

# What is an inconsistent truth table?

Zach Weber (University of Otago)

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Joint work with G Badia (Otago) and P Girard (Auckland)

Introduction: Non-classical logic, top to bottom

Elements of a (paraconsistent) metatheory

Semantics

Soundness, completeness, and non-triviality

Conclusion

Q: Can standard reasoning about logic be carried out without any appeal to classical logic?

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- A: Yes. The semantics of propositional logic can be given paraconsistently, with soundness and completeness theorems (as well as their negations).

This is evidence for a more general claim:

*Metatheory determines object theory.*

When we write down the orthodox clauses for a logic, whatever logic we presuppose in the background will be the object-level logic that obtains.

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But the syntax and semantics of paraconsistent and paracomplete logics—their grammar and truth tables—are always taken to be ‘classically behaved’, from Kripke 1974 to Field 2008.

When talking *about* a logic, must we be working in a classical metatheory?

*How far can a logician who professes to hold that [paraconsistency] is the correct criterion of a valid argument, but who freely accepts and offers standard mathematical proofs, in particular for theorems about [paraconsistent] logic itself, be regarded as sincere or serious in objecting to classical logic? [Burgess]*

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Okay ... then what is the plan for once everyone is converted to the One True (paraconsistent) Logic?

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And so the main reason for this paper is pragmatic, too—just to show the answer.

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Here, a paraconsistent set theory naturally generates a paraconsistent semantics.

# Logic (Propositional Fragment)

## Axioms

$$\vdash \varphi \rightarrow \varphi$$

$$\vdash (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$$

$$\vdash \varphi \vee \neg\varphi$$

$$\vdash \neg\neg\varphi \rightarrow \varphi$$

$$\vdash (\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$$

$$\vdash \varphi \wedge \psi \rightarrow \varphi$$

$$\vdash \varphi \wedge \psi \rightarrow \psi \wedge \varphi$$

$$\vdash \varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$$

$$\vdash \varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$\vdash (\varphi \rightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi)$$

$$\vdash \neg(\varphi \Rightarrow \psi) \Rightarrow \neg(\varphi \rightarrow \psi)$$

$$\vdash (\varphi \Rightarrow \psi) \wedge (\chi \Rightarrow \psi) \Rightarrow (\varphi \vee \chi \Rightarrow \psi)$$

$$\vdash x = y \Rightarrow (\varphi(x) \rightarrow \varphi(y))$$

# Rules

$$\begin{array}{l} \varphi, \varphi \Rightarrow \psi \vdash \psi \\ \varphi, \neg\psi \vdash \neg(\varphi \Rightarrow \psi) \end{array}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi}$$

$$\frac{\Gamma, \varphi, \chi \vdash \psi}{\Gamma, \chi, \varphi \vdash \psi}$$

$$\frac{\Gamma, \varphi, \chi \vdash \psi}{\Gamma, \varphi \wedge \chi \vdash \psi}$$

$$\frac{\Gamma \vdash \psi \quad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \varphi \wedge \psi}$$

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## Axiom (Ext)

$$\forall z((z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$$

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## Axiom (Choice)

*A unique object can be picked out from any non-empty set.*

## Axiom (Induction)

*Proofs by induction work for any recursively defined structure.*

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If this sounds (comfortingly? suspiciously?) familiar, this is prelude for what is to come.

## True vs true only: two values or three?

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Choose: *untruth-avoidance* or *truth-seeking*.

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If the original Tarski problem was insoluble, this new, three-tiered approach will be no less intractable.

The three-valued approach rather encourages a common criticism —that dialetheists have lost some important expressive power, the ability to demarcate the truths (t valued) from the true contradictions (b valued).

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A dialethic paraconsistentist should lead the discussion away from pre-Tarskian ideation, and use a formalism that does not invite or suggest such criticism.

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To reiterate, this is not really a *decision* on our part, but rather a *requirement* for any logic that can express its own metatheory.

There are two truth values,  $t$  and  $f$ , which are duals,

$$\bar{t} = f$$

$$\bar{f} = t$$

They are also exclusive, on pain of absurdity:

$$t = f \Rightarrow \varphi$$

for any  $\varphi$ .

# Relational Truth Conditions

A truth-value assignment on PROP is any relation

$$R^0 \subseteq \text{PROP} \times \{t, f\}$$

such that  $x \in \text{PROP} \Leftrightarrow \exists y (\langle x, y \rangle \in R^0)$ , and

$$\langle p, t \rangle \in R^0 \Leftrightarrow \langle p, f \rangle \notin R^0 \qquad \langle p, f \rangle \in R^0 \Leftrightarrow \langle p, t \rangle \notin R^0$$

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By the law of excluded middle,  $R^0$  is not empty:

either  $\langle p, f \rangle \in R^0$ , or else  $\langle p, f \rangle \notin R^0$ , in which case  $\langle p, t \rangle \in R^0$ .

## Definition of a model

Extend  $R^0$  to  $R \subseteq \text{FMLA} \times \{t, f\}$ :

$$\neg\varphi Rt \Leftrightarrow \varphi Rf$$

$$\neg\varphi Rf \Leftrightarrow \varphi Rt$$

$$(\varphi \wedge \psi) Rt \Leftrightarrow \varphi Rt \wedge \psi Rt$$

$$(\varphi \wedge \psi) Rf \Leftrightarrow \varphi Rf \vee \psi Rf$$

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$$(\varphi \vee \psi) R f \Leftrightarrow \varphi R f \wedge \psi R f$$

Satisfies  $\varphi R t \Leftrightarrow \neg(\varphi R f)$  and  $\varphi R f \Leftrightarrow \neg(\varphi R t)$



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i.e.  $\varphi$  is both satisfied and not in the model  $R$ .

This will be the situation with any contradiction.

## Definition

A sentence  $\psi$  is a valid consequence of  $\varphi_0, \dots, \varphi_n$ ,

$$\varphi_0, \dots, \varphi_n \models \psi$$

iff  $\varphi_0 R t \wedge \dots \wedge \varphi_n R t \Rightarrow \psi R t$  for all models  $R$ .

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## Theorem

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One exists: let  $R^0 = \{\langle p, t \rangle, \langle p, f \rangle\}$ .

Extend  $R^0$  with the lift  $R$  (which exists by comprehension):

$$\begin{aligned} R = & \quad \{\langle p, t \rangle : pR^0t\} \\ & \cup \quad \{\langle p, f \rangle : pR^0f\} \\ & \\ & \cup \quad \{\langle \neg\varphi, t \rangle : \langle \varphi, f \rangle \in R\} \\ & \cup \quad \{\langle \neg\varphi, f \rangle : \langle \varphi, t \rangle \in R\} \\ & \\ & \cup \quad \{\langle \varphi \wedge \psi, t \rangle : \langle \varphi, t \rangle \in R \wedge \langle \psi, t \rangle \in R\} \\ & \cup \quad \{\langle \varphi \wedge \psi, f \rangle : \langle \varphi, f \rangle \in R \vee \langle \psi, f \rangle \in R\} \\ & \\ & \cup \quad \{\langle \varphi \vee \psi, t \rangle : \langle \varphi, t \rangle \in R \vee \langle \psi, t \rangle \in R\} \\ & \cup \quad \{\langle \varphi \vee \psi, f \rangle : \langle \varphi, f \rangle \in R \wedge \langle \psi, f \rangle \in R\} \end{aligned}$$

# Proof that $R$ is a model

Show by induction that for any formula

$$\begin{aligned} (\star) \quad & \langle \varphi, t \rangle \in R \Leftrightarrow \langle \varphi, f \rangle \notin R \\ & \langle \varphi, f \rangle \in R \Leftrightarrow \langle \varphi, t \rangle \notin R \end{aligned}$$

For the base case, for any proposition  $p$  and  $x \in \{t, f\}$ , by definition  $pRx \Leftrightarrow pR^0x$  and  $\neg(pRx) \Leftrightarrow \neg(pR^0x)$ .

Induction: assume  $(\star)$  as the inductive hypothesis.

To avoid contraction, we don't use the very same hypothesis for each inductive case. They are rather *hypothesis schemata*, each instance used once.

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f	t

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f	f	f	

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t	f	t	t	f	t	t	t	t	t
f	t	f	f	f	f	t	t	f	f

- ▶ Such two-dimensional displays are often implicitly assumed to be functional look-up tables.
- ▶ No such assumption on the page.
- ▶ It is simply presupposing classicality to do so.
- ▶ Diagrams must be used with great care in mathematics!



	$\neg$
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f	t

The tables are read as, 'if t is *among* the values of  $\varphi$ , then f is *among* the values of  $\neg\varphi$ '.

Or more concisely, 'if  $\varphi$  is true, then  $\neg\varphi$  is false'.

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The copula—the 'is' of predication—is not univocal in general, and it is not here.

# Soundness

## Theorem

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## Corollary

*For some  $\varphi$ , it is the case that  $\vdash \varphi \Rightarrow \models \varphi$  and  $\neg(\vdash \varphi \Rightarrow \models \varphi)$ .*

## Corollary

$$(\models \varphi \Rightarrow \perp) \Rightarrow (\vdash \varphi \Rightarrow \perp).$$

# Completeness

Notation: given  $R$ ,

$$\Theta_R^\varphi = \{p_i : \langle p_i, t \rangle \in R\} \cup \{\neg p_i : \langle p_i, f \rangle \in R\}$$

$\Theta$  contains as many copies of  $p_i$  as there are in  $\varphi$ .

## Lemma

For any model  $R$  and formula  $\varphi$ ,

1.  $\langle \varphi, t \rangle \in R \Rightarrow \Theta_R^\varphi \vdash \varphi$
2.  $\langle \varphi, f \rangle \in R \Rightarrow \Theta_R^\varphi \vdash \neg \varphi$

*Proof.* If  $\langle \neg \psi, t \rangle \in R$ , then  $\langle \psi, f \rangle \in R$ , so  $\Theta \vdash \neg \psi$ . If  $\langle \neg \psi, f \rangle \in R$ , then  $\langle \neg \neg \psi, f \rangle \in R$ , so  $\Theta \vdash \neg \neg \psi$ , so  $\Theta \vdash \psi$ . Etc.  $\square$

## Theorem

$\models \varphi \Rightarrow \vdash \varphi$

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**Theorem** Naive set theory is not trivial.

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Either naive set theory is trivial or not. If not, we are done. If trivial, then, since this very proof is in naive set theory, it follows that the system is not trivial—since, after all, anything follows.  $\square$

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Using a paraconsistent metatheory constitutes no expressive loss. Rather it makes clearer and more explicit the hard facts.

Indeed, in a paraconsistent system, one *can* prove consistency and non-triviality.

This is the closest one can get to a guarantee that the proof methods themselves are reliable, by methods that are equally reliable.



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A paraconsistent substructural approach can at least match the classical textbook presentation of semantics, and may eventually be uniquely able to carry its own weight.