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Electromagnetic Nonreciprocity

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We aim at providing a global perspective on *electromagnetic nonreciprocity* and clarifying confusions that arose in recent developments of the field. We provide a general definition of nonreciprocity and classify nonreciprocal systems according to their linear time-invariant (LTI), linear time-variant (LTV), or nonlinear natures. The theory of nonreciprocal systems is established on the foundation formed by the concepts of time reversal, time-reversal symmetry, time-reversal symmetry breaking, and related Onsager-Casimir relations. Special attention is given to LTI systems, the most-common nonreciprocal systems, for which a generalized form of the Lorentz reciprocity theorem is derived. The delicate issue of loss in nonreciprocal systems is demystified and the so-called thermodynamics paradox is resolved from energy-conservation considerations. An overview of the fundamental characteristics and applications of LTI, LTV, and nonlinear nonreciprocal systems is given with the help of pedagogical examples. Finally, asymmetric structures with fallacious nonreciprocal appearances are debunked.

I. INTRODUCTION

Nonreciprocity arises in all branches of physics—classical mechanics, thermodynamics and statistical mechanics, condensed-matter physics, electromagnetism and electronics, optics, relativity, quantum mechanics, particle and nuclear physics, and cosmology—where it underpins a myriad of phenomena and applications.

Historically, the study of nonreciprocity in physics probably started with the experimental discovery by Faraday of the eponymic polarization rotation of light passing through glass in the direction of an applied magnetic field in 1845 [1], although it is unclear whether Faraday, mostly excited by his success in "magnetising a ray of light," specifically noted the nonreciprocity of this phenomenon. Reciprocity (nonreciprocity), conceived as the property of a system where a ray of light and its reverse ray encounter identical (different) optical adventures – reflection, refraction, and absorption – was first explicitly described by Stokes in 1840 [2] and Helmholtz in 1856 [3], which led to the so-called Stokes-Helmholtz reciprocity principle. The concept was later reformulated by Kirchoff in 1860, described as a consequence of propagation linearity by Rayleigh

in 1873 [4], and extensively applied by Planck in 1900 [5] in his proof of the Kirchhoff law of thermal radiation (equal blackbody-radiation emissivity and absorptivity). The development of commercial nonreciprocal systems, following some device explorations in the second half of the 19th century [6], started in the microwave regime, following the invention of the magnetron cavity at the dawn of World War II, and experienced a peak in the period from 1950 to 1965 [7]. The development of nonreciprocal systems in the optical regime lagged that of its microwave counterparts by nearly 30 years, roughly corresponding to the time lapse between the invention of the magnetron cavity and that of the laser.

In electromagnetics, nonreciprocity is now an important scientific and technological concept at both microwave [7,8] and optical [9,10] frequencies. In both regimes, nonreciprocal devices have been almost exclusively based on ferrimagnetic (dielectric) compounds, called "ferrites," such as yttrium iron garnet and materials composed of iron oxides and other elements (Al, Co, Mn, Ni) [11]. Ferrite nonreciprocity results from electron-spin precession (Landau-Lifshitz equation: $\partial \mathbf{m}/\partial t = -\gamma \mathbf{m} \times (\mathbf{B}_0 + \mu_0 \mathbf{H})$, where $\mathbf{m}$ is the magnetic dipole moment and γ is the gyromagnetic ratio) in the microwave regime [12,13] and from electron cyclotron orbiting (electron equation of

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motion: $(m_e/e)\partial\mathbf{v}_e/\partial t = \mathbf{E} + \mathbf{v}_e \times \mathbf{B}_0$, where $\mathbf{v}_e$ is electron velocity, e is electron charge, and m_e is electron mass) in the optical regime [13,14], with both effects being induced by a static magnetic field bias $\mathbf{B}_0$, which is provided by a permanent magnet, or a resistive or superconductive coil.

Unfortunately, ferrite-based systems tend to be bulky, heavy, costly, and nonamenable to integrated circuit technology, due to the incompatibility of ferrite crystal lattices with those of semiconductor materials. These issues have recently triggered an intensive quest for "magnetless" nonreciprocity (i.e., nonreciprocity requiring no ferrimagnetic materials and magnets/coils).

This quest has led to the development of a plethora of magnetless nonreciprocal systems, including metamaterials, space-time-modulated structures, and nonlinear materials. However, it has also generated some confusion [15–21], particularly pertaining to the definition of "nonreciprocity," the difference between linear and nonlinear nonreciprocity, the relation between nonreciprocity and time-reversal symmetry breaking, the handling of time reversal in lossy and open systems, the "thermodynamics paradox," and the distinction between nonreciprocal and asymmetric propagation. The objective of this paper is to clarify this confusion and provide a global perspective on electromagnetic nonreciprocity.

The paper is organized as follows. Section II defines and classifies nonreciprocal systems. Sections III–VI explain the concepts of time reversal and time-reversal symmetry breaking, in general and specifically in electromagnetics. Sections VII–IX study nonreciprocity in linear time-invariant (LTI) media, culminating with the demonstration of a generalized form of the Lorentz theorem and the derivation of the Onsager-Casimir relations. Section X points out the applicability of these relations to all nonreciprocal systems, while Sec. XI provides a finer classification of these systems into LTI, linear time-variant (LTV), and nonlinear systems, and indicates the applicability of previously seen concepts to the three categories. Sections XII and XIII clarify the delicate handling of time-reversal symmetry in lossy and open systems, respectively. On the basis of the general definition of nonreciprocity, Sec. XIV extends the concept of S-parameters to all nonreciprocal systems. This serves as the foundation for fundamental energy-conservation rules in Sec. XV and the resolution of the so-called thermodynamics paradox in Sec. XVI. Building on previously established concepts, Secs. XVIII–XXI provide an overview of the fundamental characteristics and applications of LTI, LTV, and nonlinear nonreciprocal systems. Finally, Sec. XXII describes a few exotic systems whose asymmetries might be erroneously confused with nonreciprocity. Conclusions are given in Sec. XXIII in the form of an enumeration of the main results of the paper.

II. NONRECIPROCITY DEFINITION AND CLASSIFICATION

Nonreciprocity is the absence of "reciprocity." The adjective "reciprocal" itself comes from the Latin word *reciprocus* [22], built on the prefixes "re" (backward) and "pro" (forward), which combine in the phrase *reque proque*, with the meaning of "going backward as forward." Thus, "reciprocal" etymologically means "going the same way backward as forward."

In physics and engineering, the concept of nonreciprocity/reciprocity applies to systems that encompass media or structures and components or devices. A nonreciprocal (reciprocal) system is defined as a system that exhibits different (same) received-transmitted field ratios when its source(s) and detector(s) are exchanged. In this definition, the notion of "ratio" has been added to the aforementioned etymological meaning of "reciprocity" to reflect common practice, as discussed later.

Nonreciprocal systems may be classified into two fundamentally distinct categories: linear and nonlinear nonreciprocal systems [23], as indicated in Table I, whose details are discussed throughout this paper. We see that, in both cases, nonreciprocity is based on time-reversal symmetry breaking, by an external bias in the linear case, and by a combination of self-biasing and structural asymmetry in the nonlinear case. We also see that linear nonreciprocity is stronger than nonlinear nonreciprocity, the former working for arbitrary excitations and intensities with high isolation, and the latter being restricted to nonsimultaneous excitations from different directions, specific intensity conditions, poor isolation or hysteresis. We exclude here nonreciprocity based on externally biased nonlinear media [24], because this approach has been little studied and also because it loses the most-attractive feature of nonlinear nonreciprocity; namely, the magnetless operation.

III. TIME REVERSAL AND TIME-REVERSAL SYMMETRY

The etymological meaning of "reciprocal" as "going the same way backward as forward" suggests the thought experiment depicted in Fig. 1.

In this experiment, one monitors a process (temporal evolution of a physical phenomenon) occurring in a given system, with ports, each representing a specific access point (or terminal), field mode, and frequency range, as time is reversed. Specifically, let us monitor the process between ports P_1 and P_2 of the system via the state—magnitude, phase, temporal frequency (ω), spatial frequency ($\mathbf{k}$), polarization or spin angular momentum, orbital angular momentum—vector $\boldsymbol{\Psi}(t) = [\psi_{P_1}(t), \psi_{P_2}(t)]^T$ (where T is the transpose), where $\psi_{P_1}(t)$ and $\psi_{P_2}(t)$ represent the wave state at P_1 and P_2, respectively. First, we excite P_1 at $t = 0$ and trace the response

TABLE I.   Classification and characteristics of nonreciprocal systems.

Type	Linear	Nonlinear
Time-reversal symmetry breaking	External bias	Self bias + structural asymmetry
Nonreciprocity form	Strong	Weak
Operation	Arbitrary direction	Only one direction at a time
Intensity	Arbitrary	Restricted range
Isolation	High	Poor (or hysteresis)
Media or structures	ferromagnets, ferrites, magnetized plasmas, two-dimensional electron gases and materials, nano- and transistor-loaded metamaterials, space-time modulated media	any strongly driven material (e.g. glasses, crystals, semiconductors) with spatial asymmetry, diversity of metastructures
Components or devices	isolators, phase shifters (e.g. gyrators), circulators	diodes, pseudo-isolators, power amplifiers (e.g. vacuum tubes)

of the system until the time $t = T$ of complete transmission to P_2. Then, we flip the sign of the time variable in the process [25], which results in a system, not necessarily identical to the original one (see Sec. V), being excited at $t = -T$ and evolving until the time $t = 0$ of complete transmission to P_1: this operation is called "time reversal," and it is a concept that is at the core of the work of Onsager in thermodynamics [26–31].

Mathematically, time reversal is represented by the operator $\mathcal{T}$, defined as

$$\mathcal{T}\{t\} = t' = -t \text{ or } \mathcal{T} : t \mapsto t' = -t \qquad (1)$$

when trivially applied to the time variable [32], and generally, when applied to a process, $\psi(t)$, defined as

$$\mathcal{T}\{\psi(t)\} = \psi'(t') = \psi'(-t). \qquad (2)$$

In these two relations, it is meant that the time variable value is reversed, while the time coordinate direction is maintained fixed, corresponding to symmetry with respect to the axis $t = 0$, consistently with Fig. 1.
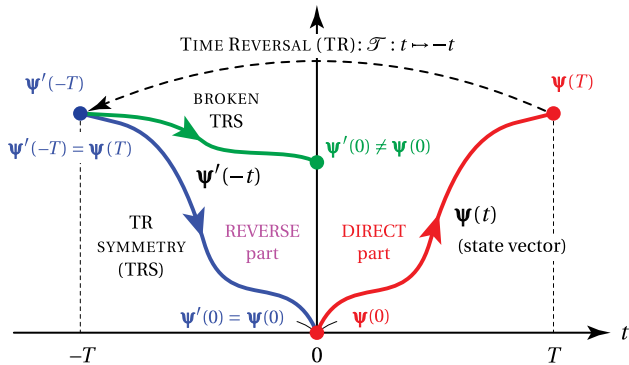

FIG. 1.   Time-reversal symmetry (TRS) (red and blue curves) and broken time-reversal symmetry (red and green curves), or time-reversal asymmetry, as a general thought experiment and mathematical criterion for nonreciprocity.

If the system remains the same (changes) under time reversal, corresponding to the red-blue (red-green) curve pairs in Fig. 1, that is,

$$\mathcal{T}\{\psi(t)\} = \psi'(-t) \begin{Bmatrix} = \\ \neq \end{Bmatrix} \psi(t), \qquad (3)$$

it is called "time-reversal symmetric (asymmetric)". Since the direct and reverse parts of the process typically describe the response of the system for opposite transmission directions, time-reversal asymmetry (symmetry) is intimately related to nonreciprocity (reciprocity). According to the definition in Sec. II, time-reversal symmetry (asymmetry) is equivalent to reciprocity (nonreciprocity) insofar as, in both cases, the system exhibits the same (a different) response when transmitting at P_1 and receiving at P_2 as (than) when transmitting at P_2 and receiving at P_1. Thus, time-reversal symmetry (asymmetry) provides a fundamental criterion for reciprocity (nonreciprocity). It is, however, pointed out in Sec. XII that this is a loose criterion, as sensu stricto equivalence requires equal direct and reverse field levels and not just equal field ratios.

IV. TIME-REVERSAL SYMMETRY IN ELECTROMAGNETICS

The basic laws of physics are classically [33] invariant under time reversal or are time-reversal symmetric [34], as may be intuitively understood by realizing that reversing time is equivalent to "flipping the movie film" of the process, as in Fig. 1. In contrast, the physical quantities involved in the laws of physics, which we generically denote by $f(t)$, may be either time-reversal symmetric or time-reversal antisymmetric; that is,

$$\mathcal{T}\{f(t)\} = f'(t') = f'(-t) = \pm f(-t), \qquad (4)$$

where the plus sign corresponds to time-reversal symmetry, or even time-reversal parity, and the minus sign corresponds to time-reversal antisymmetry, or odd time-reversal parity.

The time-reversal parity of physical quantities may be easily inferred from fundamental laws. Table II presents the case of electromagnetic quantities [34]. Realizing that charges do not change as time passes, and are hence invariant under time reversal, one can sequentially deduce all the results of the figure by successively invoking Coulomb, Ohm, Ampère, Maxwell, Poynting, impedance and Joule laws, equations, and relations. Note, particularly, that time reversal reverses the direction of wave propagation, consistently with the considerations in Sec. III: $\mathcal{T}\{\mathbf{k}(t)\} = -\mathbf{k}(-t)$.

One may straightforwardly verify the time-reversal symmetry of the Maxwell equations by applying the time-reversal rules in Table II and noting that the ∇ operator is time-reversal invariant. Specifically, replacing all the primed quantities in the time-reversed Maxwell equations by their unprimed (original) counterparts according to Table II and simplifying signs restores the original Maxwell equations, that is,

$$\nabla \times \mathbf{E}^{(\prime)} = -\partial \mathbf{B}^{(\prime)}/\partial t^{(\prime)} \ (\text{even} \equiv \text{odd/odd}), \quad (5a)$$

$$\nabla \times \mathbf{H}^{(\prime)} = \partial \mathbf{D}^{(\prime)}/\partial t^{(\prime)} + \mathbf{J}^{(\prime)} \ (\text{odd} \equiv \text{even/odd} + \text{odd}), \quad (5b)$$

with parity matching indicated in parentheses (Appendix A). The time-reversal invariance of the Maxwell equations indicates that if *all* the quantities of an electromagnetic system are time-reversed, according to the time-reversal parity rules in Table II, then the time-reversal system

TABLE II. Time-reversal parity (even/odd), or symmetry/antisymmetry, of the main electromagnetic quantities (time-harmonic dependence $e^{j\omega t}$).

$f(t)$	$\xrightarrow{\mathcal{T}}$	$f'(t') = \pm f(-t)$	FIELD $-$ PARITY
q		$q' = q$	q, ρ – even
$\mathbf{E}$		$\mathbf{E}' = \mathbf{E}$	$\mathbf{E}, \mathbf{P}, \mathbf{D}$ – even
$\mathbf{J}$		$\mathbf{J}' = -\mathbf{J}$, $\mathbf{v}' = -\mathbf{v}$	$\mathbf{v}, \mathbf{J}$ – odd
$\mathbf{J}$, $\mathbf{B}$		$\mathbf{J}'$, $\mathbf{B}' = -\mathbf{B}$	$\mathbf{B}, \mathbf{M}, \mathbf{H}$ – odd
$\mathbf{k} = \mathbf{D} \times \mathbf{B}$, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$		$\mathbf{k}', \mathbf{S}'$	$\mathbf{k}, \mathbf{S}$ – odd
$\eta = E/H$		$\eta' = E'/H' = -\eta$	η – odd
$\alpha = \Im\{k\} < 0$ loss		$\alpha' = \Im\{k'\} = -\alpha > 0$ gain	α – odd

will have the same electromagnetic solution as the direct system, whatever its complexity!

V. TIME-REVERSAL SYMMETRY BREAKING AND RELATED NONRECIPROCITY CRITERION

Time-reversal symmetry breaking is an operation that destroys, with an external or internal (due to the wave itself) bias, the time symmetry of a process, and hence makes it time-reversal asymmetric, by violating (at least) one of the time-reversal rules, such as those in Table II. Since all physical quantities are either even or odd under time reversal [Eq. (4)], time-reversal symmetry breaking requires reversing (maintaining) the sign of *at least one* of the time-reversal even (odd) quantities, or bias, involved in the system. Only the latter option (i.e., maintaining the sign of a time-reversal odd quantity, such as $\mathbf{v}$, $\mathbf{J}$, or $\mathbf{B}$ in Table I) is practically meaningful, since this is the only one that leaves the system unchanged. This fact will become clearer in Sec. IX.

The time-reversal asymmetry (symmetry) criterion for determining the nonreciprocity (reciprocity) of a given system (Sec. III) may be applied as follows. First, one fully time-reverses the system using the rules in Table II. As a result, the process retrieves its initial state. However, the time-reversal operation may have altered the nature of the system, resulting in different direct and reverse systems. In such a case, the time-reversal experiment is irrelevant, since it compares apples and pears. So one must examine whether the reversed system is identical to the given one or not. If it is identical, the process is time-reversal symmetric and the system is reciprocal. Otherwise, the system must violate a time-reversal rule to maintain its nature, or break time-reversal symmetry (or become time-reversal asymmetric), and is hence nonreciprocal. These considerations are valid only in the absence of loss/gain. The case of loss/gain (last row in Table II) requires special attention, and is separately treated in Secs. XII and XIII.

VI. ELECTROMAGNETIC EXAMPLE

To better grasp the concepts in Secs. III, IV, and V, consider Fig. 2, which involves two gyrotropic systems: a chiral system [35–40] and a Faraday system [1,8,11,41,42].

In the case of the chiral system, in Fig. 2(a), the field polarization is rotated along the chiral medium according to the handedness of the helix-shaped particles that compose it. On time reversal, the direction of propagation is reversed, according to Table II. Thus, the field polarization symmetrically returns to its original state, as the current rewinds along the particles, without any system alteration. Chiral (unbiased) gyrotropy is thus a time-reversal-symmetric process, and is therefore reciprocal.

Now consider the Faraday system, in Figs. 2(b) and 2(c). In such a system, the direction of polarization rotation is
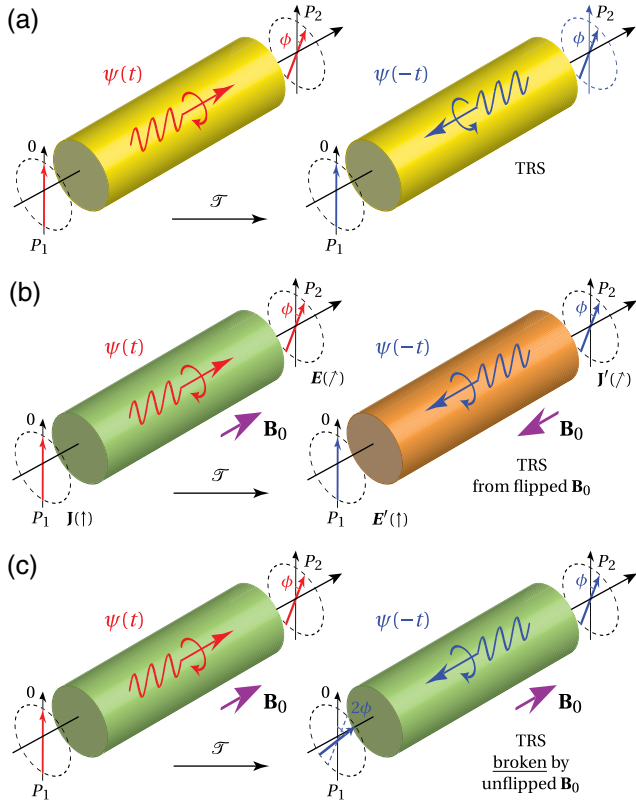
FIG. 2. Application of the time-reversal asymmetry/symmetry criterion for nonreciprocity/reciprocity (Sec. V) to two systems inducing electromagnetic-field-polarization rotation (process equivalent to gyrotropy). (a) Chiral system, without bias, and hence time-reversal symmetric (TRS) $[\psi_{P_1}(0) = E^{P_1} = E_0 \mapsto \psi'_{P_1}(0) = E_0 = \psi(0)]$ (i.e., reciprocal). (b) Faraday system altered by time-reversal odd bias $\mathbf{B}_0$ flipped according to time reversal (irrelevant time-reversal test due to altered nature of the system). (c) Same as in (b) but with unflipped $\mathbf{B}_0$, and hence an unaltered system, breaking the time-reversal symmetry of the process $[\psi'_{P_1}(0) = E_0 \cos(2\phi) \neq E_0 = \psi_{P_1}(0)]$ and hence revealing nonreciprocity.

dictated no longer by particle shape but by a static magnetic field, $\mathbf{B}_0$, or bias, provided by an external magnet and inducing specific spin states in the medium at the atomic level.

Full time reversal requires here reversing the sign of $\mathbf{B}_0$, as in Fig. 2(b), just as that of any other time-reversal odd quantity. Then, waves propagating in opposite directions see the same effective medium, by symmetry. However, the system is altered on time reversal, since its spins are reversed. Therefore, the time-reversal experiment is irrelevant to nonreciprocity! To preserve the nature of this system, and hence properly decide on its nonreciprocity (reciprocity), one must preserve its spin states by keeping the direction of $\mathbf{B}_0$ unchanged, as shown in Fig. 2(c) and done in practice. But this violates a time-reversal symmetry rule (i.e., breaks time-reversal symmetry) or

makes the process time-reversal asymmetric, revealing ipso facto that the Faraday system is nonreciprocal.

VII. LINEAR NONRECIPROCAL MEDIA

The vast majority of current nonreciprocal systems are based on LTI media or, more precisely, media whose nonreciprocity is enabled by an external bias rather than nonlinearity combined with structural asymmetry (Table I). We therefore dedicate this section, as well as Secs. VIII–IX, to the study of nonreciprocity in such media, and later generalize the discussion to the case of the LTV and nonlinear nonreciprocal systems.

The example in Sec. VI illustrates how nonreciprocity is achieved by breaking time-reversal symmetry with an external bias (i.e., a magnetic field). However, there are alternatives for time-reversal odd biases, such as the velocity and the electric current (Table II), and we therefore generically denote the bias field as $\mathbf{F}_0$.

Linear nonreciprocal media (Table I) include (a) ferromagnets (magnetic conductors) and ferrites [43], whose nonreciprocity is based on electron-spin precession (permeability tensor) [7,11,44], or cyclotron orbiting (permittivity tensor) [12,34,45], due to a magnetic field bias, (b) magnetized plasmas [14,46], two-dimensional electron gases (e.g., GaAs, GaN, InP) [47–50], and other two-dimensional materials, such as graphene [51–58], whose nonreciprocity is again based on cyclotron orbiting due to a magnetic field bias (permittivity tensor), (c) space-time-modulated media, whose nonreciprocity is based on the motion of matter/perturbation associated with a force/wave bias [59–69], and (d) transistor-loaded metamaterials, mimicking ferrites [70–74] or using twisted dipoles [75], both based on a current bias.

The constitutive relations of media are most conveniently expressed in the frequency domain, since molecules act as small oscillators with specific resonances. In the case of an LTI bianisotropic medium [40,76,77], these relations may be written, for the given medium and its time-reversed counterpart, as

$$\tilde{\mathbf{D}}^{(\prime)} = \bar{\bar{\tilde{\epsilon}}}^{(\prime)}(\pm\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^{(\prime)} + \bar{\bar{\tilde{\xi}}}^{(\prime)}(\pm\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^{(\prime)}, \quad (6a)$$

$$\tilde{\mathbf{B}}^{(\prime)} = \bar{\bar{\tilde{\zeta}}}^{(\prime)}(\pm\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^{(\prime)} + \bar{\bar{\tilde{\mu}}}^{(\prime)}(\pm\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^{(\prime)}, \quad (6b)$$

where the temporal frequency (ω) dependence is implicitly assumed everywhere, where $\bar{\bar{\tilde{\epsilon}}}$, $\bar{\bar{\tilde{\mu}}}$, $\bar{\bar{\tilde{\xi}}}$, and $\bar{\bar{\tilde{\zeta}}}$ are the frequency-domain medium permittivity, permeability, magnetic-to-electric coupling, and electric-to-magnetic coupling complex dyadic functions, respectively, and where the plus and minus signs of the (time-reversal odd) bias $\mathbf{F}_0$ correspond to the unprimed (given) and primed (time-reversal-symmetric) problems, respectively. The LTV counterparts of Eqs. (6a) and (6b) (Sec. XX) would involve integral operators with time-dependent

kernels, while its nonlinear counterpart (Sec. XXI) would involve tensors of increasing order [78].

An important example of an LTI bianisotropic medium is biased ferrite in the microwave regime. In such a medium, $\bar{\bar{\xi}} = \bar{\bar{\zeta}} = 0$, $\bar{\bar{\epsilon}} = \epsilon\bar{I}$, and, assuming $\mathbf{B}_0 \| \hat{\mathbf{z}}$, $\bar{\bar{\mu}}(\omega) = \mu_d\bar{I}_t - j\mu_o\bar{J} + \mu_0\hat{\mathbf{z}}\hat{\mathbf{z}}$ ($\bar{I}$ is a unit dyadic, $\bar{\bar{I}}_t = \bar{I} - \hat{\mathbf{z}}\hat{\mathbf{z}}$, $\bar{J} = \bar{I}_t \times \hat{\mathbf{z}} = \hat{\mathbf{y}}\hat{\mathbf{x}} - \hat{\mathbf{x}}\hat{\mathbf{y}}$), where $\mu_d = \mu_0[1 + \omega_0\omega_m/(\omega_0^2 - \omega^2)]$ and $\mu_o = \mu_0\omega\omega_m/(\omega_0^2 - \omega^2)$, with $\omega_0 = \gamma B_0$ (ferrimagnetic resonance, γ is the gyromagnetic ratio) and $\omega_m = \mu_0\gamma M_s$ (M_s is the saturation magnetization). These relations extend to the lossy case on substitution $\omega_0 \to \omega_0 + j\omega\alpha$ (α is a damping factor) [7,8,11], and if $\mathbf{B}_0 = 0$ (unbiased case), $\bar{\bar{\mu}}(\omega) = \mu_0 I$, so the ferrite reduces to a purely dielectric medium with relative permittivity $\epsilon_r \approx 10 - 15$ [7]. In the optical regime (magneto-optic effect) [45], as well as in magnetized plasmas [46], it is the permittivity that is tensorial, but its form is identical to that of the permeability tensor in the microwave regime.

VIII. TIME REVERSAL IN THE FREQUENCY DOMAIN

Equations (6a) and (6b) involve time-reversed frequency-domain fields and constitutive parameters. How do these quantities (for LTI systems) relate to their direct counterparts?

The effect of time reversal on a frequency-domain field, $\tilde{f}(\omega)$, may be found by writing this field in terms of its Fourier transform and applying the time-reversal rules (Sec. IV) as

$$\mathcal{T}\{\tilde{f}(\omega)\} = \tilde{f}'(\omega) = \mathcal{T}\left\{\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt\right\}$$
$$= \int_{+\infty}^{-\infty} [\pm f(-t)]e^{+j\omega t}(-dt) \overset{\text{Eq. (4)}}{=} \pm\tilde{f}^*(\omega), \tag{7}$$

where the plus and minus signs correspond to time-reversal even and time-reversal odd quantities, respectively (Table II). For instance, $\mathcal{T}\{\tilde{E}(\omega)\} = \tilde{E}^*(\omega)$ and $\mathcal{T}\{\tilde{B}(\omega)\} = -\tilde{B}^*(\omega)$.

One may next infer from Eq. (7) the effect of time reversal on frequency-domain constitutive parameters. For this purpose, compare the given (unprimed) medium in Eqs. (6a) and (6b) and its time-reversed (primed) counterpart with time-reversed field substitutions [Eq. (7)]. This yields (Appendix B)

$$\bar{\bar{\epsilon}}'(\mathbf{F}_0) = \bar{\bar{\epsilon}}^*(-\mathbf{F}_0), \quad \bar{\bar{\mu}}'(\mathbf{F}_0) = \bar{\bar{\mu}}^*(-\mathbf{F}_0), \tag{8a}$$

$$\bar{\bar{\xi}}'(\mathbf{F}_0) = -\bar{\bar{\xi}}^*(-\mathbf{F}_0), \quad \bar{\bar{\zeta}}'(\mathbf{F}_0) = -\bar{\bar{\zeta}}^*(-\mathbf{F}_0). \tag{8b}$$

Thus, frequency-domain time reversal implies *complex conjugation* plus *proper parity signing*. One may easily

verify that inserting Eqs. (7), (8a), and (8b) into Eqs. (5a), (5b), (6a), and (6b) transforms the time-reversed Maxwell and constitutive equations into equations that are exactly identical to their original forms with all the field and constitutive parameter quantities being complex conjugated (Appendix C), hence confirming the invariance of physical laws under time reversal (Sec. IV).

Time reversal without phase conjugation (or "∗") in Eqs. (8a) and (8b) (i.e., not transforming loss into gain and vice versa) is called "restricted time reversal" [32], and is used in the next section to derive the generalized Lorentz (non)reciprocity theorem for LTI systems.

IX. GENERALIZED LORENTZ RECIPROCITY THEOREM AND ONSAGER-CASIMIR RELATIONS

Applying the usual reciprocity manipulations of the Maxwell equations [40,77,79,80] to the frequency-domain version of Eq. (5) with the time-reversal transformations [Eq. (7)] yields (Appendices D and E)

$$\iiint_{V_J} \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}}^* dv - \iiint_{V_J} \tilde{\mathbf{J}}^* \cdot \tilde{\mathbf{E}}dv$$
$$= \oiint_S (\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* - \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}) \cdot \hat{\mathbf{n}}ds$$
$$- j\omega \iiint_V (\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}^* + \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^* - \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}})dv. \tag{9}$$

If the medium is unbounded, so that $[\hat{\mathbf{n}} \times \mathbf{E}^{(*)} = \eta\mathbf{H}^{(*)}]_S$ assuming *restricted time reversal* (η unchanged), or enclosed by an impenetrable cavity, the surface integral in this equation vanishes (Appendix F). In reciprocal systems, the left-hand side (reaction difference [40]) also vanishes, as found by first applying Eq. (9) to a vacuum, where the right-hand-side volume integral vanishes, as a fundamental reciprocity condition in terms of system ports. Inserting Eqs. (6a) and (6b), transformed according to the restricted time-reversal version of Eqs. (8a) and (8b) (no "∗"), in the resulting relation yields (Appendix G)

$$\iiint_V \{\tilde{\mathbf{E}}^* \cdot [\bar{\bar{\epsilon}}(\mathbf{F}_0) - \bar{\bar{\epsilon}}^T(-\mathbf{F}_0)] \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{H}}^* \cdot [\bar{\bar{\mu}}(\mathbf{F}_0)$$
$$- \bar{\bar{\mu}}^T(-\mathbf{F}_0)] \cdot \tilde{\mathbf{H}} + \tilde{\mathbf{E}}^* \cdot [\bar{\bar{\xi}}(\mathbf{F}_0) + \bar{\bar{\zeta}}^T(-\mathbf{F}_0)] \cdot \tilde{\mathbf{H}}$$
$$- \tilde{\mathbf{H}}^* \cdot [\bar{\bar{\zeta}}(\mathbf{F}_0) + \bar{\bar{\xi}}^T(-\mathbf{F}_0)] \cdot \tilde{\mathbf{E}}\}dv = 0, \tag{10}$$

where the identity $\mathbf{a} \cdot \bar{\bar{\chi}} \cdot \mathbf{b} = (\mathbf{a} \cdot \bar{\bar{\chi}} \cdot \mathbf{b})^T = \mathbf{b} \cdot \bar{\bar{\chi}}^T \cdot \mathbf{a}$ (T is the transpose) is used to group dyadics with opposite premultiplying/postmultiplying fields. Equation (10) represents the generalized form of the Lorentz reciprocity

theorem for LTI bianisotropic media [29,81]. Since this relation must hold for arbitrary fields, one must have

$$\bar{\bar{\epsilon}}(\mathbf{F}_0) = \bar{\bar{\epsilon}}^T(-\mathbf{F}_0) \tag{11a}$$

and

$$\bar{\bar{\mu}}(\mathbf{F}_0) = \bar{\bar{\mu}}^T(-\mathbf{F}_0), \tag{11b}$$

and

$$\bar{\bar{\xi}}(\mathbf{F}_0) = -\bar{\bar{\zeta}}^T(-\mathbf{F}_0). \tag{11c}$$

Equations (11a)–(11c) are the electromagnetic version of the Onsager-Casimir reciprocity relations. If $\mathbf{F}_0 = 0$ (no bias), Eqs. (11a)–(11c) reduce to the conventional reciprocity relations $\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}^T$, $\bar{\bar{\mu}} = \bar{\bar{\mu}}^T$, and $\bar{\bar{\xi}} = -\bar{\bar{\zeta}}^T$ [40]. If the medium is lossless, we also have $\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}^\dagger$, $\bar{\bar{\mu}} = \bar{\bar{\mu}}^\dagger$, and $\bar{\bar{\xi}} = \bar{\bar{\zeta}}^\dagger$ († is the transpose conjugate) [40], which leads to the additional constraint $\text{Im}\{\bar{\bar{\epsilon}}\} = \text{Im}\{\bar{\bar{\mu}}\} = \text{Re}\{\bar{\bar{\xi}}\} = \text{Re}\{\bar{\bar{\zeta}}\} = 0$ in reciprocal media.

If the sign of $\mathbf{F}_0$ is reversed on only one side of Eqs. (11a)–(11c), the equalities obviously do not hold anymore, because the constitutive parameters are not even functions of $\mathbf{F}_0$ [82], and therefore they transform into the *inequality* relations

$$\bar{\bar{\epsilon}}(\mathbf{F}_0) \neq \bar{\bar{\epsilon}}^T(\mathbf{F}_0) \tag{12a}$$

or

$$\bar{\bar{\mu}}(\mathbf{F}_0) \neq \bar{\bar{\mu}}^T(\mathbf{F}_0) \tag{12b}$$

or

$$\bar{\bar{\xi}}(\mathbf{F}_0) \neq -\bar{\bar{\zeta}}^T(\mathbf{F}_0). \tag{12c}$$

These relations are the electromagnetic Onsager-Casimir nonreciprocity relations, and they have been separated by "or's" [rather than "and's" as in Eqs. (11a)–(11c)], since violating one reciprocity condition is sufficient to render the system globally nonreciprocal.

Noting that the transposed dyadics in Eqs. (11a)–(11c) and (12a)–(12c) correspond to propagation in the reverse direction to that of the wave associated with the untransposed dyadic (Appendix H) leads to the following profound and enlightening physical interpretation of the Onsager-Casimir relations: reciprocity (nonreciprocity) results from reversing (not reversing) the sign of $\mathbf{F}_0$, so that waves propagating in opposite directions see the same (different) effective media. We have thus provided a rigorous demonstration of the time-reversal symmetry criterion for nonreciprocity that is described in Sec. V.

To better appreciate this interpretation, let us consider again the example of ferrite, biased by the static magnetic field $\mathbf{F}_0 = \mathbf{B}_0$ and characterized by the permeability tensor $\bar{\bar{\mu}}$, in Figs. 2(b) and 2(c). The Onsager-Casimir reciprocity relation $\bar{\bar{\mu}}(\mathbf{F}_0) = \bar{\bar{\mu}}^T(-\mathbf{F}_0)$ [Eq. (11b)] corresponds

to Fig. 2(b), where the wave propagating in the direct direction (on the left) sees the medium $\bar{\bar{\mu}}(\mathbf{F}_0)$, and the wave propagating in the reverse direction with flipped bias (on the right) sees the same effective medium $\bar{\bar{\mu}}^T(-\mathbf{F}_0) = \bar{\bar{\mu}}(\mathbf{F}_0)$ (despite the alteration of the nature of the system): time-reversal symmetry is ensured by reversing *all* the time-reversal odd quantities of the system.

The negation of Eq. (11b), given by Eq. (12b) as $\bar{\bar{\mu}}(\mathbf{F}_0) \neq \bar{\bar{\mu}}^T(\mathbf{F}_0)$, is the time-reversal asymmetric relation corresponding to Fig. 2(c), where the wave propagating in the direct direction (on the left) still sees the medium $\bar{\bar{\mu}}(\mathbf{F}_0)$, whereas the wave propagating in the reverse direction with unflipped bias (on the righ) sees a different effective medium $\bar{\bar{\mu}}^T(+\mathbf{F}_0) \neq \bar{\bar{\mu}}(\mathbf{F}_0)$ (without system alteration), corresponding to time-reversal symmetry breaking and consistent with the time-reversal symmetry criterion for nonreciprocity described in Sec. III.

It is interesting to note that reversing an even quantity (Sec. V), such as $\mathbf{E}_0$, instead of an odd quantity $\mathbf{F}_0$, would yield nonreciprocity relations of the kind $\bar{\bar{\epsilon}}(\mathbf{E}_0) \neq \bar{\bar{\epsilon}}^T(-\mathbf{E}_0)$, involving bias flipping and hence forbidding simultaneous excitations from both ends since $\mathbf{E}_0$ cannot be simultaneously pointing in opposite directions.

X. GENERALITY OF ONSAGER-CASIMIR RELATIONS

Although the frequency-domain derivation in Sec. IX assumes linear time invariance, the Onsager-Casimir relations are totally general. Onsager derived them for linear processes without any other assumption than microscopic reversibility and basic theorems from the general theory of fluctuations [27], and these relation were later extended to nonlinear systems [83]. Therefore, the Onsager-Casimir relations hold not only for LTI systems, as shown in Sec. IX, but also for LTV and nonlinear systems.

Systems that are not media, but rather are inhomogeneous structures, components, or devices, cannot, of course, be characterized by medium parameters, but the Onsager-Casimir relations still apply to them in terms of extended S-parameters and S-matrix, as shown in Sec. XIV.

XI. FINER CLASSIFICATION OF NONRECIPROCAL SYSTEMS

In Sec. II we established a gross classification of nonreciprocity in terms of linear and nonlinear nonreciprocal systems. However, in Sec. VII we revealed the necessity to further divide the linear category into LTI and LTV nonreciprocal systems, Eqs. (6a) and (6b) and Secs. VIII–IX applying to only the former type [84].

Table III reflects this fact while summarizing the applicability of the main concepts studied in this paper to the

TABLE III.   Fundamental concept applicability to the three different types of nonreciprocal systems: linear time invariant, linear time variant, and nonlinear.

	Linear		Nonlinear
	Time invariant	Time variant (space-time mod)	
	$\bar{\bar{S}}(\mathbf{F}_0) \neq \bar{\bar{S}}^T(\mathbf{F}_0)$	$\bar{\bar{S}}(\mathbf{F}_0) \neq \bar{\bar{S}}^T(\mathbf{F}_0)$	$\bar{\bar{S}}(\mathbf{F}_0) \neq \bar{\bar{S}}^T(\mathbf{F}_0)$
Common bias $\mathbf{F}_0$	Magnetic field ($\mathbf{B}_0$)	Velocity ($\mathbf{v}_0$)	Wave field ($\mathbf{E}$)
Time reversal, Secs. III–VI	✓	✓	✓
Onsager-Casimir, Eqs. (11a)–(11c) and (12a)–(12c)	✓	✓	✓
Lorentz nonreciprocity, Secs. VII–IX	✓	✗	✗
Extended S-parameters, Sec. XIV	✓	✓ ⚠	✓ ⚠⚠

resulting three different categories. In addition to mentioning the most-common bias field involved, it presents time-reversal symmetry breaking and Onsager-Casimir relations in terms of the generic S-matrix/tensor ($\bar{\bar{S}}$) relation

$$\bar{\bar{S}}(\mathbf{F}_0) \neq \bar{\bar{S}}^T(\mathbf{F}_0) \tag{13}$$

as a common descriptor for *all* nonreciprocal systems, stresses that the Lorentz theorem is applicable only to LTI systems, and indicates that extended S-parameters apply to the three types of nonreciprocal systems, with increasing restriction from the LTI case through the LTV case to the nonlinear case.

XII. RECIPROCITY DESPITE TIME-REVERSAL ASYMMETRY IN LOSSY SYSTEMS

We now consider the case of lossy nonreciprocal systems, and particularly explain how to handle time reversal and time-reversal symmetry breaking in such systems, which is not trivial and possibly prone to confusion.

Figure 3 shows the process of electromagnetic wave propagation in a simple lossy biasless waveguide system. Let us see how this system responds to time reversal (Secs. III–IV) by applying the time-reversal symmetry breaking test in Sec. V, as previously done for the Faraday system in Figs. 2(b) and 2(c).

In the direct part of the process, the wave is attenuated by dissipation as it propagates from port P_1 to port P_2 (red curve); say, from P_0 to $P_0/2$ (3-dB loss). On time reversal, the propagation direction is reversed, and loss is transformed into gain (Table II). As a result, the wave propagates back from P_2 to P_1 and its power level is restored (green curve), from $P_0/2$ to P_0. However, the system has been altered since it has become active. Maintaining it lossy leads to further attenuation on the return trip, from $P_0/2$ to $P_0/4$ (6-dB loss), and hence breaks time-reversal symmetry. According to Sec. V, this would imply nonreciprocity, which is at odds with the generalized reciprocity theorem (Sec. IX)!

This paradox originates in the looseness of the assumption that "time-reversal symmetry (asymmetry) is

equivalent to reciprocity (nonreciprocity)" in Sec. III. This assumption is made on the ground that it is the ratio definition of Sec. II—and not its restricted level form—that is commonly used in practice but it is sensu stricto incorrect, the equivalence rigorously holding only in terms of absolute field levels and not in terms of field ratios. In the case of loss, as just seen, the field/power ratios are equal $[(P_0/4)/(P_0/2) = 0.5 = (P_0/2)/P_0]$, consistent with the general definition of reciprocity in Sec. II, but the field/power levels are not ($P_0/4 \neq P_0$), in contradiction with the definition of time reversal in Sec. III. In this sense, a simple lossy system breaks time-reversal symmetry despite being perfectly reciprocal.

This time-reversal asymmetry may be seen as an expression of thermodynamical macroscopic irreversibility. Consider, for instance, an empty metallic waveguide. The transfer of charges along the waveguide's lossy walls results in electromagnetic energy being transformed into heat (Joule's first law) [30]. In theory, a Maxwell demon [85] could reverse the velocities of all the molecules of the system, which would surely reconvert that heat into electromagnetic energy. In this sense, all systems
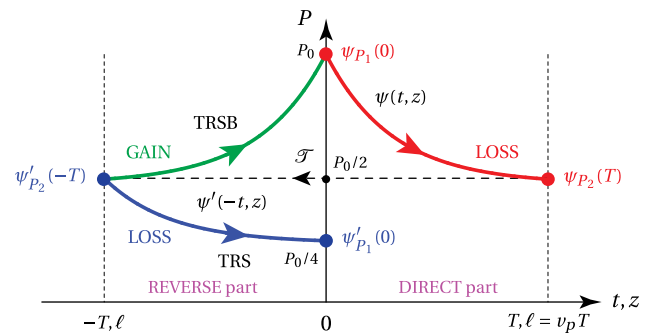

FIG. 3.   Time-reversal symmetry breaking (TRSB) (Sec. V) in a lossy reciprocal waveguide of length ℓ. Assuming that the process under consideration is the propagation of a modulated pulse between ports P_1 and P_2, we have $\boldsymbol{\Psi}'(-t) = [\psi'_{P_1}(-t), \psi'_{P_2}(-t)]^T$ (blue curve) $\neq [\psi_{P_1}(t), \psi_{P_2}(t)]^T = \boldsymbol{\Psi}(-t)$ (green curve), and in particular, with the power loss assumed in the figure, $\boldsymbol{\Psi}'(0) = [P_0/4, 0]^T \neq [P_0, 0]^T = \boldsymbol{\Psi}(0)$.

are microscopically reversible, which is the fundamental assumption underpinning Onsager reciprocity relations [27–29,86]. However, such reconversion is prohibited by the second law of thermodynamics, which stipulates that the total entropy in an isolated system cannot decrease over time. It would at the least require injection of energy from outside the system! So, such a lossy system is macroscopically—and hence practically—irreversible. Loss cannot be undone; it ever accumulates over time, as illustrated in Fig. 3.

As a corollary of this discussion, one may state that although time-reversal symmetry necessarily implies reciprocity, time-reversal asymmetry does not necessarily imply nonreciprocity!

XIII. OPEN SYSTEMS AND THEIR TIME-REVERSAL-SYMMETRY "LOSSY" BEHAVIOR

Consider the two-antenna open system in Fig. 4, showing successively the original, time-reversed, and reciprocal problems in Figs. 4(a), 4(b), and 4(c), respectively. The nature of the system is clearly altered on time reversal, from Fig. 4(a) to 4(b), where the intrinsic impedance of the surrounding medium becomes negative (Table II). This
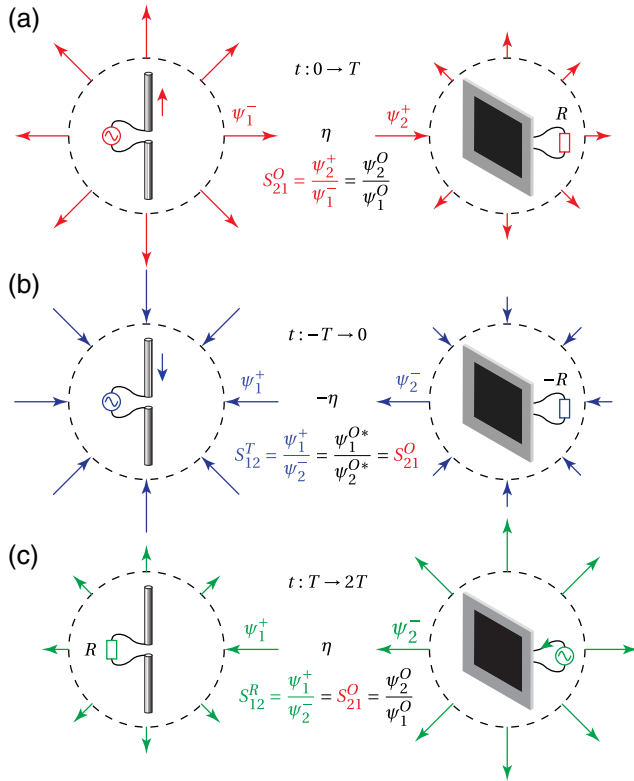

results from the fact that the radiated and scattered energy escaping the receiving antennas in the original problem is equivalent to loss relatively to the two-port system. Such loss transforms into gain on time reversal, as in Sec. XII, leading to fields emerging from infinity. On replacing time reversal by restricted time reversal (Sec. VIII) so as to avoid denaturing the system, one would find, as in the lossy case, reduced field levels but conserved field ratios. The only difference between the restricted time-reversal problem and the reciprocity problem, depicted in Fig. 4(c), would then be the fact that in the former case the field level is not reset to its initial value before propagation in the opposite direction. An open system is thus time-reversal-wise similar to a lossy system (Sec. XII).

XIV. EXTENDED SCATTERING PARAMETER MODELING

The lossy/open-system difficulty (Secs. XII and XIII), the nonreciprocity/reciprocity definition in Sec. II, and the general reciprocity relations derived by Onsager [26–28] prompt a description of nonreciprocal/reciprocal systems in terms of field ratios. This leads to the scattering parameters, or S-parameters, introduced in quantum physics in 1937 [87], used for more than 70 years in microwave engineering for LTI systems [8,88,89], and extended to power parameters for arbitrary loads in the 1960s [8,90] and to the cross-coupled matrix theory for topologically coupled resonators in the 1970s [91,92]. We attempt here an extension of these parameters to LTV and nonlinear systems.

Figure 5 defines an extended arbitrary P-port network as an electromagnetic structure delimited by a surface S with n waveguide terminals, T_n, each supporting a number of mode-frequency ports, $\mathrm{P}_p = \mathrm{P}_{\mu,\omega}^n$, with $p = 1, 2, \ldots, P$. "Frequency" refers here to new frequency sets (possibly infinite or continuous) that may be generated in LTV and nonlinear systems. For instance, if T_1 in an LTI network is a waveguide terminal with the $M_1 = 2$ modes TE_{10} and TM_{11} and the $\Omega_1 = 2$ frequencies ω and $\alpha\omega$, it includes the $M_1\Omega_1 = 4$ ports $\mathrm{P}_1 = \mathrm{P}_{\mathrm{TE}_{10},\omega}^1$, $\mathrm{P}_2 = \mathrm{P}_{\mathrm{TE}_{10},\alpha\omega}^1$, $\mathrm{P}_3 = \mathrm{P}_{\mathrm{TM}_{11},\omega}^1$, and $\mathrm{P}_4 = \mathrm{P}_{\mathrm{TM}_{11},\alpha\omega}^1$, where ω and $\alpha\omega$ could also represent the input frequency and a generated frequency in the case of a LTV or a nonlinear network.

The transverse fields in the waveguides have the frequency-domain form [89], extended here to LTV and nonlinear systems,

$$\begin{Bmatrix} \tilde{\mathbf{E}}_{t,p}(x,y,z) \\ \tilde{\mathbf{H}}_{t,p}(x,y,z) \end{Bmatrix} = \left(a_p e^{-j\beta_p z} \pm b_p e^{+j\beta_n z} \right) \begin{Bmatrix} \tilde{\mathbf{e}}_{t,p}(x,y) \\ \tilde{\mathbf{h}}_{t,p}(x,y) \end{Bmatrix},$$

(14)

where $\iint_S (\tilde{\mathbf{e}}_{t,p} \times \tilde{\mathbf{h}}_{t,q}) \cdot \hat{\mathbf{n}} ds = 2\delta_{pq}$ (also applying when p and q differ only by frequency, with the same terminal/mode, assuming narrow-band and hence independent port detectors), and where a_p (b_p) $(p = 1, \ldots, P)$ are the
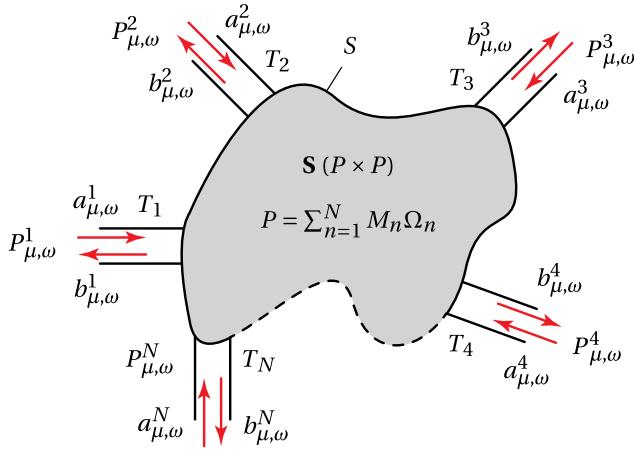
FIG. 4. Time-reversal asymmetry and apparent (false) non-reciprocity of an open system composed of a dipole antenna and a patch antenna in free space. (a) Original ("O") problem. (b) Time-reversed ("T") problem. (c) Reciprocal ("R") problem.

FIG. 5. Arbitrary P-port network and extended scattering matrix, **S**.

port input (output) complex wave amplitudes, which are related by the extended S-matrix, **S**, as

$$\mathbf{b} = \mathbf{Sa}, \quad \text{with} \quad \begin{cases} \mathbf{b} = [b_1, b_2, \ldots, b_P]^T \\ \mathbf{a} = [a_1, a_2, \ldots, a_P]^T \end{cases}. \quad (15)$$

If the system is linear, and hence superposition applies, each entry of the matrix can be expressed by the simple transfer function $S_{ij} = b_i/a_j|_{a_k=0, k\neq j}$, which corresponds to the conventional definition of the S-parameters, except for the frequency-port-definition extension in the space-time modulated (LTV) case. If the system is nonlinear, then $S_{ij} = S_{ij}(a_1, a_2, \ldots, a_P)$, and therefore *all* the (significant) input signals must be simultaneously present in the measurement of the transfer function S_{ij}, as done in broadband polyharmonic distortion, used in the Keysight microwave nonlinear vector network analyzer [93–95]. For instance, in a simple waveguide junction (without any bias), $S_{21} = b_2/a_1|_{a_2=a_3=0} \neq S_{21}(a_3)$, which allows measurement of the system with separate inputs, whereas in a mixer (or modulator), $S_{21} = S_{21}(a_3 \equiv \text{LO})$ and the local oscillator (or pump) port (P_3) must be excited simultaneously with the signal port (P_1) for proper measurement of the transfer function S_{21}.

If the system is an LTI medium its bianisotropic reciprocity relations [40,77] for the two network states $\mathbf{a}'$ and $\mathbf{a}''$ with responses $\mathbf{b}'$ and $\mathbf{b}''$ read [35]

$$\nabla \cdot (\tilde{\mathbf{E}}' \times \tilde{\mathbf{H}}'' - \tilde{\mathbf{E}}'' \times \tilde{\mathbf{H}}') = j\omega \tilde{\mathbf{E}}'' \cdot (\bar{\bar{\epsilon}} - \bar{\bar{\epsilon}}^T) \cdot \tilde{\mathbf{E}}' - \tilde{\mathbf{H}}''$$
$$\cdot (\bar{\bar{\mu}} - \bar{\bar{\mu}}^T) \cdot \tilde{\mathbf{H}}' + \tilde{\mathbf{E}}'' \cdot (\bar{\bar{\xi}} + \bar{\bar{\zeta}}^T) \cdot \tilde{\mathbf{H}}'$$
$$- \tilde{\mathbf{H}}'' \cdot (\bar{\bar{\zeta}} + \bar{\bar{\xi}}^T) \cdot \tilde{\mathbf{E}}'] = 0, \quad (16)$$

where the sources, being outside the system, do not contribute, contrary to the situation of the reciprocity theorem [Eq. (9)] (Appendix I). Inserting the sum ($\sum_{p=1}^{P}$) of fields

[Eq. (14)] into this equation, taking the volume integral of the resulting relation, applying the Gauss theorem and the orthogonality relation, and using the Onsager-Casimir relations [Eq. (11)] yields [35]

$$\sum_p (b'_p a''_p - a'_p b''_p) = \mathbf{ba}''^T - \mathbf{ab}''^T = \mathbf{a}'\mathbf{a}''^T (\mathbf{S}^T - \mathbf{S}), \quad (17)$$

where Eq. (15) is used to eliminate $\mathbf{b}^{(', '')}$ in the last equality. This leads to the reciprocity condition (Appendix J)

$$\mathbf{S} = \mathbf{S}^T, \quad (18)$$

and hence to the convenient scattering-parameter nonreciprocity condition

$$\mathbf{S} \neq \mathbf{S}^T \text{ or } \exists(i,j)|S_{ij} \neq S_{ji}, \quad i =, 1, 2, \ldots, N \quad (19)$$

(e.g., $S_{21} \neq S_{12}$), which also applies to LTV and nonlinear systems, although the current demonstration is restricted to LTI media (Appendix K).

In the microwave regime, these S-parameters can be directly measured (magnitude and phase) with a vector network analyzer [8] or with a nonlinear vector network analyzer [93–95]. In contrast, in the optical regime no specific instrumentation is available to do this, and a special setup, with nontrivial phase handling, is therefore required [19].

The symmetry reciprocity relation $\mathbf{S} = \mathbf{S}^T$ and nonreciprocity relation $\mathbf{S} \neq \mathbf{S}^T$ [Eq. (19)], assuming the extended S-parameters introduced in this section, are nothing but the general Onsager-Casimir reciprocity/ nonreciprocity relations [27–30], and may be explicitly written as

$$\bar{\bar{S}}(\mathbf{F}_0) = \bar{\bar{S}}^T(-\mathbf{F}_0) \quad \text{(reciprocity)}, \quad (20a)$$

$$\bar{\bar{S}}(\mathbf{F}_0) \neq \bar{\bar{S}}^T(\mathbf{F}_0) \quad \text{(nonreciprocity)}, \quad (20b)$$

where the matrix **S** is written in the tensor notation $\bar{\bar{S}}$ for direct comparison with Eqs. (11a)–(11c).

XV. ENERGY CONSERVATION

In a lossless system, energy conservation requires that the total output power equals the total input power, or $\sum_{p=1}^{P} |b_p|^2 = \sum_{p=1}^{P} |a_p|^2$ in Fig. 5, since no power is dissipated in the system. In terms of the S-matrix, this requirement translates into the unitary relation

$$\mathbf{SS}^\dagger = I, \quad (21)$$

where the dagger symbol represents the transpose conjugate and I the unit matrix. Many fundamental useful facts

on multiport systems straightforwardly follow from energy conservation (see, e.g., Ref. [8]).

Some immediate consequences of Eq. (21) for nonreciprocity are as follows:

1. A lossless one-port system, $\mathbf{S} = [S_{11}]$, can be only totally reflective, from $|S_{11}|^2 = 1$, even if it includes nonreciprocal materials, contrary to claims in Ref. [96].

2. A two-port system, $\mathbf{S} = [S_{11}, S_{12}; S_{21}, S_{22}]$, can be magnitude-wise nonreciprocal only if it is lossy; specifically, a purely reflective (lossless) two-port isolator $\mathbf{S} = [0, 0; 1, 1]$ is impossible, since energy conservation requires $|b_2|^2 = |a_1|^2 + |a_2|^2$, whereas the device would exhibit $b_2^2 = a_1^2 + a_2^2 + 2a_1a_2$, with the additional term $2a_1a_2$ that may cause the total energy to be larger than the input power, depending on the relative phases of a_1 and a_2.

3. A lossless two-port system can still be nonreciprocal in terms of phase, since Eq. (21) does not demand $\angle S_{21} = \angle S_{12}$ [97,98].

4. A lossless three-port system, $\mathbf{S} = [S_{11}, S_{12}, S_{13}; S_{21}, S_{22}, S_{23}; S_{31}, S_{32}, S_{33}]$, can be matched simultaneously at all ports only if it is nonreciprocal, as may be shown by manipulation of the matrix system given by Eq. (21) [8].

XVI. THE "THERMODYNAMICS PARADOX"

Case 2 in Sec. XV—the two-port isolator—raised much perplexity in the past, and led to the so-called "thermodynamics paradox." The paradox states that an isolator system would ever increase the temperature of the load at the passing end and not that of the load at the isolated end, hence violating the second law of thermodynamics, which prescribes heat transfer from hot to cold bodies.

The paradox started in 1885 with the comment by Rayleigh that the recently developed system composed of two Nicol prisms sandwiching a magnetized dielectric would be "inconsistent with the second law of thermodynamics" [99].

It was overruled 16 years later by Rayleigh himself [6], who realized then, following related studies of Wien [100], a misunderstanding of the system, which was actually *not* nonreciprocal, as the wave on the presumed stop direction could eventually exit the device after three round trips across the device (see Sec. XIX).

The paradox resurfaced in 1955, as Lax and Button [101] pointed out the existence of lossless unidirectional eigenmodes in some ferrite-loaded waveguide structures. It was eventually resolved by Ishimaru [14,102], who showed that such a waveguide would necessarily support substantial loss in its terminations, due to energy conservation (Sec. XV), even in the limit of negligible material loss. This loss would eventually heat up the isolator, and therefore ensure thermal balance through thermal emission toward the cold bath. The paradox was because

the Maxwell equations were resolved for a completely lossless medium, which represents an "improperly posed problem" since it does not correspond to physical reality. Properly solving the Maxwell equations for a realistic medium with nonzero conductivity, and letting then the conductivity go to zero completely resolves the issue [14,102].

XVII. NOTE ON EVANESCENT AND COMPLEX WAVES

We emphasize here that all the contents of this paper apply to all electromagnetic waves. Indeed, all the reasonings and derivations are completely general, since they are based on symmetry first principles and the Maxwell equations, without any restriction, unless otherwise stated. Therefore, they do not apply only to propagating waves, but also to evanescent waves. Evanescent waves are characterized by an exponential power decay in space and occur, notably, in total internal reflection, diffraction from an aperture, and dipolar radiation [103], and in complex (or inhomogeneous) waves (see, e.g., Ref. [104]). Complex waves have complex (generally real and imaginary) wavenumbers in different directions of space and play a central role in plasmonic and Zenneck surface waves, leaky waves, optical waveguides, couplers, coupled-resonator filters, and quantum tunneling [14].

The applicability of nonreciprocity concepts, and particularly of the Lorentz reciprocity theorem, to evanescent waves was explicitly verified in Ref. [105] with use of an angular spectrum electromagnetic perspective and by consideration of an arbitrary distance between the scatterer and the observation point.

Evanescent waves are rarely discussed in the context of nonreciprocity, because they are not present or essential in most nonreciprocal systems. However, interesting nonreciprocal effects specifically affecting or leveraging evanescent waves might naturally be discovered and applied in the future.

XVIII. OVERVIEW OF THE THREE FUNDAMENTAL TYPES OF NONRECIPROCAL SYSTEMS

At this point, the fundamental concepts of reciprocity have been presented to a level that we judge appropriate for such a review. Upon this foundation, we now describe the aforementioned three types of nonreciprocal systems (Table III)—namely, LTI, LTV, and nonlinear nonreciprocal systems—which are covered in Secs. XIX, XX, and XXI, respectively. In each case, we list the fundamental characteristics, enumerate the main applications, and describe a particular example of the system.

XIX. LINEAR TIME-INVARIANT NONRECIPROCAL SYSTEMS

LTI nonreciprocal systems have the following fundamental characteristics:

1. Time-reversal symmetry breaking by time reversal odd external bias $\mathbf{F}_0$, which is most often a magnetic field, $\mathbf{B}_0$, as, for instance, in Fig. 2(c).

2. Applicability, from linearity, to arbitrary excitations and intensities, corresponding to strong nonreciprocity, as pointed out in Table I.

3. Frequency conservation, also due to linearity, and hence unrestricted frequency-domain descriptibility (Secs. VII and VIII), and full applicability of the Lorentz reciprocity theorem (Sec. IX) and of the S-parameter formalism (Sec. XIV).

4. Generally based on LTI materials [7,44,45,106–111], including two-dimensional electron gases and graphene [51,53–58], or metamaterials [70–75,112–117] (Sec. VII).

The main LTI nonreciprocal systems are isolators, nonreciprocal phase shifters, and circulators [7,8,44]. Isolators ($\mathbf{S} = [0,0;1,0]$) may be of Faraday, resonance, field-displacement, or matched-port-circulator type, and may involve resistive sheets, quarter-wave plates, or polarizing grids. They are typically used to shield equipment (e.g., vector network analyzer or laser) from detuning, interfering reflections, and even destructive reflections. Nonreciprocal phase shifters ($\mathbf{S} = [0, e^{j\Delta\varphi}; 1, 0]$), with the gyrator ($\Delta\varphi = \pi$) [118] as a particular case, may be of latching (hard magnetic hysteresis) or Faraday rotation type, and may involve quarter-wave plates. They combine with couplers to form isolators or circulators, provide compact simulated inductors and filter inverters, and enable nonreciprocal pattern and scanning arrays. Circulators ($\mathbf{S} = [0,0,1;1,0,0;0,1,0]$) may be of four-port Faraday rotation or three-port junction rotation type; in the latter type, which is more common, nonreciprocal circulation results from the suppression of the degeneracy between counter-rotating modes in the presence of a perpendicular magnetic biasing field, which results in constructive and destructive interferences at the coupled and isolated ports, respectively. They are used for isolation, duplexing (radar and communication), and reflection amplifiers. All these components are dissipative, with the stopped energy being transformed into heat, since they do not satisfy Eq. (21).

Figures 2(b) and 2(c) represent a Faraday rotator, whose operation is described in Sec. VI. This system constitutes the key building brick of a Faraday isolator. A typical implementation of such an isolator involves a Faraday rotation angle of $\phi = \pi/4$ and two linear polarizers sandwiching the magnetic (generally ferrite) medium, rotated by $\phi = \pi/4$ with respect to each other. In the passing direction, the electric field at P_1 is perpendicular to the first polarizer grid and therefore fully traverses it; it is next rotated in the, say, clockwise direction by $\pi/4$ so as to emerge perpendicular to the second grid, which leads to full transmission to P_2, corresponding to $S_{21} = 1$. In the stopping direction, the wave enters the system at P_2, so as to perpendicularly face the second grid and hence completely traverse it; it is then rotated, still clockwise, by $\pi/4$. As a result, it is now parallel to the first grid, and hence fully reflected by it, which leads in principle [119] to $S_{12} \approx 0$.

XX. LINEAR TIME-VARIANT NONRECIPROCAL SYSTEMS

LTV nonreciprocal systems have the following fundamental characteristics:

1. Time-reversal symmetry breaking by time-reversal odd external bias ($\mathbf{F}_0$) velocity, $\mathbf{v}_0$, associated with spatial inversion symmetry breaking, as seen in the example below.

2. Strong nonreciprocity, as LTI systems (Sec. XIX), from linearity.

3. Generation of new, possibly anharmonic frequencies, due to assumed external application of the modulation, and hence restricted applicability of S-parameters.

4. Moving-medium (i.e., moving-matter) modulation (e.g., optomechanical) [34,40,63,120,121] or moving-wave (i.e., moving-perturbation) modulation (e.g., electro-optic, acousto-optic, or nonlinear-optic) modulation [10,59], with both producing Doppler shifts [122] and nonreciprocity, but only the former supporting Fizeau drag [34,123] and bianisotropy transformation [40], and only the latter allowing superluminality [124];

5. Pulse or periodic (i.e., crystal [125]) and abrupt or smooth medium/wave modulations.

As an illustration, Fig. 6 graphically depicts, using an extended Minkowski diagram representation [124,125], step space-time modulated systems with an interface between media of refractive indices n_1 and n_2 moving in the $-z$ direction with constant velocity $\mathbf{v}_0 = v\hat{\mathbf{z}}$ ($v < 0$) and excited by an incident wave propagating in the $+z$ direction. On full time reversal, $\mathbf{v}_0$ is reversed, which leads to identical Doppler shifts [122] in the reflected and transmitted waves [with temporal frequency (ω) and spatial frequency ($\mathbf{k}$) transitions following corresponding frequency conservation lines in the moving frame [124]], as shown in Fig. 6(a), but alters the system. The unaltered system is time-reversal asymmetric, and hence breaks time-reversal symmetry, which leads to the complex nonreciprocal scattering observed in Fig. 6(b). It clearly appears that, as mentioned above, purely temporal
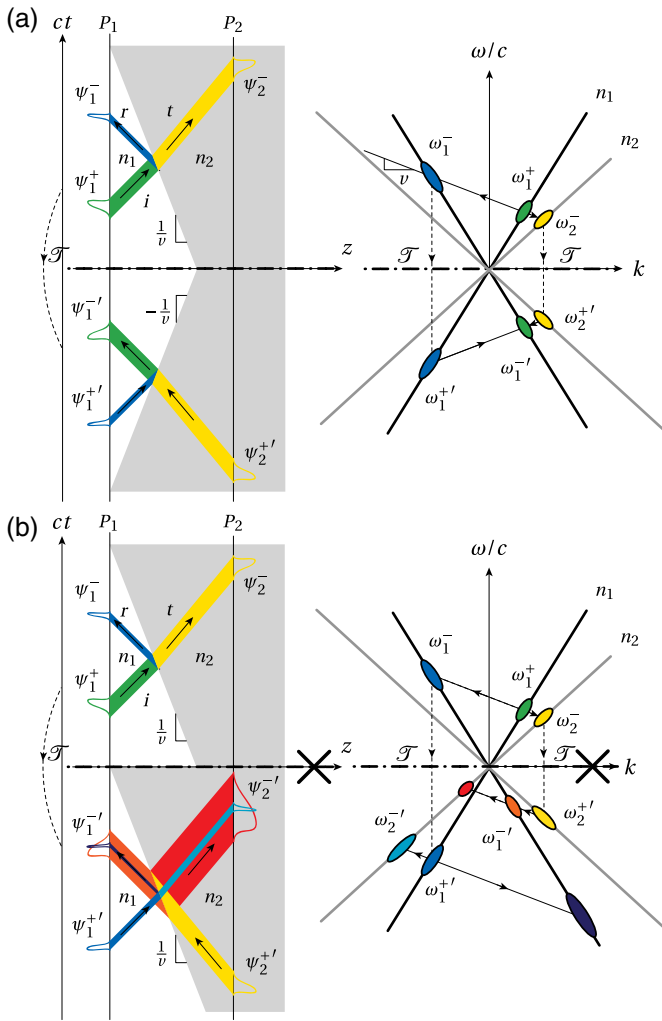
FIG. 6. Step space-time-modulated (LTV) systems ($n_2 > n_1$). (a) Original system (top $t > 0$) and its time-reversal symmetric counterpart (bottom $t < 0$); (b) Original system (top $t > 0$) and its time-reversal asymmetric and nonreciprocal counterpart (bottom $t < 0$). i, incident; r, reflected; t transmitted.

FIG. 7. Three-port nonreciprocal space-time-modulated reflective metasurface with **S** matrix. Left: reflection from port P_1 with temporal-spatial frequencies (ω_i, $\mathbf{k}_i$) to port P_2 with temporal-spatial frequencies (ω_r, $\mathbf{k}_r$). Right: reflection from port P_2 with temporal-spatial frequencies (ω_i', $\mathbf{k}_i'$) = (ω_r, $\mathbf{k}_r$) to port P_1 with temporal-spatial frequencies (ω_r', $\mathbf{k}_r'$).

developed by microwave engineers in the 1950s, primarily for amplifiers or mixers rather than nonreciprocal devices [147–153]. However, these systems will probably lead to many novel structures and applications, especially when more than one dimension of space is involved.

Figure 7 shows such a multidimensional system, specifically a nonreciprocal metasurface reflector [140, 154] based on the space-time modulation $n(x) = n_0 + n_m \cos(\beta_m x + \omega_m t)$, where $(\omega_m/\beta_m)\hat{\mathbf{z}} = \mathbf{v}_0$. The space-time-modulated metasurface breaks reciprocity and hence provides a quite unique nonreciprocal device by adding the spatial and temporal momenta $\mathbf{K}_{MS}$ and ω_{MS} to those of the incident wave. This system includes infinitely many ports, with scattering parameters $S_{n,n+1} = 0$ and $S_{n+1,n} \neq 0$, and with frequencies $\omega_{n+1} > \omega_n$ ($n = 1, \ldots \infty$), but its *functional* reduction in Fig. 7 is meaningful if the power transfer beyond P_3 is of no interest.

XXI. NONLINEAR NONRECIPROCAL SYSTEMS

Nonlinear nonreciprocal systems have the following fundamental characteristics:

1. Time-reversal symmetric breaking by spatial asymmetry and nonlinear self-biasing (nonlinearity triggering by the signal wave itself) [83,155], as is seen in the example below.

2. Limitation to restricted excitations, intensities, and isolation, which represents the first aspect of the weak non-reciprocity given in Table I, as will also be understood by the example.

3. Limitation to excitation of one direction only at a time, which represents the second aspect of weak nonreciprocity, still to be understood from the example, and which unfortunately precludes most practical isolator applications (Sec. XIX) [20].

4. Generation of new, only harmonic frequencies (assuming self-biasing) and inapplicability of superposition, and hence very restricted applicability of S-parameters.

5. Large diversity of possible time-reversal symmetry-breaking approaches.

modulation, which would here have the form of a horizontal interface between the two (white and gray) media [126–128], would be insufficient for nonreciprocity; spatial inversion symmetry breaking, provided here by the moving modulation (oblique interface), is also required to break time-reversal symmetry: $\bar{\bar{\chi}}(\mathbf{v}_0) \neq \bar{\bar{\chi}}^T(\mathbf{v}_0)$. Spatial inversion asymmetry must always accompany time variance for nonreciprocity in LTV systems, since nonreciprocity requires asymmetry between different points in space, which cannot be provided by time modulation alone.

A great diversity of useful space-time-modulated nonreciprocal systems have been reported in recent years [64–67,129–146]. They are all based on the production of different traveling phase gradients in opposite directions and are therefore all, in that sense, more or less lumped/distributed [8] variations of parametric systems
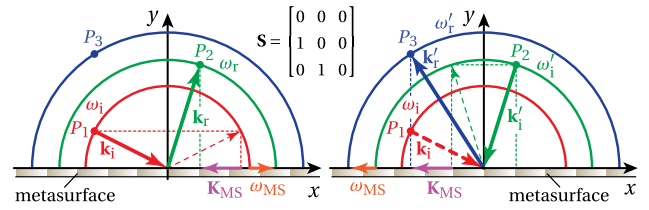
Figure 8 shows a simple way to achieve nonlinear nonreciprocity consisting in asymmetrically pairing a linear medium and a nonlinear lossy medium. The two media are strongly mismatched, with reflection Γ. A wave injected at port P_1 experiences a transmittance of $|T|^2 = 1 - |\Gamma|^2 \ll 1$, yielding a much smaller field level in the nonlinear medium. If this level is insufficient to trigger nonlinear loss, all the power transmitted through the interface ($|T|^2$) reaches port P_2, so $|S_{21}| = |T|$ and $|S_{11}| = |\Gamma|$. The same wave injected at P_2, assuming sufficient intensity to trigger nonlinear loss, undergoes exponential attenuation $e^{-\alpha \ell_{NL}}$ (ℓ_{NL} is the nonlinear length), so $|S_{12}| = |T|e^{-\alpha \ell_{NL}} \approx 0$. The system is thus nonreciprocal, but it is a pseudoisolator because (a) it is restricted to a small range of intensities; (b) it works only for one excitation direction ($P_1 \rightarrow P_2$ *or* $P_2 \rightarrow P_1$) at a time, since the $P_2 \rightarrow P_1$ wave, triggering nonlinear loss, would also extinct a simultaneous $P_1 \rightarrow P_2$ wave; (c) it suffers from poor isolation ($|S_{21}|/|S_{12}| = e^{\alpha \ell_{NL}}$) and poor isolation to insertion loss ratio (($|S_{21}|/|S_{12}|/)/|S_{11}| = e^{\alpha \ell_{NL}}/|\Gamma|$)—typically smaller than 20/25 dB in nonreciprocal nonlinear structures, while commonly exceeding 45/50 dB in nonreciprocal LTI isolators, or hysteresis (memory) dependence in the case of bistable systems [156]; and (d) it is reciprocal to noise, since noise simultaneously propagates in both directions [20]. Such a device is also not a diode [157], whose nonreciprocity consists of different forward/backward spectra due to positive/negative wave-cycle clipping, rather than different forward/backward signal levels.

Ingenious variations of the nonlinear nonreciprocal device in Fig. 8 have been reported [156,158–171]. Some of them mitigate some of the aforementioned issues, but these improvements are severely restricted by fundamental limitations of nonlinear nonreciprocity [19,20,172].

XXII. DISTINCTION WITH ASYMMETRIC PROPAGATION

A nonreciprocal system is a system that exhibits time-reversal asymmetric field ratios between well-defined ports
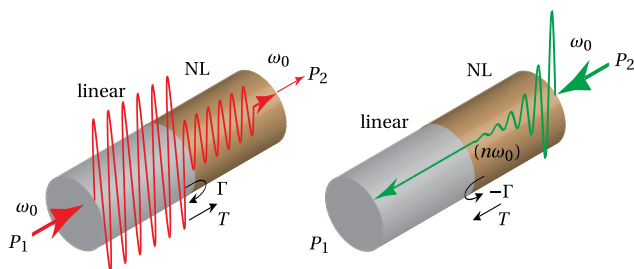

FIG. 8. Principle of nonlinear (NL) nonreciprocity with the asymmetric cascade of a linear medium, $\epsilon_L = \epsilon_1$, and a nonlinear lossy medium, $\epsilon_{NL} = \epsilon_2 + [\epsilon'(E) - j\epsilon''(E)]$, with strong mismatch, $|\Gamma| = |(\sqrt{\epsilon_{NL}} - \sqrt{\epsilon_L})/(\sqrt{\epsilon_{NL}} + \sqrt{\epsilon_L})| \gg 0$. Left: propagation from P_1 to P_2. Right: propagation from P_2 to P_1.
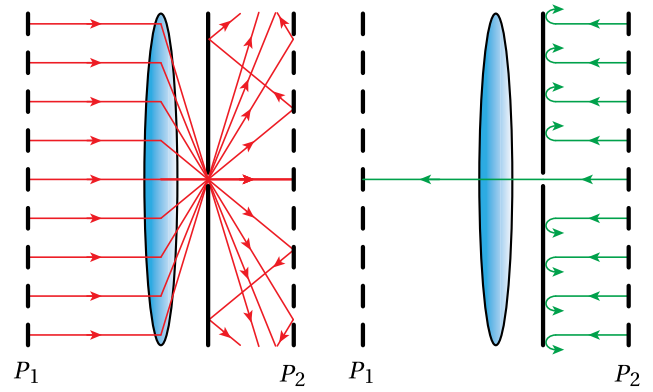

FIG. 9. Asymmetric reciprocal system formed by a lens and a mirror sandwiched between two antenna arrays. Left: propagation from P_1 to P_2. Right: propagation from P_2 to P_1.

[Eq. (19)], which is possible only under external biasing (linear nonreciprocity) or self-biasing plus spatial asymmetry (nonlinear nonreciprocity). Any system not satisfying this condition is necessarily reciprocal, despite possible fallacious transmission asymmetries [15–21,173,174].

For instance, the system in Fig. 9 exhibits asymmetric ray propagation, but it is fully reciprocal since only the horizontal ray gets transmitted between the array ports, the $P_1 \rightarrow P_2$ oblique waves symmetrically canceling out on the right array due to opposite phase gradients.

Other deceptively nonreciprocal cases include symmetric field rotation filtering (e.g., $\pi/2$ reciprocal rotator + polarizer), where $S_{21}^{yx} \neq S_{21}^{xy}$, but $S_{21}^{yx} = S_{12}^{xy}$; an asymmetric waveguide junction (e.g., step-width variation), with full transmission to the larger side distributed over multiple modes and small transmission from the same mode to the smaller (single-mode) side, but reciprocal mode-to-mode transmission (e.g., $S_{51} = S_{15}$, 1: port corresponding to mode 1 at small-side single-mode terminal, 5: port corresponding to mode 5 at large-side multi-mode terminal) [19]; asymmetric mode conversion (e.g., waveguide with nonuniform load), where an even mode transmits in opposite directions with and without excitation of an odd mode, without breaking reciprocity, since $S_{21}^{ee} = S_{12}^{ee}$ and $S_{21}^{oe} = 0 = S_{12}^{eo}$ [21]. In all cases, reciprocity is verified on exchange of the source and detector.

XXIII. CONCLUSION

We present, in the context of recent magnetless nonreciprocal systems aiming at repelling the frontiers of nonreciprocity technology, a global perspective of nonreciprocity, with the following main conclusions:

1. Nonreciprocal systems are defined as systems that exhibit different received-transmitted field ratios – and not just field levels – when their source(s) and detector(s) are exchanged.

2. Time reversal symmetry breaking is always a necessary condition for nonreciprocity, since the field ratios cannot differ if the field levels do not.

3. Nonreciprocity is equivalent to time-reversal symmetry breaking in lossless systems or, by commonsense extension, in systems with negligible loss.

4. So, systems with significant loss or gain are time-reversal asymmetric even when they are reciprocal; in such systems, reciprocity should be expressed in the restricted sense of equal field ratios (rather than field levels), which is consistent with the definition in the previous point.

5. Time reversal/time-reversal symmetry breaking is a fundamental and powerful common descriptor for all nonreciprocal systems.

6. In the case of LTI media, time reversal leads to a generalized Lorentz reciprocity theorem, itself leading the powerful Onsager-Casimir relations, according to which nonreciprocity follows from the fact that waves propagating in opposite directions see different effective media when the time-reversal odd bias is fixed.

7. However, Onsager-Casimir relations are completely general; they also apply to LTV and nonlinear nonreciprocal systems, most generally in terms of field ratios or generalized transfer functions.

8. Nonreciprocal systems may be classified into linear (LTI and LTV) and nonlinear systems on the basis of time-reversal symmetry breaking by external biasing and self-biasing plus spatial asymmetry, respectively; in the LTV case, time modulation must be accompanied by spatial symmetry breaking, as nonlinearity.

9. Nonlinear nonreciprocity is a weaker form of nonreciprocity than linear nonreciprocity, as it suffers from restricted intensities, one-way-at-a-time excitations, and poor isolation or hysteresis.

10. S-parameters can be advantageously generalized to all types of nonreciprocal systems, including, in addition to LTI systems, LTV and nonlinear systems, with significant and major restrictions in the former and latter cases, respectively.

11. Care must be exercised to avoid confusing asymmetric transmission with nonreciprocity in some fallacious systems.

Nonreciprocity is a rich and fascinating concept that will continue to open new scientific and technological horizons in the forthcoming decades.

APPENDIX A: TIME-REVERSAL SYMMETRY OF THE MAXWELL EQUATIONS: EQS. (5A) AND (5B)

Time-reversed version of the Maxwell equations ($\nabla' = \nabla$):

$$\nabla \times \mathbf{E}' = -\partial \mathbf{B}'/\partial t', \tag{A1a}$$

$$\nabla \times \mathbf{H}' = \partial \mathbf{D}'/\partial t' + \mathbf{J}'. \tag{A1b}$$

Applying the time-reversal rules in Table II to Eqs. (A1a) and (A1b):

$$\nabla \times (\mathbf{E}) = -\partial(-\mathbf{B})/\partial(-t), \tag{A2a}$$

$$\nabla \times (-\mathbf{H}) = \partial(\mathbf{D})/\partial(-t) + (-\mathbf{J}). \tag{A2b}$$

Simplification of Eqs. (A2a) and (A2b):

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t, \tag{A3a}$$

$$\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t + \mathbf{J}. \tag{A3b}$$

These are identical to the original Maxwell equations, equivalent to the proof of Eqs. (5a) and (5b).

APPENDIX B: FREQUENCY-DOMAIN EXPRESSIONS OF THE TIME-REVERSED CONSTITUTIVE PARAMETERS: EQS. (8A) AND (8B)

Frequency-domain constitutive relations for a biased LTI bianisotropic medium with bias $\mathbf{F}_0$ [Eqs. (6a) and (6b) without primes and positive signs]:

$$\tilde{\mathbf{D}} = \bar{\bar{\tilde{\epsilon}}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} + \bar{\bar{\tilde{\xi}}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}}, \tag{B1a}$$

$$\tilde{\mathbf{B}} = \bar{\bar{\tilde{\zeta}}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} + \bar{\bar{\tilde{\mu}}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}}. \tag{B1b}$$

The same for the corresponding time-reversed medium [Eqs. (6a) and (6b) with primes and negative signs]:

$$\tilde{\mathbf{D}}' = \bar{\bar{\tilde{\epsilon}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}' + \bar{\bar{\tilde{\xi}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}', \tag{B2a}$$

$$\tilde{\mathbf{B}}^{(\prime)} = \bar{\bar{\tilde{\zeta}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}' + \bar{\bar{\tilde{\mu}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}', \tag{B2b}$$

where the sign of $\mathbf{F}_0$ is reversed because this quantity is assumed to be time reversal odd (Sec. VII).

Applying Eq. (7) with the time-reversal rules in Table II to Eqs. (B2a) and (B2b):

$$\tilde{\mathbf{D}}^* = \bar{\bar{\tilde{\epsilon}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* + \bar{\bar{\tilde{\xi}}}'(-\mathbf{F}_0) \cdot (-\tilde{\mathbf{H}}^*), \tag{B3a}$$

$$(-\tilde{\mathbf{B}}^*) = \bar{\bar{\tilde{\zeta}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* + \bar{\bar{\tilde{\mu}}}'(-\mathbf{F}_0) \cdot (-\tilde{\mathbf{H}}^*). \tag{B3b}$$

Simplifying Eqs. (B3a) and (B4b):

$$\tilde{\mathbf{D}}^* = \bar{\bar{\tilde{\epsilon}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* - \bar{\bar{\tilde{\xi}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^*, \tag{B4a}$$

$$\tilde{\mathbf{B}}^* = -\bar{\bar{\tilde{\zeta}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* + \bar{\bar{\tilde{\mu}}}'(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^*. \tag{B4b}$$

Complex conjugating Eqs. (B4a) and (B4b):

$$\tilde{\mathbf{D}} = \bar{\bar{\epsilon}}'^{*}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}} - \bar{\bar{\xi}}'^{*}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}, \tag{B5a}$$

$$\tilde{\mathbf{B}} = -\bar{\bar{\zeta}}'^{*}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}} + \bar{\bar{\mu}}'^{*}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}. \tag{B5b}$$

Comparing Eqs. (B5a) and (B5b) and Eqs. (B1a) and (B1b):

$$\bar{\bar{\epsilon}}(\mathbf{F}_0) = \bar{\bar{\epsilon}}'^{*}(-\mathbf{F}_0) \quad \bar{\bar{\xi}}(\mathbf{F}_0) = -\bar{\bar{\xi}}'^{*}(-\mathbf{F}_0), \tag{B6a}$$

$$\bar{\bar{\zeta}}(\mathbf{F}_0) = -\bar{\bar{\zeta}}'^{*}(-\mathbf{F}_0) \quad \bar{\bar{\mu}}(\mathbf{F}_0) = \bar{\bar{\mu}}'^{*}(-\mathbf{F}_0). \tag{B6b}$$

Complex conjugating Eqs. (B6a) and (B6b) and applying the dummy variable change $\mathbf{F} \to -\mathbf{F}_0$:

$$\bar{\bar{\epsilon}}'(\mathbf{F}_0) = \bar{\bar{\epsilon}}^{*}(-\mathbf{F}_0) \quad \bar{\bar{\xi}}'(\mathbf{F}_0) = -\bar{\bar{\xi}}^{*}(-\mathbf{F}_0), \tag{B7a}$$

$$\bar{\bar{\zeta}}'(\mathbf{F}_0) = -\bar{\bar{\zeta}}^{*}(-\mathbf{F}_0) \quad \bar{\bar{\mu}}'(\mathbf{F}_0) = \bar{\bar{\mu}}^{*}(-\mathbf{F}_0), \tag{B7b}$$

which are equivalent to Eqs. (8a) and (8b).

Restricted time-reversed version of Eqs. (B7a) and (B7b) (same without *):

$$\bar{\bar{\epsilon}}'(\mathbf{F}_0) = \bar{\bar{\epsilon}}(-\mathbf{F}_0) \quad \bar{\bar{\xi}}'(\mathbf{F}_0) = -\bar{\bar{\xi}}(-\mathbf{F}_0), \tag{B8a}$$

$$\bar{\bar{\zeta}}'(\mathbf{F}_0) = -\bar{\bar{\zeta}}(-\mathbf{F}_0) \quad \bar{\bar{\mu}}'(\mathbf{F}_0) = \bar{\bar{\mu}}(-\mathbf{F}_0). \tag{B8b}$$

APPENDIX C: FREQUENCY-DOMAIN EXPRESSIONS OF THE TIME-REVERSED MAXWELL EQUATIONS (SEC. VIII)

Fourier transforming Eqs. (A3a) and (A3b) (dependence $e^{j\omega t}$):

$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}, \tag{C1a}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega \tilde{\mathbf{D}} + \tilde{\mathbf{J}}, \tag{C1b}$$

Substituting $t' = -t$ in Eqs. (A1a) and (A1b):

$$\nabla \times \mathbf{E}' = \partial \mathbf{B}'/\partial t, \tag{C2a}$$

$$\nabla \times \mathbf{H}' = -\partial \mathbf{D}'/\partial t + \mathbf{J}'. \tag{C2b}$$

Fourier transforming Eqs. (C2a) and (C2b):

$$\nabla \times \tilde{\mathbf{E}}' = j\omega \tilde{\mathbf{B}}', \tag{C3a}$$

$$\nabla \times \tilde{\mathbf{H}}' = -j\omega \tilde{\mathbf{D}}' + \tilde{\mathbf{J}}'. \tag{C3b}$$

Applying Eq. (7) with the time-reversal rules in Table II to Eqs. (C3a) and Eqs. (C3b):

$$\nabla \times \tilde{\mathbf{E}}^{*} = -j\omega \tilde{\mathbf{B}}^{*}, \tag{C4a}$$

$$\nabla \times \tilde{\mathbf{H}}^{*} = j\omega \tilde{\mathbf{D}}^{*} + \tilde{\mathbf{J}}^{*}. \tag{C4b}$$

Comparing Eqs. (4a) and (4b) and Eqs. (1a) and (1b): The frequency-domain Maxwell equations of the time-reversed problem are identical to those of the original problem with all the field quantities conjugated.

APPENDIX D: FREQUENCY-DOMAIN EXPRESSIONS OF THE CONSTITUTIVE RELATIONS (SEC. VIII)

Frequency-domain constitutive relations [Eqs. (B1a) and (B1b)]:

$$\tilde{\mathbf{D}} = \bar{\bar{\epsilon}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} + \bar{\bar{\xi}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}}, \tag{D1a}$$

$$\tilde{\mathbf{B}} = \bar{\bar{\zeta}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} + \bar{\bar{\mu}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}}. \tag{D1b}$$

Substituting frequency-domain restricted time-reversed constitutive relations [Eqs. (B8a) and (8b)] into Eqs. (B4a) and (B4b):

$$\tilde{\mathbf{D}}^{*} = \bar{\bar{\epsilon}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^{*} + \bar{\bar{\xi}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^{*}, \tag{D2a}$$

$$\tilde{\mathbf{B}}^{*} = \bar{\bar{\zeta}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^{*} + \bar{\bar{\mu}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^{*}. \tag{D2b}$$

Comparing Eqs. (D2a) and (D2b) and Eqs. (D1a) and (D1b): The frequency-domain constitutive relations of the time-reversed problem are identical to those of the original problem with all the field (but not the constitutive parameter!) quantities conjugated and $\mathbf{F}_0$ changed to $-\mathbf{F}_0$.

APPENDIX E: DERIVATION OF EQ. (9)

Frequency-domain Maxwell equations for the set of original (excitation-response) fields [Eqs. (C1a) and (C1b)]:

$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}, \tag{E1a}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega \tilde{\mathbf{D}} + \tilde{\mathbf{J}}. \tag{E1b}$$

Same for time-reversed fields [Eqs. (C4a) and (C4b)]:

$$\nabla \times \tilde{\mathbf{E}}^* = -j\omega \tilde{\mathbf{B}}^*, \tag{E2a}$$

$$\nabla \times \tilde{\mathbf{H}}^* = j\omega \tilde{\mathbf{D}}^* + \tilde{\mathbf{J}}^*. \tag{E2b}$$

Subtracting Eq. (E1a) dot-multiplied by $\mathbf{H}^*$ from Eq. (E2b) dot-multiplied by $\mathbf{E}$, and doing the same with swapped unconjugate and conjugate terms:

$$\tilde{\mathbf{E}} \cdot \nabla \times \tilde{\mathbf{H}}^* - \tilde{\mathbf{H}}^* \cdot \nabla \times \tilde{\mathbf{E}} = j\omega \tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}^* + \tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}^*$$
$$+ j\omega \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}}, \tag{E3a}$$

$$\tilde{\mathbf{E}}^* \cdot \nabla \times \tilde{\mathbf{H}} - \tilde{\mathbf{H}} \cdot \nabla \times \tilde{\mathbf{E}}^* = j\omega \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} + \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{J}}$$
$$+ j\omega \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^*, \tag{E3b}$$

and applying the identity $\mathbf{A} \cdot \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \mathbf{A} = -\nabla \cdot (\mathbf{A} \times \mathbf{B})$:

$$-\nabla \cdot (\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*) = j\omega \tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}^* + \tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}^* + j\omega \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}}, \tag{E4a}$$

$$-\nabla \cdot (\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}) = j\omega \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} + \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{J}} + j\omega \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^*. \tag{E4b}$$

Subtracting Eq. (E4a) from Eq. (E4b):

$$\nabla \cdot (\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* - \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}})$$
$$= j\omega \left(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}^* + \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^* - \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}} \right)$$
$$+ \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{J}} - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}^* \tag{E5}$$

or

$$\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{J}} - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}^*$$
$$= \nabla \cdot (\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* - \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}})$$
$$- j\omega \left(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}^* + \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^* - \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}} \right). \tag{E6}$$

Integrating over the volume V formed by the surface S and applying the Gauss theorem:

$$\iiint_{V \to V_J} \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}}^* dv - \iiint_{V \to V_J} \tilde{\mathbf{J}}^* \cdot \tilde{\mathbf{E}} dv$$
$$= \oiint_S \left(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* - \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}} \right) \cdot \hat{\mathbf{n}} ds$$
$$- j\omega \iiint_V \left(\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}^* + \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^* - \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}} \right) dv, \tag{E7}$$

which is equivalent to Eq. (9)

APPENDIX F: VANISHING OF THE SURFACE INTEGRAL IN EQ. (9)

The surface integral in Eq. (9) reads

$$I_S = \oiint_S \left(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* - \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}} \right) \cdot \hat{\mathbf{n}} ds. \tag{F1}$$

From the identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$:

$$(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*) \cdot \hat{\mathbf{n}} = \tilde{\mathbf{E}} \cdot (\tilde{\mathbf{H}}^* \times \hat{\mathbf{n}}) = \tilde{\mathbf{H}}^* \cdot (\hat{\mathbf{n}} \times \tilde{\mathbf{E}}), \tag{F2a}$$

$$(\tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}) \cdot \hat{\mathbf{n}} = \tilde{\mathbf{E}}^* \cdot (\tilde{\mathbf{H}} \times \hat{\mathbf{n}}) = \tilde{\mathbf{H}} \cdot (\hat{\mathbf{n}} \times \tilde{\mathbf{E}}^*). \tag{F2b}$$

Impenetrable [perfect electric conductor (PEC) or perfect magnetic conductor (PMC) or combination of the two] cavity:

$$\left[\tilde{\mathbf{H}} \times \hat{\mathbf{n}} \right]_{S=\mathrm{PMC}} = \left[\tilde{\mathbf{H}}^* \times \hat{\mathbf{n}} \right]_{S=\mathrm{PMC}} = 0 \tag{F3a}$$

or

$$\left[\hat{\mathbf{n}} \times \tilde{\mathbf{E}} \right]_{S=\mathrm{PEC}} = \left[\hat{\mathbf{n}} \times \tilde{\mathbf{E}}^* \right]_{S=\mathrm{PEC}} = 0. \tag{F3b}$$

Inserting Eqs. (F3a) and (F3b) into Eqs. (F2a) and (F2b) and substituting the result into Eq. (F1):

$$I_S = 0. \tag{F4}$$

In an unbounded medium, at an infinite distance from the source(s), the field is a plane wave:

$$\left[\hat{\mathbf{n}} \times \tilde{\mathbf{E}} = \eta \mathbf{H} \right]_{S=\infty} \quad \text{and} \quad \left[\hat{\mathbf{n}} \times \tilde{\mathbf{E}}^* = \eta \mathbf{H}^* \right]_{S=\infty}, \tag{F5}$$

where the restricted time reversal assumed in Sec. IX is used in the latter equation by not changing η into η^*

Inserting Eq. (F5) into Eqs. (F2a) and (F2b) and substituting the result into the integrand of Eq. (F1):

$$
\begin{aligned}
&\left[(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* - \tilde{\mathbf{E}}^* \times \tilde{\mathbf{H}}) \cdot \hat{\mathbf{n}}\right]_{S=\infty} \\
&= \left[\tilde{\mathbf{H}}^* \cdot (\hat{\mathbf{n}} \times \tilde{\mathbf{E}}) - \tilde{\mathbf{H}} \cdot (\hat{\mathbf{n}} \times \tilde{\mathbf{E}}^*)\right]_{S=\infty} \\
&\overset{\text{by Eq. (F5)}}{=\!=} \left[\tilde{\mathbf{H}}^* \cdot \eta \tilde{\mathbf{H}} - \tilde{\mathbf{H}} \cdot \eta \tilde{\mathbf{H}}^*\right]_{S=\infty} \\
&= |\mathbf{H}|^2(\eta - \eta) = 0.
\end{aligned}
\tag{F6}
$$

Inserting Eq. (F6) inserted into Eq. (F1):

$$
I_S = 0. \tag{F7}
$$

APPENDIX G: DERIVATION OF THE GENERALIZED LORENTZ THEOREM: EQ. (10)

Using the result of the vanishing of the surface integral [Eqs. (F4) and (F7)], and the fact that the left-hand side (reaction difference) in Eq. (9) vanishes under reciprocity, Eq. (9) reduces to

$$
\iiint_V X \, dv = 0, \tag{G1}
$$

with

$$
\mathbf{X} = \tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{D}} - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{D}}^* + \tilde{\mathbf{H}} \cdot \tilde{\mathbf{B}}^* - \tilde{\mathbf{H}}^* \cdot \tilde{\mathbf{B}}. \tag{G2}
$$

Substituting Eqs. (D1a) and (D1b) and Eqs. (D2a) and (D2b) into Eq. (G2):

$$
\begin{aligned}
\mathbf{X} &= \tilde{\mathbf{E}}^* \cdot \left[\bar{\bar{\epsilon}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} + \bar{\bar{\xi}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}}\right] \\
&\quad - \tilde{\mathbf{E}} \cdot \left[\bar{\bar{\epsilon}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* + \bar{\bar{\xi}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^*\right] \\
&\quad + \tilde{\mathbf{H}} \cdot \left[\bar{\bar{\zeta}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* + \bar{\bar{\mu}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^*\right] \\
&\quad - \tilde{\mathbf{H}}^* \cdot \left[\bar{\bar{\zeta}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} + \bar{\bar{\mu}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}}\right] \\
&= \tilde{\mathbf{E}}^* \cdot \bar{\bar{\epsilon}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{E}} \cdot \bar{\bar{\epsilon}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* \\
&\quad - \tilde{\mathbf{H}}^* \cdot \bar{\bar{\mu}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}} + \tilde{\mathbf{H}} \cdot \bar{\bar{\mu}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^* \\
&\quad + \tilde{\mathbf{E}}^* \cdot \bar{\bar{\xi}}(\mathbf{F}_0) \cdot \tilde{\mathbf{H}} + \tilde{\mathbf{H}} \cdot \bar{\bar{\zeta}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{E}}^* \\
&\quad - \tilde{\mathbf{H}}^* \cdot \bar{\bar{\zeta}}(\mathbf{F}_0) \cdot \tilde{\mathbf{E}} - \tilde{\mathbf{E}} \cdot \bar{\bar{\xi}}(-\mathbf{F}_0) \cdot \tilde{\mathbf{H}}^*.
\end{aligned}
\tag{G3}
$$

Applying the tensor identity $\mathbf{a} \cdot \bar{\bar{\chi}} \cdot \mathbf{b} = (\mathbf{a} \cdot \bar{\bar{\chi}} \cdot \mathbf{b})^T = \mathbf{b} \cdot \bar{\bar{\chi}}^T \cdot \mathbf{a}$ (scalar quantity) to the terms at the right in the last

equality of Eq. (G3):

$$
\begin{aligned}
\mathbf{X} &= \tilde{\mathbf{E}}^* \cdot \left[\bar{\bar{\epsilon}}(\mathbf{F}_0) - \bar{\bar{\epsilon}}^T(-\mathbf{F}_0)\right] \cdot \tilde{\mathbf{E}} \\
&\quad - \tilde{\mathbf{H}}^* \cdot \left[\bar{\bar{\mu}}(\mathbf{F}_0) - \bar{\bar{\mu}}^T(-\mathbf{F}_0)\right] \cdot \tilde{\mathbf{H}} \\
&\quad + \tilde{\mathbf{E}}^* \cdot \left[\bar{\bar{\xi}}(\mathbf{F}_0) + \bar{\bar{\zeta}}^T(-\mathbf{F}_0)\right] \cdot \tilde{\mathbf{H}} \\
&\quad - \tilde{\mathbf{H}}^* \cdot \left[\bar{\bar{\zeta}}(\mathbf{F}_0) + \bar{\bar{\xi}}^T(-\mathbf{F}_0)\right] \cdot \tilde{\mathbf{E}}.
\end{aligned}
\tag{G4}
$$

Inserting Eq. (G4) into Eq. (G1) and considering that the resulting relation must hold for any fields:

$$
\bar{\bar{\epsilon}}(\mathbf{F}_0) = \bar{\bar{\epsilon}}^T(-\mathbf{F}_0), \tag{G5a}
$$

$$
\bar{\bar{\mu}}(\mathbf{F}_0) = \bar{\bar{\mu}}^T(-\mathbf{F}_0), \tag{G5b}
$$

$$
\bar{\bar{\xi}}(\mathbf{F}_0) = -\bar{\bar{\zeta}}^T(-\mathbf{F}_0), \tag{G5c}
$$

$$
\bar{\bar{\zeta}}(\mathbf{F}_0) = -\bar{\bar{\xi}}^T(-\mathbf{F}_0), \tag{G5d}
$$

which are equivalent to Eq. (10).

APPENDIX H. INTERPRETATION OF THE TRANSVERSE CONSTITUTIVE TENSORS

The transverse tensors in Eqs. (G5a)–(G5d) are clearly seen in Appendix G, to stem from the fields $\tilde{\mathbf{D}}^*$ and $-\tilde{\mathbf{B}}^*$, which, according to Eq. (7), are related to their time-reversed counterparts by $\tilde{\mathbf{D}}^* = \mathscr{T}\{\tilde{\mathbf{D}}\}$ and $-\tilde{\mathbf{B}}^* = \mathscr{T}\{\tilde{\mathbf{B}}\}$. Therefore, the transverse dyadics may be interpreted as corresponding to the medium seen in the time-reversed problem.

APPENDIX I. DERIVATION OF THE LINEAR TIME-INVARIANT MEDIUM NONRECIPROCITY/RECIPROCITY CONDITION: EQ. (16)

Eq. (E5) with unconjugated quantities primed and conjugated quantities double-primed:

$$
\begin{aligned}
&\nabla \cdot (\tilde{\mathbf{E}}' \times \tilde{\mathbf{H}}'' - \tilde{\mathbf{E}}'' \times \tilde{\mathbf{H}}') \\
&= j\omega \left(\tilde{\mathbf{E}}'' \cdot \tilde{\mathbf{D}}' - \tilde{\mathbf{E}}' \cdot \tilde{\mathbf{D}}'' + \tilde{\mathbf{H}}' \cdot \tilde{\mathbf{B}}'' - \tilde{\mathbf{H}}'' \cdot \tilde{\mathbf{B}}'\right) \\
&\quad + \tilde{\mathbf{E}}'' \cdot \tilde{\mathbf{J}}' - \tilde{\mathbf{E}}' \cdot \tilde{\mathbf{J}}''.
\end{aligned}
\tag{I1}
$$

Considering that the domain of interest (i.e., the integration domain) will not include the sources (Sec. XIV), and hence

dropping the source terms in Eq. (I1):

$$\nabla \cdot (\tilde{\mathbf{E}}' \times \tilde{\mathbf{H}}'' - \tilde{\mathbf{E}}'' \times \tilde{\mathbf{H}}')$$
$$= j\omega \left(\tilde{\mathbf{E}}'' \cdot \tilde{\mathbf{D}}' - \tilde{\mathbf{E}}' \cdot \tilde{\mathbf{D}}'' + \tilde{\mathbf{H}}' \cdot \tilde{\mathbf{B}}'' - \tilde{\mathbf{H}}'' \cdot \tilde{\mathbf{B}}'. \right) \quad (I2)$$

Applying in Eq. (I2) substitutions similar to those in Appendix H:

$$\nabla \cdot \left(\tilde{\mathbf{E}}' \times \tilde{\mathbf{H}}'' - \tilde{\mathbf{E}}'' \times \tilde{\mathbf{H}}' \right)$$
$$= j\omega \left[\tilde{\mathbf{E}}'' \cdot \left(\bar{\bar{\epsilon}} - \bar{\bar{\epsilon}}^T \right) \cdot \tilde{\mathbf{E}}' - \tilde{\mathbf{H}}'' \cdot \left(\bar{\bar{\mu}} - \bar{\bar{\mu}}^T \right) \cdot \tilde{\mathbf{H}}' \right.$$
$$\left. + \tilde{\mathbf{E}}'' \cdot \left(\bar{\bar{\xi}} + \bar{\bar{\zeta}}^T \right) \cdot \tilde{\mathbf{H}}' - \tilde{\mathbf{H}}'' \cdot \left(\bar{\bar{\zeta}} + \bar{\bar{\xi}}^T \right) \cdot \tilde{\mathbf{E}}' \right] = 0,$$
$$(I3)$$

which is equivalent to Eq. (16)

APPENDIX J: DERIVATION OF THE S-MATRIX NONRECIPROCITY/RECIPROCITY CONDITION: EQ. (18)

In Eq. (I3), elimination of the right-had side from general Onsager-Casimir relations [Eqs. (11a)–(11c)], integration over volume V of the surface S defining the system (Fig. 5), and application of Gauss theorem on the left-hand side:

$$\oiint_S \left(\tilde{\mathbf{E}}' \times \tilde{\mathbf{H}}'' - \tilde{\mathbf{E}}'' \times \mathbf{H}' \right) \cdot \hat{\mathbf{n}} ds = 0. \quad (J1)$$

Total field as the sum of fields at all the ports [Eq. (14)], with terminal local reference planes placed at $z = 0$ on the surface of the network:

$$\tilde{\mathbf{E}}_t(x,y,z) = \sum_p \left(a_p + b_p \right) \tilde{\mathbf{e}}_{t,p}(x,y), \quad (J2a)$$

$$\tilde{\mathbf{H}}_t(x,y,z) = \sum_p \left(a_p - b_p \right) \tilde{\mathbf{h}}_{t,p}(x,y). \quad (J2b)$$

Inserting single-primed and double-primed instances of Eqs. (J2a) and (J2b) into Eq. (J1) yields

$$\oiint_S \left(\tilde{\mathbf{E}}' \times \tilde{\mathbf{H}}'' - \tilde{\mathbf{E}}'' \times \mathbf{H}' \right) \cdot \hat{\mathbf{n}} ds = 0 = I_1 - I_2, \quad (J3)$$

$$I_1 = \oiint_S \left[\sum_p \left(a_p' + b_p' \right) \tilde{\mathbf{e}}_{t,p}'(x,y) \right.$$
$$\left. \times \sum_q \left(a_q'' - b_q'' \right) \tilde{\mathbf{h}}_{t,q}''(x,y) \right] \cdot \hat{\mathbf{n}} ds,$$
$$= \sum_p \sum_q \left(a_p' a_q'' - a_p' b_q'' + b_p' a_q'' - b_p' b_q'' \right)$$
$$\times \underbrace{\oiint_S \left[\tilde{\mathbf{e}}_{t,p}'(x,y) \times \tilde{\mathbf{h}}_{t,q}''(x,y) \right] \cdot \hat{\mathbf{n}} ds}_{=2\delta_{pq} \text{ by orthogonality (Sec. XIV)}}$$
$$= 2 \sum_p \left(a_p' a_p'' - a_p' b_p'' + b_p' a_p'' - b_p' b_p'' \right), \quad (J4a)$$

$$I_2 = \oiint_S \left[\sum_p \left(a_p'' + b_p'' \right) \tilde{\mathbf{e}}_{t,p}''(x,y) \right.$$
$$\left. \times \sum_q \left(a_q' - b_q' \right) \tilde{\mathbf{h}}_{t,q}'(x,y) \right] \cdot \hat{\mathbf{n}} ds$$
$$= \sum_p \sum_q \left(a_p'' a_q' - a_p'' b_q' + b_p'' a_q' - b_p'' b_q' \right)$$
$$\times \underbrace{\oiint_S \left[\tilde{\mathbf{e}}_{t,p}''(x,y) \times \tilde{\mathbf{h}}_{t,q}'(x,y) \right] \cdot \hat{\mathbf{n}} ds}_{=2\delta_{pq} \text{ by orthogonality (Sec. XIV)}}$$
$$= 2 \sum_p \left(a_p'' a_p' - a_p'' b_p' + b_p'' a_p' - b_p'' b_p' \right), \quad (J4b)$$

so that

$$I_1 - I_2$$
$$= 2 \sum_p \left(a_p' a_p'' - a_p' b_p'' + b_p' a_p'' - b_p' b_p'' \right)$$
$$- \left(a_p'' a_p' - a_p'' b_p' + b_p'' a_p' - b_p'' b_p' \right)$$
$$= 4 \sum_p \left(b_p' a_p'' - a_p' b_p'' \right). \quad (J4c)$$

Inserting Eq. (J4C) into Eq. (J3):

$$\sum_p \left(b_p' a_p'' - a_p' b_p'' \right) = 0, \quad (J5)$$

which may be alternatively written in the matrix form

$$\sum_p \left(b_p' a_p'' - a_p' b_p'' \right)$$

$$= [b'_1, b'_2, \ldots][a''_1, a''_2, \ldots]^T - [a'_1, a'_2, \ldots][b''_1, b''_2, \ldots]$$

$$= \mathbf{b}'\mathbf{a}''^T - \mathbf{a}'\mathbf{b}''^T = 0. \tag{J6}$$

Recalling the definition of the S-matrix [Eq. (15)]:

$$\mathbf{b} = \mathbf{Sa}, \tag{J7a}$$

for all $\quad \mathbf{b} = [b_1, b_2, \ldots]^T \quad$ and $\quad \mathbf{a} = [a_1, a_2, \ldots]^T.$
                                                                      (J7b)

Eliminating, using Eq. (J7), $\mathbf{b}'$ and $\mathbf{b}''$ in the last equality of (J6):

$$\begin{aligned}
\mathbf{b}'\mathbf{a}''^T - \mathbf{a}'\mathbf{b}''^T &= (\mathbf{Sa}')\mathbf{a}''^T - \mathbf{a}'(\mathbf{Sa}'')^T \\
&= \mathbf{Sa}'\mathbf{a}''^T - \mathbf{a}'\mathbf{a}''^T\mathbf{S}^T \\
&= \mathbf{a}'\mathbf{a}''^T\mathbf{S} - \mathbf{a}'\mathbf{a}''^T\mathbf{S}^T \\
&= \mathbf{a}'\mathbf{a}''^T(\mathbf{S} - \mathbf{S}^T). \\
&= 0
\end{aligned} \tag{J8}$$

Since the last equality must hold true for any source sets $\mathbf{a}'$ and $\mathbf{a}''$, one must have

$$\mathbf{S} - \mathbf{S}^T = 0 \quad \text{or} \quad \mathbf{S} = \mathbf{S}^T, \tag{J9}$$

which is equivalent to Eq. (18)

APPENDIX K: INEXISTENCE OF A MEDIUM-BASED GENERALIZED RECIPROCITY RELATION [EQ. (16)] FOR AN LINEAR TIME-VARIANT (AND ALSO NONLINEAR) MULTIPORT NETWORK

Maxwell equations for set of prime [175] (excitation-response) fields:

$$\nabla \times \mathbf{E}' = -\partial \mathbf{B}'/\partial t, \tag{K1a}$$

$$\nabla \times \mathbf{H}' = \partial \mathbf{D}'/\partial t + \mathbf{J}'. \tag{K1b}$$

Maxwell equations for set of double-prime (excitation-response) fields:

$$\nabla \times \mathbf{E}'' = -\partial \mathbf{B}''/\partial t. \tag{K2a}$$

$$\nabla \times \mathbf{H}'' = \partial \mathbf{D}''/\partial t + \mathbf{J}''. \tag{K2b}$$

Subtracting Eq. (K1a) dot-multiplied by $\mathbf{H}''$ from Eq. (K2b) dot-multiplied by $\mathbf{E}'$ and doing the same with swapped prime and double-prime quantities,

$$\begin{aligned}
\mathbf{E}' \cdot \nabla &\times \mathbf{H}'' - \mathbf{H}'' \cdot \nabla \times \mathbf{E}' \\
&= \mathbf{E}' \cdot \partial \mathbf{D}''/\partial t + \mathbf{E}' \cdot \mathbf{J}'' + \mathbf{H}'' \cdot \partial \mathbf{B}'/\partial t, \quad \text{(K3a)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}'' \cdot \nabla &\times \mathbf{H}' - \mathbf{H}' \cdot \nabla \times \mathbf{E}'' \\
&= \mathbf{E}'' \cdot \partial \mathbf{D}'/\partial t + \mathbf{E}'' \cdot \mathbf{J}' + \mathbf{H}' \cdot \partial \mathbf{B}''/\partial t, \quad \text{(K3b)}
\end{aligned}$$

and applying the identity $\mathbf{A} \cdot \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \mathbf{A} = -\nabla \cdot (\mathbf{A} \times \mathbf{B})$:

$$-\nabla \cdot (\mathbf{E}' \times \mathbf{H}'') = \mathbf{E}' \cdot \partial \mathbf{D}''/\partial t + \mathbf{E}' \cdot \mathbf{J}'' + \mathbf{H}'' \cdot \partial \mathbf{B}'/\partial t, \tag{K4a}$$

$$-\nabla \cdot (\mathbf{E}'' \times \mathbf{H}') = \mathbf{E}'' \cdot \partial \mathbf{D}'/\partial t + \mathbf{E}'' \cdot \mathbf{J}' + \mathbf{H}' \cdot \partial \mathbf{B}''/\partial t. \tag{K4b}$$

Subtracting Eq. (K4a) from Eq. (K4b):

$$\begin{aligned}
\nabla \cdot (\mathbf{E}' &\times \mathbf{H}'' - \mathbf{E}'' \times \mathbf{H}') \\
&= \mathbf{E}'' \cdot \partial \mathbf{D}'/\partial t - \mathbf{E}' \cdot \partial \mathbf{D}''/\partial t \\
&\quad + \mathbf{H}' \cdot \partial \mathbf{B}''/\partial t - \mathbf{H}'' \cdot \partial \mathbf{B}'/\partial t \\
&\quad + \mathbf{E}'' \cdot \mathbf{J}' - \mathbf{E}' \cdot \mathbf{J}''. \quad \text{(K5)}
\end{aligned}$$

Eliminating the terms involving currents in Eq. (K5), since the currents are assumed to be outside the network (Fig. 5):

$$\begin{aligned}
\nabla \cdot (\mathbf{E}' &\times \mathbf{H}'' - \mathbf{E}'' \times \mathbf{H}') \\
&= (\mathbf{E}'' \cdot \partial \mathbf{D}'/\partial t - \mathbf{E}' \cdot \partial \mathbf{D}''/\partial t \\
&\quad + \mathbf{H}' \cdot \partial \mathbf{B}''/\partial t - \mathbf{H}'' \cdot \partial \mathbf{B}'/\partial t). \quad \text{(K6)}
\end{aligned}$$

Inserting the time-domain version of the constitutive relations of Eq. (B1) to accommodate LTV [$\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}(t)$, etc.] and nonlinear [$\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}(\mathbf{E}, \mathbf{H})$, etc.] media,

$$\mathbf{D} = \bar{\bar{\epsilon}}(\mathbf{F}_0) * \mathbf{E} + \bar{\bar{\xi}}(\mathbf{F}_0) * \mathbf{H}, \tag{K7a}$$

$$\mathbf{B} = \bar{\bar{\zeta}}(\mathbf{F}_0) * \mathbf{E} + \bar{\bar{\mu}}(\mathbf{F}_0) * \mathbf{H}, \tag{K7b}$$

into Eq. (K6):

$$\begin{aligned}
\nabla \cdot (\mathbf{E}' &\times \mathbf{H}'' - \mathbf{E}'' \times \mathbf{H}') \\
&= \mathbf{E}'' \cdot \partial \left[\bar{\bar{\epsilon}}(\mathbf{F}_0) * \mathbf{E}' + \bar{\bar{\xi}}(\mathbf{F}_0) * \mathbf{H}' \right]/\partial t \\
&\quad - \mathbf{E}' \cdot \partial \left[\bar{\bar{\epsilon}}(\mathbf{F}_0) * \mathbf{E}'' + \bar{\bar{\xi}}(\mathbf{F}_0) * \mathbf{H}'' \right]/\partial t \\
&\quad + \mathbf{H}' \cdot \partial \left[\bar{\bar{\zeta}}(\mathbf{F}_0) * \mathbf{E}'' + \bar{\bar{\mu}}(\mathbf{F}_0) * \mathbf{H}'' \right]/\partial t \\
&\quad - \mathbf{H}'' \cdot \partial \left[\bar{\bar{\zeta}}(\mathbf{F}_0) * \mathbf{E}' + \bar{\bar{\mu}}(\mathbf{F}_0) * \mathbf{H}' \right]/\partial t. \quad \text{(K8)}
\end{aligned}$$

Further applying the rule $\partial(f * g)/\partial t = \partial f /\partial t * g = f * \partial g/\partial t$:

$$\nabla \cdot \left[\mathbf{E}' \times \mathbf{H}'' - \mathbf{E}'' \times \mathbf{H}' \right]$$
$$= \mathbf{E}'' \cdot \left(\bar{\bar{\epsilon}}(\mathbf{F}_0) * \partial\mathbf{E}'/\partial t + \bar{\bar{\xi}}(\mathbf{F}_0) * \partial\mathbf{H}'/\partial t \right)$$
$$- \mathbf{E}' \cdot \left[\bar{\bar{\epsilon}}(\mathbf{F}_0) * \partial\mathbf{E}''/\partial t + \bar{\bar{\xi}}(\mathbf{F}_0) * \partial\mathbf{H}''/\partial t \right]$$
$$+ \mathbf{H}' \cdot \left[\bar{\bar{\zeta}}(\mathbf{F}_0) * \partial\mathbf{E}''/\partial t + \bar{\bar{\mu}}(\mathbf{F}_0) * \partial\mathbf{H}''/\partial t \right]$$
$$- \mathbf{H}'' \cdot \left[\bar{\bar{\zeta}}(\mathbf{F}_0) * \partial\mathbf{E}'/\partial t + \bar{\bar{\mu}}(\mathbf{F}_0) * \partial\mathbf{H}'/\partial t \right] \quad \text{(K9)}$$

or

$$\nabla \cdot \left(\mathbf{E}' \times \mathbf{H}'' - \mathbf{E}'' \times \mathbf{H}' \right)$$
$$= \mathbf{E}'' \cdot \bar{\bar{\epsilon}}(\mathbf{F}_0) * \partial\mathbf{E}'/\partial t + \mathbf{E}'' \cdot \bar{\bar{\xi}}(\mathbf{F}_0) * \partial\mathbf{H}'/\partial t$$
$$- \mathbf{E}' \cdot \bar{\bar{\epsilon}}(\mathbf{F}_0) * \partial\mathbf{E}''/\partial t - \mathbf{E}' \cdot \bar{\bar{\xi}}(\mathbf{F}_0) * \partial\mathbf{H}''/\partial t$$
$$+ \mathbf{H}' \cdot \bar{\bar{\zeta}}(\mathbf{F}_0) * \partial\mathbf{E}''/\partial t + \mathbf{H}' \cdot \bar{\bar{\mu}}(\mathbf{F}_0) * \partial\mathbf{H}''/\partial t$$
$$- \mathbf{H}'' \cdot \bar{\bar{\zeta}}(\mathbf{F}_0) * \partial\mathbf{E}'/\partial t - \mathbf{H}'' \cdot \bar{\bar{\mu}}(\mathbf{F}_0) * \partial\mathbf{H}'/\partial t \quad \text{(K10)}$$

and grouping terms with the same constitutive parameters:

$$\nabla \cdot \left(\mathbf{E}' \times \mathbf{H}'' - \mathbf{E}' \times \mathbf{H}' \right)$$
$$= \mathbf{E}'' \cdot \bar{\bar{\epsilon}}(\mathbf{F}_0) * \partial\mathbf{E}'/\partial t - \mathbf{E}' \cdot \bar{\bar{\epsilon}}(\mathbf{F}_0) * \partial\mathbf{E}''/\partial t$$
$$+ \mathbf{H}' \cdot \bar{\bar{\mu}}(\mathbf{F}_0) * \partial\mathbf{H}''/\partial t - \mathbf{H}'' \cdot \bar{\bar{\mu}}(\mathbf{F}_0) * \partial\mathbf{H}'/\partial t$$
$$+ \mathbf{E}'' \cdot \bar{\bar{\xi}}(\mathbf{F}_0) * \partial\mathbf{H}'/\partial t - \mathbf{E}' \cdot \bar{\bar{\xi}}(\mathbf{F}_0) * \partial\mathbf{H}''/\partial t$$
$$+ \mathbf{H}' \cdot \bar{\bar{\zeta}}(\mathbf{F}_0) * \partial\mathbf{E}''/\partial t - \mathbf{H}'' \cdot \bar{\bar{\zeta}}(\mathbf{F}_0) * \partial\mathbf{E}'/\partial t. \quad \text{(K11)}$$

There is no way, even approximate, to reduce Eq. (K11) to an equation of the type of Eq. (I3) [or Eq. (16)]. So this relation is used erroneously in Ref. [19] in reference to nonlinear systems.

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