WHAT IS...

a Leavitt path algebra?

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The first postcalculus theorem you encountered as an undergraduate may well have been this: any two bases of a finitely generated real vector space contain the same number of vectors, called the dimension. The standard verification of this result relies on being able to clear nonzero coefficients, which is possible here because $\mathbb{R}$ is a field. When you take the direct sum $V \oplus W$ of two such vector spaces, you get another one. A set with some such associative addition (but not necessarily subtraction) like $\oplus$, which has a zero element, is called a monoid. Also, the dimensions of the vector spaces (a.k.a. "ranks") add under $\oplus$. So, recast somewhat more formally, the theorem establishes that the monoid of finitely generated free modules over $\mathbb{R}$ behaves just like the monoid of nonnegative integers $\mathbb{Z}^+$, with $V \leftrightarrow \text{rank}(V)$. (We view $\{0\}$ as a vector space of dimension 0.)

Any ring R whose finitely generated free modules behave just like $\mathbb{Z}^+$ is said to have the IBN property (for Invariant Basis Number). Many rings fail to have this property. For example, let S consist of infinite real matrices with rows and columns indexed by the positive integers and all but finitely many entries in each column equalling 0 (so that we still have matrix multiplication). Then $S \cong S \oplus S := S^2$ by letting the odd columns correspond to the first summand and the even columns to the second. Using this, we easily get $S^n \cong S$ for *all n*, i.e., maximally epic failure of IBN.

Then the natural question arises: are there rings for which the behavior of the finitely generated free modules lies somewhere in between the $\mathbb{R}$ and S extremes? Does there exist, for example, a ring R for which (as free modules) $R^2 \ncong R$ but $R^3 \cong R$? Well, suppose you have a ring R containing six elements $x_1, x_2, x_3, y_1, y_2, y_3$ that multiply as follows:

(1) $y_i x_i = 1, y_j x_i = 0 \ (j \neq i)$ and $x_1 y_1 + x_2 y_2 + x_3 y_3 = 1.$

Then the maps $R \to R^3$ via $r \mapsto (rx_1, rx_2, rx_3)$ and $R^3 \to R$ via $(r_1, r_2, r_3) \mapsto r_1 y_1 + r_2 y_2 + r_3 y_3$ are easily shown to be inverses of each other, so that $R^3 \cong R$. (For one direction: $r \mapsto (rx_1, rx_2, rx_3) \mapsto rx_1 y_1 + rx_2 y_2 + rx_3 y_3 = r(x_1 y_1 + x_2 y_2 + x_3 y_3) = r \cdot 1_R = r$.) So your ring R would be a good candidate for such an "in between" ring. How to find an example of such an R? EASY, just rig a ring that contains elements which behave this way, e.g., by taking the free associative algebra in the six variables and imposing (modding out by) the relations (1). Then $R^3 \cong R$. But how could you show that $R^2 \ncong R$? THAT'S NOT SO EASY. (Even showing that $R \neq \{0\}$ is not so easy.)

In fact $R^2 \ncong R$ (and much, much more) was established by Bill Leavitt [2] in 1962. This R is now called the Leavitt algebra of type $(1, 3)$. There is an analogous Leavitt algebra of type $(1, n)$ for each integer $n \geq 2$.

In deep, fundamental work from 1974, George Bergman described an explicit general construction which starts with any appropriate monoid (along with some additional data about that monoid) and produces a corresponding algebra. The resulting algebra has the property that the monoid of finitely generated projective modules with operation $\oplus$ (which contains, and possibly equals, the monoid of finitely generated free modules) for this algebra behaves just like the given monoid. A special case of the construction yields the Leavitt algebra of type $(1, n)$ by starting from the monoid

(2) $$M_n = \{0, x, 2x, \dots, (n-1)x\}$$

with the relation $nx = x$.

We now switch gears. Let Γ be a finite directed graph with vertex set V. Consider the commutative monoid M_Γ generated by V, modulo relations of the form $v = \sum\{r(e) : e \text{ is an edge from } v \text{ to } r(e)\}$ (assuming that set is nonempty). For example, if Γ is the "rose with n petals" of Figure 1, then M_Γ is the monoid M_n of (2). For general Γ, Bergman's corresponding algebra for M_Γ

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(with germane additional data) is called the *Leavitt path algebra* of Γ. Familiar examples of Leavitt path algebras include the algebra of $n \times n$ matrices and the Laurent polynomial algebra (generated by x and x^{-1}). More interesting examples have perhaps unexpected behavior.

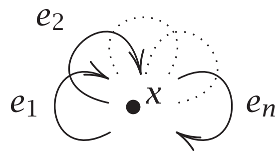


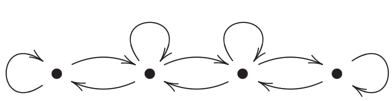
Figure 1. This "rose" graph with n petals yields the monoid M_n with relation $nx = x$ because the edges which start at x end back at x with multiplicity n.

Since the introduction of Leavitt path algebras in 2005, the research effort into their structure has included a number of lines, e.g., the discovery of conditions on the graph which are equivalent to various ring conditions on the associated Leavitt path algebra, such as simplicity (no nontrivial two-sided ideals), finite dimensionality, so-called von Neumann regularity, and primeness.

Leavitt path algebras have been used to settle long-standing questions about apparently unrelated structures

There is a tight (but not yet completely well-understood) connection between the Leavitt path algebra and a C^*-algebra associated with a graph. This connection was an initial motivation for the study of Leavitt path algebras, and it continues to drive one of the research lines. As well, there is an extremely close connection between certain Leavitt path algebras and structures arising in symbolic dynamics. Results established about Leavitt path algebras and their generalizations have been used to settle long-standing questions about apparently unrelated structures, for example, infinite simple groups.

The key open question is tantalizingly easy to state: if E_4 denotes the graph



then is the Leavitt path algebra of E_4 isomorphic to the Leavitt algebra of type $(1, 2)$ generated from the monoid M_2 given above in (2)? A more general version of this question (the Algebraic Kirchberg Phillips Question) currently lies at the heart of the subject. Many algebraists, analysts, and dynamicists are working on its resolution. Perhaps you'd like to join in?

See Abrams [1] for a fuller description of the subject.

Credit

Photo of Gene Abrams, courtesy of Gene Abrams.

References

[1] G. Abrams, Leavitt path algebras: The first decade, *Bull. Math. Sci.* **5** (2015), no. 1, 59–120. MR 3319981

[2] W. G. Leavitt, The module type of a ring, *Trans. Amer. Math. Soc.* **103** (1962), 113–130. MR 0132764

Gene Abrams

ABOUT THE AUTHOR

Gene Abrams enjoys trying to keep up with his wife, Mickey, while skiing and biking in the great outdoors of Colorado. All things baseball dominate the vast majority of his remaining free time.

"[I]t's easy to [lose sight of the reader] in writing nonfiction, when you're trying to convey knowledge. You have to vividly conjure up someone who doesn't know what you know. That's hard. It's hard, once you've understood something, to remember what it's like not to understand it. Your whole sense of what's obvious shifts, and you come, over time, to forget that there ever was a shift, and you have difficulty recalling your pre-shift state of mind. But that's the state of mind of your readers, and you have to work to make it vivid to yourself."

— Rebecca Newberger Goldstein,
in an interview with Rachel Toor,
Chronicle of Higher Education,
October 6, 2015