurrent relation, i.e., $\nabla C^* = \pi \otimes C^*$ and prove that

Theorem 4.2. *If a non-Sasakian (κ, μ) -contact manifold M , ($n > 1$) satisfies $\nabla C^* = \pi \otimes C^*$, then M is locally isometric to the trivial bundle $E^{n+1} \times S^n(4)$.*

Proof. By hypothesis we have

$$(\nabla_W C^*)(X, Y)Z = \pi(W)C^*(X, Y)Z. \quad (4.10)$$

Contracting (4.10) over W provide

$$(\operatorname{div}C^*)(X, Y)Z = C^*(X, Y, Z, P), \quad (4.11)$$

where P is the recurrence vector metrically associated to the recurrence form π . Taking covariant differentiation of (4.9) along W and then contracting the resultant equation over W and using $\operatorname{Tr}h = \operatorname{Tr}\varphi h = 0$, we entails that

$$\begin{aligned} (\operatorname{div}C^*)(X, Y)Z &= g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) - \frac{n\mu + \kappa}{2n - 1} \{2g(X, \varphi Y)\eta(Z) \\ &\quad + g(X, \varphi Z + h\varphi Z)\eta(Y) - g(Y, \varphi Z + h\varphi Z)\eta(X)\}. \end{aligned} \quad (4.12)$$

Combining (4.11) and (4.12), we find that

$$\begin{aligned} g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) &= \frac{n\mu + \kappa}{2n - 1} \{2g(X, \varphi Y)\eta(Z) \\ &\quad + g(X, \varphi Z + h\varphi Z)\eta(Y) - g(Y, \varphi Z + h\varphi Z)\eta(X)\} + C^*(X, Y, Z, P). \end{aligned} \quad (4.13)$$

Next, differentiating covariantly (4.4) along an arbitrary vector field X and using (2.3) we get

$$(\nabla_X Q)\xi = Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX). \quad (4.14)$$

Substituting Z by ξ in (4.13) and making use of (4.14) we find that

$$\begin{aligned} &g(Q\varphi X + \varphi QX + Q\varphi hX + h\varphi QX - 4n\kappa\varphi X, Y) \\ &= \frac{2(n\mu + \kappa)}{2n - 1} g(X, \varphi Y) + C^*(X, Y, \xi, P). \end{aligned} \quad (4.15)$$

Replacing X by φX , Y by φY and Z by ξ in (4.9) and by virtue of (4.1), it follows that $C^*(\varphi X, \varphi Y)\xi = 0$. Thus setting $X = \varphi X$, $Y = \varphi Y$ in (4.15) and making use of last equality we obtain

$$Q\varphi X + \varphi QX - \varphi QhX - hQ\varphi X - 4n\kappa\varphi X + \frac{2(n\mu + \kappa)}{2n - 1}\varphi X = 0.$$

By virtue of (4.2), the foregoing equation reduces to

$$\kappa\mu - n\mu - 2\kappa - \mu + \frac{2(n\mu + \kappa)}{2n - 1} = 0. \quad (4.16)$$

Taking the covariant differentiation of (4.2) and making use of (2.3) gives

$$\begin{aligned} (\nabla_X Q)Y &= [2(n - 1) + \mu](\nabla_X h)Y - [2(1 - n) + n(2\kappa + \mu)] \\ &\quad \{g(\varphi X - \varphi hX, Y)\xi + \eta(Y)(\varphi X + \varphi hX)\}. \end{aligned} \quad (4.17)$$

Interchanging X and Y in (4.17) and subtracting the resultant equation with (4.17) and by virtue of (4.3) and (4.16) we find that

$$\begin{aligned} g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) &= \frac{2(n\mu + \kappa)}{2n - 1} \{2g(X, \varphi Y)\eta(Z) \\ &+ \eta(X)g(\varphi Y, Z) - \eta(Y)g(\varphi X, Z)\} + (3\mu - \mu^2 - n\mu + 2n\kappa) \\ &\quad \{\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z)\}. \end{aligned} \quad (4.18)$$

By virtue of (4.13), the foregoing equation reduces to

$$\begin{aligned} C^*(X, Y, Z, P) &= \frac{n\mu + \kappa}{2n - 1} \{2g(X, \varphi Y)\eta(Z) + \eta(X)g(\varphi Y, Z) \\ &- \eta(Y)g(\varphi X, Z)\} + \{(3\mu - \mu^2 - n\mu + 2n\kappa) - \frac{n\mu + \kappa}{2n - 1}\} \\ &\quad \{\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z)\}. \end{aligned} \quad (4.19)$$

Replacing X by φX , Y by φY and Z by ξ in (4.19), it follows that

$$n\mu + \kappa = 0, \quad (4.20)$$

where we used $C^*(\varphi X, \varphi Y)\xi = 0$. Setting $Z = P$ and $X = \xi$ in (4.19) and by virtue of (4.20) yields

$$(3\mu - \mu^2 - n\mu + 2n\kappa)g(\varphi hY, P) = 0.$$

Thus we have two possible cases:

$$(i) \quad 3\mu - \mu^2 - n\mu + 2n\kappa = 0, \quad (4.21)$$

$$(ii) \quad h\varphi P = 0. \quad (4.22)$$

Case(i). Keeping in mind that $n\mu + \kappa = 0$. Solving (4.16) and (4.21) we have the following solutions

$$\kappa = \mu = 0, \quad \kappa = \mu = n + 3 \quad \text{or} \quad \kappa = \frac{n^2 - 1}{n}, \mu = 2(1 - n).$$

When $\kappa = \mu = 0$, we obtain from (4.1) that $R(X, Y)\xi = 0$ and applying Blair's theorem we see that M is locally isometric to the product $E^{n+1} \times S^n(4)$. Since $n > 1$, the last two solutions leads to a contradiction as $\kappa < 1$.

Case(ii). Operating (4.22) by h and making use of (4.5) it follows that $P = \pi(\xi)\xi$. Together this with the condition (4.10) gives $(\nabla_W C^*)(X, Y)Z = \pi(\xi)\eta(W)C^*(X, Y)Z$. Substituting W by $\varphi^2 W$ in the last equality and contracting the resultant equation over W gives

$$(\text{div} C^*)(X, Y)Z = g((\nabla_\xi C^*)(X, Y)Z, \xi). \quad (4.23)$$

Taking covariant differentiation of (4.1) along ξ provides

$$(\nabla_{\xi}R)(X, Y)\xi = \mu^2\{\eta(Y)h\varphi X - \eta(X)h\varphi Y\}. \quad (4.24)$$

On the other hand from (4.9) and together with the help of (4.24) we have

$$g((\nabla_{\xi}C^*)(X, Y)Z, \xi) = -\mu^2\{\eta(Y)g(h\varphi X, Z) - \eta(X)g(h\varphi Y, Z)\}. \quad (4.25)$$

In view of (4.23) and (4.25) it follows that

$$(\operatorname{div}C^*)(X, Y)Z = -\mu^2\{\eta(Y)g(h\varphi X, Z) - \eta(X)g(h\varphi Y, Z)\}.$$

Making use of (4.12) in the foregoing equation yields

$$\begin{aligned} g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) &= \frac{n\mu + \kappa}{2n - 1}\{2g(X, \varphi Y)\eta(Z) \\ &+ g(X, \varphi Z + h\varphi Z)\eta(Y) - g(Y, \varphi Z + h\varphi Z)\eta(X)\} \\ &- \mu^2\{\eta(Y)g(h\varphi X, Z) - \eta(X)g(h\varphi Y, Z)\}. \end{aligned} \quad (4.26)$$

Setting $Y = \xi$ in the above equation and making use of (2.2), (4.6) and (4.14), we find that

$$\begin{aligned} Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX) + \frac{n\mu + \kappa}{2n - 1}\varphi X \\ - [\mu(2(n - 1) - n\mu) + \frac{n\mu + \kappa}{2n - 1} - \mu^2]h\varphi X = 0. \end{aligned} \quad (4.27)$$

By virtue of (4.2), the foregoing equation reduces to

$$\begin{aligned} \left\{\kappa\mu - n\mu - 2\kappa - \frac{n\mu + \kappa}{2n - 1}\right\}g(\varphi X, Y) + \{(3\mu + 2n\kappa - n\mu) \\ - \frac{n\mu + \kappa}{2n - 1}\}g(h\varphi X, Y) = 0. \end{aligned} \quad (4.28)$$

Interchanging X and Y in (4.28) and adding the resultant equation with (4.28) and by virtue of (2.2) we find that

$$(3\mu + 2n\kappa - n\mu) - \frac{n\mu + \kappa}{2n - 1} = 0. \quad (4.29)$$

Solving (4.29) and (4.16) it follows that

$\kappa = \mu = 0$ or $\kappa = \frac{(n-1)(n+3)}{n}$, $\mu = \frac{2(n-1)(n+3)}{n-3}$, where we used $n\mu + \kappa = 0$, (in the last solution $n \neq 3$, because if $n = 3$, then from (4.29) it follows that $\kappa = 0$ and hence $\mu = 0$). The first solution shows that M is locally isometric to the product $E^{n+1} \times S^n(4)$. The last solution leads to a contradiction as $\kappa < 1$. This completes the proof. \square

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