

WEYL GROUPS, THE HARD LEFSCHETZ THEOREM, AND THE SPERNER PROPERTY*

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Abstract. Techniques from algebraic geometry, in particular the hard Lefschetz theorem, are used to show that certain finite partially ordered sets Q^X derived from a class of algebraic varieties X have the k -Sperner property for all k . This in effect means that there is a simple description of the cardinality of the largest subset of Q^X containing no $(k+1)$ -element chain. We analyze, in some detail, the case when $X = G/P$, where G is a complex semisimple algebraic group and P is a parabolic subgroup. In this case, Q^X is defined in terms of the "Bruhat order" of the Weyl group of G . In particular, taking P to be a certain maximal parabolic subgroup of $G = SO(2n+1)$, we deduce the following conjecture of Erdős and Moser: Let S be a set of $2\ell+1$ distinct real numbers, and let T_1, \dots, T_k be subsets of S whose element sums are all equal. Then k does not exceed the middle coefficient of the polynomial $2(1+q)^2(1+q^2)^2 \cdots (1+q^\ell)^2$, and this bound is best possible.

1. The Sperner property. Let P be a finite partially ordered set (or *poset*, for short), and assume that every maximal chain of P has length n . We say that P is *graded of rank n* . Thus P has a unique *rank function* $\rho: P \rightarrow \{0, 1, \dots, n\}$ satisfying $\rho(x) = 0$ if x is a minimal element of P , and $\rho(y) = \rho(x) + 1$ if y covers x in P (i.e., if $y > x$ and no $z \in P$ satisfies $y > z > x$). If $\rho(x) = i$, then we say that x has *rank i* . Define $P_i = \{x \in P: \rho(x) = i\}$ and set $p_i = p_i(P) = \text{card } P_i$. The polynomial $F(P, q) = p_0 + p_1q + \cdots + p_nq^n$ is called the *rank-generating function* of P . We say that P is *rank-symmetric* if $p_i = p_{n-i}$ for all i , and that P is *rank-unimodal* if $p_0 \leq p_1 \leq \cdots \leq p_i \geq p_{i+1} \geq \cdots \geq p_n$ for some i .

An *antichain* (also called a *Sperner family* or *clutter*) is a subset A of P , such that no two distinct elements of A are comparable. The poset P is said to have the *Sperner property* (or *property S_1*) if the largest size of an antichain is equal to $\max \{p_i: 0 \leq i \leq n\}$. More generally, if k is a positive integer then P is said to have the *k -Sperner property* (or *property S_k*) if the largest subset of P containing no $(k+1)$ -element chain has cardinality $\max \{p_{i_1} + \cdots + p_{i_k}: 0 \leq i_1 < \cdots < i_k \leq n\}$. If P has property S_k for all $k \leq n$, then following [21] we say that P has *property S*. For further information concerning the Sperner property and related concepts, see for instance [15], [16], [17].

Using some results from algebraic geometry, we will give several new classes of graded posets which have property S. These posets will all be rank-symmetric and rank-unimodal. First we must consider a property of posets related to property S. Suppose P is graded of rank n and is rank-symmetric. Again following [21], we say that P has *property T* if for all $0 \leq i \leq [n/2]$, there exist p_i pairwise disjoint saturated chains $x_i < x_{i+1} < \cdots < x_{n-i}$ where $x_j \in P_j$. It is clear that P is then rank-unimodal.

LEMMA 1.1. *Let P be a finite graded rank-symmetric poset of rank n . The following three conditions are equivalent:*

- (i) P is rank-unimodal and has property S.
 - (ii) P has property T.
 - (iii) Let V_i be the complex vector space with basis P_i . Then for $0 \leq i < n$, there exist linear transformations $\varphi_i: V_i \rightarrow V_{i+1}$ satisfying the following two properties:
 - (a) If $0 \leq i \leq [n/2]$, then the composite transformation $\varphi_{n-i-1}\varphi_{n-i-2} \cdots \varphi_{i+1}\varphi_i: V_i \rightarrow V_{n-i}$ is invertible.
 - (b) Let $x \in P_i$ and $\varphi_i(x) = \sum_{y \in P_{i+1}} c_y y$. Then $c_y = 0$ unless $x < y$.
- Proof.* (i) \Leftrightarrow (ii). This is a special case of [21, Thms. 2 and 3].

* Received by the editors June 1, 1979.

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(iii) \Rightarrow (ii). (I am grateful to Joseph Kung for supplying the following argument, which is considerably simpler than my original proof.) Assume (iii). Identify φ_i with its matrix with respect to the bases P_i and P_{i+1} . If φ is a matrix whose rows are indexed by a set S and whose columns are indexed by T , and if $S' \subset S$ and $T' \subset T$, then let $\varphi[S', T']$ denote the submatrix of φ with rows indexed by S' and columns by T' . By the Binet-Cauchy theorem (e.g., [1, § 36]) we have

$$\det(\varphi_{n-i-1} \cdots \varphi_i) = \sum (\det \varphi_i[Q_i, Q_{i+1}]) \cdot (\det \varphi_{i+1}[Q_{i+1}, Q_{i+2}]) \cdots (\det \varphi_{n-i-1}[Q_{n-i-1}, Q_{n-i}]),$$

where the sum is over all sequences of subsets $Q_i = P_i, Q_{i+1} \subset P_{i+1}, Q_{i+2} \subset P_{i+2}, \dots, Q_{n-i-1} \subset P_{n-i-1}, Q_{n-i} = P_{n-i}$ such that $|Q_{i+1}| = |Q_{i+2}| \cdots = |Q_{n-i-1}| = p_i$. By (a), some term in the above sum is nonzero. Hence, the expansion of each factor $\det \varphi_k[Q_k, Q_{k+1}]$ in this term contains a nonzero term. By (b), this nonzero term defines a map $\sigma: Q_k \rightarrow Q_{k+1}$ such that $x < \sigma(x)$ for all $x \in Q_k$. Piecing together these two-element chains over all k yields (ii).

(ii) \Rightarrow (iii). The steps of the above argument can be reversed, provided we pick the φ_i 's as generically as possible, i.e., all the entries of the matrices $\varphi_0, \varphi_1, \dots, \varphi_{n-1}$ should be chosen to be algebraically independent over \mathbb{Q} , except for entries forced to equal 0 by condition (b). This completes the proof. \square

2. Varieties with cellular decompositions. We now are in a position to invoke algebraic geometry. Let X be a complex projective variety of complex dimension n . Suppose that there are finitely many pairwise-disjoint subsets C_i of X , each isomorphic as an algebraic variety to complex affine space of some dimension n_i , such that (i) the union of the C_i 's is X , and (ii) $\bar{C}_i - C_i$ is a union of some of the C_j 's. (Here \bar{C}_i denotes the closure of C_i either in the Hausdorff or Zariski topology—under the present circumstances the two closures coincide.) Following [4, p. 500], we then say that the C_i 's form a *cellular decomposition* of X . The simplest and most familiar example is complex projective space \mathbb{P}^n itself. Recall that \mathbb{P}^n may be regarded as the set of nonzero $(n+1)$ -tuples $x = (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$, modulo the equivalence relation $x \sim \lambda x$ ($\lambda \in \mathbb{C}^*$). The set of elements of \mathbb{P}^n of the form $(0, \dots, 0, 1, x_{n-i+1}, \dots, x_n)$ forms a subvariety isomorphic to \mathbb{C}^i . Hence we have the cellular decomposition $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \dots \cup \mathbb{C}^0$.

If X is any complex projective variety and Y is a closed subvariety, then e.g., by [4] or [18, Chap. 5, § 4], Y represents an element (cocycle) $[Y]$ of the cohomology group $H^*(X, \mathbb{C})$. If X is irreducible of (complex) dimension n , and Y is irreducible of dimension m , then in fact $[Y] \in H^{2(n-m)}(X, \mathbb{C})$. If X is irreducible of dimension n and has a cellular decomposition $\{C_i\}$, it follows that the closures \bar{C}_i represent cohomology classes $[\bar{C}_i] \in H^{2(n-m)}(X, \mathbb{C})$ where $C_i \cong \mathbb{C}^m$. (For this fact, we don't need condition (ii) in our definition of cellular decomposition.) The following fundamental result concerning varieties with a cellular decomposition appears in [4, p. 501], [22, § 6] in the case when X is nonsingular. The extension to singular varieties follows from [14]. (Again, condition (ii) is not actually necessary.)

THEOREM 2.1. *Let X be a complex projective variety of complex dimension n , and suppose that X has a cellular decomposition $\{C_i\}$. Then the cohomology classes $[\bar{C}_i]$ form a basis (over \mathbb{C}) for $H^*(X, \mathbb{C})$. In particular, $H^{2m+1}(X, \mathbb{C}) = 0$ for all $m \in \mathbb{Z}$, while if X is irreducible then $H^{2(n-m)}(X, \mathbb{C})$ has a basis consisting of those classes $[\bar{C}_i]$ for which $C_i \cong \mathbb{C}^m$. \square*

Now given a cellular decomposition $\{C_i\}$ of X , define a partial ordering $Q^X = Q^X(C_1, C_2, \dots)$ on the C_i 's by setting $C_i \cong C_j$ in Q^X if $C_i \subset \bar{C}_j$. If X is irreducible of

dimension n , then it can be shown, using standard techniques from algebraic geometry, that Q^X is graded of rank n , with the rank function given by $\rho(C) = n - \dim C$. If, moreover, X is nonsingular, then Poincaré duality implies that Q^X is rank-symmetric. Theorem 2.1 then implies that we may identify the vector space V_i of Lemma 1.1 (iii) with $H^{2i}(X, \mathbb{C})$ by identifying $C \in Q_i^X$ with $[\bar{C}] \in H^{2i}(X, \mathbb{C})$.

We now wish to define linear transformations $\varphi_i: V_i \rightarrow V_{i+1}$ (or equivalently, $\varphi_i: H^{2i}(X, \mathbb{C}) \rightarrow H^{2(i+1)}(X, \mathbb{C})$) satisfying conditions (a) and (b) of Lemma 1.1 (iii). This will enable us to conclude that Q^X has property *S*. Let Y be a hyperplane section of X , i.e., the intersection of X (regarded as being imbedded in some projective space \mathbb{P}^N) with a hyperplane of \mathbb{P}^N . If X is irreducible, then Y is a closed subvariety of X of dimension $n - 1$ which represents a cohomology class $[Y] \in H^2(X, \mathbb{C})$. The cup product operation on cohomology then yields a linear transformation $\varphi_i: H^{2i}(X, \mathbb{C}) \rightarrow H^{2(i+1)}(X, \mathbb{C})$ defined as multiplication by $[Y]$. In other words, $\varphi_i(K) = [Y] \cdot K$. We now verify that when X is nonsingular and irreducible (so Q^X is graded and rank-symmetric), then these linear transformations φ_i satisfy conditions (a) and (b) of Lemma 1.1 (iii). First we dispose of condition (b). I am grateful to Steve Kleiman for providing a proof of this result.

LEMMA 2.2. *Let X be a complex projective variety with a cellular decomposition $\{C_i\}$, and let Y be a hyperplane section (or in fact any closed subvariety) of X . If $[Y] \cdot [\bar{C}_i] = \sum \alpha_j [\bar{C}_j]$ in $H^*(X, \mathbb{C})$, then $\alpha_j = 0$ unless $C_j \subset \bar{C}_i$.*

Proof. Let $A(W)$ denote the Chow group of the variety W , i.e., the group of cycles modulo rational equivalence. If W is nonsingular and has a cellular decomposition $\{D_i\}$, then it is mentioned in [22, § 6] that the cycles \bar{D}_i form a basis for $A(W)$, and that the corresponding map $A(W) \rightarrow H^*(W, \mathbb{Z})$ is an isomorphism of groups. It follows from [14] that this result continues to hold when W is singular. Now returning to our hypotheses, the C_j 's contained in \bar{C}_i form a cellular decomposition of \bar{C}_i . Hence a hyperplane section of \bar{C}_i is rationally equivalent to a linear combination of the \bar{C}_j that are contained in \bar{C}_i . A priori, the rational equivalence is on \bar{C}_i , but it may be considered as a rational equivalence on X . Hence $\alpha_j = 0$ unless $C_j \subset \bar{C}_i$ because the $[\bar{C}_i]$ are linearly independent in $H^*(X, \mathbb{C})$. \square

Lemma 2.2 shows that condition (b) of Lemma 1.1 (iii) holds for Q^X (assuming X is nonsingular and irreducible, so we know Q^X is graded and rank-symmetric). Condition (a) is implied by the following basic result, known as the "hard Lefschetz theorem" (although the first rigorous proof was given by Hodge). See [34] for a brief history and survey of this theorem, and for its extension to characteristic p . Other references include [24, p. 187], [29], [10, Corollary, p. 75], [30, p. 44], [19, Chap. 0, § 7].

LEMMA 2.3 (the hard Lefschetz theorem). *Let X be a nonsingular irreducible complex projective variety of complex dimension n . Let Y be a hyperplane section of X . If $0 \leq i \leq n$, then the linear transformation $H^i(X, \mathbb{C}) \rightarrow H^{2n-i}(X, \mathbb{C})$ given by multiplication by $[Y]^{n-i}$ is an isomorphism.*

Putting Lemmas 1.1, 2.2, and 2.3 together, we obtain the main result of this paper.

THEOREM 2.4. *Let X be a nonsingular irreducible complex projective variety of complex dimension n with a cellular decomposition $\{C_i\}$. Then Q^X is graded of rank n , rank-symmetric, rank-unimodal, and has property *S*.*

For future use, we record the following simple result. The proof is evident.

PROPOSITION 2.5. *Let X and Y be complex projective varieties, with cellular decompositions $\{C_i\}$ and $\{D_j\}$ respectively. Then the product variety $X \times Y$ has a cellular decomposition with cells $C_i \times D_j$, and $Q^{X \times Y} \cong Q^X \times Q^Y$.*

It follows from Theorem 2.4 and Proposition 2.5 that if $P = Q^X$ and $P' = Q^Y$ for nonsingular irreducible complex projective varieties X and Y , each having a cellular

decomposition, then $P \times P'$ has property S. More generally, Canfield [7] and independently Proctor, Saks, and Sturtevant [36] have shown that the product $P \times P'$ of any two graded, rank-symmetric, rank-unimodal posets P and P' , each with property S, also has property S. (An even more general result has subsequently been proved by Saks [37].) For our purposes, however, it suffices to consider only Proposition 2.5.

3. Weyl groups. It remains to find interesting examples of varieties X with cellular decompositions and to describe the resulting posets Q^X . The best known examples of such varieties are the following. Let G be a complex semisimple algebraic group, and let P be a *parabolic subgroup* of G (i.e., a closed subgroup which contains a maximal solvable subgroup B of G . B is known as a *Borel subgroup*.) Then the coset space G/P has the structure of a non-singular irreducible complex projective variety, and the Bruhat decomposition of G affords a cellular decomposition $\{C_i\}$ of G/P . The cells C_i are known as *generalized Schubert cells*. See [5, § 3] for further details.

When $X = G/P$, a description of the poset Q^X can be given in terms of the Weyl groups W of G , and W_J of P [5, § 3], [11] as follows. Every Weyl group W is a finite Coxeter group, i.e., W is a finite group with a finite set $S = \{s_1, \dots, s_m\}$ of generators such that for all $1 \leq k \leq m$, $1 \leq i < j \leq m$ and certain integers $n_{ij} \geq 2$, W is defined by the relations $s_k^2 = 1$ and $(s_i s_j)^{n_{ij}} = 1$. The pair (W, S) is called a *Coxeter system*.

A *parabolic subgroup* of W (with respect to S) is any subgroup W_J generated by a subset J of S . Thus $W_\emptyset = \{1\}$ and $W_S = W$. The *length* $\ell(w)$ of an element $w \in W$ is the smallest integer $q \geq 0$ for which w is a product of q elements of S . Define a partial order, called the *Bruhat order*, on W as follows. We say $w \leq w'$ if there exist conjugates t_1, \dots, t_j of the elements of S such that $w' = wt_1 t_2 \cdots t_j$ and $\ell(wt_1 t_2 \cdots t_{i+1}) > \ell(wt_1 t_2 \cdots t_i)$ for all $0 \leq i < j$. The following properties (among others) of the Bruhat order of a finite Coxeter group W are known:

1. The Bruhat order makes W into a graded poset (which we still call W).
2. The function ℓ is the rank function of W , and the rank-generating function of W is given by

$$(1) \quad F(W, q) = \prod_{i=1}^m (1 + q + q^2 + \cdots + q^{e_i})$$

for certain positive integers e_i known as the *exponents* of W . One may regard (1) as the definition of the exponents. For other equivalent definitions, see, e.g., [6, Chap. 5, § 6.2] or [8, Chap. 10]. Note that (1) implies the well-known fact that $|W| = \prod (e_i + 1)$, and that W has rank $e_1 + \cdots + e_m$.

3. If $J \subset S$, then each coset wW_J of W_J in W contains a unique element w_J of minimal length. For any $v \in W_J$ we have $\ell(w_J v) = \ell(w_J) + \ell(v)$.

4. Let W^J be the set of minimal length coset representatives w_J . Then W^J is a graded subset of W such that the rank function of W^J is the restriction of the rank function of W .

5. (W_J, J) is itself a finite Coxeter system, say with exponents f_1, \dots, f_r . Then W^J has the rank-generating function

$$(2) \quad F(W^J, q) = \frac{F(W, q)}{F(W_J, q)} = \frac{\prod_{i=1}^m (1 + q + q^2 + \cdots + q^{e_i})}{\prod_{j=1}^r (1 + q + q^2 + \cdots + q^{f_j})}.$$

For proofs of these results and further information on Coxeter groups, see e.g., [6], [8], [11]. For a connection between the posets W^J and combinatorics, different from the one given here, see [23].

Now we return to the varieties $X = G/P$, where G is a complex semisimple algebraic group and P a parabolic subgroup of G . It is known [6, p. 29], [5, § 3] that the parabolic subgroups of G containing a given Borel subgroup B are in one-to-one

correspondence with the parabolic subgroups W_J of the Weyl group W of G (with respect to a fixed set S of Coxeter generators of W). Moreover, the poset Q^X corresponding to the cellular decomposition of $X = G/P$ obtained from the Bruhat decomposition of G is isomorphic to the partial order on W^J defined above. Hence from Theorem 2.4 we conclude:

THEOREM 3.1. *Let (W, S) be a Coxeter system for which W is a Weyl group. Let $J \subset S$ and let W^J be the poset defined above. Then W^J is rank-symmetric, rank-unimodal, and has property S.*

A Coxeter system (W, S) is *irreducible* if one cannot write S as a nontrivial disjoint union $T \cup T'$ such that $W = W_T \times W_{T'}$. If (W, S) is reducible, say $W = W_T \times W_{T'}$, then we also have $W = W_T \times W_{T'}$ as posets, and similarly for W^J . Thus by Proposition 2.5 nothing is lost by considering only irreducible Coxeter systems. Now all finite irreducible Coxeter systems are known (e.g., [6, p. 193]). There are the infinite families of type $A_n (n \geq 1)$, $B_n (n \geq 2)$, and $D_n (n \geq 4)$, together with seven "exceptional" systems $E_6, E_7, E_8, F_4, G_2, H_3, H_4$ and the dihedral groups $I_2(p)$ of order $2p$ for $p = 5$ or $p \geq 7$. ($I_2(3)$ coincides with A_2 , $I_2(4)$ with B_2 , and $I_2(6)$ with G_2 .) For all of these systems (W, S) , W is a Weyl group except for $H_3, H_4, I_2(p), p = 5$ or $p \geq 7$. It is easy to check that Theorem 3.1 remains valid for the dihedral groups $I_2(p)$, and for H_3 . Presumably the remaining case H_4 can also be checked directly, so in fact one could determine those finite Coxeter systems (probably all of them) for which Theorem 3.1 remains valid.

4. Type A_n . We now want to describe the posets W^J in greater detail, for the types A_n, B_n, D_n . First consider A_{n-1} . Then W is the symmetric group \mathfrak{S}_n of all permutations of $\{1, 2, \dots, n\}$. The exponents are $1, 2, \dots, n-1$, and as Coxeter generators we may take the "adjacent transpositions" $s_i = (i, i+1), 1 \leq i \leq n-1$. Regard a permutation $\pi \in \mathfrak{S}_n$ as a linear array $a_1 a_2 \dots a_n$, where $\pi(i) = a_i$. Then a direct translation of the definition of the Bruhat order yields the following: $\pi \leq \sigma$ in W if σ can be obtained from π by a sequence of operations which interchange i and j in a permutation $a_1 a_2 \dots a_n$ provided i appears to the left of j and $i < j$. We abbreviate this operation as

$$(3) \quad i < j \longrightarrow j > i.$$

Thus the notation " $i < j$ " in (3) means that i and j appear in the given order (i.e., i to the left of j) and $i < j$. For instance, $213 \leq 312$ (obtained by $2 < 3 \longrightarrow 3 > 2$) and $24153 \leq 35241$ (obtained, e.g., by $2 < 3 \rightarrow 3 < 2, 1 < 2 \rightarrow 2 > 1, 4 < 5 \rightarrow 5 > 4$). The rank $\ell(\pi)$ of $\pi = a_1 a_2 \dots a_n \in W$ is equal to the number $i(\pi)$ of *inversions* of π , i.e., the number of pairs (i, j) for which $i < j$ and $a_i > a_j$. Thus $12 \dots n$ is the unique permutation of rank 0 and $n \dots 21$ is the unique permutation of highest rank $\binom{n}{2}$. It is well-known (e.g., [9, § 6.4]) that

$$\sum_{\pi \in \mathfrak{S}_n} q^{i(\pi)} = (1+q)(1+q+q^2) \dots (1+q+\dots+q^{n-1}),$$

which of course agrees with (1). Figure 1 depicts the Bruhat order of \mathfrak{S}_3 .

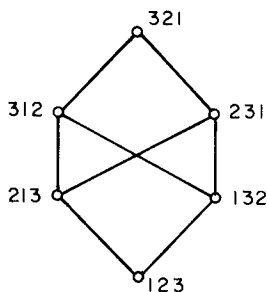


FIG. 1

Now let $J \subset S = \{s_1, \dots, s_{n-1}\}$ where $s_i = (i, i + 1)$. If we let $\mathfrak{S}(a, b)$ denote the group of all permutations of $\{a, a + 1, \dots, b\}$, then it is clear that $W_J = \mathfrak{S}(1, c_1) \times \mathfrak{S}(c_1 + 1, c_2) \times \dots \times \mathfrak{S}(c_{j-1} + 1, n)$ for some integers $1 \leq c_1 < c_2 < \dots < c_{j-1} < n$, where $j = n - |J|$. If $\pi = a_1 a_2 \dots a_n \in W$, then the coset πW_J consists of all $c_1!(c_2 - c_1)! \dots (n - c_{j-1})!$ permutations obtained from π by permuting among themselves the elements within the sets $N_1 = \{1, 2, \dots, c_1\}$, $N_2 = \{c_1 + 1, \dots, c_2\}$, \dots , $N_j = \{c_{j-1} + 1, \dots, n\}$. The coset representative $\pi_J \in \pi W_J$ with the least number of inversions is that element of πW_J for which the elements of the above sets N_i appear in their natural order. Hence W^J consists of those $n!/c_1!(c_2 - c_1)! \dots (n - c_{j-1})!$ permutations for which the elements of each of the sets N_i appear in their natural order; or, as it is sometimes called, the set of *shuffles* of N_1, \dots, N_j . The rank-generating function of W^J is given by

$$(4) \quad F(W^J, q) = \frac{(\mathbf{n})!}{(\mathbf{c}_1)!(\mathbf{c}_2 - \mathbf{c}_1)! \dots (\mathbf{n} - \mathbf{c}_{j-1})!}$$

where $(\mathbf{k})! = (1 - q)(1 - q^2) \dots (1 - q^k)$. The right-hand side of (4) is known as a *q-multinomial coefficient* and is commonly denoted $\left[\begin{matrix} n \\ c_1, c_2 - c_1, \dots, n - c_{j-1} \end{matrix} \right]$. Figure 2 illustrates the poset W^J in the case $n = 4$, $J = \{(12)\}$.

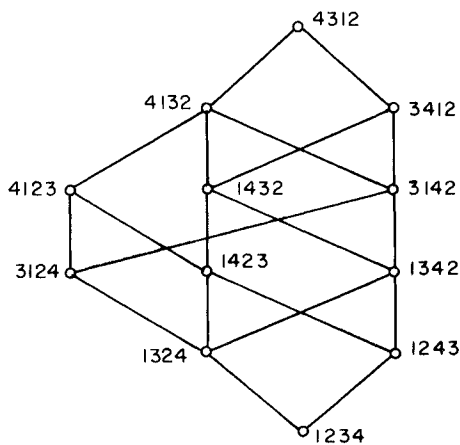


FIG. 2

If we take W_J to be a *maximal* parabolic subgroup above, i.e., $|J| = n - 2$, then the poset W^J has an interesting alternative description. Suppose $J = S - \{(n - k, n - k + 1)\}$, so $N_1 = \{1, 2, \dots, n - k\}$ and $N_2 = \{n - k + 1, \dots, n\}$. If $\pi = a_1 a_2 \dots a_n \in W^J$ and $1 \leq i \leq k$, then set

$$(5) \quad \ell_i(\pi) = \text{card} \{j : j \text{ appears to the right of } n - i + 1 \text{ and } j < n - i + 1\}.$$

Clearly $\ell(\pi) = \sum_{i=1}^k \ell_i(\pi)$. The mapping $\pi \mapsto (\ell_1(\pi), \dots, \ell_k(\pi))$ is a bijection between W^J and all integer sequences $0 \leq \ell_1 \leq \dots \leq \ell_k \leq n - k$. Moreover, $\pi \leq \pi'$ in W^J if and only if $\ell_i \leq \ell'_i$ for $1 \leq i \leq k$. Hence, W^J is isomorphic to the poset of all partitions of integers into at most k parts, with largest part at most $n - k$, i.e., a partition whose Ferrers diagram (e.g., [9, § 2.4]) fits into a $k \times (n - k)$ rectangle. These partitions are ordered by inclusion of their Ferrers diagrams. Since the union and intersection of Ferrers diagrams is again a Ferrers diagram, it follows that the poset W^J is actually a distributive lattice, which we will denote by $L(k, n - k)$. Figure 3 depicts $L(2, 3)$.

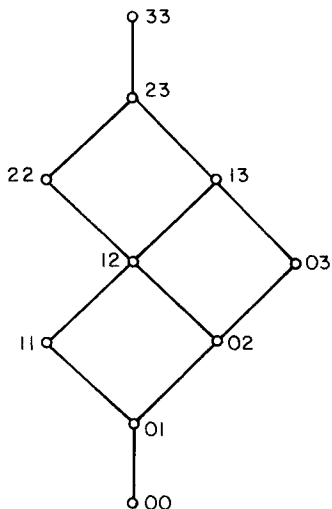


FIG. 3

In terms of the characterization [3, Thm. 3, p. 46] of a finite distributive lattice L as the lattice 2^P of semi-ideals (also called “order ideals” or “decreasing subsets”) of a poset P , we have $L(k, n - k) = 2^{k \times (n - k)}$, where i denotes an i -element chain. The rank-generating function of this lattice is the q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(n)!}{(k)!(n - k)!}$. It is by no means a priori obvious that W^J is rank-unimodal; this was first shown essentially by Sylvester in 1878 (see [40] for historical details) and no combinatorial proof is known. I am grateful to Tony Iarrobino for originally calling to my attention that the hard Lefschetz theorem implies the unimodality of the coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}$. It was my attempt to understand this fact which eventually led to the present paper.

By applying Theorem 3.1 to the lattice $L(k, m)$, we can deduce a “multiset analogue” to a conjecture of Erdős and Moser [13, (12)]. (Regarding their actual conjecture, see Corollary 5.3 below.) I am grateful to Raneen Gupta for her comments on this result.

COROLLARY 4.1. *Fix positive integers k, m , and j . Let $A = \{a_0, a_1, \dots, a_m\}$ be a set of $m + 1$ distinct real numbers. Let B_1, \dots, B_r be subsets of A with exactly k elements **with repeated elements allowed**. (One may think of B_s as being an $m + 1$ -tuple $(\alpha_0, \dots, \alpha_m)$ of nonnegative integers such that $\sum \alpha_i = k$, where α_i is the number of repetitions of a_i .) Let $\sum B_s$ denote the sum of the elements of B_s , i.e., $\sum B_s = \sum \alpha_i a_i$. Suppose that there are at most j distinct numbers among $\sum B_1, \dots, \sum B_r$. Then r is less than or equal to the sum of the j middle coefficients of the polynomial $\begin{bmatrix} m + k \\ k \end{bmatrix}$. Moreover, this value of r is achieved by taking $A = \{0, 1, \dots, m\}$ and B_1, \dots, B_r to have element sums consisting of the j middle elements of the set $\{0, 1, \dots, km\}$. (If $km - j$ is even, then there are two equivalent choices of the “ j middle coefficients” and “ j middle elements.”)*

Proof. Regarding $B_s = (\alpha_0, \dots, \alpha_m)$ associate with B_s the sequence $\lambda_s = (\ell_1, \dots, \ell_k) \in L(k, m)$ defined by setting exactly α_i of the ℓ_k 's equal to i . It is easy to see that the subset $\{\lambda_1, \dots, \lambda_r\}$ of $L(k, m)$ contains no $(j + 1)$ -element chain provided there are only j distinct numbers among $\sum B_1, \dots, \sum B_r$. The proof now follows from Theorem 3.1 and the fact that the rank-generating function of $L(k, m)$ is $\begin{bmatrix} k + m \\ k \end{bmatrix}$. \square

As a variation of the preceding corollary, we have

COROLLARY 4.2. Fix positive integers k, m , and j . Let $A' = \{a_1, \dots, a_m\}$ be a set of m distinct nonzero real numbers. Let B_1, \dots, B_r be subsets of A' with **at most** k elements with repeated elements allowed. Suppose that there are at most j distinct numbers among $\sum B_1, \dots, \sum B_r$. Then r is less than or equal to the sum of the j middle coefficients of the polynomial $\begin{bmatrix} m+k \\ k \end{bmatrix}$. Moreover, this value of r is achieved by taking $A' = \{1, \dots, m\}$ and B_1, \dots, B_r to have element sums consisting of the j middle elements of the set $\{0, 1, \dots, km\}$.

Proof. Apply Corollary 4.1 to the set $A = A' \cup \{0\}$. \square

Remark. The cellular decomposition of G/P in the case $W(G) = \mathfrak{S}_n$ and $W(P) = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$ can be described quite concretely. The group G is given by $SL(n, \mathbb{C})$, which acts linearly on n -dimensional complex projective space \mathbb{P}^{n-1} . Let V be a $(k-1)$ -dimensional subspace (or $(k-1)$ -plane) of \mathbb{P}^{n-1} , and let P be the subgroup of G leaving V invariant. (Then P is a maximal parabolic subgroup of G .) The coset ϕP transforms V into the subspace ϕV , and this sets up a one-to-one correspondence between $X = G/P$ and the $(k-1)$ -planes in \mathbb{P}^{n-1} . Hence X is the *Grassmann manifold* $G(k-1, n-1)$ of all $(k-1)$ -planes in \mathbb{P}^{n-1} . Regard the elements of \mathbb{P}^{n-1} as (equivalence classes of) n -tuples $(x_1, \dots, x_n) \in \mathbb{C}^n - \{0\}$. A $(k-1)$ -plane V in \mathbb{P}^{n-1} has a unique ordered basis

w_1, \dots, w_k for which the matrix $\begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}$ is in row-reduced echelon form. Choose integers

$0 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n-k$, and suppose we specify that for each i , the first 1 in w_i occurs in coordinate $a_i + i$. The set of all such V forms a subset $C(a_1, \dots, a_k)$ of $G(k-1, n-1)$ isomorphic to $\mathbb{C}^{k(n-k)-a_1-\dots-a_k}$; indeed, there are $n-k-a_i$ coordinates in w_i which can be specified arbitrarily, and the remaining coordinates are pre-determined. By considering all sequences $0 \leq a_1 \leq \dots \leq a_k \leq n-k$, we obtain a cellular decomposition of $G(k-1, n-1)$. Thus the cells $C(a_1, \dots, a_k)$ are in one-to-one correspondence with the elements (a_1, \dots, a_k) of $L(k, n-k)$. For instance, when $k=2$ and $n=4$ the cells correspond to the following row-reduced echelon matrices:

$$\begin{matrix} \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, & \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, & \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ C(0, 0) & C(0, 1) & C(0, 2) \\ \\ \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, & \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \\ C(1, 1) & C(1, 2) & C(2, 2) \end{matrix}$$

A little thought shows that $\overline{C(a_1, \dots, a_k)} \supset C(b_1, \dots, b_k)$ if and only if $a_i \leq b_i$ for $1 \leq i \leq k$. Thus we see directly that $Q^X \cong L(k, n-k)$. The closure of the cell $C(a_1, \dots, a_k)$ is called a *Schubert variety*, and its cohomology class is called a *Schubert cycle*, which we shall denote by $\Omega(a_1, \dots, a_k)$. (A more common notation is $\Omega(a'_1, \dots, a'_k)$ where $a'_i = n-k+i-1-a_{k-i+1}$.) The Schubert cycle $\omega = \Omega(0, 0, \dots, 0, 1) \in H^2(X, \mathbb{C})$ turns out to be the class of a hyperplane section. According to a special case of Pieri's formula in the Schubert calculus, the product of $\Omega(a_1, \dots, a_k)$ with ω in $H^*(X, \mathbb{C})$ is equal to the sum of all $\Omega(b_1, \dots, b_k)$ such that $b_i \geq a_i$ and $\sum b_i = 1 + \sum a_i$. In other words, $\omega \cdot \Omega(a_1, \dots, a_k) = \sum \Omega(b_1, \dots, b_k)$, where the sum is over all sequences (b_1, \dots, b_k) covering (a_1, \dots, a_k) in $L(k, n-k)$. Thus we

have a direct verification of Lemma 2.2. For further information on these matters, see, for example, [26], [27], [41].

5. Type B_n . We next turn our attention to type B_n . In this case W is the group of all $n \times n$ signed permutation matrices (i.e., matrices with entries $0, \pm 1$ with one nonzero entry in every row and column). W has order $2^n n!$ and exponents $1, 3, 5, \dots, 2n - 1$. Identify the matrix $(m_{ij}) \in W$ with the ordered pair (π, ε) , where $\pi \in \mathfrak{S}_n$ is given by $m_{i, \pi(i)} = \pm 1$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$ by $\varepsilon_i = m_{i, \pi(i)}$. We then have the multiplication rule $(\pi, \varepsilon)(\pi', \varepsilon') = (\pi\pi', \delta)$, where $\delta_i = \varepsilon_{\pi'(i)}\varepsilon'_i$. We sometimes will abbreviate a group element such as $(24513, (-1, 1, -1, -1, 1))$ by $\bar{2} \ 4 \ \bar{5} \ \bar{1} \ 3$, and thus regard W as consisting of all “barred permutations” of $\{1, 2, \dots, n\}$. For the Coxeter generators of W we take the set $S = \{s_1, \dots, s_n\}$, where s_i is the adjacent transposition $(i, i + 1)$, $1 \leq i \leq n - 1$, and $s_n = \bar{1} \ 2 \ 3 \ \dots \ n$. A little thought shows that $\pi \leq \sigma$ in W if σ can be obtained from π by a sequence of the following seven types of operations on barred permutations:

- a) $i \longrightarrow \bar{i}$,
- b) $i < j \longrightarrow j > i$,
- c) $\bar{i} < j \longrightarrow j > \bar{i}$,
- d) $\bar{i} < j \longrightarrow \bar{j} > i$,
- e) $\bar{i} > j \longrightarrow j < \bar{i}$,
- f) $i > \bar{j} \longrightarrow j < \bar{i}$,
- g) $\bar{i} > \bar{j} \longrightarrow \bar{j} < \bar{i}$.

For instance, Fig. 4 illustrates W when $n = 2$.

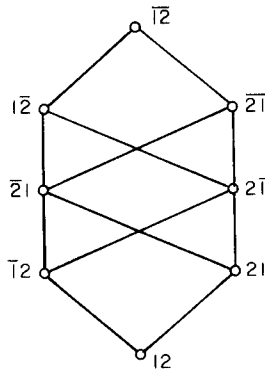


FIG. 4

If $(\pi, \varepsilon) \in W$, then one can check that

$$(6) \quad \ell(\pi) = i(\pi) + \sum_j (2d_j + 1),$$

where $i(\pi)$ is the number of inversions of π , j ranges over all integers for which $\varepsilon_j = -1$, and d_j is the number of k 's appearing in $\pi = a_1 a_2 \dots a_n$ to the left of a_j for which $k < a_j$. For instance, $\ell(\bar{3} \ 1 \ 5 \ \bar{4} \ 2) = 11$, since $i(\pi) = 5$, $d_1 = 0$, $d_4 = 2$. It is easy to give a direct combinatorial proof that

$$\sum_{\pi \in W} q^{\ell(\pi)} = \prod_{i=1}^n (1 + q + q^2 + \dots + q^{2i-1}),$$

agreeing with (1).

Now let $J \subset S$. Let $\bar{\mathfrak{S}}(a, b)$ denote the group of all signed permutations of $\{a, a + 1, \dots, b\}$. Then W_J has the form

$$(7) \quad W_J = \bar{\mathfrak{S}}(1, c_1) \times \mathfrak{S}(c_1 + 1, c_2) \times \mathfrak{S}(c_2 + 1, c_3) \times \dots \times \mathfrak{S}(c_{j-1} + 1, n),$$

where $0 \leq c_1 < c_2 < \dots < c_{j-1} < n$. The case $c_1 = 0$ corresponds to $s_n \notin J$. If $c_1 = 0$ then $j = n - |J|$; otherwise $j = n - |J| + 1$. Set $N_1 = \{1, 2, \dots, c_1\}$, $N_2 = \{c_1 + 1, \dots, c_2\}, \dots, N_j = \{c_{j-1} + 1, \dots, n\}$. One can check that W^J consists of all $(a_1 a_2 \dots a_n, \varepsilon) \in W$ satisfying:

- (i) $\varepsilon_i = 1$ if $a_i \in N_1$.
- (ii) If $a_r, a_s \in N_i$ with $r < s$ and $\varepsilon_r = \varepsilon_s = 1$, then $a_r < a_s$.
- (iii) If $a_r, a_s \in N_i$ with $r < s$ and $\varepsilon_r = \varepsilon_s = -1$, then $a_r > a_s$.
- (iv) If $a_r, a_s \in N_i$ and $\varepsilon_r = 1, \varepsilon_s = -1$, then $a_r > a_s$.

For instance, if $W_J = \bar{\mathfrak{S}}(1, 2) \times \mathfrak{S}(3, 7) \times \mathfrak{S}(8, 9)$, then a typical element of W^J is $5 \bar{4} 1 \bar{8} 6 2 7 9 \bar{3}$. The letters 1, 2 are unbarred and appear in increasing order. Similarly 3, 4 are barred and decrease, 5, 6, 7 are unbarred and increase, 8 is barred and "decreases," and 9 is unbarred and "increases."

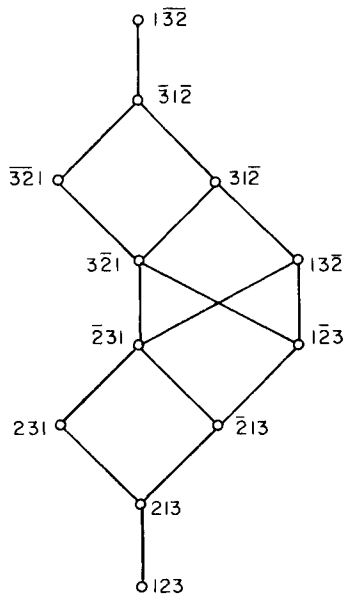


FIG. 5

Figure 5 illustrates W^J when $n = 3$ and $J = \{s_1, s_3\}$. We see that, unlike the situation for A_n , W^J need not be a distributive lattice (or even just a lattice) when J is a maximal subset of S . There is one case, however, in which W^J is a distributive lattice, viz., $J = \{s_1, s_2, \dots, s_{n-1}\}$, so $W_J = \mathfrak{S}(1, n)$. In this case we will denote W^J by $M(n)$. To see that $M(n)$ is indeed a distributive lattice, observe that for every sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n$, there is a unique $\pi \in \mathfrak{S}_n$ for which $(\pi, \varepsilon) \in M(n)$. Identify ε with the subset of $\{1, 2, \dots, n\}$ consisting of those integers t for which $\varepsilon_t = -1$. Then the partial order on $M(n)$ is given by $\{a_1, \dots, a_j\} \leq \{b_1, \dots, b_k\}$ if $a_1 < \dots < a_j, b_1 < \dots < b_k, j \leq k$, and $a_{j-i} \leq b_{k-i}$ for $0 \leq i \leq j - 1$. It is then easily seen that $M(n)$ is a distributive lattice. The poset P for which $M(n) = 2^P$ is given by $P = 2^{2 \times (n-1)}$. Figure 6 illustrates $M(4)$.

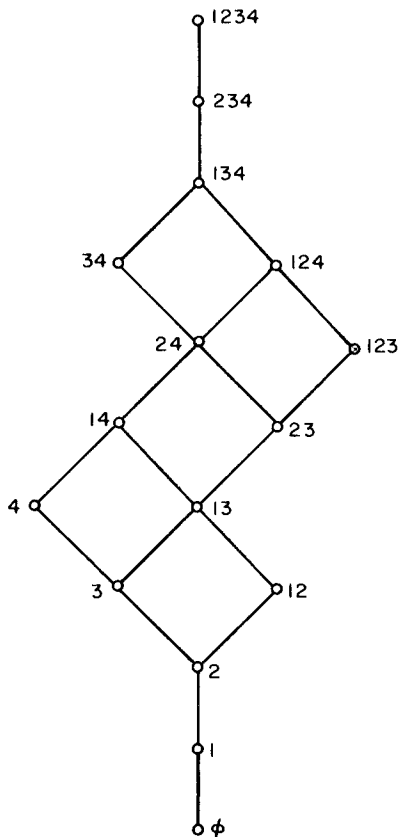


FIG. 6

Lindström [30] conjectured that $M(n)$ has property S_1 , while in fact we now know that $M(n)$ has property S and is rank-unimodal. (I am grateful to Larry Harper for calling my attention to Lindström’s conjecture.) The rank-generating function of $M(n)$ is $(1 + q)(1 + q^2) \cdots (1 + q^n)$. The unimodality of the coefficients of this polynomial was first explicitly proved by Hughes [25], based on a result of Dynkin (see [40] for further information). Presumably, however, this result could also be proved analytically using the methods of [12]. Lindström [30], [31] shows that the structure of $M(n)$ is related to a conjecture [13, (12)] of Erdős and Moser (see also [12], [38], [42]). In fact, Corollary 5.3 below provides a more general result. I am grateful to Ranee Gupta for pointing out an error in my original treatment of the Erdős–Moser conjecture.

COROLLARY 5.1. *Let A be a set of distinct real numbers. Assume that ν elements of A are negative, ζ are equal to 0 (so $\zeta = 0$ or 1), and π are positive. Let B_1, \dots, B_r be subsets of A whose element sums take on at most k distinct values. Then r does not exceed the sum of the k middle coefficients of the polynomial*

$$G_{\nu, \zeta, \pi}(q) = 2^\zeta (1 + q)(1 + q^2) \cdots (1 + q^\nu) \cdot (1 + q)(1 + q^2) \cdots (1 + q^\pi)$$

(there being two equivalent choices of the “ k middle coefficients” when $\binom{\nu + 1}{2} + \binom{\pi + 1}{2} - k$ is even). Moreover, this value of r is achieved by taking $A = \{-1, -2, \dots, -\nu\} \cup \{1, 2, \dots, \pi\} \cup Z$, where $Z = \phi$ or $\{0\}$ depending on whether $\zeta = 0$ or 1.

Proof. Since 0 can be adjoined to a set without affecting its element sum we may assume $\zeta = 0$. Let $M(\nu)^*$ denote the order-dual of $M(\nu)$. (The elements of $M(\nu)$ and $M(\nu)^*$ coincide, but $C \subseteq C'$ in $M(\nu)^*$ if and only if $C \supseteq C'$ in $M(\nu)$.) Regard elements of the product $M(\nu)^* \times M(\pi)$ as consisting of pairs (C, D) , where C is a subset of $\{1, 2, \dots, \nu\}$, and D is a subset of $\{1, 2, \dots, \pi\}$. Suppose that the elements of A are $\alpha_\nu < \dots < \alpha_1 < 0 < \beta_1 < \dots < \beta_\pi$ and that $B_s = \{\alpha_{i_1}, \dots, \alpha_{i_h}, \beta_{j_1}, \dots, \beta_{j_m}\}$. Associate with B_s the set $(C_s, D_s) = (\{i_1, \dots, i_h\}, \{j_1, \dots, j_m\}) \in M(\nu)^* \times M(\pi)$. It is easy to see that the subset $\{(C_1, D_1), \dots, (C_r, D_r)\}$ of $M(\nu)^* \times M(\pi)$ contains no $(k + 1)$ -element chain provided there are most k distinct element sums of B_1, \dots, B_r . Now it is not difficult to see that $M(\nu)^* \cong M(\nu)$. (For instance, given the set $T = \{i_1, \dots, i_h\} \in M(\nu)$ with $1 \leq i_1 < \dots < i_h \leq \nu$, define T^* to be the set of nonzero parts of the partition λ which is conjugate (in the sense of [9, p. 100]) to the partition whose parts are $\nu - i_h, \nu - 1 - i_{h-1}, \dots, \nu - h + 1 - i_1, \nu - h, \nu - h - 1, \dots, 1$. Then the mapping $T \rightarrow T^*$ is an isomorphism $M(\nu) \rightarrow M(\nu)^*$. See also § 7 for a more general result.) The proof now follows from Theorem 3.1 and Proposition 2.5 (or from Theorem 3.1 alone applied to the appropriate *reducible* Weyl group) and the fact that the rank-generating function of $M(\nu)^* \times M(\pi)$ is $G_{\nu,0\pi}(q)$. \square

We now want to consider the situation where $\nu + \zeta + \pi$ is fixed, but ν, ζ , and π can vary. First we need:

LEMMA 5.2. *Let $G(q)$ be a polynomial of degree d with symmetric unimodal coefficients. Fix positive integers j and k . Then the sum of the middle k coefficients of $G(q)(1 + q^{j+1})$ does not exceed the sum of the middle k coefficients of $G(q)(1 + q^j)$.*

Proof. Let $G(q) = \alpha(0) + \alpha(1)q + \dots + \alpha(d)q^d$. For simplicity of notation we assume $d = 2d', j = 2j', k = 2k'$. The other cases are done similarly. The middle k coefficients of $G(q)(1 + q^j)$ are

$$\alpha(d' + j' - k' + i) + \alpha(d' - j' - k' + i), \quad 0 \leq i \leq k - 1.$$

The middle k coefficients of $G(q)(1 + q^{j+1})$ are

$$\alpha(d' + j' - k' + i + 1) + \alpha(d' - j' - k' + i), \quad 0 \leq i \leq k - 1.$$

(Here we set $\alpha(t) = 0$ if $t < 0$.) If Ω applied to a polynomial denotes the sum of its middle k coefficients, then

$$\Omega G(q)(1 + q^j) - \Omega G(q)(1 + q^{j+1}) = \alpha(d' + j' - k') - \alpha(d' + j' + k').$$

Since $\alpha(i) = \alpha(d - i)$ and $\alpha(0) \leq \alpha(1) \leq \dots \leq \alpha(d')$, it follows that $\alpha(d' + j' - k') \geq \alpha(d' + j' + k')$, completing the proof. \square

COROLLARY 5.3. *Let A be a set of n distinct real numbers, and let B_1, \dots, B_r be subsets of A whose element sums take on at most k distinct values. Let $\nu = [(n - 1)/2]$ and $\pi = [n/2]$. Then r does not exceed the sum of the middle k coefficients of the polynomial*

$$2(1 + q)(1 + q^2) \cdots (1 + q^\nu) \cdot (1 + q)(1 + q^2) \cdots (1 + q^\pi).$$

Moreover, this value of r is achieved by choosing $A = \{-\nu, -\nu + 1, \dots, \pi\}$.

Proof. For fixed $n = \nu + \zeta + \pi$, it follows from Lemma 5.2 that the sum of the middle k coefficients of $G_{\nu,\zeta,\pi}(q)$ is maximized by choosing $\zeta = 1, \nu = [(n - 1)/2], \pi = [n/2]$. The proof follows from Corollary 5.1. \square

The actual conjecture [13, (12)] of Erdős and Moser is equivalent to the case $k = 1$, and n odd, of Corollary 5.3. A purely combinatorial derivation of the Erdős–Moser conjecture from the fact that $M(n)$ has property S appears in [35].

6. Type D_n . If (W, S) is a Coxeter system of type D_n , then W is the subgroup of the group W' of type B_n consisting of all (π, ε) such that $\prod_{i=1}^n \varepsilon_i = +1$. W has order $2^{n-1}n!$ and exponents $1, 3, 5, \dots, 2n-5, 2n-3, n-1$. We may take $S = \{s_1, \dots, s_n\}$ where $s_i = (i, i+1)$ if $1 \leq i \leq n-1$ (as in type B_n) and $s_n = \bar{2} \bar{1} 3 4 \dots n$. We then have the following seven transformation rules for obtaining w' from w when $w \leq w'$ in W :

- a) $i < j \longrightarrow \bar{j} > \bar{i}$,
- b) $i < j \longrightarrow j > i$,
- c) $i < j \longrightarrow j > \bar{i}$,
- d) $\bar{i} < j \longrightarrow \bar{j} > i$,
- e) $\bar{i} > j \longrightarrow j < \bar{i}$,
- f) $i > \bar{j} \longrightarrow j < \bar{i}$,
- g) $\bar{i} > \bar{j} \longrightarrow \bar{j} < \bar{i}$.

Note that rules b–g coincide with those for B_n , and that rule a for D_n is obtained by applying rule b and rule a twice for B_n . It follows that if $\pi \leq \sigma$ in W then $\pi \leq \sigma$ in W' . The converse, however, is false. For instance, $21 < \bar{2}\bar{1}$ in W' but 21 and $\bar{2}\bar{1}$ are incomparable in W . Figure 7 depicts W when $n = 2$.

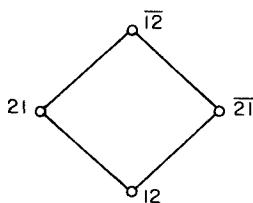


FIG. 7

If $(\pi, \varepsilon) \in W$, then

$$\ell(\pi) = i(\pi) + 2 \sum_j d_j,$$

where $i(\pi)$ and d_j have the same meaning as in (6). For instance, $\ell(\bar{3} \ 1 \ 5 \ \bar{4} \ 2) = 9$ for D_5 , while $\ell(\bar{3} \ 1 \ 5 \ \bar{4} \ 2) = 11$ for B_5 .

Now let $J \subset S$. In so far as describing the poset W^J is concerned, we may assume that if $s_n = \bar{2}\bar{1}34 \dots n \in J$ then also $s_1 = 213 \dots n \in J$, since interchanging s_1 and s_n induces an automorphism of the Coxeter system (W, S) . Thus if we let $\mathfrak{S}(a, b)$ denote the group of all signed permutations of $\{a, a+1, \dots, b\}$ with an even number of -1 's, then W_J has the form

$$W_J = \mathfrak{S}(1, c_1) \times \mathfrak{S}(c_1+1, c_2) \times \dots \times \mathfrak{S}(c_{j-1}+1, n),$$

where $0 \leq c_1 < c_2 < \dots < c_{j-1} < n$ and $c_1 \neq 1$. The case $c_1 = 0$ corresponds to $s_n \notin J$. Defining $N_1 = \{1, 2, \dots, c_1\}$, $N_2 = \{c_1+1, \dots, c_2\}$, \dots , $N_i = \{c_{i-1}+1, \dots, n\}$, one can check that W^J consists of all $(a_1 a_2 \dots a_n, \varepsilon) \in W$ satisfying:

- (i) $\varepsilon_1 = 1$ if $a_i \in N_1$ and $a_i > 1$.
- (ii)–(iv) Same as for type B_n .
- (v) 1 precedes every other element of N_1 (even if 1 is barred).

For instance, Fig. 8 depicts W^J when $n = 3$ and $J = \{12\}$, i.e., $W_J = \mathfrak{S}(1, 2) \times \mathfrak{S}(3, 3)$, so $N_1 = \emptyset$, $N_2 = \{1, 2\}$, $N_3 = \{3\}$. Note that this poset is isomorphic to that of Fig. 2; this is no accident since Coxeter systems of types A_3 and D_3 are isomorphic. (Recall that to obtain nonisomorphic systems, one may take A_n for $n \geq 1$, B_n for $n \geq 2$, and D_n for $n \geq 4$.)

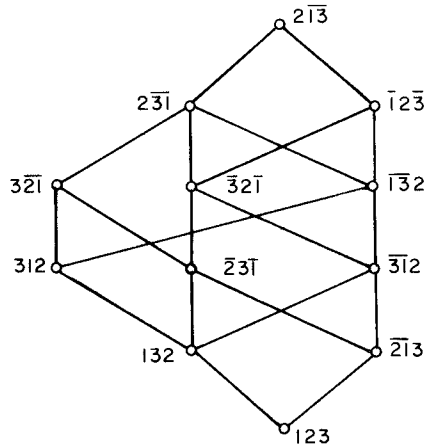


FIG. 8

As in the case of B_n , W^J need not be a distributive lattice when J is maximal. For instance, take $n = 4$ and $J = \{s_1, s_3, s_4\} = S - \{(23)\}$, so $W_J = \mathfrak{S}(1, 2) \times \mathfrak{S}(3, 4)$. Then the rank-generating function of W^J is given by

$$F(W^J, q) = 1 + q + 3q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6 + 3q^7 + q^8 + q^9,$$

and it is easy to check that there does not exist a distributive lattice with this rank-generating function. As in the situation for B_n , there is one special case for which W^J is a distributive lattice. Take $J = \{s_1, s_2, \dots, s_{n-1}\}$, so $W_J = \mathfrak{S}(1, n)$. If we regard $M(n)$ (as defined in the previous section) as consisting of all subsets of $\{1, 2, \dots, n\}$, then W^J turns out to be the subposet of $M(n)$ consisting of all sets of even cardinality. But it is easily seen that this subposet is isomorphic to $M(n - 1)$, so nothing new is obtained.

7. Final comments. In view of the examples $L(m, n)$ and $M(n)$, it is natural to ask under what circumstances is W^J a distributive lattice. I am grateful to Robert Proctor for supplying the following answer to this question. The Coxeter generators S of an irreducible Weyl group W correspond to the *fundamental representations* $\lambda_i (1 \leq i \leq n)$ of a certain complex simple Lie algebra \mathfrak{g} . By direct computation facilitated by representation theory, Proctor has shown that (except for the representations λ_1 and λ_2 of G_2) W^J is distributive if and only if the irreducible representation of \mathfrak{g} with highest weight $\sum_{i \in J} \lambda_i$ is *miniscule*, as defined in [6, p. 226]. These representations have special significance in other contexts; see [39] and more generally [28]. It turns out that for all the distributive W^J 's except $L(m, n)$ and $M(n)$, it is easy to check Property S directly.

Proctor has also shown that if W is a Weyl group with largest element v (in the Bruhat order) and if W^J (for any $J \subset S$) has largest element y , then the bijection from W^J to W^J given by $w \rightarrow vwy^{-1}v^{-1}$ is an anti-automorphism of W^J . Thus W^J is self-dual whenever W is a Weyl group. We do not know whether the more general posets Q^X of Theorem 2.4 need always be self-dual.

We conclude with an open problem. Let P be a finite graded rank-symmetric poset of rank n , with rank function ρ . P is called a *symmetric chain order* (e.g., [17, §3], [20], [21]) if it can be partitioned into pairwise disjoint saturated chains $x_i < x_{i+1} < \dots < x_{n-i}$ such that $\rho(x_j) = j$. It is easy to see that a symmetric chain order satisfies Property T and hence is rank-unimodal. Easy examples show that a rank-symmetric poset satisfying Property T need not be a symmetric chain order.

Our open problem is the following: Are all the posets Q^X of Theorem 2.4 (or at least the special cases W^J of Theorem 3.1) symmetric chain orders? Since any poset Q^X given by Theorem 2.4 has property T , there are pairwise disjoint chains connecting all of Q_i^X to Q_{i+1}^X when $i < n/2$, and all of Q_i^X to Q_{i-1}^X when $i > n/2$. Piecing together these chains yields a partition of Q^X into saturated chains all of which pass through the middle rank (when n is even) or middle two ranks (when n is odd). However, it is by no means clear whether these chains may be chosen to be symmetric about the middle.

Emden Gansner has pointed out to me that for type A_n , there is a rank-preserving, order-preserving bijection $1 \times 2 \times \cdots \times n \xrightarrow{\varphi} W = \mathfrak{S}_n$, where $1 \times 2 \times \cdots \times n = \{(b_1, \dots, b_n) : 0 \leq b_i < i\}$. Namely, $\varphi(b_1, \dots, b_n)$ is that permutation $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ such that b_i is the number of elements j appearing in π to the right of i and satisfying $j < i$. Since any product of chains is a symmetric chain order (e.g., [17, pp. 30–31]), it follows that \mathfrak{S}_n (with the Bruhat order) is also a symmetric chain order. A similar argument for types B_n and D_n produces rank-preserving order-preserving bijections $2 \times 4 \times \cdots \times 2n \rightarrow \tilde{\mathfrak{S}}_n$ and $2 \times 4 \times \cdots \times 2(n-1) \times n \rightarrow \hat{\mathfrak{S}}_n$. Hence $\tilde{\mathfrak{S}}_n$ and $\hat{\mathfrak{S}}_n$ are also symmetric chain orders. However, we do not know for instance whether $L(m, n)$ and $M(n)$ are always symmetric chain orders. Lindström [32] has shown that $L(3, n)$ is a symmetric chain order, and D. West [44] has shown that $L(4, n)$ is a symmetric chain order. Littlewood [33, pp. 193–203] claims to prove that $L(m, n)$ is indeed a symmetric chain order for all m and n . However, his proof is invalid. Specifically, it relies on the “method of chains” of Aitken [45], and this method is not correct as stated by Aitken. For the reader’s benefit we will discuss the nature of Aitken’s error in more detail. Let $P = \{x_1, \dots, x_n\}$ be a finite poset, and let $\Phi = (a_{ij})$ be the $n \times n$ matrix defined by $a_{ij} = 0$ unless $x_i < x_j$ in P ; otherwise the a_{ij} ’s are independent indeterminates over \mathbb{Q} . Remove a chain C_1 of maximum cardinality c_1 from P , then remove a chain C_2 of maximum cardinality c_2 from $P - C_1$, etc. Aitken essentially claims first that the numbers c_1, c_2, \dots , are independent of the choice of chains C_1, C_2, \dots , and second that the numbers c_1, c_2, \dots are the sizes of the Jordan blocks of Φ . The first claim is clearly false. However, Littlewood’s proof would still be valid if there were *some* way of choosing C_1, C_2, \dots so that the second claim is true. Even this weaker result is false. Let P be the poset of Fig. 9. We have no choice but to take $c_1 = 4, c_2 = 1, c_3 = 1$. However, the Jordan block sizes of Φ are 4 and 2. A corrected version of Aitken’s result appears in [37]. If this corrected result is used in conjunction with Littlewood’s method, it yields the result that $L(m, n)$ has property T. Thus we have an alternative proof, avoiding the hard Lefschetz theorem (though actually Littlewood’s method essentially proves the hard Lefschetz theorem for the Grassmann variety), that $L(m, n)$ has property T.

A further property of posets which implies the Sperner property is the LYM property [17, § 4]. However, Griggs has observed that $L(4, 3)$ fails to satisfy the LYM property.

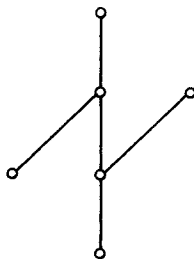


FIG. 9

Note added in proof. A proof that $L(3, m)$ and $L(4, m)$ have symmetric chain decompositions was first given by W. Riess, *Zwei Optimierungsprobleme auf Ordnungen*, Arbeitsberichte des Institute für Mathematische Maschinen und Datenverarbeitung (Informatik) 11, Number 5, Erlangen, April 1978.

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