

## WEYL'S TYPE THEOREMS FOR ALGEBRAICALLY $(p, k)$ -QUASIHYPONORMAL OPERATORS

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ABSTRACT. For a bounded linear operator  $T$  we prove the following assertions: (a) If  $T$  is algebraically  $(p, k)$ -quasihyponormal, then  $T$  is  $a$ -isoloid, polaroid, reguloid and  $a$ -polaroid. (b) If  $T^*$  is algebraically  $(p, k)$ -quasihyponormal, then  $a$ -Weyl's theorem holds for  $f(T)$  for every  $f \in Hol(\sigma(T))$ , where  $Hol(\sigma(T))$  is the space of all functions that analytic in an open neighborhoods of  $\sigma(T)$  of  $T$ . (c) If  $T^*$  is algebraically  $(p, k)$ -quasihyponormal, then generalized  $a$ -Weyl's theorem holds for  $f(T)$  for every  $f \in Hol(\sigma(T))$ . (d) If  $T$  is a  $(p, k)$ -quasihyponormal operator, then the spectral mapping theorem holds for semi- $B$ -essential approximate point spectrum  $\sigma_{SBF_+^-}(T)$ , and for left Drazin spectrum  $\sigma_{lD}(T)$  for every  $f \in Hol(\sigma(T))$ .

### 1. Introduction

Throughout this paper let  $\mathbf{B}(\mathcal{H})$ , denote, the algebra of bounded linear operators acting on an infinite dimensional separable Hilbert space  $\mathcal{H}$ . If  $T \in \mathbf{B}(\mathcal{H})$  we shall write  $\ker(T)$  and  $\mathcal{R}(T)$  for the null space and range of  $T$ , respectively. Also, let  $\alpha(T) := \dim \ker(T)$ ,  $\beta(T) := \dim \mathcal{R}(T)$ , and let  $\sigma(T), \sigma_a(T), \sigma_p(T)$  denote the spectrum, approximate point spectrum and point spectrum of  $T$ , respectively. An operator  $T \in \mathbf{B}(\mathcal{H})$  is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

$T$  is called *Weyl* if it is Fredholm of index 0, and *Browder* if it is Fredholm “of finite ascent and descent”.

Recall that the *ascent*,  $a(T)$ , of an operator  $T$  is the smallest non-negative integer  $p$  such that  $\ker(T^p) = \ker(T^{p+1})$ . If such integer does not exist we put  $a(T) = \infty$ . Analogously, the *descent*,  $d(T)$ , of an operator  $T$  is the smallest non-negative integer  $q$  such that  $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$ , and if such integer does

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not exist we put  $d(T) = \infty$ . The essential spectrum  $\sigma_F(T)$ , the Weyl spectrum  $\sigma_W(T)$  and the Browder spectrum  $\sigma_b(T)$  of  $T$  are defined by

$$\sigma_F(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T) \subseteq \sigma_W(T) \subseteq \sigma_b(T) \subseteq \sigma_F(T) \cup \text{acc}\sigma(T),$$

where we write  $\text{acc}K$  for the accumulation points of  $K \subseteq \mathbb{C}$ .

Following [13], we say that *Weyl's theorem* holds for  $T$  if  $\sigma(T) \setminus \sigma_W(T) = E_0(T)$ , where  $E_0(T)$  is the set of all eigenvalues  $\lambda$  of finite multiplicity isolated in  $\sigma(T)$ . And *Browder's theorem* holds for  $T$  if  $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$ , where  $\pi_0$  is the set of all poles of  $T$  of finite rank.

Let  $\Phi_+(\mathcal{H})$  be the class of all upper semi-Fredholm operators,  $\Phi_+(\mathcal{H})$  be the class of all  $T \in \Phi_+(\mathcal{H})$  with  $i(T) \leq 0$ , and for any  $T \in \mathbf{B}(\mathcal{H})$  let

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+(\mathcal{H})\}.$$

Let  $E_0^a$  be the set of all eigenvalues of  $T$  of finite multiplicity which are isolated in  $\sigma_a(T)$ . According to [27], we say that  $T$  satisfies *a-Weyl's theorem* if  $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T)$ . It follows from [27, Corollary 2.5] *a-Weyl's theorem* implies Weyl's theorem.

In [12] Berkani define the class of *B-Fredholm* operators as follows. For each integer  $n$ , define  $T_n$  to be the restriction of  $T$  to  $\mathcal{R}(T^n)$  viewed as a map from  $\mathcal{R}(T^n)$  into  $\mathcal{R}(T^n)$  (in particular  $T_0 = T$ ). If for some  $n$  the range  $\mathcal{R}(T^n)$  is closed and  $T_n$  is a Fredholm (resp. semi-Fredholm) operator, then  $T$  is called a *B-Fredholm* (resp. *semi-B-Fredholm*) operator. In this case and from [6]  $T_m$  is a Fredholm operator and  $i(T_m) = i(T_n)$  for each  $m \geq n$ . The index of a *B-Fredholm* operator  $T$  is defined as the index of the Fredholm operator  $T_n$ , where  $n$  is any integer such that the range  $\mathcal{R}(T^n)$  is closed and  $T_n$  is a Fredholm operator (see [12]).

Let  $BF(\mathcal{H})$  be the class of all *B-Fredholm* operators. In [6] Berkani has studied this class of operators and has proved that an operator  $T \in \mathbf{B}(\mathcal{H})$  is *B-Fredholm* if and only if  $T = T_0 \oplus T_1$ , where  $T_0$  is a Fredholm operator and  $T_1$  is a nilpotent operator.

Recall that an operator  $T \in \mathbf{B}(\mathcal{H})$  is called a *B-Weyl* operator (see [8]) if it is a *B-Fredholm* operator of index 0. The *B-Weyl* spectrum  $\sigma_{BW}(T)$  of  $T$  is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a } B\text{-Weyl operator}\}.$$

In the case of a normal operator  $T$  acting on a Hilbert space  $\mathcal{H}$ , Berkani [12, Theorem 4.5] showed that  $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ , where  $E(T)$  is the set of

all eigenvalues of  $T$  which are isolated in the spectrum of  $T$ . This result gives a generalization of the classical Weyl's theorem.

Let  $SBF_+(\mathcal{H})$  be the class of all upper semi- $B$ -Fredholm operators, and  $SBF_+^-(\mathcal{H})$  the class of all  $T \in SBF_+(\mathcal{H})$  such that  $i(T) \leq 0$ , and

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{H})\}.$$

Recall that an operator  $T \in \mathbf{B}(\mathcal{H})$  satisfies the *generalized  $a$ -Weyl's theorem* if  $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$ , where  $E^a(T)$  is the set of all eigenvalues of  $T$  which are isolated in  $\sigma_a(T)$ . Note that generalized  $a$ -Weyl's theorem implies  $a$ -Weyl's theorem (see [11]).

Recall that an operator  $T \in \mathbf{B}(\mathcal{H})$  is *Drazin invertible* if and only if it has a finite ascent and descent, which is also equivalent to the fact that  $T = T_0 \oplus T_1$ , where  $T_0$  is a nilpotent operator and  $T_1$  is an invertible operator (see [23, Proposition A]). The Drazin spectrum is given by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

We observe that  $\sigma_D(T) = \sigma(T) \setminus \pi(T)$ , where  $\pi(T)$  is the set of all poles.

An operator  $T \in \mathbf{B}(\mathcal{H})$  is called *left Drazin invertible* if  $a(T) < \infty$  and  $\mathcal{R}(T^{a(T)+1})$  is closed (see [9, Definition 2.4]). The left Drazin spectrum is given by

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}.$$

Recall [9, Definition 2.5] that  $\lambda \in \sigma_a(T)$  is a left pole of  $T$  if  $T - \lambda I$  is a left Drazin invertible operator and  $\lambda \in \sigma_a(T)$  is a left pole of finite rank if  $\lambda$  is a left pole of  $T$  and  $\alpha(T - \lambda) < \infty$ . We will denote  $\pi^a(T)$  the set of all left poles of  $T$ , and by  $\pi_0^a(T)$  the set of all left poles, of  $T$  of finite rank. We have  $\sigma_{LD}(T) = \sigma_a(T) \setminus \pi^a(T)$ .

Note that if  $\lambda \in \pi^a(T)$ , then it is easily seen that  $T - \lambda$  is an operator of topological uniform descent. Therefore it follows from ([11, Theorem 2.5]) that  $\lambda$  is isolated in  $\sigma_a(T)$ . Following [9] if  $T \in \mathbf{B}(\mathcal{H})$  and  $\lambda \in \mathbb{C}$  is isolated in  $\sigma_a(T)$ , then  $\lambda \in \pi^a(T)$  if and only if  $\lambda \notin \sigma_{SBF_+^-}(T)$  and  $\lambda \in \pi_0^a(T)$  if and only if  $\lambda \notin \sigma_{SF_+^-}(T)$ .

For the sake of simplicity of notation we introduce the abbreviations  $gaW$ ,  $aW$ ,  $gW$  and  $W$  to signify that an operator  $T \in \mathbf{B}(\mathcal{H})$  obeys generalized  $a$ -Weyl's theorem,  $a$ -Weyl's theorem, generalized Weyl's theorem and Weyl's theorem, respectively. Analogous meaning is attached to the abbreviations  $gaB$ ,  $aB$ ,  $gB$  and  $B$  with respect to Browder's theorem.

In the following diagram, arrows signify implications between various Weyl and Browder type theorems. It is known from [1, 3, 7, 11, 19, 20, 27] that if

$T \in \mathbf{B}(\mathcal{H})$ , then we have:

$$\begin{array}{ccccc}
 & & gW & \longrightarrow & gB & \longleftarrow & B \\
 & \nearrow & & & & & \nearrow \\
 gaW & \longrightarrow & aW & \longrightarrow & aB & & \\
 & \searrow & & & & & \searrow \\
 & & & & gaB & & 
 \end{array}$$

The quasinilpotent part  $H_0(T - \lambda)$  and the analytic core  $K(T - \lambda)$  of  $T - \lambda$  are defined by

$$H_0(T - \lambda) := \{x \in \mathcal{H} : \lim_{n \rightarrow \infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$K(T - \lambda) = \{x \in \mathcal{H} : \text{there exists a sequence } \{x_n\} \subset \mathcal{H} \text{ and } \delta > 0 \text{ for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \dots\}.$$

We note that  $H_0(T - \lambda)$  and  $K(T - \lambda)$  are generally non-closed hyper-invariant subspaces of  $T - \lambda$  such that  $(T - \lambda)^{-p}(0) \subseteq H_0(T - \lambda)$  for all  $p = 0, 1, \dots$  and  $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ . Recall that if  $\lambda \in \text{iso}(\sigma(T))$ , then  $H_0(T - \lambda) = \chi_T(\{\lambda\})$ , where  $\chi_T(\{\lambda\})$  is the global spectral subspace consisting of all  $x \in \mathcal{H}$  for which there exists an analytic function  $f : \mathbb{C} \setminus \{\lambda\} \rightarrow \mathcal{H}$  that satisfies  $(T - \mu)f(\mu) = x$  for all  $\mu \in \mathbb{C} \setminus \{\lambda\}$  (see [17]).

Let  $\text{Hol}(\sigma(T))$  be the space of all functions that analytic in an open neighborhoods of  $\sigma(T)$ . Following [18] we say that  $T \in \mathbf{B}(\mathcal{H})$  has the single-valued extension property (SVEP) at point  $\lambda \in \mathbb{C}$  if for every open neighborhood  $U_\lambda$  of  $\lambda$ , the only analytic function  $f : U_\lambda \rightarrow \mathcal{H}$  which satisfies the equation  $(T - \mu)f(\mu) = 0$  is the constant function  $f \equiv 0$ . It is well-known that  $T \in \mathbf{B}(\mathcal{H})$  has SVEP at every point of the resolvent  $\rho(T) := \mathbb{C} \setminus \sigma(T)$ . Moreover, from the identity theorem for analytic function it easily follows that  $T \in \mathbf{B}(\mathcal{H})$  has SVEP at every point of the boundary  $\partial\sigma(T)$  of the spectrum. In particular,  $T$  has SVEP at every isolated point of  $\sigma(T)$ . In [25, Proposition 1.8], Laursen proved that if  $T$  is of finite ascent, then  $T$  has SVEP.

**Proposition 1.1** ([24]). *Let  $T \in \mathbf{B}(\mathcal{H})$ .*

- (i) *If  $T$  has the SVEP, then  $i(T - \lambda I) \leq 0$  for every  $\lambda \in \rho_{SBF}(T)$ .*
- (ii) *If  $T^*$  has the SVEP, then  $i(T - \lambda I) \geq 0$  for every  $\lambda \in \rho_{SBF}(T)$ .*
- (iii) *If  $T^*$  has the SVEP, then*

$$(a) \quad \sigma_{SF_+^-}(T) = \omega(T) \quad \text{and} \quad (b) \quad \sigma_{SBF_+^-}(T) = \sigma_{B\omega}(T).$$

In [36] H. Weyl examined the spectra of all compact perturbations of a hermitian operator  $T$  on a Hilbert space and proved that their intersection coincides with the isolated point of the spectrum  $\sigma(T)$  which are the eigenvalues of finite multiplicity. Weyl's theorem has been extended to several classes of

Hilbert space operators including seminormal operators [4, 5]. In [7] M. Berkani introduced the concepts of the generalized Weyl's theorem and generalized Browder's theorem, and they showed that  $T$  satisfies the generalized Weyl's theorem whenever  $T$  is a normal operator on Hilbert space. More recently, [10] extended this result to hyponormal operators. In [32] extended this result to log-hyponormal operators. Recently, Rashid et al. [31] showed that if  $T$  is quasi-class  $A$ , then generalized Weyl's theorem holds  $f(T)$  for every  $f \in Hol(\sigma(T))$ . More recently, in [26] Mecheri showed that generalized Weyl's theorem holds for algebraically  $(p, k)$ -quasihyponormal operators.

In this paper, we study generalized  $a$ -Weyl's theorem for algebraically  $(p, k)$ -quasihyponormal operators. Among other things, we prove that the spectral mapping theorem holds for semi- $B$ -essential approximate point spectrum  $\sigma_{SBF_+^-}(T)$ , and for left Drazin spectrum for every  $f \in Hol(\sigma(T))$ .

## 2. Properties of algebraically $(p, k)$ -quasihyponormal operators

**Definition 2.1** ([22]). An operator  $T \in \mathbf{B}(\mathcal{H})$  is said to be  $(p, k)$ -quasihyponormal if

$$T^{k*}((T^*T)^p - (TT^*)^p)T^k \geq 0,$$

where  $0 \leq p \leq 1$  and  $k$  is a positive integer. Especially, when  $p = 1, k = 1, p = k = 1$ ,  $T$  is called  $k$ -quasihyponormal,  $p$ -quasihyponormal, quasihyponormal, respectively.

**Definition 2.2.** An operator  $T \in \mathbf{B}(\mathcal{H})$  is said to be algebraically  $(p, k)$ -quasihyponormal if there exists a non-constant complex polynomial  $\mathcal{P}$  such that  $\mathcal{P}(T)$  is a  $(p, k)$ -quasihyponormal operator.

In general, the following implications hold:

$$\begin{aligned} p\text{-hyponormal} &\Rightarrow p\text{-quasihyponormal} \Rightarrow \text{algebraically } p\text{-quasihyponormal} \\ &\Rightarrow \text{algebraically } (p, k)\text{-quasihyponormal.} \end{aligned}$$

An operator  $T \in \mathbf{B}(\mathcal{H})$  is called isoloid if every isolated point of  $\sigma(T)$  is an eigenvalue of  $T$ . An operator  $T \in \mathbf{B}(\mathcal{H})$  is called normaloid if  $r(T) = \|T\|$ , where  $r(T)$  is the spectral radius of  $T$ .  $X \in \mathbf{B}(\mathcal{H})$  is called a quasiaffinity if it has trivial kernel and dense range.  $S \in \mathbf{B}(\mathcal{H})$  is said to be a quasiaffine transform of  $T \in \mathbf{B}(\mathcal{H})$  (notation:  $S \prec T$ ) if there is a quasiaffinity  $X \in \mathbf{B}(\mathcal{H})$  such that  $XS = TX$ . If both  $S \prec T$  and  $T \prec S$ , then we say that  $S$  and  $T$  are quasisimilar.

The following facts follow from the above definition and some well known facts about  $(p, k)$ -quasihyponormal operators.

(i) If  $T \in \mathbf{B}(\mathcal{H})$  is an algebraically  $(p, k)$ -quasihyponormal operator, then so is  $T - \lambda I$  for each  $\lambda \in \mathbb{C}$ .

(ii) If  $T \in \mathbf{B}(\mathcal{H})$  is an algebraically  $(p, k)$ -quasihyponormal operator and  $M$  is a closed  $T$ -invariant subspace of  $\mathcal{H}$ , then  $T|_M$  is an algebraically  $(p, k)$ -quasihyponormal operator.

**Lemma 2.3.** *Let  $T \in \mathbf{B}(\mathcal{H})$  be a  $p$ -quasihyponormal operator for  $0 < p \leq 1$ . Then the following assertions hold.*

- (1)  $\|T^n x\|^2 \leq \|T^{n-1} x\| \|T^{n+1} x\|$  for all unit vector  $x \in \mathcal{H}$  and all positive integer  $n$ .
- (2)  $\|T^n\|^n \leq \|T^{n-1}\|^n r(T^n)$  for all positive integer  $n$ , where  $r(T^n)$  denote the spectral radius of  $T^n$ . Hence  $T$  is normaloid.
- (3)  $T$  is a paranormal operator.

*Proof.* (1) It is obvious that if  $T$  is  $p$ -quasihyponormal, then it is a  $(p, n)$ -quasihyponormal operator for each positive integer  $n$ , since

$$\begin{aligned} & \langle T^{*n} (TT^*)^p T^n x, x \rangle \\ &= \langle T^{*n} T (T^* T)^{p-1} T^* T^n x, x \rangle \\ &= \langle (T^* T)^{p+1} T^{n-1} x, T^{n-1} x \rangle \\ &\geq \|T^{n-1} x\|^{-2p} \langle T^* T T^{n-1} x, T^{n-1} x \rangle^{p+1} \quad (\text{by Hölder-McCarthy inequality}) \\ &= \|T^{n-1} x\|^{-2p} \|T^n x\|^{2p+2} \end{aligned}$$

and

$$\begin{aligned} & \langle T^{*n} (T^* T)^p T^n x, x \rangle \\ &= \langle (T^* T)^p T^n x, T^n x \rangle \\ &\leq \|T^n x\|^{2-2p} \langle T^* T T^n x, T^n x \rangle \quad (\text{Hölder-McCarthy inequality}) \\ &= \|T^n x\|^{2-2p} \|T^{n+1} x\|^{2p}. \end{aligned}$$

But  $T$  is a  $p$ -quasihyponormal operator. Then

$$\langle T^{*n} ((T^* T)^p - (TT^*)^p) T^n x, x \rangle \geq 0.$$

Hence

$$\|T^n x\|^2 \leq \|T^{n-1} x\| \|T^{n+1} x\|.$$

(2) If  $T^n = 0$  for some  $n > 1$ , then  $T = 0$ , and in this case  $r(T) = 0$ . Hence (2) is obvious. Hence we may assume  $T^n \neq 0$  for all  $n \geq 1$ . Then

$$\frac{\|T^n\|}{\|T^{n-1}\|} \leq \frac{\|T^{n+1}\|}{\|T^n\|} \leq \cdots \leq \frac{\|T^{mn}\|}{\|T^{mn-1}\|}$$

by (1), and we have

$$\left( \frac{\|T^n\|}{\|T^{n-1}\|} \right)^{mn-n-1} \leq \frac{\|T^{n+1}\|}{\|T^n\|} \times \cdots \times \frac{\|T^{mn}\|}{\|T^{mn-1}\|} = \frac{\|T^{mn}\|}{\|T^{n-1}\|}.$$

Hence

$$\left( \frac{\|T^n\|}{\|T^{n-1}\|} \right)^{n - \frac{n}{m} - \frac{1}{n}} \leq \frac{\|T^{mn}\|^{1/m}}{\|T^{mn-1}\|^{1/m}}.$$

Now letting  $m \rightarrow \infty$ . We get

$$\|T^n\|^n \leq \|T^{n-1}\|^n r(T^n).$$

Put  $n = 1$ , we have  $\|T\| \leq r(T)$ . So  $\|T\| = r(T)$ , i.e.,  $T$  is normaloid.

(3) Put  $n = 1$  in (1), we have  $\|Tx\|^2 \leq \|T^2x\|$ , that is,  $T$  is paranormal.  $\square$

**Definition 2.4** ([17]). An operator  $T \in \mathbf{B}(\mathcal{H})$  is said to be totally hereditarily normaloid,  $T \in THN$  if every part of  $T$  (i.e., its restriction to an invariant subspace), and  $T_p^{-1}$  for every invertible part  $T_p$  of  $T$ , is normaloid.

**Lemma 2.5.** *Let  $T \in THN$ , let  $\lambda \in \mathbb{C}$ . Assume that  $\sigma(T) = \{\lambda\}$ . Then  $T = \lambda I$ .*

*Proof.* We consider two cases:

case I. ( $\lambda = 0$ ): Since  $T$  is normaloid. Therefore  $T = 0$ .

case II. ( $\lambda \neq 0$ ): Here  $T$  is invertible, and since  $T \in THN$ , we see that  $T, T^{-1}$  are normaloid. On the other hand  $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$ , so  $\|T\|\|T^{-1}\| = |\lambda|\frac{1}{|\lambda|} = 1$ . It follows that  $T$  is convexoid, so  $W(T) = \{\lambda\}$ . Therefore  $T = \lambda I$ .  $\square$

In [14], Curto and Han proved that quasinilpotent algebraically paranormal operators are nilpotent. We now establish a similar result for algebraically  $(p, k)$ -quasihyponormal operators.

**Proposition 2.6.** *Let  $T$  be a quasinilpotent  $(p, k)$ -quasihyponormal operator. Then  $T$  is nilpotent.*

*Proof.* Assume that  $p(T)$  is a totally hereditarily normaloid operator for some nonconstant polynomial  $p$ . Since  $\sigma(p(T)) = p(\sigma(T))$ , the operator  $p(T) - p(0)$  is quasinilpotent. Thus Lemma 2.5 would imply that

$$cT^m(T - \lambda_1 I) \cdots (T - \lambda_n I) \equiv p(T) - p(0) = 0,$$

where  $m \geq 1$ . Since  $T - \lambda_j I$  is invertible for every  $\lambda_j \neq 0$ , we must have  $T^m = 0$ .  $\square$

**Lemma 2.7.** *Let  $T$  be an invertible  $p$ -quasihyponormal operator. Then  $\mathcal{H} = \mathcal{R}(T) \oplus \ker(T)$ . Moreover  $T_1$ , the restriction of  $T$  to  $\mathcal{R}(T)$  is one-one and onto.*

*Proof.* Suppose that  $y \in \mathcal{R}(T) \cap \ker(T)$  then  $y = Tx$  for some  $x \in \mathcal{H}$  and  $Ty = 0$ . It follows that  $T^2x = 0$ . However,  $d(T) = 1$  and so  $x \in \ker(T^2) = \ker(T)$ . Hence  $y = Tx = 0$  and so  $\mathcal{R}(T) \cap \ker(T) = \{0\}$ . Also,  $T\mathcal{R}(T) = \mathcal{R}(T)$ .

If  $x \in \mathcal{H}$ , there is  $u \in \mathcal{R}(T)$  such that  $Tu = Tx$ . Now if  $z = x - u$ , then  $Tz = 0$ . Hence

$$\mathcal{H} = \mathcal{R}(T) \oplus \ker(T).$$

Since  $a(T) = 1$ ,  $T$  maps  $\mathcal{R}(T)$  onto itself. If  $y \in \mathcal{R}(T)$  and  $Ty = 0$ , then  $y \in \mathcal{R}(T) \cap \ker(T) = \{0\}$ . Hence  $T_1$  is one-one and onto.  $\square$

Observe that  $\{\lambda_0\}$  is a clopen subset of  $\sigma(T)$ . Let  $T \in \mathbf{B}(\mathcal{H})$ . The  $R_\lambda(T) = (T - \lambda)^{-1}$  is analytic on  $\rho(T)$ , and an isolated point  $\lambda_0$  of  $\sigma(T)$  is an isolated singular point of the resolvent of  $T$ . Here there is a Laurent expansion of this function in powers of  $\lambda - \lambda_0$ . We write this in the form

$$(T - \lambda)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n.$$

The coefficients  $A_n$  and  $B_n$  are members of  $\mathbf{B}(\mathcal{H})$  and given by the standard formulas

$$(2.1) \quad A_n = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \lambda_0)^{-n-1} (\lambda - T)^{-1} d\lambda,$$

$$(2.2) \quad B_n = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \lambda_0)^{n-1} (\lambda - T)^{-1} d\lambda,$$

where  $\Gamma$  is any circle  $|\lambda - \lambda_0| = \rho$  with  $0 < \rho < \delta$  described once counter-clockwise.

The function  $f_n$  defined by

$$f_n(\lambda) = \begin{cases} (\lambda - \lambda_0)^{n-1}, & \text{if } |\lambda - \lambda_0| \leq \rho < \delta, \\ 0, & \text{otherwise.} \end{cases}$$

is in  $Hol(\sigma(T))$  and moreover

$$B_n = f_n(T), \quad n = 1, 2, \dots$$

For each positive integer  $n$ , we have

$$(\lambda - \lambda_0)f_n(\lambda) = f_{n+1}.$$

So

$$(2.3) \quad (T - \lambda_0)B_n = B_{n+1}$$

and by induction

$$(2.4) \quad (T - \lambda_0)^n B_1 = B_{n+1}.$$

We note in passing that

$$(2.5) \quad B_1 = E(\lambda_0)$$

the spectral projection corresponding to the clopen set  $\lambda_0$  of  $\sigma(T)$ .

Consider for each non-negative integer  $n$  the function  $g_n$  defined by

$$g_n(\lambda) = \begin{cases} 0, & \text{if } |\lambda - \lambda_0| \leq \rho < \delta, \\ (\lambda - \lambda_0)^{-n-1}, & \text{otherwise.} \end{cases}$$

is in  $Hol(\sigma(T))$ . Moreover,

$$A_n = -g_n(T)$$

for each non-negative integer  $n$ . We have

$$(2.6) \quad (\lambda - \lambda_0)g_{n+1}(\lambda) = g_n(\lambda)$$

and so

$$(2.7) \quad (\lambda - \lambda_0)A_{n+1} = A_n.$$

Similarly  $(\lambda - \lambda_0)g_0(\lambda) + f_1(\lambda) = 1$  and so

$$(2.8) \quad (T - \lambda)A_0 = B_0 - 1.$$



Recall that if  $T \in \mathbf{B}(\mathcal{H})$  and  $\lambda_0$  is an isolated point of  $\sigma(T)$ , then  $\lambda_0$  is called a pole of order  $m$  if and only if  $E(\lambda_0)(\lambda_0 - T)^m = 0$  and  $E(\lambda_0)(\lambda_0 - T)^{m-1} \neq 0$ .

**Lemma 2.8.** *Let  $T$  be a  $(p, k)$ -quasihyponormal operator and  $\lambda_0 \in \text{iso}\sigma(T)$ . Let  $\tau = \sigma(T) \setminus \{\lambda_0\}$ . Then  $\lambda_0$  is an eigenvalue of  $T$ . The ascent and descent of  $T - \lambda_0$  are both equal to  $k$ . Also*

$$\begin{aligned}\mathcal{R}(E(\lambda_0)) &= \ker((T - \lambda_0)^k), \\ \mathcal{R}(E(\tau)) &= \mathcal{R}((T - \lambda_0)^k).\end{aligned}$$

*Proof.* For convenience we denote the null-space and range of  $(\lambda_0 - T)^k$  by  $\ker_k$  and  $\mathcal{R}_k$ , respectively. If  $x \in \ker_k$ , where  $k \geq 1$ , we see by (2.7), induction and (2.8) that

$$0 = A_{k-1}(T - \lambda_0)^k x = (T - \lambda_0)^k A_{k-1} x = (T - \lambda_0) A_0 x = B_1 x - x.$$

So that by (2.5), we have  $x = B_1 x \in \mathcal{R}(E(\lambda_0))$ . Thus  $\ker_k \subseteq \mathcal{R}(E(\lambda_0))$  if  $k \geq 1$ . On the other hand, it follows from (2.4) that if  $x \in \mathcal{R}(E(\lambda_0))$ , then  $x = B_1 x$  and  $(T - \lambda_0)^k x = B_{k+1} x$ . Since  $B_{n+1} x = 0$  if  $n \geq k$ . It follows that  $\mathcal{R}(E(\lambda_0)) \subseteq \ker_k$  and  $\ker_n = \mathcal{R}(E(\lambda_0))$  if  $n \geq k$ . However,  $\ker_{k-1}$  is a proper subset of  $\ker_k$  because  $B_k \neq 0$ . The equations  $\ker_{k-1} = \ker_k = \mathcal{R}(E(\lambda_0))$  imply that  $B_k = 0$  in view of the relation  $B_k = (T - \lambda_0)^{k-1} B_1$ . We have now proved that the ascent of  $\lambda_0 - T$  is  $k$  and  $\ker_k = \mathcal{R}(E(\lambda_0))$ . In particular, since  $k > 0$ ,  $\lambda_0$  is an eigenvalue of  $T$ .

Now let  $T_1$  and  $T_2$  be the restrictions of  $T$  to  $\mathcal{R}(E(\tau))$  and  $\mathcal{R}(E(\lambda_0))$ , respectively.  $\lambda_0 \in \sigma(T_2)$  but  $\lambda_0 \notin \sigma(T_1)$ . Hence, the descent of  $\lambda_0 - T_1$  is 0 and  $\mathcal{R}((\lambda_0 - T_1)^k) = \mathcal{R}(E(\tau))$  when  $k \geq 1$ . Thus  $\mathcal{R}(E(\tau)) \subseteq \mathcal{R}_k$ . Now if  $n \geq k$ , the only point common to  $\mathcal{R}_n$  and  $\ker_n$  is 0. For, if  $x \in \mathcal{R}_n \cap \ker_n$ , then  $(\lambda_0 - T)^n x = 0$  and there is  $y \in \mathcal{H}$  such that  $x = (\lambda_0 - T)^n y$ . Hence  $y \in \ker_{2n} = \ker$  and so  $x = 0$ . Now suppose that  $n \geq k$  and  $x \in \mathcal{R}_n$ . Let  $x_1 = E(\tau)x$  and  $x_2 = E(\lambda_0)x$ , then  $x_2 = x - x_1 \in \mathcal{R}_n$  because  $\mathcal{R}(E(\tau)) \subseteq \mathcal{R}_n$ . However,  $x_2 \in \mathcal{R}(E(\lambda_0)) = \ker_n$ , and so  $x_2 = 0$  whence  $x = x_1 \in \mathcal{R}(E(\tau))$ . Thus  $\mathcal{R}_n \subseteq \mathcal{R}(E(\tau))$  if  $n \geq k$  and therefore that the descent of  $\lambda_0 - T$  is less than or equal to  $k$ . Then by [15, Proposition 1.49] shows that the descent is exactly  $k$ , which know to be the ascent.  $\square$

**Corollary 2.9.** *Let  $T \in \mathbf{B}(\mathcal{H})$  be a  $(p, k)$ -quasihyponormal operator. Then  $T$  is of finite ascent.*

An operator  $T \in \mathbf{B}(\mathcal{H})$  is said to be polaroid if  $\text{iso}\sigma(T) \subseteq \pi(T)$ , where  $\pi(T)$  is the set of all poles of  $T$ . In general, if  $T$  is polaroid, then it is isoloid. However, the converse is not true. Consider the following example. Let  $T \in \ell^2(\mathbb{N})$  be defined by

$$T(x_1, x_2, \dots) = \left( \frac{x_2}{2}, \frac{x_3}{3}, \dots \right).$$

Then  $T$  is a compact quasinilpotent operator with  $\alpha(T) = 1$ , and so  $T$  is isoloid. However, since  $T$  does not have finite ascent,  $T$  is not polaroid.

**Proposition 2.10.** *Let  $T$  be an algebraically  $(p, k)$ -quasihyponormal operator. Then  $T$  is polaroid.*

*Proof.* Suppose  $T$  is an algebraically  $(p, k)$ -quasihyponormal operator. Then  $p(T)$  is  $(p, k)$ -quasihyponormal for some nonconstant polynomial  $p$ . Let  $\lambda \in \text{iso}(\sigma(T))$ . Using the spectral projection  $P := \frac{1}{2i\pi} \int_{\partial D} (\mu - T)^{-1} d\mu$ , where  $D$  is a closed disk of center  $\lambda$  which contains no other points of  $\sigma(T)$ , we can represent  $T$  as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{and} \quad \sigma(T_1) = \{\lambda\} \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since  $T_1$  is algebraically  $(p, k)$ -quasihyponormal and  $\sigma(T_1) = \{\lambda\}$ . But  $\sigma(T_1 - \lambda I) = \{0\}$  it follows from Proposition 2.6 that  $T_1 - \lambda I$  is nilpotent. Therefore  $T_1 - \lambda$  has finite ascent and descent. On the other hand, since  $T_2 - \lambda I$  is invertible, clearly it has finite ascent and descent. Therefore  $T - \lambda I$  has finite ascent and descent. Therefore  $\lambda$  is a pole of the resolvent of  $T$ . Thus if  $\lambda \in \text{iso}(\sigma(T))$  implies  $\lambda \in \pi(T)$ , and so  $\text{iso}(\sigma(T)) \subset \pi(T)$ . Hence  $T$  is polaroid.  $\square$

**Corollary 2.11.** *Let  $T$  be an algebraically  $(p, k)$ -quasihyponormal operator. Then  $T$  is isoloid.*

For  $T \in \mathbf{B}(\mathcal{H})$ ,  $\lambda \in \sigma(T)$  is said to be a regular point if there exists  $S \in \mathbf{B}(\mathcal{H})$  such that  $T - \lambda I = (T - \lambda I)S(T - \lambda I)$ .  $T$  is called reguloid if every isolated point of  $\sigma(T)$  is a regular point. It is well known [19, Theorems 4.6.4 and 8.4.4] that  $T - \lambda I = (T - \lambda I)S(T - \lambda I)$  for some  $S \in \mathbf{B}(\mathcal{H}) \iff T - \lambda I$  has a closed range.

**Theorem 2.12.** *Let  $T$  be an algebraically  $(p, k)$ -quasihyponormal operator. Then  $T$  is reguloid.*

*Proof.* Suppose  $T$  is an algebraically  $(p, k)$ -quasihyponormal operator. Then  $p(T)$  is a  $(p, k)$ -quasihyponormal operator for some nonconstant polynomial  $p$ . Let  $\lambda \in \text{iso}(\sigma(T))$ . Using the spectral projection  $P := \frac{1}{2i\pi} \int_{\partial D} (\mu - T)^{-1} d\mu$ , where  $D$  is a closed disk of center  $\lambda$  which contains no other points of  $\sigma(T)$ , we can represent  $T$  as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{and} \quad \sigma(T_1) = \{\lambda\} \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since  $T_1$  is algebraically  $(p, k)$ -quasihyponormal and  $\sigma(T_1) = \{\lambda\}$ , it follows from Lemma 2.5 that  $T_1 = \lambda I$ . Therefore by [34, Theorem 6],

$$(2.9) \quad \mathcal{H} = E(\mathcal{H}) \oplus E(\mathcal{H})^\perp = \ker(T - \lambda I) \oplus \ker(T - \lambda I)^\perp.$$

Relative to decomposition 2.9,  $T = \lambda I \oplus T_2$ . Therefore  $T - \lambda I = 0 \oplus T - \lambda I$  and hence  $\text{ran}(T - \lambda I) = (T - \lambda I)(\mathcal{H}) = 0 \oplus (T_2 - \lambda I)(\ker(T - \lambda I)^\perp)$ . Since  $T_2 - \lambda I$  is invertible,  $T - \lambda I$  has closed range.  $\square$

**Theorem 2.13.** *Let  $T^* \in \mathbf{B}(\mathcal{H})$  be an algebraically  $(p, k)$ -quasihyponormal operator. Then  $T$  is a-isoloid.*

*Proof.* Suppose  $T^*$  is algebraically  $(p, k)$ -quasihyponormal. Since  $T^*$  has SVEP, then  $\sigma(T) = \sigma_a(T)$ . Let  $\lambda \in \text{iso}(\sigma_a(T)) = \text{iso}(\sigma(T))$ . But  $T^*$  is polaroid, hence  $T$  is also polaroid. Therefore it is isoloid, and hence  $\lambda \in \sigma_p(T)$ . Thus  $T$  is  $a$ -isoloid.  $\square$

### 3. Weyl's type theorem

**Lemma 3.1.** *If  $T$  is a  $(p, k)$ -quasihyponormal operator and  $S \prec T$ , then  $S$  has SVEP.*

*Proof.* Since  $T$  is a  $(p, k)$ -quasihyponormal operator, then it has a SVEP. So the result follows from [14, Lemma 3.1].  $\square$

**Theorem 3.2.** *Let  $S, T \in \mathbf{B}(\mathcal{H})$ . If  $T$  has SVEP and  $S \prec T$ , then  $f(S) \in gaB$  for every  $f \in \text{Hol}(\sigma(T))$ . In particular, if  $T$  has SVEP, then  $T \in gaB$ .*

*Proof.* Suppose that  $T$  has SVEP. Since  $S \prec T$ , it follows from the proof of [14] that  $S$  has SVEP. We now show that  $S \in gaB$ . Let  $\lambda \in \sigma_a(S) \setminus \sigma_{SBF_+^-}(S)$ ; then  $S - \lambda I \in SBF_+^-(S)$  but not bounded below. Since  $S - \lambda I \in SBF_+^-(S)$ , it follows from [11, Corollary 2.10] that  $S - \lambda I = S_1 \oplus S_2$ , where  $S_1$  is an upper semi-Fredholm operator with  $i(S_1) \leq 0$ , and  $S_2$  is nilpotent. Since  $S$  has SVEP,  $S_1$  and  $S_2$  also have SVEP. Therefore  $a$ -Browder's theorem holds for  $S_1$ , and hence  $\sigma_{ab}(S_1) = \sigma_{SF_+^-}(S_1)$ . Since  $S_1$  is semi-Fredholm with  $i(S_1) \leq 0$ ,  $S_1$  is  $a$ -Browder's. Hence  $\lambda$  is an isolated point of  $\sigma_a(S)$ . It follows that  $S \in gaB$ .

Now let  $f \in \text{Hol}(\sigma(T))$ . Since the SVEP is stable under the functional calculus, then  $f(S)$  has the SVEP. Therefore  $f(S) \in gaB$ , by the first part of the proof.  $\square$

We now recall that the generalized  $a$ -Weyl's theorem may not hold for quasinilpotent operators, and that it does not necessarily transfer to or from adjoints.

**Example 3.3.** Let  $T \in \mathbf{B}(\mathcal{H})$  defined on  $\ell^2$  by

$$T(x_1, x_2, \dots) = \left( \frac{x_2}{2}, \frac{x_3}{3}, \dots \right).$$

Then  $T$  is a quasinilpotent operator and  $\sigma(T) = \sigma_{SBF_+^-}(T) = E^a(T) = \{0\}$ . Thus  $T$  does not obey generalized  $a$ -Weyl's theorem.

Now  $\sigma(T^*) = \sigma_{SBF_+^-}(T^*) = \{0\}$  and  $E^a(T^*) = \emptyset$ . Therefore  $T^* \in gaW$ .

As a consequence of [17, Theorem 2.4] and [16, Lemma 2.5] we have:

**Theorem 3.4.** *Let  $T \in \mathbf{B}(\mathcal{H})$  be a  $(p, k)$ -quasihyponormal operator. Then  $T$  is of stable index.*

Let  $T \in \mathbf{B}(\mathcal{H})$ . It is well known that the inclusion  $\sigma_{SF_+^-}(f(T)) \subseteq f(\sigma_{SF_+^-}(T))$  holds for every  $f \in \text{Hol}(\sigma(T))$  with no restriction on  $T$  [29]. The next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically  $(p, k)$ -quasihyponormal operator.

**Theorem 3.5.** *Suppose  $T^*$  or  $T$  is an algebraically  $(p, k)$ -quasihyponormal operator. Then*

$$\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T)).$$

*Proof.* Assume first that  $T$  is an algebraically  $(p, k)$ -quasihyponormal operator and let  $f \in \text{Hol}(\sigma(T))$ . It suffices to show that  $\sigma_{SF_+^-}(f(T)) \supseteq f(\sigma_{SF_+^-}(T))$ . Suppose that  $\lambda \notin \sigma_{SF_+^-}(f(T))$ . Then  $f(T) - \lambda I \in SF_+^-(\mathcal{H})$  and

$$f(T) - \lambda I = c(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)g(T),$$

where  $c, \mu_1, \mu_2, \dots, \mu_n \in \mathbb{C}$ , and  $g(T)$  is invertible. Since  $T$  is an algebraically  $(p, k)$ -quasihyponormal operator, it has SVEP. It follows from [2, Theorem 2.6] that  $i(T - \mu_j) \leq 0$  for each  $j = 1, 2, \dots, n$ . Therefore  $\lambda \notin f(\sigma_{SF_+^-}(T))$ , and hence  $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$ . Suppose now that  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator. Then  $T^*$  has SVEP, and so by [2, Theorem 2.6]  $i(T - \mu_j I) \geq 0$  for each  $j = 1, 2, \dots, n$ . Since

$$0 \leq \sum_{j=1}^n i(T - \mu_j I) = i(f(T) - \lambda I) \leq 0,$$

$T - \mu_j I$  is Weyl for each  $j = 1, 2, \dots, n$ . Hence  $\lambda \notin f(\sigma_{SF_+^-}(T))$ , and so  $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$ . This completes the proof.  $\square$

**Theorem 3.6.** *Suppose  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator. Then  $a$ -Weyls theorem holds for  $f(T)$  for every  $f \in \text{Hol}(\sigma(T))$ .*

*Proof.* Suppose  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator. We first show that  $a$ -Weyls theorem holds for  $T$ . Suppose that  $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ . Then  $T - \lambda I$  is upper semi-Fredholm and  $i(T - \lambda I) \leq 0$ . Since  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator,  $T^*$  has SVEP. Therefore by [2, Theorem 2.6] that  $i(T - \lambda I) \geq 0$ , and hence  $T - \lambda I$  is Weyl. Since  $T^*$  has SVEP, it follows from [18, Corollary 7] that  $\sigma_a(T) = \sigma(T)$ . Also, since Weyls theorem holds for  $T$  by [26],  $\lambda \in \pi_0^a(T)$ .

Conversely, suppose that  $\lambda \in \pi_0^a(T)$ . Since  $T^*$  has SVEP, it follows from [18, Corollary 7] that  $\sigma_a(T) = \sigma(T)$ . Therefore  $\lambda$  is an isolated point of  $\sigma(T)$ , and hence  $\bar{\lambda}$  is an isolated point of  $\sigma(T^*)$ . But  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator, hence by Proposition 2.10 that  $\bar{\lambda} \in \pi(T^*)$ . Therefore there exists a natural number  $n_0$  such that  $n_0 = a(T^* - \bar{\lambda}I) = d(T^* - \bar{\lambda}I)$ . Hence we have  $\mathcal{H} = \ker((T^* - \bar{\lambda}I)^{n_0}) \oplus \text{ran}((T^* - \bar{\lambda}I)^{n_0})$  and  $\text{ran}((T^* - \bar{\lambda}I)^{n_0})$  is closed. Therefore  $\text{ran}((T - \lambda I)^{n_0})$  is closed and  $\mathcal{H} = \ker((T^* - \bar{\lambda}I)^{n_0})^\perp \oplus \text{ran}((T^* - \bar{\lambda}I)^{n_0})^\perp = \ker((T - \lambda I)^{n_0}) \oplus \text{ran}((T - \lambda I)^{n_0})$ . So  $\lambda \in \sigma_p(T)$ , and hence  $T - \lambda I$  is Weyl. Consequently,  $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ . Thus  $a$ -Weyls theorem holds for  $T$ .

Now we show that  $T$  is  $a$ -isoloid. Let  $\lambda$  be an isolated point of  $\sigma_a(T)$ . Since  $T^*$  has SVEP,  $\lambda$  is an isolated point of  $\sigma(T)$ . But  $T^*$  is polaroid, hence  $T$  is also polaroid. Therefore it is isoloid, and hence  $\lambda \in \sigma_p(T)$ . Thus  $T$  is  $a$ -isoloid.

Finally, we shall show that  $a$ -Weyls theorem holds for  $f(T)$  for every  $f \in Hol(\sigma(T))$ . Let  $f \in Hol(\sigma(T))$ . Since  $a$ -Weyls theorem holds for  $T$ , it satisfies  $a$ -Browders theorem. Therefore  $\sigma_{ab}(T) = \sigma_{SF_+^-}(T)$ . It follows from Theorem 3.5 that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{SF_+^-}(T)) = \sigma_{SF_+^-}(f(T)),$$

and hence  $a$ -Browders theorem holds for  $f(T)$ . So  $\sigma_a()f(T) \setminus \sigma_{SF_+^-}(f(T)) \subset \pi_0^a(T)$ . Conversely, suppose that  $\lambda \in \pi_0^a(f(T))$ . Then  $\lambda$  is an isolated point of  $\sigma_a(f(T))$  and  $0 < \alpha(f(T) - \lambda I) < 1$ . Since  $\lambda$  is an isolated point of  $f(\sigma_a(T))$ , if  $\mu_j \in \sigma_a(T)$ , then  $\mu_j$  is an isolated point of  $\sigma_a(T)$ . Since  $T$  is  $a$ -isoloid,  $0 < \alpha(T - \mu_j) < 1$  for each  $j = 1, 2, \dots, n$ . Since  $a$ -Weyls theorem holds for  $T$ ,  $T - \mu_j$  is upper semi-Fredholm and  $i(T - \mu_j) \leq 0$  for each  $j = 1, 2, \dots, n$ . Therefore  $f(T) - \lambda I$  is upper semi-Fredholm and  $f(T) - \lambda I = \sum_{j=1}^n i(T - \mu_j I) \leq 0$ . Hence  $\lambda \in \sigma_a()f(T) \setminus \sigma_{SF_+^-}(f(T))$ , and so  $a$ -Weyls theorem holds for  $f(T)$  for each  $f \in Hol(\sigma(T))$ . This completes the proof.  $\square$

**Theorem 3.7.** *Let  $T$  be an algebraically  $(p, k)$ -quasihyponormal operator. Then  $\sigma_{lD}(T) = \sigma_{SBF_+^-}(T) \cup acc(\sigma_a(T))$ .*

*Proof.* Suppose that  $\lambda \in \sigma_a(T) \setminus \sigma_{lD}(T)$ . Then  $T - \lambda I$  is left Drazin invertible but not bounded below. In particular,  $T - \lambda I$  is semi-B-Fredholm. Therefore  $d = a(T - \lambda) < \infty$  and  $\text{ran}((T - \lambda I)^{d+1})$  is closed. On the other hand, since  $d = a(T - \lambda I) < \infty$  and  $\text{ran}(T - \lambda)^{d+1}$  is closed,  $\lambda$  is an isolated point of  $\sigma_a(T)$ . Hence  $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF_+^-}(T) \cup acc(\sigma_a(T)))$ .

Conversely, suppose that  $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF_+^-}(T) \cup acc(\sigma_a(T)))$ . Then  $T - \lambda I$  is semi-B-Fredholm and  $\lambda$  is an isolated point of  $\sigma_a(T)$ . Since  $T - \lambda I$  is upper semi-Fredholm, it follows from [11, Corollary 2.10] that  $T - \lambda I$  can be decompose as  $T - \lambda I = T_1 \oplus T_2$ , where  $T_1$  is an upper semi-Fredholm operator with  $i(T_1) \leq 0$  and  $T_2$  is nilpotent. We consider two cases.

Case I. Suppose that  $T_1$  is bounded below. Then  $T - \lambda I$  is left Drazin invertible, and so  $\lambda \notin \sigma_{lD}(T)$ .

Case II. Suppose that  $T_1$  is not bounded below. Then 0 is an isolated point of  $\sigma_a(T_1)$ . But  $T_1$  is an upper semi-Fredholm operator, hence it follows from the punctured neighborhood theorem that  $T_1$  is  $a$ -Browder. Therefore there exists a finite rank operator  $S_1$  such that  $T_1 + S_1$  is bounded below and  $T_1 S_1 = S_1 T_1$ . Put  $F := S_1 \oplus 0$ . Then  $F$  is a finite rank operator,  $TF = FT$  and  $T - \lambda I + F = T_1 \oplus T_2 + S_1 \oplus 0 = (T_1 + S_1) \oplus T_2$  is left Drazin invertible. Hence  $\lambda \notin \sigma_{lD}(T)$ .  $\square$

As shown in [12] that the spectral mapping theorem holds for the Drazin spectrum. We prove here the spectral mapping theorem holds for left Drazin spectrum.

**Theorem 3.8.** *Let  $T$  be an algebraically  $(p, k)$ -quasihyponormal operator and let  $f \in \text{Hol}(\sigma(T))$ . Then  $\sigma_{lD}(f(T)) = f(\sigma_{lD}(T))$ .*

*Proof.* Suppose that  $\mu \notin f(\sigma_{lD}(T))$  and set  $h(\lambda) = f(\lambda) - \mu I$ . Then  $h$  has no zeros in  $\sigma_{lD}(T)$ . Since  $\sigma_{lD}(T) = \sigma_{SBF_+^-}(T) \cup \text{acc}(\sigma_a(T))$  by Theorem 3.7, we conclude that  $h$  has finitely many zeros in  $\sigma_a(T)$ . Now we consider two cases.

Case I. Suppose that  $h$  has no zeros in  $\sigma_a(T)$ . Then  $h(T) = f(T) - \mu I$  is bounded below, and so  $\mu \notin \sigma_{lD}(f(T))$ .

Case II. Suppose that  $h$  has at least one zero in  $\sigma_a(T)$ . Then

$$h(\lambda) = c(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)g(\lambda),$$

where  $c, \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$  and  $g(\lambda)$  is a nonvanishing analytic function on an open neighborhood. Therefore

$$h(T) = c(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)g(T),$$

where  $g(T)$  is bounded below. Since  $\mu \notin f(\sigma_{lD}(T))$ ,  $\lambda_1, \lambda_2, \dots, \lambda_n \notin \sigma_{lD}(T)$ . Therefore  $T - \lambda_j I$  is left Drazin invertible, and hence each  $T - \lambda_j I \in SBF_+^-(r)$ ,  $j = 1, 2, \dots, n$ . But each  $\lambda_j$  is an isolated point of  $\sigma_a(T)$ , it follows from [11, Theorem 2.8] that each  $\lambda_j$  is a left pole of the resolvent of  $T$ . Therefore  $a(T - \lambda_j I) = d < \infty$  and  $\text{ran}(T - \lambda_j I)^{d+1}$  is closed ( $j = 1, 2, \dots, n$ ), so  $a((T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)) = s < \infty$  and  $\text{ran}((T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I))^{s+1}$  is closed. Since  $g(T)$  is bounded below,  $a(h(T)) = t < \infty$  and  $\text{ran}((h(T))^{t+1})$  is closed. Therefore  $h(T)$  is left Drazin invertible, and so  $0 \notin \sigma_{lD}(h(T))$ . Hence  $\mu \notin \sigma_{lD}(f(T))$ . It follows from Cases I and II that  $\sigma_{lD}(f(T)) \subseteq f(\sigma_{lD}(T))$ .

Conversely, suppose that  $\lambda \notin \sigma_{lD}(f(T))$ . Then  $f(T) - \lambda I$  is left Drazin invertible. We again consider two cases.

Case I. Suppose that  $f(T) - \lambda I$  is bounded below. Then  $\lambda \notin \sigma_a(f(T)) = f(\sigma_a(T))$ , and hence  $\lambda \notin f(\sigma_{lD}(T))$ .

Case II. Suppose that  $\lambda \in \sigma_a(f(T)) \setminus \sigma_{lD}(f(T))$ . Write

$$f(T) = c(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)g(T),$$

where  $c, \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$  and  $g(T)$  is bounded below. Since  $f(T) - \lambda I$  is left Drazin invertible,  $f(T) = c(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)g(T)$  has finite ascent say  $r$  and  $\text{ran}(f(T))^{r+1}$  is closed. Hence  $T - \lambda_j I$  has finite ascent say  $r_j$  and  $\text{ran}(T - \lambda_j I)^{r_j+1}$  is closed for every  $j = 1, 2, \dots, n$ . Therefore each  $T - \lambda_j I$  is left Drazin invertible, and so  $\lambda_1, \dots, \lambda_n \notin \sigma_{lD}(T)$ .

We now wish to prove that  $\lambda \notin f(\sigma_{lD}(T))$ . Assume not; then there exists  $\mu \in \sigma_{aD}(T)$  such that  $f(\mu) = \lambda$ . Since  $g(\mu) \neq 0$ , we must have  $\mu = \mu_j$  for some  $j = 1, 2, \dots, n$ , which implies  $\mu_j \in \sigma_{lD}(T)$ , a contradiction. Hence  $\lambda \notin f(\sigma_{lD}(T))$ , and so  $f(\sigma_{lD}(T)) \subseteq \sigma_{lD}(f(T))$ . This completes the proof.  $\square$

**Theorem 3.9.** *Suppose  $T$  or  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator. Then  $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$  for all  $f \in \text{Hol}(\sigma(T))$ .*

*Proof.* Let  $\lambda \notin \sigma_{SBF_+^-}(f(T))$ . Then  $f(T) - \lambda I \in SBF_+^-(T)$  and

$$f(T) - \lambda I = \prod_{j=1}^m (T - \lambda_j I)g(T),$$

where  $\lambda_1, \dots, \lambda_m \in \mathbb{C}$  and  $g(T)$  is invertible. Since  $f(T) - \lambda I$  is an upper semi- $B$ -Fredholm operator, it follows from [7, Theorem 3.2] that  $T - \lambda_j I$  is upper semi- $B$ -Fredholm for each  $1 \leq j \leq m$ . Hence

$$i(f(T) - \lambda I) = \sum_{j=1}^m i(T - \lambda_j I) \leq 0.$$

Now from [7, Remark A] there exists some integer  $k$  such that for each,  $1 \leq j \leq m$ ,  $T - (\lambda_j + \frac{1}{k})I$  is an upper semi- $B$ -Fredholm operator and  $i(T - (\lambda_j + \frac{1}{k})I) = i(T - \lambda_j I)$ . If  $T$  is an algebraically  $(p, k)$ -quasihyponormal operator, then it follows from Proposition 1.1 that  $i(T - \lambda_j I) \leq 0$ . Hence  $\lambda \notin f(\sigma_{SBF_+^-}(T))$ .

Now if  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator, then we have from Proposition 1.1 that  $i(T - \lambda_j I) = 0$  and so  $T - \lambda_j I$  is a  $B$ -Fredholm operator of index 0. Thus  $\lambda \notin f(\sigma_{SBF_+^-}(T))$ .

For the converse inclusion. Let  $\lambda \in \sigma_{SBF_+^-}(f(T)) \setminus f(\sigma_{SBF_+^-}(T))$ . Suppose that

$$f(T) - \lambda I = \prod_{j=1}^m (T - \lambda_j I)g(T),$$

where  $\lambda_1, \dots, \lambda_m \in \mathbb{C} \setminus \sigma_{SBF_+^-}(T)$  and  $g(T)$  is invertible. Hence  $f(T) - \lambda I$  is upper semi- $B$ -Fredholm and  $i(f(T) - \lambda I) = \sum_{j=1}^m i(T - \lambda_j I) \leq 0$ . Therefore  $\lambda \notin \sigma_{SBF_+^-}(f(T))$ , so a contradiction.  $\square$

**Lemma 3.10.** *Suppose that  $T \in \mathbf{B}(\mathcal{H})$  is algebraically  $(p, k)$ -quasihyponormal. Then for any  $f \in \text{Hol}(\sigma(T))$  we have*

$$\sigma_a(f(T)) \setminus E^a(f(T)) = f(\sigma_a(T)) \setminus E^a(T).$$

*Proof.* Let  $\lambda \in \sigma_a(f(T)) \setminus E^a(f(T))$ . Then  $\lambda \in \sigma_a(f(T)) = f(\sigma_a(T))$ . We distinguish two cases:

Case I.  $\lambda \notin \text{iso}(f(\sigma_a(T)))$ , then there is an infinite sequence  $\{\eta_n\}_{n \in \mathbb{N}} \in \sigma_a(T)$  such that  $\lambda = f(\eta_0)$  and  $\eta_n \rightarrow \eta_0$ . But  $f \in \text{Hol}(\sigma(T))$ , therefore  $f(\eta_n) \rightarrow f(\eta_0) = \lambda$  and  $\lambda \in f(\sigma_a(T) \setminus E^a(T))$ .

Case II.  $\lambda \in \text{iso}(f(\sigma_a(T)))$ , since  $\lambda \notin E^a(f(T))$  then  $\lambda$  is not an eigenvalue of  $f(T)$ . Then

$$f(T) - \lambda I = (T - \eta_1 I)^{t_1} (T - \eta_2 I)^{t_2} \cdots (T - \eta_m I)^{t_m} g(T),$$

where  $\eta_1, \dots, \eta_m$  are scalars and  $g$  is invertible. Since  $\lambda$  is not an eigenvalue of  $f(T)$ , then for each  $j \in \{1, \dots, m\}$ ,  $\eta_j$  is not an eigenvalue of  $T$ . Hence  $\eta_j \in \sigma_a(T) \setminus E^a(T)$  and  $\lambda = f(\eta_j) \in f(\sigma_a(T) \setminus E^a(T))$ .

Conversely, Let  $\lambda \in f(\sigma_a(T) \setminus E^a(T))$  then  $\lambda \in \sigma_a(f(T)) = f(\sigma_a(T))$ . Assume that  $\lambda \in E^a(f(T))$ . Then

$$f(T) - \lambda I = (T - \eta_1 I)^{t_1} (T - \eta_2 I)^{t_2} \dots (T - \eta_m I)^{t_m} g(T),$$

where  $\eta_1, \dots, \eta_m$  are scalars and  $g$  is invertible. If  $\eta_j \in \sigma_a(T)$ , then  $\eta_j \in \text{iso}(\sigma_a(T))$ . Since  $T$  is  $a$ -isoloid,  $\eta_j$  is an eigenvalue of  $T$ . Hence  $\eta_j \in E^a(T)$ . So  $\lambda = f(\eta_j)$  this leads a contraction to the fact that  $\lambda \in f(\sigma_a(T) \setminus E^a(T))$ .  $\square$

**Theorem 3.11.** *Let  $T^* \in \mathbf{B}(\mathcal{H})$  be algebraically  $(p, k)$ -quasihyponormal. Then generalized  $a$ -Weyl's theorem holds for  $f(T)$ , for every  $f \in \text{Hol}(\sigma(T))$ .*

*Proof.* If  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator, then  $T^*$  has SVEP  $\sigma(T) = \sigma_a(T)$  and consequently  $E(T) = E^a(T)$ .

Let  $\lambda \notin \sigma_{SBF_+^-}(T)$  be given. Then  $T - \lambda$  is semi- $B$ -Fredholm and  $i(T - \lambda) \leq 0$ . Then Proposition 1.1 implies that  $i(T - \lambda) = 0$  and consequently  $T - \lambda$  is  $B$ -Weyl's. Hence  $\lambda \notin \sigma_{B\omega}(T)$ . Hence it follows from [37, Theorem 3.1] that  $\lambda \in E(T) = E^a(T)$ .

For the converse, let  $\lambda \in E^a(T)$ . Then  $\lambda \in \text{iso}\sigma_a(T)$ . Since  $T^*$  has SVEP, we have  $\sigma(T) = \sigma_a(T)$ . Hence  $\bar{\lambda} \in \sigma(T^*)$ . Now we represent  $T^*$  as the direct sum  $T^* = T_1 \oplus T_2$ , where  $\sigma(T_1) = \{\bar{\lambda}\}$  and  $\sigma(T_2) = \sigma(T) \setminus \{\bar{\lambda}\}$ . Since  $T \in \mathbf{Y}(\mathcal{H})$  then so does  $T_1$ , and so we have two cases:

Case I. ( $\bar{\lambda} = 0$ ): then  $T_1$  is quasinilpotent. Hence it follows that  $T_1$  is nilpotent. Since  $T_2$  is invertible, Then  $T^*$  is  $B$ -Weyl's.

Case II. ( $\bar{\lambda} \neq 0$ ): Since  $\sigma(T_1) = \{\bar{\lambda}\}$ , then  $T_1 - \bar{\lambda}$  is nilpotent and  $T_2 - \bar{\lambda}$  is invertible, it follows from [37, Theorem 3.1] that  $T^* - \bar{\lambda}$  is  $B$ -Weyl's. Thus in any case  $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ .

Let  $f \in \text{Hol}(\sigma(T))$ . Since  $T$  is  $a$ -isoloid, then it follows from Theorem 3.9 that  $\sigma_{SBF_+^-}(f(T)) = f(\sigma_{SBF_+^-}(T)) = f(\sigma_a(T) \setminus E^a(T)) = \sigma_a(f(T)) \setminus E^a(f(T))$ . Thus generalized  $a$ -Weyl's theorem holds for  $f(T)$ .  $\square$

**Corollary 3.12.** *Let  $T^* \in \mathbf{B}(\mathcal{H})$  be an algebraically  $(p, k)$ -quasihyponormal. Then  $E^a(T) = \pi^a(T)$ .*

*Proof.* If  $T^*$  is an algebraically  $(p, k)$ -quasihyponormal operator, then  $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$ . Let  $\lambda \in E^a(T)$ . Then  $\lambda$  is isolated in  $\sigma_a(T)$ , and  $\lambda \notin \sigma_{SBF_+^-}(T)$ . So  $T - \lambda I$  is in  $SBF_+^-(\mathcal{H})$ . It follows from [11, Theorem 2.8] that  $\lambda$  is a left pole of  $T$ , and so  $\lambda \in \pi^a(T)$ . As we have always  $\pi^a(T) \subset E^a(T)$ , then  $E^a(T) = \pi^a(T)$ .  $\square$

**Definition 3.13.** Let  $T \in \mathbf{B}(\mathcal{H})$  and let  $k \in \mathbb{N}$ . Then  $T$  has a uniform descent for  $n \geq k$  if  $\mathcal{R}(T) + \ker(T^n) = \mathcal{R}(T) + \ker(T^k)$  for all  $n \geq k$ . If, in addition,



$\mathcal{R}(T) + \ker(T^k)$  is closed, then  $T$  is said to have a topological uniform descent for  $n \geq k$ .

An operator  $T \in \mathbf{B}(\mathcal{H})$  is called  $a$ -polaroid if  $iso\sigma_a(T) \subset \pi^a(T)$ . In general, if  $T$  is  $a$ -polaroid, then it is polaroid. However, the converse is not true. Consider the following example.

**Example 3.14.** Let  $R$  be the unilateral right shift on  $\ell^2(\mathbb{N})$  and define

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \dots) \quad \text{for all } x_n \in \ell^2(\mathbb{N}).$$

Clearly,  $U$  is a quasi-nilpotent operator. Let  $T := R \oplus U$ . We have  $\sigma(T) = \mathbf{D}$ ,  $\mathbf{D}$  is the unit disc of  $\mathbb{C}$ , so  $iso(\sigma(T)) = E_0(T) = \emptyset$  and hence  $T$  is polaroid. Moreover,  $\sigma_a(T) = \partial\mathbf{D} \cup \{0\}$ . Since  $\sigma_a(T)$  does not cluster at 0, then  $T$  has the SVEP at 0, as well as at the points  $\lambda \notin \sigma_a(T)$ . Since  $T$  has SVEP at all points  $\lambda \in \partial\sigma(T)$  it then follows that  $T$  has SVEP. Finally,  $\sigma_{SBF_+^-}(T) = \partial\sigma(T)$  so  $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{0\}$ . Hence  $T$  is not  $a$ -polaroid.

**Theorem 3.15.** Let  $T^* \in \mathbf{B}(\mathcal{H})$  be an algebraically  $(p, k)$ -quasihyponormal operator. Then  $T$  is  $a$ -polaroid.

*Proof.* Suppose  $T^*$  is algebraically  $(p, k)$ -quasihyponormal. Since  $T^*$  has the SVEP, then  $\sigma_a(T) = \sigma(T)$ . Let  $\lambda \in iso(\sigma_a(T)) = iso(\sigma(T))$ . Since  $a$ -Weyl's theorem holds for  $T$  by Theorem 3.6, then  $\lambda$  is a left pole of finite rank of  $T$ . Therefore  $T - \lambda I$  has a finite ascent  $k = a(T - \lambda I)$  and  $\mathcal{R}(T - \lambda I)^{k+1}$  is closed. Since  $T - \lambda I$  is also an operator of topological uniform descent for  $n \geq 0$ , then it follows from [9, Lemma 2.8] that  $T - \lambda I$  is injective. So  $a(T - \lambda I) = 0$  and  $\mathcal{R}(T - \lambda I)$  is closed. Since  $\pi^a(T) = E^a(T)$ , we see that  $\lambda$  is a left pole of  $T$ . That is, all isolated points of the approximate point spectrum of  $T$  are left poles of the resolvent of  $T$ .  $\square$

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