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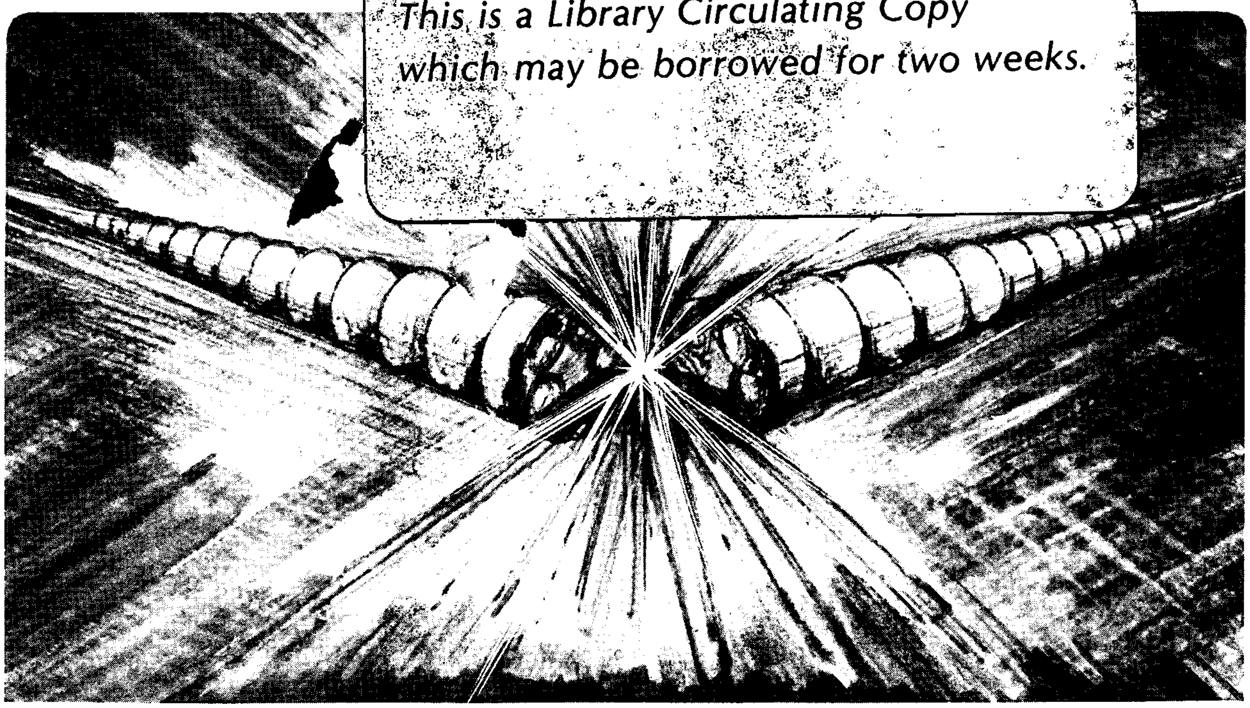

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THE WEYL REPRESENTATION FOR ELECTROMAGNETIC WAVES: THE WAVE KINETIC EQUATION*

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ABSTRACT

A formal derivation of the wave kinetic equation governing the wave action density for electromagnetic waves propagating in a weakly inhomogeneous and nonstationary medium is presented. The method is based on the Weyl representation of a general vector wave equation (including local sources and dissipation) governing the evolution of the field spectral tensor in the phase space of the (geometric optics) rays. This equation is systematically approximated with ordering assumptions consistent with conventional eikonal methods, leading to a natural definition of the wave action density.

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The action density of a wave propagating in a medium is a concept which is central to the analysis of a variety of wave processes. For waves of uniform amplitude in a uniform, stationary medium, the action density $J(\underline{k})$ on $(\underline{x}, \underline{k})$-space of a mode at wavevector $\underline{k}$ and frequency $\omega(\underline{k})$ is given by the energy density $U(\underline{k})$ of the mode divided by its frequency, $J(\underline{k}) = U(\underline{k})/\omega(\underline{k})$. The extension of this definition to nonuniform amplitude waves in a uniform medium or to the case of a weakly inhomogeneous, nonstationary medium is often made by simply considering these quantities to be slowly varying local functions of space and time. In particular, the wave action density $J(\underline{x}, \underline{k}; t)$ becomes a function on the *phase space* of the (geometrical optics) rays which are generated by the eikonal (or WKB) methods that are traditionally applied in this regime.

The wave kinetic equation which governs the evolution of $J(\underline{x}, \underline{k}; t)$ in a weakly nonuniform, nonstationary medium is

$$\frac{d}{dt}J(\underline{x}, \underline{k}; t) \equiv [\partial_t + \dot{\underline{x}} \cdot \partial_{\underline{x}} + \dot{\underline{k}} \cdot \partial_{\underline{k}}]J(\underline{x}, \underline{k}; t) = S(\underline{x}, \underline{k}; t) \tag{1}$$

Here, $(\dot{\underline{x}}, \dot{\underline{k}})$ are given by Hamilton's equations for the rays in phase space, determined by the *local dispersion relation* $\omega = \Omega(\underline{x}, \underline{k}; t)$. The local source $S(\underline{x}, \underline{k}; t)$ represents the effects of dissipation, external sources, nonlinear interactions, *etc.* The most common method of derivation[1] proceeds from the quantum field-theoretic concept of the occupation number $n_{\underline{k}}$ of a mode and the changes in that number due to nonlinear interactions with other modes; the classical limit then assumes large occupation numbers with smooth (as opposed to discrete) variation in time. As $n_{\underline{k}}$ is proportional to the square of the amplitude of the wave, the connection between the relation $J = U/\omega$ and the usual quantum electrodynamic model of modes as oscillators provides the identification of the occupation number as the wave action in the classical limit. The primary focus of those methods is the form of the source terms S which are taken to represent nonlinear couplings among modes and the approximations which can be made to simplify them. The classical, irreversible aspect of dissipation is generally just inserted into the kinetic equations which appear, and the extension to nonuniform nonstationary medium is typically achieved by simply assuming a local spatial dependence of the occupation numbers and postulating the replacement of $(\partial/\partial t)$ by the full convective operator $(\partial_t + \dot{\underline{x}} \cdot \partial_{\underline{x}} + \dot{\underline{k}} \cdot \partial_{\underline{k}})$.

Other non-quantum derivations have been given based on conventional WKB treatments of wave propagation. As such, these approaches introduce the ray trajectories at the lowest order approximation and arrive at an evolution equation for the amplitude of the wave at next order. This equation, which adequately describes the transport of wave energy, momentum and action in nonuniform dissipative medium (including external sources), is naturally set in $\underline{x}$-space (or $\underline{k}$-space) and these quantities are densities on $\underline{x}$- or $\underline{k}$-space. In order to obtain an equation for $J(\underline{x}, \underline{k}; t)$ in the $(\underline{x}, \underline{k})$ ray phase space, the $\underline{x}$-space action density is *lifted* into a phase space density using various procedures. One technique[2] is to label the contribution to the amplitude at a point due to a single ray by the initial value of the wavevector of that ray; in this way the amplitude becomes an implicit function of $\underline{k}$. Another method,[3] applied to scalar fields, relies on an assumption for the asymptotic form of the *Wigner function*[4] which corresponds to a wave field that is a superposition of eikonal wavelets. Both of these schemes for introducing phase space representations into fundamentally $\underline{x}$-space equations tacitly assume a relationship $\underline{k}(\underline{x})$ (either through initial conditions or the eikonal phase) and this poses difficulties in each case: either quantities appearing in the resulting equations are tied to initial conditions (requiring the inversion of all trajectories) or the eikonal phase label may be continous (invalidating the assumption used for the Wigner function). In addition, neither method incorporates the possibility of local nonlinear sources.

The procedure we shall employ in the present derivation of the wave kinetic equation for electromagnetic waves has several advantages. In contrast to the conventional eikonal approach, this will be inherently a phase space method from the outset; thus, we shall treat $\underline{x}, t$ and $\underline{k}, \omega$ as independent variables, with all functions on phase space well-defined through the *Weyl transform.*[5] The exact phase space equation which governs the evolution of the electric field spectral tensor will be derived and solved under an ordering hierarchy consistent with the customary eikonal approximations. This leads to a natural definition for the wave action density in a weakly nonuniform, nonstationary medium which is a reasonable extension of the wave action in a uniform medium. Furthermore, our procedure allows one to include nonlinearities and to proceed to higher order in a straightforward manner.

We begin with a general integral equation for electromagnetic waves in an inhomogeneous, time-varying medium,

$$\int d^3x'\, dt'\; \underset{\sim}{D}(\underline{x}, t; \underline{x}', t') \cdot \underline{E}(\underline{x}', t') = \underline{j}(\underline{x}, t) \tag{2}$$

where $\underline{E}(\underline{x}, t)$ is the wave electric field. The two-point space-time dispersion kernel $\underset{\sim}{D}$ represents Maxwell's equations with a linear response model of the medium (*i.e.*, the current density associated with the wave is given by a linear conductivity law $\delta \underline{j}(\underline{x}, t) = \int d^3x'\, dt'\; \underset{\sim}{\sigma}(\underline{x}, t; \underline{x}', t') \cdot \underline{E}(\underline{x}', t')$); we allow $\underset{\sim}{D}$ to correspond to a non-hermitian operator. The current-source $\underline{j}(\underline{x}, t)$ on the right-hand side accounts for *departures* from the linear treatment of the wave (such as external sources, discreteness effects or nonlinear wave processes). We view (2) as the space-time representation of the abstract equation

$$\underset{\sim}{\mathbf{D}} \cdot \underline{E} = \underline{j} \tag{3a}$$

The product of (3a) with its adjoint equation

$$\underline{E}^\dagger \cdot \underset{\sim}{\mathbf{D}}^\dagger = \underline{j}^\dagger \tag{3b}$$

gives

$$\underset{\sim}{\mathbf{D}} \cdot (\underline{E}\underline{E}^\dagger) = (\underline{j}\underline{j}^\dagger) \cdot (\underset{\sim}{\mathbf{D}}^\dagger)^{-1} \tag{4}$$

This is an equation for the *spectral tensor* of the electric field $(\underline{E}\underline{E}^\dagger)$ in terms of the dispersion operator $\underset{\sim}{\mathbf{D}}$ and the spectral tensor $(\underline{j}\underline{j}^\dagger)$ of the current-source field. Furthermore, while (3) is naturally viewed in either its space-time representation (2) or the wavevector-frequency $(\underline{k}, \omega)$ basis (obtained by Fourier transform), the operator equation (4) is appropriate for introducing a *joint* space-time/wavevector-frequency representation. Thus we define

$$\underset{\sim}{D}(\underline{x}, t, \underline{k}, \omega) \equiv \int d^3s\, d\tau\; \underset{\sim}{D}(\underline{x} + \tfrac{1}{2}\underline{s}, t + \tfrac{1}{2}\tau; \underline{x} - \tfrac{1}{2}\underline{s}, t - \tfrac{1}{2}\tau)\, e^{-i(\underline{k}\cdot\underline{s} - \omega\tau)} \tag{5a}$$

$$(\underline{E}\underline{E}^\dagger)(\underline{x}, t, \underline{k}, \omega) \equiv \int d^3s\, d\tau\; \underline{E}(\underline{x} + \tfrac{1}{2}\underline{s}, t + \tfrac{1}{2}\tau)\underline{E}^*(\underline{x} - \tfrac{1}{2}\underline{s}, t - \tfrac{1}{2}\tau)\, e^{-i(\underline{k}\cdot\underline{s} - \omega\tau)} \tag{5b}$$

This transformation may be viewed as a *local* Fourier transform of the space-time representation $(\underset{\sim}{D}(\underline{x}, t; \underline{x}', t')$ or $\underline{E}(\underline{x}, t)\underline{E}^*(\underline{x}', t'))$ of the corresponding operator ($\underset{\sim}{\mathbf{D}}$ or $(\underline{E}\underline{E}^\dagger)$). In particular, the transform produces the local wavevector-frequency spectrum of the wave field $\underline{E}$ (hence the appelation *spectral tensor* for $(\underline{E}\underline{E}^\dagger)$). We shall refer to this description of the operators as a *phase space representation* since the quantities defined in (5) are functions on the eight-dimensional *ray phase space* (including time and frequency). We note that this wave operator/phase space function correspondence is not unique; this formalism is the *Weyl representation*,[5] and the phase space functions defined in (5) are the *Weyl symbols* of the operators. In its scalar form, the Weyl symbol of the wave field (5b) is commonly called the *Wigner function*.[4]

In addition to providing a correspondence between wave operators and phase space functions, the Weyl formalism includes a *symbol calculus*. In particular, we have the follow-

ing theorem:[7] if **A**, **B** and **C** are abstract operators which satisfy

$$\mathbf{AB} = \mathbf{C} \tag{6}$$

then the Weyl symbols A, B and C of the operators satisfy

$$A(z^8)\exp(\tfrac{i}{2}\overleftrightarrow{\mathcal{L}})B(z^8) = C(z^8) \tag{7}$$

Here, A, B and C are defined as in (5) with the notation that z^8 represents the point $(\underline{x}, t, \underline{k}, \omega)$ in the eight-dimensional phase space. The exponentiated *Janus operator* $\overleftrightarrow{\mathcal{L}}$ acting on *both* $A(z^8)$ and $B(z^8)$ is

$$\overleftrightarrow{\mathcal{L}} \equiv \overleftarrow{\partial_{\underline{x}}} \cdot \overrightarrow{\partial_{\underline{k}}} - \overleftarrow{\partial_{\underline{k}}} \cdot \overrightarrow{\partial_{\underline{x}}} + \overleftarrow{\partial_\omega}\overrightarrow{\partial_t} - \overleftarrow{\partial_t}\overrightarrow{\partial_\omega}, \tag{8}$$

precisely the *Poisson bracket operator* on the extended ray phase space.[6] Applying this theorem (extended to tensor operators) to (4), we directly obtain[7] our basic phase space equation

$$\underline{\underline{D}}(z^8)\exp(\tfrac{i}{2}\overleftrightarrow{\mathcal{L}}) \cdot (\underline{E}\underline{E}^\dagger)(z^8) = (\underline{j}\underline{j}^\dagger)(z^8)\exp(\tfrac{i}{2}\overleftrightarrow{\mathcal{L}}) \cdot (\underline{\underline{D}}^\dagger)^{-1}(z^8) \tag{9}$$

That this is indeed the equation which governs the wave spectral tensor $(\underline{E}\underline{E}^\dagger)(z^8)$ can be verified by substituting the definitions (5) into (9), expanding the exponential operator in power series, and recovering the space-time representation (2) of (3).

Having presented the *exact* equation for the wave spectral tensor, we now wish to treat this equation in the short wavelength, high frequency regime. We shall assume that the waves described by (2) have a wavelength λ which is short compared with the scalelength L of the spatial variation of the medium (as represented by $\underline{\underline{D}}$), $(\lambda/L) \ll 1$; similarly, we assume that the temporal wave period is short compared with the timescale T of variation, $\omega T \gg 1$. Unlike traditional eikonal methods[8] for treating wave equations such as (2) in this regime, however, we do not postulate an explicit form for the solution $(\underline{E}\underline{E}^\dagger)(z^8)$. Instead, we begin by expanding the exponential operator in (9) as $\exp(i\overleftrightarrow{\mathcal{L}}/2) = 1 + (i\overleftrightarrow{\mathcal{L}}/2) + \ldots$ and estimate the order of the resulting terms. Specifically, we assume that derivatives acting on $\underset{\sim}{D}$ and $(\underline{EE}^\dagger)$ are of the order $\partial_{\underline{x}} \sim L^{-1}$, $\partial_t \sim T^{-1}$, $\partial_{\underline{k}} \sim \lambda$ and $\partial_\omega \sim \omega^{-1}$. These requirements on the variation of the medium are standard, but the assumption for the behavior of the solution would have to be justified *a posteriori*. Introducing the hermitian and anti-hermitian parts of the dispersion tensor $(\underset{\sim}{D}', \underset{\sim}{D}'')$, we assume that the medium is only slightly dissipative, $\underset{\sim}{D}'' \sim \mathcal{O}(\lambda/L)$. We also assume that the waves are only weakly driven (weak sources or small nonlinearities), $|(\underline{jj}^\dagger)| \sim \mathcal{O}(\lambda/L)$.

With these assumptions in (9), we collect terms by powers of $(\lambda/L) \sim (\omega T)^{-1}$; the lowest two orders are

$$\underset{\sim}{D}'(z^8) \cdot (\underline{EE}^\dagger)(z^8) = 0 \tag{10a}$$

$$\tfrac{i}{2}\underset{\sim}{D}'\overleftrightarrow{\mathcal{L}} \cdot (\underline{EE}^\dagger) + i\underset{\sim}{D}'' \cdot (\underline{EE}^\dagger) = (\underline{jj}^\dagger) \cdot (\underset{\sim}{D}^\dagger)^{-1} \tag{10b}$$

As we shall see, the lowest order equation (10a) is a "constraint" on $(\underline{EE}^\dagger)(z^8)$, while the $\mathcal{O}(\lambda/L)$ equation[9] (10b) governs the evolution of $(\underline{EE}^\dagger)$.

We consider first the lowest order equation (10a). Unlike the usual dispersion equation in a homogeneous medium (or the corresponding lowest order result in conventional eikonal methods), this is a matrix equation. Therefore, the standard result $\det\underset{\sim}{D}' = 0$ does not immediately follow from (10a). Since both $\underset{\sim}{D}'$ and $(\underline{EE}^\dagger)$ are manifestly hermitian, however, the adjoint of (10a) implies that these tensors commute (in this approximation.) We can thus simultaneously diagonalize the matrices:

$$D'_{\mu\nu} \equiv \sum_{\alpha=1}^{3} D^\alpha e^\alpha_\mu \bar{e}^\alpha_\nu, \qquad (\underline{EE}^\dagger)_{\mu\nu} \equiv \sum_{\alpha=1}^{3} W^\alpha e^\alpha_\mu \bar{e}^\alpha_\nu, \tag{11}$$

where we have introduced the basis of the *local polarization vectors* $\hat{e^\alpha}(z^8)$. The scalars $D^\alpha(z^8)$ are the local eigenvalues of $\underset{\sim}{D}'$, and we shall refer to $W^\alpha(z^8)$ as the Wigner function corresponding to the component of the wave field in the $\hat{e^\alpha}$ polarization. In this basis, the lowest order equation becomes three equations for the diagonal elements

$$D^\alpha(z^8)W^\alpha(z^8) = 0 \tag{12}$$

This constraint must be satisfied at each point (z^8) in phase space; that is, at each point either D^α or W^α (or both) must vanish. Thus, we have the condition that $W^\alpha = 0$ everywhere except on the seven-dimensional *dispersion manifold* defined by $D^\alpha(z^8) = 0$. The restriction $D^\alpha(z^8) = 0$ is then equivalent to the usual eikonal condition $\det \underset{\sim}{D}' = 0$, and this in turn implicitly defines the *local dispersion relation* $\omega = \Omega^\alpha(\underline{x}, \underline{k}; t)$. More generally, the vanishing of a single eigenvalue D^α may yield multiple solutions (or branches) for $\omega = \Omega^\alpha$, all with the same polarization. In addition, more than one eigenvalue D^α may be zero at a particular point (or on some manifold) in phase space; this possibility introduces coupling between the linear modes at the next order and requires special treatment. We do not consider further such degeneracies.[10]

The dispersion manifold defined by $\omega = \Omega^\alpha(\underline{x}, \underline{k}; t)$ is the surface on which the rays generated in conventional eikonal methods propagate. According to (12), this is also the only region of phase space where $W^\alpha(z^8)$ is nonzero. While this apparently establishes a connection between the Wigner function W^α and the rays of geometrical optics, we point out that these rays are not directly induced by this lowest order equation. [In standard eikonal treatments, the local dispersion relation $\omega = \Omega(\underline{x}, \underline{k}; t)$ arises as a partial differential (Hamilton-Jacobi) equation for the wave phase $\phi(\underline{x}, t)$, with $(\underline{k}, \omega) \equiv (\partial_{\underline{x}}\phi, -\partial_t \phi)$; this is then solved by introducing the characteristic ray trajectories. It is important to remember that in our method $(\underline{k}, \omega)$ are independent of $(\underline{x}, t)$.] Furthermore, (12) does not require W^α to be nonzero everywhere that D^α vanishes, so that it is possible for the Wigner function to be concentrated on a submanifold of the dispersion surface. In view of these considerations, we take as an appropriate solution for the lowest order constraint (10a,12) the form

$$W^{\alpha}(\underline{x}, t, \underline{k}, \omega) = J^{\alpha}(\underline{x}, \underline{k}; t)\, \delta(D^{\alpha}(\underline{x}, t, \underline{k}, \omega)) \tag{13}$$

We shall see that the function $J^{\alpha}(\underline{x}, \underline{k}; t)$ giving structure to the Wigner function on the dispersion manifold can be identified as the wave action density.

The next order equation (10b) will determine the evolution of J^{α} with the use of (13). First, however, we express (10b) in the basis of the polarization vectors $\hat{e^{\alpha}}$. In doing so, we make the simplifying assumption that the waves in the system are of only one polarization: that is, we take only W^{s} to be nonzero ($W^{\alpha} = 0$ everywhere for $\alpha \neq s$). Furthermore, due to the form of (13), the derivatives of W^{s} also vanish where $D^{s} \neq 0$. As previously stated, we assume that there are no degeneracies: in the region where W^{s} is nonzero, $D^{s} = 0$ but $D^{\alpha} \neq 0$ for $\alpha \neq s$. Under these conditions, (10b) becomes[7] the evolution equation for $W^{s}(z^{8})$

$$\{D^{s}, W^{s}\}_{8} = -2(D'')^{ss} W^{s} - \frac{2i(\underline{j}\underline{j}^{\dagger})^{ss}}{D^{s} - i(D'')^{ss}} \tag{14}$$

In this expression, the superscript ss denotes the ss-component of a tensor expressed in the polarization-vector basis. The bracket notation $\{\cdot, \cdot\}_{8}$ stands for the standard Poisson bracket on the eight-dimensional phase space: $\{A(z^{8}), B(z^{8})\}_{8} = A\overleftrightarrow{\mathcal{L}}B$. This equation is derived by casting all the terms in (10b) in the polarization-vector basis and then comparing the expressions produced by extracting the ss-component and the trace.

Equation (14) governs the evolution of the Wigner function W^{s} for the polarization s on the dispersion manifold $D^{s} = 0$. Therefore, the quantities $(\underline{j}\underline{j}^{\dagger})^{ss}(z^{8})$ and $(D'')^{ss}(z^{8})$ as well as the derivatives of D^{s} and W^{s} must be evaluated on this surface by setting $\omega = \Omega^{s}(\underline{x}, \underline{k}; t)$. This can be incorporated into (14) to some extent by inserting the solution (13) of the lowest order constraint which explicitly exhibits this restriction. Using the antisymmetry of the Poisson bracket, we have $\{D^s, J^s\delta(D^s)\}_8 = \{D^s, J^s\}_8\delta(D^s)$ so that (14) is

$$[(\partial_\omega D^s)\partial_t J^s(\underline{x},\underline{k};t) + \{D^s, J^s\}_6]\delta(D^s) = -2(D'')^{ss}J^s\delta(D^s) - \frac{2i(\underline{j}\underline{j}^\dagger)^{ss}}{D^s - i(D'')^{ss}} \tag{15}$$

Now the bracket notation $\{\cdot,\cdot\}_6$ denotes the standard Poisson bracket on the six-dimensional $(\underline{x},\underline{k}) \equiv (z^6)$ phase space; *i.e.*, $\{A(z^6;t), B(z^6;t)\}_6 \equiv A(\overleftarrow{\partial_{\underline{x}}} \cdot \overrightarrow{\partial_{\underline{k}}} - \overleftarrow{\partial_{\underline{k}}} \cdot \overrightarrow{\partial_{\underline{x}}})B$. We have dropped the term proportional to $\partial_\omega J^s$ since by definition (13) the action density $J^s(\underline{x},\underline{k};t)$ is independent of ω. The appearance of δ-functions in (15) indicates that this expression is to be interpreted as a density; integrating with respect to D^s (the local direction transverse to the dispersion manifold) we obtain

$$(\partial_\omega D^s)\partial_t J^s + \{D^s, J^s\}_6 = -2(D'')^{ss}J^s + 2\pi(\underline{j}\underline{j}^\dagger)^{ss} \tag{16}$$

In arriving at this equation, we have replaced the denominator of the current-source term in (15) with its limiting form as $(D'')^{ss}$ tends to zero: $(D^s - i(D'')^{ss})^{-1} \to i\pi\delta(D^s) + P(D^s)^{-1}$, where P denotes the Cauchy principal value. The principal value integral vanishes because of the antisymmetry of the integrand. Now, using the standard relations

$$\begin{aligned}(\partial_{\underline{x}} D^s)_{t,\underline{k},\omega} &= -(\partial_\omega D^s)_{\underline{x},t,\underline{k}}(\partial_{\underline{x}}\Omega^s)_{\underline{k},t} \\ (\partial_{\underline{k}} D^s)_{\underline{x},t,\omega} &= -(\partial_\omega D^s)_{\underline{x},t,\underline{k}}(\partial_{\underline{k}}\Omega^s)_{\underline{x},t}\end{aligned} \tag{17}$$

which hold on the surface given by $D^s(z^8) = 0$, and defining the local growth rate $\gamma^s(\underline{x},\underline{k};t)$ in the usual manner

$$\gamma^s(\underline{x},\underline{k};t) \equiv -\left[\frac{(D'')^{ss}}{\partial_\omega D^s}\right]_{\omega=\Omega}(\underline{x},\underline{k};t) \tag{18}$$

we have our main result

$$\partial_t J^s(\underline{x},\underline{k};t) + \{J^s, \Omega^s\}_6 = 2\gamma^s(\underline{x},\underline{k};t)J^s + \left[\frac{2\pi(\underline{j}\underline{j}^\dagger)^{ss}}{(\partial_\omega D^s)}\right]_{\omega=\Omega} \tag{19}$$

This equation governs the evolution of $J^s(\underline{x},\underline{k};t)$ on the dispersion manifold. In that the left-hand side is reminiscent of the Liouville operator of classical mechanics, we observe that this $O(\lambda/L)$ approximation to the exact phase space equation (9) introduces the notion of the ray trajectories in phase space. Specifically, defining the Hamiltonian ray system which corresponds to the underlying wave equation (2,9) by

$$\dot{\underline{x}} = \partial_{\underline{k}}\Omega^s \qquad \dot{\underline{k}} = -\partial_{\underline{x}}\Omega^s \tag{20}$$

we see that, as in (1), the left-hand side of (19) is the total derivative dJ^s/dt of $J^s(\underline{x}, \underline{k}; t)$ along the ray trajectories (20) in phase space.

We now turn to a discussion of (19) and the nature of our definition of $J^s(\underline{x}, \underline{k}; t)$ in order to identify it as the wave action density.

In the absence of dissipation and sources, the evolution equation (19) simply states that J^s is constant along the ray trajectories. In analogy with a similar result of classical mechanics, this suggests that J^s should be interpreted as a kind of Liouville phase space density for the propagation of waves in the short wavelength regime. Allowing for dissipation, the solution of (19) is

$$J^s(\underline{x}(t), \underline{k}(t), t) = J^s(\underline{x}_0, \underline{k}_0, 0) \exp[2 \int^t \gamma^s(\underline{x}(t'), \underline{k}(t'), t')\, dt'] \tag{21}$$

which explicitly conveys the non-Hamiltonian damping (or growth) of this phase space density from its initial value depending on the local value of γ^s. The factor of two is appropriate as J is quadratic in the field amplitude, yet it arises naturally here from the $(i\overleftrightarrow{\mathcal{L}}/2)$ in (10b) and our approximation scheme.

As stated earlier, the source contribution $(\underline{jj}^\dagger)^{ss}$ may represent given external currents, discreteness effects, nonlinearly generated currents, *etc.* If this term is independent of J^s, then (19) is a linear inhomogeneous equation for J^s. If, however, the current $\underline{j}(\underline{x}, t)$ depends nonlinearly on the field $\underline{E}(\underline{x}, t)$, then using (5) and its inverse, the source $(\underline{jj}^\dagger)^{ss}$ may be written as a nonlinear functional of J^s; in this case, (19) becomes a nonlinear equation.

To see that $J^s(\underline{x}, \underline{k}; t)$ is properly identified as the wave action density, we integrate (13) with respect to D^s:

$$
\begin{aligned}
J^s(\underline{x}, \underline{k}; t) &= \int dD^s \; W^s(\underline{x}, t, \underline{k}, \omega) \\
&= \int d\omega \; (\partial_\omega D^s)(\underline{x}, t, \underline{k}, \omega) W^s(\underline{x}, t, \underline{k}, \omega)
\end{aligned}
\tag{22}
$$

In this integral transverse to the dispersion manifold, the slowly varying term $(\partial_\omega D^s)$ may be approximated by its value at $\omega = \Omega^s$ due to the singular behavior of W^s across this surface. Thus, we have

$$
\begin{aligned}
J^s(\underline{x}, \underline{k}; t) &\approx (\partial_\omega D^s)(\underline{x}, t, \underline{k}, \Omega^s) \int d\omega \; W^s(\underline{x}, t, \underline{k}, \omega) \\
&\equiv (\partial_\omega D^s)(\underline{x}, t, \underline{k}, \Omega^s) \widetilde{W^s}(\underline{x}, \underline{k}; t)
\end{aligned}
\tag{23}
$$

which defines $\widetilde{W^s}$ as the integral of (5) over ω. In a stationary uniform medium, (23) reduces to the usual expression for the energy of the mode at $(\underline{k}, \omega)$ divided by the frequency

$$
J^s(\underline{k}) = |\underline{E}^s(\underline{k})|^2 \left(\frac{\partial D^s(\underline{k}, \omega)}{\partial \omega} \right)_{\omega = \Omega(\underline{k})} = \frac{U^s(\underline{k})}{\omega^s(\underline{k})}
\tag{24}
$$

Thus, our definition of $J^s(\underline{x}, \underline{k}; t)$ reduces to the standard one in the uniform medium case. Furthermore, we have seen that in the absence of dissipation or sources, $J^s(\underline{x}, \underline{k}; t)$ is an invariant under the flow of the rays in phase space. For these reasons, we suggest that $J(\underline{x}, \underline{k}; t)$ as defined in (13,22) is an appropriate extension of the wave action density to the case of a weakly nonuniform, nonstationary medium.

In summary, the derivation presented here offers a classical wave (as opposed to quanta) phase space (instead of $\underline{x}$-space) treatment, which results in the wave kinetic equation in a nonuniform, nonstationary medium as an approximation to the exact tensor phase space equation governing the local spectral tensor of the wave field in the presence of dissipation and sources. The steps in the development of this equation were as follows: It is assumed that one is given the form of the linear dispersion operator $\underline{\underline{D}}$ from which one constructs the local dispersion tensor $\underset{\approx}{D}$ in the Weyl representation (5) as a function on $(\underline{x}, t, \underline{k}, \omega)$ phase space, and its hermitian ($\underset{\approx}{D}'$) and anti-hermitian ($\underset{\approx}{D}''$) parts are identified. In the Weyl formalism, the wave field is represented by the local spectral tensor $(\underline{EE^{\dagger}})$ which is related to the field by (5b); a similar definition is used to construct the phase space representation of the current source contributions $(\underline{jj^{\dagger}})$ not included in the usual linear treatment of wave propagation. The general form of the wave equation in configuration space (2) is then translated directly (using the Weyl calculus) into the equation in phase space (9) which connects these phase space functions. At lowest order, the basis of the local polarization vectors is introduced in order to simplify the tensor equation, and the local dispersion relation (although *not* the rays) emerges along with the solution (13). This solution defines the quantity $J(\underline{x}, \underline{k}; t)$ which is shown to be the wave action density on $(\underline{x}, \underline{k})$ space in a nonuniform medium. Substitution of the lowest order solution into the next higher order equation results in the wave kinetic equation which governs J (and also serves to define the rays in this treatment). Dissipation (due to $\underset{\approx}{D}''$) and general sources are included while the possibility of linear mode coupling (degenerate eigenvalues of $\underset{\approx}{D}'$) is neglected for simplicity; this situation could be treated in much the same manner as used in traditional eikonal methods.

Appendix

In this Appendix, we examine the validity of the ordering scheme and the approximations we have used. Specifically, the solution (13) of the lowest order constraint and the result (19) at next order should be compared with the assumptions made for the magnitude of the $(\underline{x}, t)$- and $(\underline{k}, \omega)$- derivatives of the spectral tensor $(\underline{E}\underline{E}^{\dagger})(\underline{x}, t, \underline{k}, \omega)$. The derivatives of (11,13) involve contributions from differentiating the action density J, the polarization vectors $\hat{e}$ and the δ-function. The presence of the δ-function, however, would appear to render all of these terms to be singular so that our assumption is violated. Therefore, in order to justify this solution, we wish to impose the following interpretation: Since only a one-dimensional δ-function appears here, restricting the support of W^s to the seven-dimensional dispersion manifold in the eight-dimensional extended phase space, local coordinates in the neighborhood of this surface may be constructed so that one "direction" (evidently, the D^s direction) is "perpendicular" to the surface. It is along this direction that the solution has singular derivatives while derivatives in the other directions (lying "in" the manifold) act only on the amplitude of W^s, that is, on the action density J. Thus, the derivatives which appear in the wave kinetic equation should be understood as the pieces of those derivatives "parallel" to the surface $D^s = 0$. In fact, this has already been incorporated into (19) with the substitution of (17) into (16): the trajectories which convect J evolve on this manifold.

Now the question arises as to the order of the derivatives on the action density. By the hypothesis of a weakly dissipative medium with weak sources (or coupling), the right hand side of (19) drives changes in J which are of a magnitude consistent with the assumed order of the derivatives of J on the left side. The foregoing argument requires the action density to be somewhat smooth on the dispersion manifold and in particular, this assumption would be violated if J were concentrated on a submanifold of this surface; in that case, the pieces of the derivatives in (19) along directions transverse to that submanifold would be large. In this regard, it can be shown[11] that the rays corresponding to the propagation of monochromatic waves evolve on *Lagrangian manifolds*, which are three-dimensional surfaces in $(\underline{x}, \underline{k})$ phase space. Since J is convected by the rays, it must in these cases vanish everywhere except on these submanifolds of the dispersion surface, in contradiction to the smoothness assumption stated above. In the case of propagating waves, however, the possibility exists that the Lagrangian manifold may become so convoluted that it nearly fills the dispersion surface; allowing for a small wave-like spreading or broadening of the action density off the manifold (due to higher order corrections), a smooth variation of J on the dispersion surface may be achieved as these "diffraction edges" from neighboring "leaves" of the Lagrangian manifold coalesce.[12] This circumstance would imply the existence of many leaves "above" each point in $\underline{x}$-space and hence, in terms of the traditional eikonal description of the wave, many contributions to the field at that point; the convoluted nature of the rays also might be expected to produce a decorrelation of the phases of each contribution. In view of these considerations, it would seem that the smoothness assumptions on the action density imposed in the present derivation require that the wave system under consideration be *incoherent*. With these qualifications then, the derivation given for the wave kinetic equation is a justifiable procedure for approximating the exact equation governing the local spectral tensor.

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[9]In general, one should expand the solution in a power series $(\underline{EE}^{\dagger}) = \sum_n (\lambda/L)^n (\underline{EE}^{\dagger})_n$, in which case the $\mathcal{O}(\lambda/L)$ equation (10b) would contain the term $\underset{\sim}{D}' \cdot (\underline{EE}^{\dagger})_1$ on the left-hand side (and all other quantities in that equation would be labelled zeroeth-order). However, since all of the tensors which appear in (10b) are hermitian, one finds that the anti-hermitian part of (10b) produces the equation shown (due to the presence of i), whereas the hermitian part gives $\underset{\sim}{D}' \cdot (\underline{EE}^{\dagger})_1 = 0$. Thus, the first-order quantity $(\underline{EE}^{\dagger})_1$ does not appear here, and we have dropped the subscript notation for the zeroeth-order tensors shown.

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