

# ? WHAT IS...

## Nash Equilibrium?

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In game theory, a *Nash equilibrium* is an array of strategies, one for each player, such that no player can obtain a higher payoff by switching to a different strategy while the strategies of all other players are held fixed. The concept is named after John Forbes Nash Jr.

For example, if Chrysler, Ford, and GM choose production levels for pickup trucks, a commodity whose market price depends on aggregate production, an equilibrium is an array of production levels, one for each firm, such that none can raise its profits by making a different choice.

Formally, an  $n$ -player game consists of a set  $I = \{1, \dots, n\}$  of players, a set  $S_i$  of strategies for each player  $i \in I$ , and a set of goal functions  $g_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$  that represent the preferences of each player  $i$  over the  $n$ -tuples, or profiles, of strategies chosen by all players. A strategy profile has a higher goal-function value, or *payoff*, than another if and only if the player prefers it to the other. Let  $S = S_1 \times \dots \times S_n$  denote the set of all strategy profiles, with generic element  $s$ , and let  $(t_i, s_{-i})$  denote the strategy profile  $(s_1, \dots, s_{i-1}, t_i, s_{i+1}, \dots, s_n)$  obtained from  $s$  by switching player  $i$ 's strategy to  $t_i \in S_i$  while leaving all other strategies unchanged. An equilibrium point of such a game is a strategy profile  $s^* \in S$  with the property that, for each player  $i$  and each strategy  $t_i \in S_i$ ,

$$g_i(s^*) \geq g_i(t_i, s_{-i}^*).$$

That is, a strategy profile is an equilibrium point if no player can gain from a unilateral deviation to a different strategy.

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The invention and succinct formulation of this concept, along with the establishment of its existence under very general conditions, reshaped the landscape of research in economics and other social and behavioral sciences.

Nash's existence theorem pertains to games in which the strategies  $S_i$  available to each player are probability distributions over a finite set of alternatives. Typically, each alternative specifies what action to take under each and every circumstance that the player may encounter during the play of the game. The alternatives are referred to as *pure strategies* and the probability distributions over these as *mixed strategies*. Players' randomizations, according to their chosen probability distributions over their own set of alternatives, are assumed to be statistically independent. Any  $n$ -tuple of mixed strategies then induces a probability distribution or lottery over  $n$ -tuples of pure strategies. Provided that a player's preferences over such lotteries satisfy certain completeness and consistency conditions—previously identified by John von Neumann and Oskar Morgenstern—there exists a real-valued function with the  $n$ -tuples of *pure strategies* as its domain such that the expected value of this function represents the player's preferences over  $n$ -tuples of *mixed strategies*. Given only this restriction on preferences, Nash was able to show that every game has at least one equilibrium point in mixed strategies.

*Nash equilibrium reshaped the landscape of research in economics.*

Emile Borel had a precursory idea, concerning symmetric pure conflicts of interest between two parties with very few alternatives at hand. In 1921 he defined the notion of a finite and symmetric zero-sum two-player game. In such a game

each player has the same number of pure strategies, the gain for one player equals the loss to the other, and they both have the same probability of winning whenever

they use the same pure strategy. Borel also formalized the concept of a mixed strategy, and for games in which each player has three pure strategies, proved the existence of what would later come to be called a *maxmin pair* of mixed strategies. This is a pair of strategies such that one player's strategy maximizes his own gain while his opponent simultaneously minimizes this gain. He subsequently extended this result to the case of five strategies per player, but seems to have doubted that general existence results could be achieved.

A few years later, and apparently unaware of Borel's partial results, von Neumann formalized the notion of finite zero-sum games with an arbitrary (finite) number of players, where each player has an arbitrary (finite) number of pure strategies. For all such games involving two players he proved the existence of a maxmin strategy pair, presented the result in Göttingen in 1927, and published it in 1928.

In comes Nash, a young doctoral student in mathematics at Princeton University. Nash defined a much more general class of games and a more general equilibrium concept. He allowed for any (finite) number of players, each having an arbitrary (finite) number of pure strategies at his or her disposal and equipped with any goal function. In particular, players may be selfish, altruistic, spiteful, moralistic, fair-minded, or have any goal function whatsoever. His definitions and his existence result contain those of Borel and von Neumann as special cases. Previously restricted to pure conflicts of interest, game theory could now be addressed to any (finite) number of parties with arbitrary goal functions in virtually any kind of strategic interaction. Nash published this in a one-page article in the *Proceedings of the National Academy of Sciences* in 1950.

His existence proof—merely sketched in this short paper—is based upon Kakutani's fixed-point theorem (established some years earlier). Kakutani's theorem states that if a subset  $X$  of  $\mathbb{R}^m$  is nonempty, compact and convex, and a (set-valued) correspondence  $\Gamma : X \rightrightarrows X$  is nonempty-valued, convex-valued and has a closed graph, then there exists  $x \in X$  such that  $x \in \Gamma(x)$ . That is, there exists a fixed point of the correspondence. Nash's existence proof relies on the construction of what today is called the *best-reply correspondence*, which can then be shown to satisfy the conditions of Kakutani's theorem.

Given any  $n$ -tuple of mixed strategies, Nash defined a *countering*  $n$ -tuple as a mixed-strategy profile that obtains for each player the highest payoff given the strategies chosen by other players in the original, *countered*  $n$ -tuple. By associating with each  $n$ -tuple of mixed strategies the set of all countering  $n$ -tuples, one obtains a self-correspondence on the set of all mixed-strategy profiles. Since any  $n$ -tuple of mixed strategies is a point in the product space  $S$  obtained by taking the Cartesian product of the individual strategy spaces  $S_i$ , the domain of this correspondence is a nonempty, compact and convex subset of  $\mathbb{R}^m$  for some  $m$ . In fact, it is a polyhedron, the Cartesian product of finitely many unit simplexes. Furthermore, the correspondence thus constructed is

convex-valued, since a convex combination of countering  $n$ -tuples must itself be a countering  $n$ -tuple. And since the payoff functions are all continuous (in fact, polynomial) functions with closed domain, the correspondence has a closed graph. The existence of a fixed point follows from Kakutani's theorem, and any such fixed point is a *self-countering*  $n$ -tuple, or an equilibrium point of the game.

A year later Nash published an alternative existence proof in the *Annals of Mathematics* that instead is based on Brouwer's fixed-point theorem. Since Kakutani's theorem is derived from Brouwer's, Nash was more satisfied with the latter. This second proof has a touch of genius. It is simple and intuitive in retrospect but completely unexpected beforehand.

In order to use Brouwer's theorem, Nash needed to construct a self-map on the space of mixed-strategy profiles with the property that a strategy profile is an equilibrium point if and only if it is a fixed point of this map. But the best-reply correspondence could not be used for this purpose, since it need not be single-valued and does not permit a continuous selection in general.

This is how he did it. Consider any  $n$ -tuple of mixed strategies  $s$ , and recall that the payoff to a player  $i$  at this strategy profile is  $g_i(s)$ . Let  $g_{ih}(s)$  denote the payoff that player  $i$  would receive if he were to switch to the *pure* strategy  $h$  while all other players continued to use the strategies specified in  $s$ . Define the continuous function

$$\phi_{ih}(s) = \max\{0, g_{ih}(s) - g_i(s)\}.$$

Each function value  $\phi_{ih}(s)$  represents the "excess payoff" obtained by pure strategy  $h \in S_i$ , as compared with the payoff obtained under strategy profile  $s$ . Letting  $s_{ih}$  denote the probability with which pure strategy  $h$  is played under  $s$ , the function  $\phi$  may be used to obtain a new  $n$ -tuple of mixed strategies,  $s'$ , from  $s$  by setting

$$s'_{ih} = T_{ih}(s) = \frac{s_{ih} + \phi_{ih}(s)}{1 + \sum_h \phi_{ih}(s)}.$$

This defines a self-map  $T$  on the space of mixed strategy profiles. As long as there exists a pure strategy with positive excess payoff,  $T$  lowers the probabilities with which pure strategies having zero excess payoff are played. It is clear that if  $s$  is an equilibrium point, it must be a fixed point of  $T$ , since no pure strategy  $h$  can yield player  $i$  a higher payoff, forcing  $\phi_{ih}(s) = 0$  for all  $i$  and  $h$ . It is easily verified that the converse is also true: if  $s$  is a fixed point of  $T$ , so that  $\phi_{ih}(s) = 0$  for all  $i$  and  $h$ , then  $s$  must be an equilibrium point of the game.

To complete the proof, one need only use the fact that  $T$  is a continuous self-map on the compact and convex set of mixed-strategy  $n$ -tuples. This is sufficient, from Brouwer's theorem, for the existence of a fixed point.

Nash's equilibrium concept lies at the heart of contemporary theoretical research on strategic interactions in

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*The second proof has a touch of genius.*

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# THE GRADUATE STUDENT SECTION

economics and other fields. One especially fruitful area of application has been to auction theory, as the following example illustrates. Many strategic interactions—including lobbying, arms races, contests, and wars of attrition—can be modeled as *all-pay auctions* in which the highest bidder obtains an object of value but all players must pay their bids. (If there are multiple highest bidders they each get the object with the same probability.) Consider an object with value  $v > 0$  and  $n \geq 2$  bidders, each of whom is constrained to bid from the nonnegative integers. Players submit their bids simultaneously, without knowledge of any opponent's bid. This is a symmetric  $n$ -player game with countably infinite pure-strategy sets. However, Nash's existence result still applies, since no bid above  $v$  is ever a best reply to the bids of others, and hence the game has the same set of Nash equilibria as the finite game in which bids are bounded from above by  $v$ .

Nash's result tells us that there must be an equilibrium in pure or mixed strategies in this game. For instance, if  $n = 2$  and  $v = 5/2$ , then it can be shown that no pure strategy equilibrium exists, but if each player chooses the distribution  $(1/5, 3/5, 1/5)$  over the bids  $\{0, 1, 2\}$ , then neither can obtain a higher payoff by deviating unilaterally to any other strategy. Furthermore, each player's expected payoff in equilibrium is  $1/4$ , which is lower than the  $5/4$  that each could secure if they colluded to bid zero. This example illustrates that equilibrium behavior, while individually optimal, can cause players to impose costs on each other that are wasteful in the aggregate.

The 1994 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel was awarded to Nash, along with Reinhard Selten and John C. Harsanyi, for their "pioneering analysis of equilibria in the theory of noncooperative games."

## About the Authors

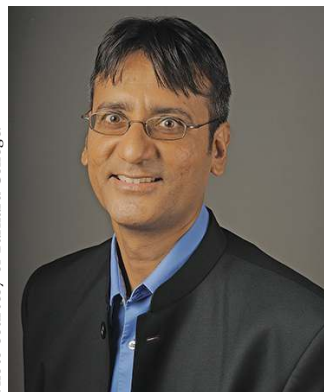


Photo courtesy of Barnard College.

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Rajiv Sethi's research interests include evolutionary game theory and applications, financial economics, and the economics of inequality.

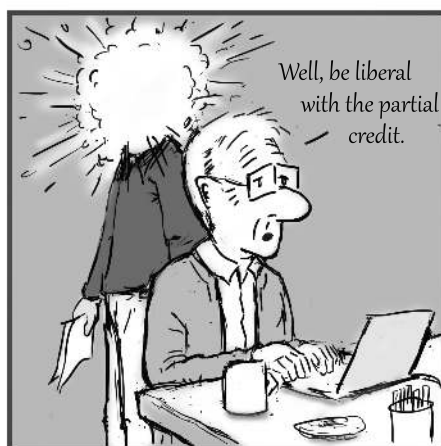
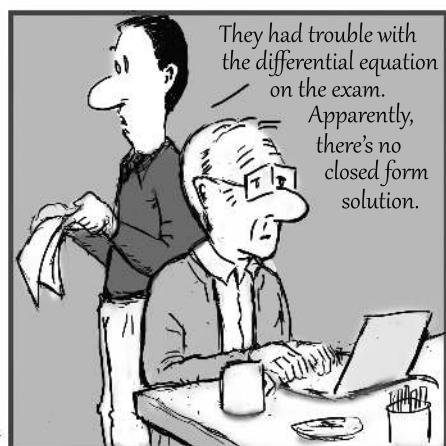


Photo courtesy of the Stockholm School of Economics.

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Jörgen Weibull's main field of research is noncooperative and evolutionary game theory, with applications to economics, political science, and evolutionary biology. He is a member of the Royal Swedish Academy of Sciences and Fellow of the Econometric Society.

## MY PROFESSOR



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