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WHAT IS THE SMALLEST POSSIBLE CONSTANT IN CÉA'S LEMMA?*

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Abstract. We consider finite element approximations of a second order elliptic problem on a bounded polytopic domain in $\mathbb{R}^d$ with $d \in \{1, 2, 3, \ldots\}$. The constant $C \geqslant 1$ appearing in Céa's lemma and coming from its standard proof can be very large when the coefficients of an elliptic operator attain considerably different values. We restrict ourselves to regular families of uniform partitions and linear simplicial elements. Using a lower bound of the interpolation error and the supercloseness between the finite element solution and the Lagrange interpolant of the exact solution, we show that the ratio between discretization and interpolation errors is equal to $1 + \mathcal{O}(h)$ as the discretization parameter h tends to zero. Numerical results in one and two-dimensional case illustrating this phenomenon are presented.

Keywords: supercloseness, Lagrange finite elements, Lagrange remainder, lower estimates, elliptic problems, d-simplex, uniform partitions

MSC 2000: 65N30

1. Introduction

The famous Céa's lemma plays an important role in finite element theory, because it enables us to transform the question of convergence of the finite element method (and a priori estimation of the discretization error) to the investigation of approximation properties of relevant finite element spaces.

More precisely, let V be a real Hilbert space with a norm $\|\cdot\|$, F a linear continuous form on V, and let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$. Assume that $a(\cdot,\cdot)$ is

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continuous, i.e., a constant M exists such that

$$|a(v,w)| \leqslant M\|v\|\,\|w\| \quad \forall\, v,w \in V, \tag{1.1}$$

and V-elliptic, i.e., a constant $m > 0$ exists such that

$$a(v,v) \geqslant m\|v\|^2 \quad \forall\, v \in V. \tag{1.2}$$

We see that $m \leqslant M$. Consider now a nonempty finite dimensional subspace $V_h \subset V$. Then, by the Lax-Milgram lemma, the problems: Find $u \in V$ such that

$$a(u,v) = F(v) \quad \forall\, v \in V, \tag{1.3}$$

and: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h, \tag{1.4}$$

have exactly one solution each. The function u_h is called the *Galerkin approximation.* Céa's lemma says (see [5] and for a historical note also [9, p. 109]) that there exists a constant C such that

$$\|u - u_h\| \leqslant C \inf_{v_h \in V_h} \|u - v_h\|. \tag{1.5}$$

The knowledge of the best possible value of C is thus important in obtaining reliable a priori bounds of the discretization error.

A standard proof of (1.5) follows directly from (1.1)–(1.4). Indeed, for every $v_h \in V_h$ we find that

$$m\|u-u_h\|^2 \leqslant a(u-u_h, u-u_h) = a(u-u_h, u-v_h) \leqslant M\|u-u_h\|\,\|u-v_h\|. \tag{1.6}$$

Therefore, $C = M/m$ is a constant that may stand in (1.5). However, the ratio M/m can be very large especially when $a(\cdot,\cdot)$ corresponds to an elliptic equation with coefficients which attain considerably different values. E.g., for highly oscillating coefficients coming from real-life technical problems we have $M/m \approx 100$ for a heat conduction problems described in [15, p. 209] and $M/m \approx 10\,000$ for a magnetic field problem involving ferromagnetic media, see [15, pp. 134–138]. In [9, p. 105], a simple procedure is presented for reducing the constant C in (1.5) to $\sqrt{M/m}$ when the bilinear form $a(\cdot,\cdot)$ is symmetric.

Using the theory of superconvergence, we show that the ratio between discretization and interpolation errors equals $1 + \mathcal{O}(h)$ as $h \to 0$ for a class of second order elliptic problems with variable coefficients. Therefore, we need not examine the

Galerkin approximation u_h, but only the approximation properties of the spaces V_h, to get a reliable bound of the discretization error. Note that the right-hand side of (1.5) can be estimated from above by $C\|u - u_h\|$. From this we find the lower bound $1 \leqslant C$. In [22, p. 197] Xu and Zikatanov derived a slight reduction of the constant in (1.5) for the Petrov-Galerkin method as it is presented in [1].

The outline of this paper is as follows. In Section 2 we prove that the lower bound $1 \leqslant C$ is attainable in a special case. Then we shall consider a bounded polytopic domain and describe regular families of uniform d-simplectic partitions which enable us to get a supercloseness property. Section 3 contains results on the gradient supercloseness between the finite element solution and the Lagrange interpolant of the exact solution. In Section 4 we prove the main theorem on the estimation of the discretization error. Section 5 is devoted to a lower bound of the interpolation error. Finally, in Section 6, numerical results are presented to illustrate the main theoretical result of Section 4.

Throughout the paper the standard Sobolev space notation is used (see [9]). The symbol C (possibly with subscripts) stands for a positive generic constant independent of the discretization parameter h, which nonetheless may depend on the solution u or on a given fixed function. The generic constant may attain different values at different occurrences.

2. Discretization

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3, \ldots\}$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Consider a class of second order elliptic problems described by the equations

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here $f \in L^2(\Omega)$ and $A = A(x) = (a_{ij}(x))_{i,j=1}^d$ is a matrix (in general nonsymmetric) whose entries are in $L^\infty(\Omega)$ and for which there exists $C > 0$ such that

$$\xi^T A(x)\xi \geqslant C\xi^T\xi \quad \forall\, \xi \in \mathbb{R}^d \quad \text{and a.a. } x \in \Omega. \tag{2.2}$$

This guarantees that the associated bilinear form $a(u, v) = (A\,\nabla u, \nabla v)$ is continuous and V-elliptic for

$$V = H_0^1(\Omega).$$

To get the supercloseness property (3.5) below, we suppose in addition that a_{ij} are Lipschitz continuous functions. This assumption is essential in superconvergence theory, see [4, p. 34] for details.

The upper bound (1.5) is usually enlarged by an interpolation error, i.e.,

$$\|u-u_h\|_1 \leqslant C \inf_{v_h\in V_h} \|u-v_h\|_1 \leqslant C\|u-I_h u\|_1, \tag{2.3}$$

where $I_h\colon V\to V_h\subset V$ is an interpolation operator, i.e., a linear continuous operator such that $I_h(v_h)=v_h$ for all $v_h\in V_h$. From Section 1 we already know that if the coefficients a_{ij} attain considerably different values, then $m\ll M$ in (1.6) and the standard proof of Céa's lemma leads to a very large constant in (1.5). On the other hand, for the Laplace operator we have

Proposition 2.1. *If A is the unit matrix, then*

$$|u-u_h|_1 = \inf_{v_h\in V_h} |u-v_h|_1$$

and

$$|u-u_h|_1 < |u-I_h u|_1 \tag{2.4}$$

whenever $u_h\neq I_h u$ on a set with a positive measure.

Proof. The associated bilinear form $a(v,w)=(\nabla v,\nabla w)$ is symmetric and therefore, by the Poincaré-Friedrichs inequality, it is a scalar product on V. From (1.3) and (1.4) we get the orthogonality relation

$$a(u-u_h,u_h-v_h)=0 \quad \forall\, v_h\in V_h.$$

Consequently, if $u_h\neq v_h$ on a set with a positive measure, then

$$\begin{aligned} |u-u_h|_1^2 &< |u-u_h|_1^2+|u_h-v_h|_1^2 \\ &= a(u-u_h,u-u_h)+2a(u-u_h,u_h-v_h)+a(u_h-v_h,u_h-v_h) \\ &= a(u-u_h,u-v_h)+a(u-v_h,u_h-v_h) \\ &= a(u-u_h+u_h-v_h,u-v_h)=|u-v_h|_1^2. \end{aligned}$$

Setting $v_h=I_h u$, we obtain (2.4). □

Note that by the Poincaré-Friedrichs inequality the seminorm $|\cdot|_1$ is also a norm in the space V. Thus for the Poisson problem and $\|\cdot\|=|\cdot|_1$, the constant in Céa's lemma (1.5) is equal by Proposition 2.1 to 1 (cf. Example 6.2) for any dimension d and any interpolation operator (linear, quadratic, cubic, etc.).

Next we will show that the constant C appearing on the right-hand side of (2.3) can be arbitrarily close to 1 as $h\to 0$ also for variable coefficients contained in the matrix

$A = A(x)$ (see (2.1)) when linear elements and uniform simplicial partitions are used. Such partitions produce gradient superconvergence of finite elements (see, e.g., [2], [3], [4], [21]).

To this end we moreover assume that $\Omega \subset \mathbb{R}^d$ is a polytopic domain, i.e., its boundary $\partial\Omega$ is contained in a finite number of $(d-1)$-dimensional hyperplanes. By a *polytope* we mean a closure of a bounded polytopic domain. We shall consider only face-to-face partitions T_h of the polytope $\overline{\Omega}$ into simplices (more precisely d-simplices). A family $\mathcal{F} = \{T_h\}_{h\to 0}$ of such partitions is said to be *regular* if there exists a constant $C > 0$ such that

$$\operatorname{meas} S \geqslant C h_S^d \tag{2.5}$$

for any $T_h \in \mathcal{F}$ and any simplex $S \in T_h$, where $h_S = \operatorname{diam} S$.

Define a reference partition $\hat{T}$ of $\mathbb{R}^d$ using the well-known Kuhn simplicial partition (see [16]) of d-dimensional unit cubes that tile the space $\mathbb{R}^d$. Each cube is partitioned into $d!$ simplices of the form

$$S_\pi = \{x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d\colon\ 0 \leqslant x_{\pi(1)} \leqslant \ldots \leqslant x_{\pi(d)} \leqslant 1\},$$

where π denotes a permutation of the numbers $1, \ldots, d$.

Let d linearly independent unit vectors $\chi_1, \ldots, \chi_d$ independent of h be given. We shall introduce a family of $d \times d$ matrices B_h whose normalized columns form the directions $\chi_1, \ldots, \chi_d$. Consider only such polytopes $\overline{\Omega}$ and such linear affine mappings $F_h\colon\ \mathbb{R}^d \to \mathbb{R}^d$,

$$F_h(\hat{x}) = B_h\hat{x} + c_h, \quad \hat{x}, c_h \in \mathbb{R}^d,$$

that for any simplex $\hat{S} \in \hat{T}$ we have either $F_h(\hat{S}) \subset \overline{\Omega}$ or $F_h(\hat{S}) \subset \mathbb{R}^d \setminus \Omega$, i.e., each simplex $S = F_h(\hat{S})$ is either inside $\overline{\Omega}$ or outside Ω.

The set

$$T_h = \{S \subset \mathbb{R}^d\colon\ S = F_h(\hat{S}) \subset \overline{\Omega},\ \ \hat{S} \in \hat{T}\}$$

will be called a *uniform simplectic partition* of $\overline{\Omega}$. The discretization parameter h is the largest norm of all columns of B_h. Let us point out that all elements in T_h have the same volume and thus (2.5) may be replaced by

$$\operatorname{meas} S \geqslant C h^d \quad \forall\, S \in T_h. \tag{2.6}$$

Define the finite element space V_h over the uniform simplectic partition T_h as

$$V_h = \{v \in V\colon\ v|_S \in P_1(S)\ \ \forall\, S \in T_h\}, \tag{2.7}$$

where $P_1(S)$ is the space of linear polynomials. Then the weak formulation and the corresponding finite element approximation for model problem (2.1) are described by (1.3) and (1.4), respectively.

3. Supercloseness

In 1969 Oganesjan and Ruhovec (see [19]) examined the convergence of the finite element method for solving a second order elliptic problem with Dirichlet boundary conditions. They considered linear triangular elements over uniform partitions (see Fig. 1). They did not apply Céa's lemma, but used the triangle inequality

$$\|u - u_h\|_1 \leqslant \|u - L_h u\|_1 + \|u_h - L_h u\|_1,$$

where L_h is the standard Lagrange linear interpolation operator. When estimating the last term they discovered a remarkable phenomenon of approximation theory, namely that

$$\|u_h - L_h u\|_1 \leqslant Ch^2 \|u\|_3 \quad \text{for } h \in (0, h_0) \tag{3.1}$$

on a rectangular domain, where $h_0 > 0$ is a constant. Later this phenomenon was called *supercloseness* (see [21]), since

$$\|u - u_h\|_1 \leqslant Ch|u|_2 \tag{3.2}$$

and

$$\|u - L_h u\|_1 \leqslant Ch|u|_2 \tag{3.3}$$

are the optimal error estimates, i.e., their approximation order cannot be improved (see Theorem 5.1 and Remark 5.4). A suitable postprocessing operator based on the supercloseness phenomenon (3.1) can be developed to improve the approximation order of the gradients of finite element solutions from $\mathcal{O}(h)$ to $\mathcal{O}(h^2)$. From then on, the supercloseness property (3.1) between the finite element solution and the interpolant of the exact solution has played an important and essential role in getting many superconvergence results (see [3], [4], [7], [11], [13], [18], [21], and references therein).

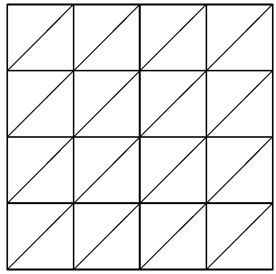

Figure 1. Uniform partition of a square.

In one space dimension, the supercloseness of the finite element solution was considered by Tong in 1969 (see [20]). For linear tetrahedral elements, the first supercloseness phenomenon was discovered by C. M. Chen (see [7]) in 1980. It was also independently rediscovered by Kantchev and Lazarov (see [12]) in 1986, and later generalized by L. Chen [8] to quasi-uniform partitions. In 1987 Hlaváček and Křížek [11] extended (3.1) to a general system of elliptic equations with variable coefficients (which includes, e.g., linear elasticity equations, see also [10]).

In 1981 Zhu [23] proved superconvergence of the gradient of quadratic triangular elements at the two Gaussian points of each edge. In a recent paper [4] his result was extended to tetrahedral elements for problem (2.1) with Lipschitz continuous coefficients. In particular, the supercloseness result for derivatives of the quadratic elements on tetrahedral partitions holds as well:

$$|u_h - Q_h u|_1 \leqslant C(u)h^3, \tag{3.4}$$

where Q_h is the standard quadratic Lagrange interpolation operator. Lin and Yan in [18, p. 251] approximated the Poisson problem by rectangular block elements (possibly smoothly deformed). Using special integral identities, they proved supercloseness which gives superconvergence by means of an appropriate postprocessing on quasi-uniform partitions.

In what follows, we will employ finite element spaces (2.7) to approximate problem (2.1) with Lipschitz continuous coefficients satisfying (2.2). For this case Brandts and Křížek in [3, p. 498] proved the following result:

Theorem 3.1. *Let $\mathcal{F} = \{T_h\}$ be a regular family of uniform simplectic partitions of a polytope $\overline{\Omega} \subset \mathbb{R}^d$, $d < 6$, and let $u \in H^3(\Omega)$. Then there exist positive constants C and h_0 such that for any $h \in (0, h_0)$ we have*

$$|a(u - L_h u, v_h)| \leqslant Ch^2 \|u\|_3 |\nabla v_h|_0 \quad \forall\, v_h \in V_h,$$

where $L_h u \in V_h$ is the standard Lagrange linear interpolant.

R e m a r k 3.2. By the Sobolev imbedding theorem, $u \in H^3(\Omega)$ is continuous for $d < 6$ and thus, the interpolant $L_h u$ is well defined. To extend Theorem 3.1 also to $d \geqslant 6$ we have to assume a higher regularity of u in order that $L_h u$ be well defined (see [3, p. 502]). We could also consider the Clément linear interpolation operator [6], whose approximation properties are the same and which is defined on noncontinuous functions via local regularizations.

Theorem 3.3. *Under the assumptions of Theorem* 3.1 *we have supercloseness between the finite element solution and the interpolant of the exact solution, namely, there exist positive constants C and h_0 such that*

$$|u_h - L_h u|_1 \leqslant Ch^2 \|u\|_3 \quad \text{for } h \in (0, h_0). \tag{3.5}$$

P r o o f. According to Theorem 3.1, we get

$$a(u - L_h u, u_h - L_h u) \leqslant Ch^2 \|u\|_3 |u_h - L_h u|_1. \tag{3.6}$$

Using (1.2) and the relation $a(u - u_h, v_h) = 0$ for all $v_h \in V_h$, we obtain from (3.6) that

$$\begin{aligned} |u_h - L_h u|_1^2 &\leqslant \frac{1}{m} a(u_h - L_h u, u_h - L_h u) = \frac{1}{m} a(u - L_h u, u_h - L_h u) \\ &\leqslant \frac{C}{m} h^2 \|u\|_3 |u_h - L_h u|_1, \end{aligned}$$

which proves the theorem. □

4. The main theorem

Now we give sufficient conditions that enable us to reduce the constant C appearing on the right-hand side of (2.3) to $1 + \mathcal{O}(h)$ in the case of variable coefficients of problem (2.1). Compare (2.3) with (4.3) below.

Theorem 4.1. *Let the assumptions of Theorem* 3.1 *hold and let there exist positive constants C and h_0 such that*

$$|u - L_h u|_1 \geqslant Ch \quad \text{for } h \in (0, h_0). \tag{4.1}$$

Then

$$|u - u_h|_1 \leqslant (1 + C_1 h) |u - L_h u|_1 \tag{4.2}$$

and

$$\|u - u_h\|_1 \leqslant (1 + C_2 h) \|u - L_h u\|_1. \tag{4.3}$$

P r o o f. According to the Poincaré-Friedrichs inequality, Theorem 3.3 and (4.1), we get

$$C_3\|u_h - L_h u\|_1 \leqslant |u_h - L_h u|_1 \leqslant C_4 h^2 \|u\|_3 \leqslant C_1 h |u - L_h u|_1,$$

where C_1 depends on u and $C_3 > 0$. From this and the triangle inequality, we find that

$$|u - u_h|_1 \leqslant |u - L_h u|_1 + |u_h - L_h u|_1 \leqslant (1 + C_1 h)|u - L_h u|_1$$

and

$$\|u - u_h\|_1 \leqslant \|u - L_h u\|_1 + \|u_h - L_h u\|_1 \leqslant (1 + C_2 h)\|u - L_h u\|_1.$$

□

R e m a r k 4.2. Assumption (4.1) says that the bound (3.3) and the standard upper bound $|u - L_h u|_1 \leqslant Ch$ are optimal, i.e., the approximation order of linear elements cannot be improved when u is not linear everywhere in Ω. Since we were not able to find any lower bound of the interpolation error in the literature, we will examine the validity of (4.1) in the next section.

R e m a r k 4.3. Theorem 4.1 can be generalized also to higher order elements that have the supercloseness property (see, e.g., (3.4)) provided an analogue of (4.1) holds. However, we have to recall here a surprising result by Bo Li. In [17] he found that for simplicial P_k-type elements with $k > d > 1$ the standard Lagrange interpolant and the finite element solution are not superclose in the H^1-norm.

5. Lower bound of the interpolation error for linear elements

First we examine in detail assumption (4.1) in the one-dimensional case.

Theorem 5.1. *Let $d = 1$ and let $v \in C^2(\overline{\Omega}) \setminus P_1(\overline{\Omega})$. Then for any family of uniform partitions there exist positive constants C and h_0 such that*

$$|v - L_h v|_1 \geqslant Ch \quad \text{for } h \in (0, h_0), \tag{5.1}$$

where C depends on the second derivatives of v.

P r o o f. Let $z_0 < z_1 < \ldots < z_N$ be a uniform partition of the interval $\overline{\Omega}$ such that $z_{j+1} - z_j = h$, and let $z_{-1} = z_0 - h$ and $z_{N+1} = z_N + h$. Set

$$C_1 = \max_{x \in \overline{\Omega}} |v''(x)| > 0. \tag{5.2}$$

Without loss of generality we may suppose that $C_1 = \max_{\overline{\Omega}} v''$, since the case $C_1 = \max_{\overline{\Omega}}(-v'')$ can be treated similarly by setting $v := -v$. Due to the continuity of v'' there exists a fixed subinterval $\Omega_0 \subset \Omega$ of positive length such that

$$v''(x) \geqslant C_0 := \sqrt{\frac{7}{8}} C_1 \quad \forall x \in \overline{\Omega}_0. \tag{5.3}$$

Furthermore, for sufficiently small h there exist indices $j_0 = j_0(h)$ and $j_1 = j_1(h)$ such that $z_j \in \overline{\Omega}_0$ for all $j = j_0, \dots, j_1$, and z_{j_0-1} and z_{j_1+1} are outside $\overline{\Omega}_0$. Set

$$w_h = v - L_h v \tag{5.4}$$

and let $j \in \{j_0, \dots, j_1 - 1\}$ be given. For simplicity, let the symbol $w_h'(z_j)$ denote the derivative of w_h from the right at the point z_j. Using the Taylor expansion with the Lagrange form remainder, we get

$$0 = w_h(z_{j+1}) - w_h(z_j) = h w_h'(z_j) + \frac{h^2}{2} w_h''(\xi),$$

where $\xi \in (z_j, z_{j+1}) \subset \Omega_0$. Hence, from (5.4), we obtain $|w_h'(z_j)| = \frac{1}{2} h |v''(\xi)|$ and thus by (5.2),

$$(w_h'(z_j))^2 \leqslant \frac{C_1^2}{4} h^2. \tag{5.5}$$

Moreover, for $x \in (z_j, z_{j+1})$ we see from (5.4) and (5.3) that

$$w_h'(x) - w_h'(z_j) = \int_{z_j}^{x} w_h''(t)\,\mathrm{d}t = \int_{z_j}^{x} v''(t)\,\mathrm{d}t \geqslant C_0(x - z_j) > 0,$$

i.e.,

$$(w_h'(x) - w_h'(z_j))^2 \geqslant C_0^2 (x - z_j)^2.$$

We shall integrate this inequality over (z_j, z_{j+1}). Since $w_h(z_{j+1}) = w_h(z_j) = 0$, we find by (5.5), (5.3) and (5.2) that

$$\begin{aligned} \int_{z_j}^{z_{j+1}} & (w_h'(x))^2\,\mathrm{d}x \\ & \geqslant C_0^2 \int_{z_j}^{z_{j+1}} (x - z_j)^2\,\mathrm{d}x + 2w_h'(z_j) \int_{z_j}^{z_{j+1}} w_h'(x)\,\mathrm{d}x - h(w_h'(z_j))^2 \\ & = \frac{C_0^2}{3}(z_{j+1} - z_j)^3 - h(w_h'(z_j))^2 \geqslant \frac{C_0^2}{3} h^3 - \frac{C_1^2}{4} h^3 \\ & = \Big(\frac{7}{3 \cdot 8} - \frac{1}{4}\Big) C_1^2 h^3 = \frac{C_1^2}{24} h^3 \geqslant \frac{h^2}{24} \int_{z_j}^{z_{j+1}} (v''(x))^2\,\mathrm{d}x. \end{aligned} \tag{5.6}$$

Summing these inequalities over all elements which entirely belong to $\overline{\Omega}_0$, we get by (5.4) for sufficiently small h that

$$|v - L_h v|_1^2 = |w_h'|_0^2 \geqslant \int_{z_{j_0}}^{z_{j_1}} (w_h'(x))^2 \, \mathrm{d}x$$
$$\geqslant \frac{h^2}{24} \int_{z_{j_0}}^{z_{j_1}} (v''(x))^2 \, \mathrm{d}x \geqslant \frac{h^2}{24} |v|_{2,\Omega_1}^2 \geqslant \frac{h^2}{24} C_0^2 \operatorname{meas} \Omega_1,$$

where $\Omega_1 \neq \emptyset$ is an open fixed interval such that $\overline{\Omega}_1 \subset \Omega_0$. The reason for introducing Ω_1 is that the end-points of Ω_0 need not coincide with z_j, in general. $\square$

R e m a r k 5.2. The required H^3-regularity of u appearing in Theorems 3.1, 3.3, and 4.1 implies that $u \in C^2(\overline{\Omega})$ for $d = 1$ due to the Sobolev imbedding theorem. Therefore, Theorem 5.1 can be applied for $v = u$ to fulfil assumption (4.1).

R e m a r k 5.3. The proof of Theorem 5.1 can be easily extended to families of quasiuniform partitions whose elements S all satisfy $0 < C_1 h \leqslant \operatorname{meas} S \leqslant C_2 h$.

R e m a r k 5.4. We will briefly sketch how to obtain a lower bound of the interpolation error for the case $d \geqslant 2$, since a detailed proof would be too long. We employ the standard technique which uses linear affine transformations from the reference simplex $\hat{S}$ with vertices $\hat{c}_0 = (0, \ldots, 0)^\top$, $\hat{c}_1 = (1, 0, \ldots, 0)^\top$, …, $\hat{c}_d = (0, \ldots, 0, 1)^\top$ to an arbitrary simplex $S \in T_h$ with vertices $c_i = c_i^S \in \mathbb{R}^d$, $i = 0, 1, \ldots, d$, considered as column vectors. Define an affine one-to-one mapping $F_S \colon \hat{S} \to S$ by

$$F_S(\hat{x}) = B_S \hat{x} + c_0, \quad \hat{x} = (\hat{x}_1, \ldots, \hat{x}_d)^\top \in \hat{S},$$

where

$$B_S = (c_1 - c_0, c_2 - c_0, \ldots, c_d - c_0) \tag{5.7}$$

is a nonsingular $d \times d$-matrix, since $F_S(\hat{c}_i) = c_i$ for $i = 0, \ldots, d$. For every $v \in L^2(S)$ and $\hat{x} \in \hat{S}$ we set

$$\hat{v}(\hat{x}) = v(x), \tag{5.8}$$

where $x = F_S(\hat{x})$. Thus we have a one-to-one correspondence between $\hat{v}$ and v. By the substitution theorem, for any $n \in \{0, 1, 2, \ldots\}$ there exists a constant $C > 0$ such that for any simplex S, any $\hat{v} \in H^n(\hat{S})$ and the associated $v \in H^n(S)$ satisfying (5.8), we have (see [9, p. 118] or [14, p. 46])

$$|\hat{v}|_{n,\hat{S}} \leqslant C \|B_S\|^n |\det B_S|^{-1/2} |v|_{n,S}, \tag{5.9}$$
$$|v|_{n,S} \leqslant C \|B_S^{-1}\|^n |\det B_S|^{1/2} |\hat{v}|_{n,\hat{S}}, \tag{5.10}$$

where $\|\cdot\|$ stands for the spectral norm. For a regular family of partitions we may derive by (5.7) and (2.5) that

$$\|B_S\| \leqslant Ch_S \quad \text{and} \quad \|B_S^{-1}\| \leqslant Ch_S^{-1}. \tag{5.11}$$

Now let $v \in C^2(\overline{\Omega}) \setminus P_1(\overline{\Omega})$ and let

$$C_1 = \max_{\overline{\Omega}} \max_{i,j} |\partial_{ij} v(x)| > 0,$$

where $\partial_{ij} v(x) = \partial^2 v/(\partial x_i \partial x_j)$. We may suppose without loss of generality that (cf. (5.2))

$$C_1 = \max_{x \in \overline{\Omega}} \partial_{k\ell} v(x)$$

for some $k, \ell \in \{1, \dots, d\}$. Then there exist a positive constant $C_0 < C_1$ (sufficiently close to C_1 like in (5.3)) and a subdomain $\overline{\Omega}_0 \subset \overline{\Omega}$ with positive measure such that

$$\partial_{k\ell} v(x) \geqslant C_0 \quad \forall\, x \in \overline{\Omega}_0.$$

For any $S \in T_h$ define a linear interpolation function $L_S v \in P_1(S)$ by $(L_S v)(c_i) = v(c_i)$ for all $i = 0, 1, \dots, d$. Similarly we define $\hat{L}\hat{v} \in P_1(\hat{S})$ and on the reference simplex we set (cf. (5.4))

$$\hat{w} = \hat{v} - \hat{L}\hat{v}.$$

Then $\hat{w}(\hat{c}_i) = 0$ for all $i = 0, 1, \dots, d$, and for the Hessian matrices we have

$$\operatorname{hes} \hat{w} = \operatorname{hes} \hat{v}. \tag{5.12}$$

For simplicity we furthermore assume that B_S are diagonal matrices. Then the directions χ_i introduced in Section 2 are parallel to the coordinate axes and for the second derivatives we have (cf. (5.10))

$$|\partial_{k\ell} v|_{0,S} \leqslant C \|B_S^{-1}\|^2 |\det B_S|^{1/2} |\partial_{k\ell} \hat{v}|_{0,\hat{S}}. \tag{5.13}$$

The seminorm $|\hat{w}|_{1,\hat{S}}$ can be bounded from below by the L^2-norm of the second derivative $\partial_{k\ell} \hat{w}$ like in (5.6), cf. also [15, p. 68]. Using (5.9), the fact that $\widehat{L_S v} = \hat{L}\hat{v}$, (5.12), (5.13) and (5.11), we find that

$$\begin{aligned}
|v - L_S v|_{1,S} &\geqslant C_2 \|B_S\|^{-1} |\det B_S|^{1/2} |\hat{v} - \hat{L}\hat{v}|_{1,\hat{S}} \\
&\geqslant C_3 \|B_S\|^{-1} |\det B_S|^{1/2} |\partial_{k\ell} \hat{w}|_{0,\hat{S}} \\
&= C_4 \|B_S\|^{-1} |\det B_S|^{1/2} |\partial_{k\ell} \hat{v}|_{0,\hat{S}} \\
&\geqslant C_5 \|B_S\|^{-1} |\det B_S|^{1/2} \|B_S^{-1}\|^{-2} |\det B_S|^{-1/2} |\partial_{k\ell} v|_{0,S} \\
&\geqslant C_6 h_S |\partial_{k\ell} v|_{0,S}.
\end{aligned}$$

Squaring these inequalities and summing them over all elements $S \in T_h$ that entirely belong to some $\overline{\Omega}_1 \subset \Omega_0$ and using (2.6), we get

$$|v - L_h v|_1^2 \geqslant |v - L_h v|_{1,\Omega_1}^2 \geqslant Ch^2,$$

which guarantees (4.1) for $v = u \in C^2(\overline{\Omega}) \setminus P_1(\overline{\Omega})$.

6. Numerical tests

For numerical illustration of the theoretical result presented in Theorem 4.1 and the supercloseness property described in Theorem 3.3, we approximate problem (2.1) by linear finite elements on uniform partitions. The exact solution u and the entries of the matrix A were taken polynomial. Then the associated stiffness matrices and the right-hand sides can be evaluated exactly (up to rounding errors), since we use higher order numerical quadrature formulae. The corresponding systems of linear algebraic equations are solved by the Gaussian elimination (to avoid the iteration error). Tabs. 1–6 illustrate the results described in (4.2) and (4.3), as well as in (3.1), (3.2), (3.3) and (3.5).

Example 6.1. Consider first the Poisson equation $-((x^2+1)u')' = f$ in $\Omega = (0,1)$ with homogeneous Dirichlet boundary conditions. The exact solution is chosen as $u(x) = x(x-1)$. Then the right-hand side corresponding to u is given by $f(x) = -6x^2 + 2x - 2$. Numerical results are presented in Tabs. 1–2.

$1/h$	$\|u - u_h\|_1$	$\|u - L_h u\|_1$	$\|u_h - L_h u\|_1$	$\|u - u_h\|_1 / \|u - L_h u\|_1$
4	0.144404218	0.144337568	0.004386858	1.000461762
8	0.072177522	0.072168784	0.001123109	1.000121085
16	0.036085497	0.036084392	0.000282383	1.000030620
32	0.018042335	0.018042196	0.000070696	1.000007677
64	0.009021115	0.009021098	0.000017680	1.000001921
128	0.004510551	0.004510549	0.000004420	1.000000480

Table 1.

In the next two examples we use the triangulation from Fig. 1.

Example 6.2. Consider the Poisson equation $-\Delta u = f$ in $\Omega = (0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions. The exact solution is chosen as follows: $u(x_1, x_2) = (x_1 - x_1^2)(x_2 - x_2^2)$. Then the corresponding right-hand side is

$1/h$	$\|u-u_h\|_1$	$\|u-L_hu\|_1$	$\|u_h-L_hu\|_1$	$\|u-u_h\|_1/\|u-L_hu\|_1$
4	0.144773606	0.144787921	0.004588179	0.999901132
8	0.072223310	0.072225144	0.001176216	0.999974601
16	0.036091208	0.036091439	0.000295837	0.999993607
32	0.018043048	0.018043077	0.000074070	0.999998399
64	0.009021205	0.009021208	0.000018524	0.999999600
128	0.004510562	0.004510563	0.000004632	0.999999900

Table 2.

given by $f(x_1, x_2) = 2(x_1 - x_1^2 + x_2 - x_2^2)$. We observe that in this case estimate (2.4) holds. All integrals were calculated by the quadrature formula from [14, p. 58] which is exact for all quintic polynomials on triangles. For numerical results see Tabs. 3–4.

$1/h$	$\lvert u-u_h\rvert_1$	$\lvert u-L_hu\rvert_1$	$\lvert u_h-L_hu\rvert_1$	$\lvert u-u_h\rvert_1/\lvert u-L_hu\rvert_1$
4	0.058775737	0.059199680	0.007072123	0.992838753
8	0.030161134	0.030221195	0.001904372	0.998012617
16	0.015180770	0.015188520	0.000485122	0.999489788
32	0.007603031	0.007604008	0.000121857	0.999871588

Table 3.

$1/h$	$\|u-u_h\|_1$	$\|u-L_hu\|_1$	$\|u_h-L_hu\|_1$	$\|u-u_h\|_1/\|u-L_hu\|_1$
4	0.059027858	0.059353209	0.007222771	0.994518390
8	0.030195558	0.030241036	0.001948652	0.998496154
16	0.015185172	0.015191020	0.000496651	0.999614983
32	0.007603585	0.007604321	0.000124768	0.999903162

Table 4.

E x a m p l e 6.3. Let again $\Omega = (0,1) \times (0,1)$. For the nonsymmetric matrix

$$A(x_1, x_2) = \begin{pmatrix} 2 + x_1 & -1 + x_2 \\ -1 - x_1 & 2 \end{pmatrix}$$

that satisfies (2.2) the associated equation (2.1) is

$$-(2 + x_1)\partial_{11}u + (2 + x_1 - x_2)\partial_{12}u - 2\partial_{22}u - \partial_1 u = f.$$

The exact solution is chosen as in the above example: $u(x_1, x_2) = (x_1 - x_1^2)(x_2 - x_2^2)$. Then the corresponding right-hand side is given by $f(x_1, x_2) = 2 + x_1 - 2x_2 - 6x_1^2 +$

$12x_1x_2 - x_2^2 + 4x_1^2x_2 - 8x_1x_2^2$. In Tabs. 5 and 6 we see numerical results similar to those in Example 6.2.

$1/h$	$\lvert u - u_h\rvert_1$	$\lvert u - L_hu\rvert_1$	$\lvert u_h - L_hu\rvert_1$	$\lvert u - u_h\rvert_1/\lvert u - L_hu\rvert_1$
4	0.059043100	0.059199680	0.012630950	0.997355048
8	0.030211424	0.030221195	0.003612123	0.999676693
16	0.015187941	0.015188520	0.000939795	0.999961922
32	0.007603964	0.007604008	0.000237630	0.999994297

Table 5.

$1/h$	$\lVert u - u_h\rVert_1$	$\lVert u - L_hu\rVert_1$	$\lVert u_h - L_hu\rVert_1$	$\lVert u - u_h\rVert_1/\lVert u - L_hu\rVert_1$
4	0.059396267	0.059353209	0.012898294	1.00072545
8	0.030263783	0.030241036	0.003695387	1.00075220
16	0.015194820	0.015191020	0.000961874	1.00025011
32	0.007604836	0.007604321	0.000243233	1.00006770

Table 6.

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