What Shape Is Your Conjugate? A Survey of Computational Convex Analysis and Its Applications — Source link

Yves Lucet

Published on: 01 Mar 2009 - Siam Journal on Optimization (Society for Industrial and Applied Mathematics)

Topics: Convex analysis, Convex hull, Proper convex function, Convex optimization and Numerical range

WHAT SHAPE IS YOUR CONJUGATE?
A SURVEY OF COMPUTATIONAL CONVEX ANALYSIS AND ITS APPLICATIONS

YVES LUCET

Abstract. Computational Convex Analysis algorithms have been rediscovered several times in the past by researchers from different fields. To further communications between practitioners, we review the field of Computational Convex Analysis, which focuses on the numerical computation of fundamental transforms arising from convex analysis. Current models use symbolic, numeric, and hybrid symbolic-numeric algorithms. Our objective is to disseminate widely the most efficient numerical algorithms useful for applications in image processing (computing the distance transform, the generalized distance transform, and mathematical morphology operators), partial differential equations (solving Hamilton-Jacobi equations, and using differential equations numerical schemes to compute the convex envelope), max-plus algebra (computing the equivalent of the Fast Fourier Transform), multi-fractal analysis, etc. The fields of applications include, among others, computer vision, robot navigation, thermodynamics, electrical networks, medical imaging, and network communication.

Introduction

The objective of the present paper is twofold. First, we summarize the state of the art in Computational Convex Analysis for researchers interested in computer-aided convex analysis to build their intuition, or generate nontrivial examples through a combination of convex transforms. Current algorithms allow symbolic, numerical, and hybrid symbolic-numeric computations, and have already been instrumental in discovering and illustrating several new results in Convex Analysis.

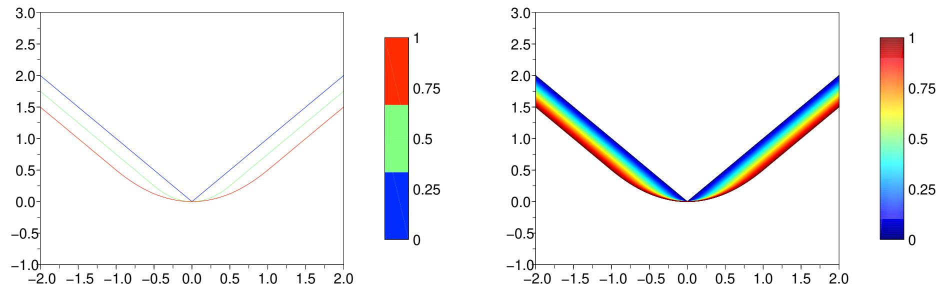
Then we present several applications benefiting from such efficient algorithms. Here we want to show Convex Analysis researchers the rich and varied set of applications they can contribute to. In addition, we want to connect the various specialized researchers with one another, by pointing out that they all use techniques related to Convex Analysis, often unknowingly, and encouraging them to consider the most recent algorithms in Computational Convex Analysis. We hope that the resulting awareness will result in new advances for all the fields involved.

While the impact of Convex Analysis in optimization is well-known, its applications to discrete problems are less understood. For example, the fact that Convex Analysis can be seen as operating on the max-plus algebra (instead of our usual plus-times algebra) in which the Fenchel conjugate plays a similar role as the FFT, is not widely known [83, p. 43]. Although they have a very wide range of applications, the most efficient numerical algorithms for computing convex transforms are still only familiar to Convex Analysis researchers, e.g. the Fast Legendre Transform is still widely used instead of the faster and simpler Linear-time Legendre Transform algorithm.

The present article is concerned with the numerical computation of transforms like the Moreau envelope. However, contrary to [106] we do not consider computing its value at one point but instead we tackle the problem of computing the Moreau envelope on a grid. In other words, we are interested in computing the shape (or graph) of the Moreau envelope and other transforms. Figure 1 illustrates typical shapes: the graph of the operator is plotted for several values of a parameter.

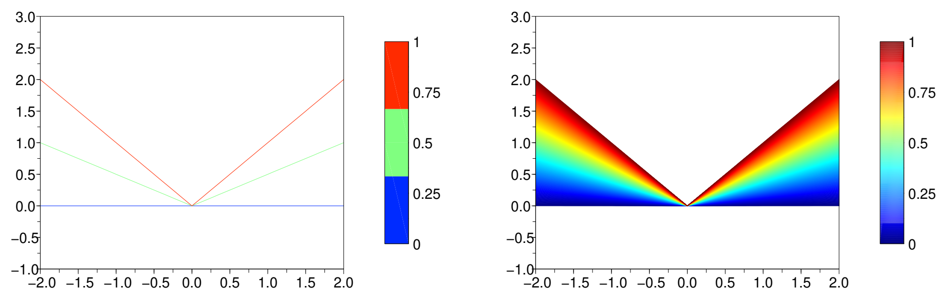
The connection between Convex Analysis, image processing, differential calculus, and dynamical systems was noted by Maragos who named the resulting area differential morphology [168, 169]. Image Processing has long been using operators closely connected to Convex Analysis: the distance transform (a special case of the Moreau envelope [162, 164]), generalized distance transforms [82, 83] (regularization with nonquadratic kernels), and morphology operators like the dilation (resp. erosion) which corresponds to the inf-convolution

Date: November 28, 2008.
2000 *Mathematics Subject Classification.* 52B55,65D99.
This work was partly supported by the author NSERC Discovery grant.



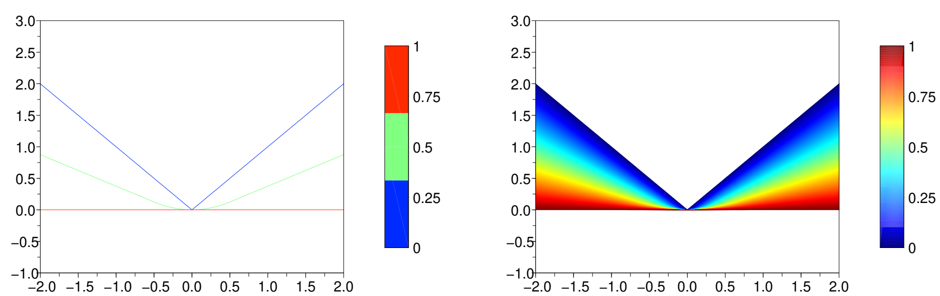
(a) Moreau envelope of $|\cdot|$ for $\lambda \in \{0, 0.5, 1\}$.

(b) Same as 1(a) for λ taking 512 values.



(c) Pasch-Hausdorff envelope of $|\cdot|$ for $\lambda \in \{0, 0.5, 1\}$.

(d) Same as 1(c) for λ taking 512 values.



(e) Proximal Average of $|\cdot|$ and $g(x) := 0$ for $\lambda \in \{0, 0.5, 1\}$.

(f) Same as 1(e) for λ taking 512 values.

Figure 1. Shapes of some operators of Convex Analysis applied to the function $f(x) = |x|$ with $0 \le \lambda \le 1$. When $0 < \lambda < 1$, the Moreau envelope and the proximal average are smooth while the Pasch-Hausdorff envelope is only Lipschitz.

(resp. deconvolution) operator of Convex Analysis [168, 169]. Partial Differential Equations (PDE) have also found applications in Image Processing e.g. the image segmentation with the Fast Marching and Level Set methods [234, 235]. The Lax and Hopf functions [122, 123, 231], which express the solution of a Hamilton-Jacobi PDE using Convex Analysis operators, are an example of the link between Convex Analysis and PDE. The computation of the convex envelope, motivated by the study of phase transition [109, 217, 180] and of the analysis of the distribution of chemical compounds [147, 148], is another example of how closely related these two fields are. Another well-known relation is the parallel between the Fourier transform and the Legendre conjugate [4, 48, 146, 157, 17, 101, 102, 60, 5]. In fact, the later plays the same role in a different algebra: the max-plus algebra. That framework has seen increased interest motivated by applications in network communication, neural networks, and discrete event systems. Classical linear and convex theory have been ported to the max-plus algebra [58, 60, 59] generating new results fundamentally related to Convex Analysis.

The applications presented in the present paper give a partial and personal overview of the wide range of fields benefiting from Computational Convex Analysis algorithms. In many instances, the same algorithm has been found independently by several authors working in different disciplines. One goal of the present paper is to point out the various connections so that future work can build on the present state-of-the-art instead of re-inventing existing algorithms.

The paper is organized as follow: Section 1 introduces the transforms: the Fenchel conjugate, inf-convolution and deconvolution operators, the Moreau envelope, the proximal average, and other related operators. Section 2 presents efficient algorithms to compute them: symbolic algorithms, numerical algorithms similar to the Fast Fourier Transform, and hybrid symbolic-numeric algorithms founded on piecewise linear-quadratic functions. Section 3 lists several applications in a wide variety of fields: Finite convex integration, network flow, phase transition, electrical networks, and robot navigation. Section 4 presents applications in image processing, computer vision, and differential morphology. Section 5 shows the link with Partial Differential Equations (PDE), while Section 6 puts the convex operators in the general framework of extremal algebra focusing on multifractal analysis, network communication, and discrete event systems. Section 7 lists additional fields that can benefit from Computational Convex Analysis, mentioning in particular medical imaging and morphology neural networks. Finally, Section 8 concludes the paper.

1. Fundamental Convex Transforms

We first recall the most fundamental operators in Convex Analysis.

1.1. **The Fenchel Conjugate.**

The Fenchel conjugate (also named Legendre-Fenchel transform, Young-Fenchel transform, the maximum transform [30, 31, 33], or Legendre-Fenchel conjugate)

$$f^*(s) = \sup_{x \in \mathbb{R}^n} [\langle s, x\rangle - f(x)] \tag{1}$$

has long been studied in a wide range of fields for its duality properties.

Consider the following (Primal) optimization problem

$$p = \inf_{x \in \mathbb{R}^n} \{f(x) + g(Ax)\},$$

where $A \in \mathbb{R}^{mn}$, f (resp. g) is convex and lower semi-continuous on $\mathbb{R}^n$ (resp. on $\mathbb{R}^m$). Problem p is naturally associated, through Fenchel conjugation, to the dual problem

$$d = \sup_{z \in \mathbb{R}^m} \{-f^*(A^T z) - g^*(-z)\},$$

where A^T is the transpose of A. The Fenchel duality Theorem links both problems (see [41, Theorem 3.3.5], [228, Theorem 31.1], [26], [230, Example 11.41]).

Theorem (Fenchel's Duality Theorem)**.** *Assume $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ with f, g and A as above. Then the following hold:*

1. *Weak duality: $p \geq d$.*
2. *Strong duality: If $A(\operatorname{Dom} f) \cap \mathit{int} \operatorname{Dom} g \neq \emptyset$, then $p = d$ and the supremum defining d is attained.*
3. *Primal solutions: If z is a solution to the dual, then the solutions to the primal are equal to the (possibly empty) set*
$$A^{-1}\partial g^*(z) \cap \partial f^*(A^T z),$$
where $\partial f(x) = \{s \in \mathbb{R}^m : \forall y \in \mathbb{R}^n,\ f(y) \geq f(x) + \langle s, y - x\rangle\}$ is the convex subdifferential.

Formulas to compute the conjugate for the main operations of Convex Analysis like addition, inf-convolution, maximum under a linear mapping, scalar multiplication, etc. have been investigated giving a complete conjugate calculus for Convex Analysis. Smoothness results linking strict convexity to differentiability are also known, making the conjugate an invaluable tool in Convex Analysis. We refer to [228, 117] for general references on Convex Analysis (and to [230] for its generalization to variational analysis), the study of the Fenchel conjugate, and different formulations of the Fenchel Duality Theorem.

Remark 1. *The name Legendre-Fenchel transform for the Fenchel conjugate above comes from the fact it is a generalization of the Legendre transform*

$$f^*(s) = \langle s, \nabla^{-1} f(s)\rangle - f(\nabla^{-1} f(s))$$

when the gradient of f is invertible. When it is not, the Fenchel conjugate or its generalization: the slope transform (see references in Section 3.5), are used.

The parallel between the Fenchel conjugate and the Fourier transform has long been known [48] (see Section 6.3 for more references).

1.2. **Inf-convolution and Deconvolution.** The inf-convolution [197, 201, 240, 241] (also called epi-addition) of two functions f and g is defined by

$$(f \oplus g)(x) := \inf_y [f(y) + g(x-y)].$$

It provides a very general transform giving rise to several regularization operators. Geometrically, it corresponds to the Minkowski addition of the epigraphs of the two functions. Under appropriate assumptions (the functions f and g are convex lower semi-continuity (lsc) proper and $\operatorname{ri}\operatorname{Dom} f^* \cap \operatorname{ri}\operatorname{Dom} g^* \neq \emptyset$, where ri denotes the relative interior [117, Corollary X.2.1.3 and Theorems X.2.3.1, X.2.3.2]), the infimal convolution reduces to several Fenchel conjugacy computations

$$(f \oplus g) = (f^* + g^*)^*. \tag{2}$$

The inverse of the inf-convolution operator is called the deconvolution [178, 116] of f by g and is defined by

$$(f \ominus g)(x) := \sup_y [f(x-y) - g(y)].$$

Under appropriate assumptions (either assume f and g are proper lsc convex functions with $\operatorname{Dom} g^* = \mathbb{R}^n$ and use [76, Proposition 2.1]; or allow unproper functions by extending the usual subtraction law and invoke [177, Proposition I.11]), the deconvolution of two convex functions reduces to computing several conjugates: $(f \ominus g) = (f^* - g^*)^*$. Mathematical Morphology has long been using erosion and dilation operators, which amounts to deconvolution and inf-convolution respectively (see Section 4.3 for details).

The Pasch-Hausdorff envelope [230, Chapter 9], also called Lipschitz regularization, is a special case of inf-convolution with the norm function

$$(f \oplus c\|\cdot\|)(x) = \inf_y [f(y) + c\|x-y\|].$$

It has been studied for its Lipschitz regularization and Lipschitz extension properties [114, 115].

1.3. **Moreau Envelope.** The Moreau envelope of an extended real-valued function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, (also called the Moreau–Yosida approximate, Yosida Approximate [14] or Moreau–Yosida regularization) corresponds to the inf-convolution with half the norm square

$$M_\lambda(f)(x) := (f \oplus \frac{\|\cdot\|^2}{2\lambda})(x) = \inf_{u\in\mathbb{R}^d} [f(u) + \frac{\|x-u\|^2}{2\lambda}]. \tag{3}$$

(We will denote it M_λ when there is no ambiguity on the function f under consideration.) It has been studied extensively both theoretically and algorithmically for its regularization properties. Its origin goes back to the work of Yosida [258] on maximal monotone operators (it is also related to Tikhonov regularization [246]), and its behavior is well known in the field of convex analysis [198, 199, 200, 228] and variational analysis [230, Chapter 12]. Under general conditions (f is prox-regular and prox-bounded [230, Proposition 13.37 p. 617]), M_λ is C^1 with Lipschitz continuous gradient, and critical points of f are fixed points of the proximal mapping

$$P_\lambda(x) := \operatorname*{Argmin}_{u\in\mathbb{R}^d} [f(u) + \frac{\|x-u\|^2}{2\lambda}]. \tag{4}$$

When f is convex lower semi-continuous and proper, the proximal mapping is a maximal monotone operator and its fixed points are the minimum of f. More precise smoothness of M_λ is known under various hypotheses on f [57, 105, 183, 187, 182, 216]. More recent developments have focused on extending the results to nonconvex functions through the notion of prox-regularity [21, 35, 34, 36, 213, 212, 181].

Considering that $M_\lambda(f)(x)$ converges to $f(x)$ when λ decreases to 0, and shares the same critical points of f, the Moreau envelope is an attractive regularization transform. On the practical side, the proximal point algorithm exploits the fixed point property of the proximal mapping to converge to a minimum of f [229]. Its convergence properties are well known [149, 100], and variants have been introduced to speed up its convergence (see [49] and references therein). Extensions to non-quadratic kernels like entropy methods and Bregman distances have also been studied [74, 124, 244, 210, 42]. Bundle methods are intrinsically linked to the Moreau envelope (see [186], and [117, Chapter XV]). Recent developments in that direction focus on $\mathcal{VU}$-decomposition [152, 154, 153, 151, 150, 188, 189, 190, 191, 192, 193, 194, 195] to take advantage of both Newtonian and bundle algorithms.

We note that the computation of the Moreau envelope is equivalent to the computation of the Legendre–Fenchel conjugate as the following formulas shows [162]

$$M_\lambda(f)(x) = \frac{\|x\|^2}{2\lambda} - \frac{1}{\lambda}\left(\frac{\|\cdot\|^2}{2} + \lambda f\right)^*(x), \tag{5}$$

$$f^*(s) = \frac{\|s\|^2}{2} - \lambda M_\lambda\left(\frac{1}{\lambda}f - \frac{\|\cdot\|^2}{2\lambda}\right)(s), \tag{6}$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and $\lambda > 0$. So algorithms for computing one transform are trivially extended to compute the other.

1.4. **Other transforms.** The Lasry-Lions double envelope [144, 13] $h_{\mu,\lambda}$ is defined as several Moreau envelopes

$$h_{\mu,\lambda}(f)(x) = -M_\mu(-M_\lambda(f))(x).$$

It is a smooth function [230, Proposition 12.62 p. 566]. Similarly the proximal hull (the proximal hull is different from the proximal mapping) can be written

$$g_\lambda(f)(x) = h_{\lambda,\lambda}(f)(x) = -M_\lambda(-M_\lambda(f))(x),$$

and so is also reducible to Moreau envelope computations.

More recently, the proximal average [25, 23, 24, 22, 165] of n functions $f_1, \dots, f_n$ is defined with combinations of Moreau envelopes

$$p_\mu(\boldsymbol{f}, \boldsymbol{\lambda}) = -M_\mu(-(\lambda_1 M_\mu f_1 + \cdots + \lambda_n M_\mu f_n)),$$

where $\boldsymbol{f} = (f_1, \dots, f_n)$, $\boldsymbol{f}^* = (f_1^*, \dots, f_n^*)$, and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$. It can also be computed as a combination of several Fenchel conjugates

$$p_\mu(\boldsymbol{f}, \boldsymbol{\lambda}) = \left(\lambda_1(f_1 + \mu^{-1}\,\mathfrak{q})^* + \cdots + \lambda_n(f_n + \mu^{-1}\,\mathfrak{q})^*\right)^* - \mu^{-1}\,\mathfrak{q},$$

where $\mathfrak{q} = \frac{1}{2}\|\cdot\|^2$. Its key properties include been an homotopy between convex functions, and inheriting smoothness. It has been used to build counter-examples [25], and compute primal-dual symmetric antiderivative methods [24] (see also Section 3.1). The proximal average has also been generalized to a kernel average [27] with current research focusing on generalization to a Bregman average based on Bregman distances.

Another key property of the proximal average is its behaviour with respect to Fenchel conjugacy: the conjugate of the proximal average is the proximal average of the conjugate. Other transforms that satisfy such compatibility with conjugacy include Ghoussoub's anti-selfdual Lagrangians [91, 92, 93], and Atteia's square root of a convex function [12].

Generalization of the Fenchel conjugate such as the c-conjugate [178] can also be considered within our framework. In the theory of optimal transport problems, c-convexity is linked to cyclical monotonicity and allows the formulation of Kantorovich duality [254, Chapter 5].

Other generalizations involve considering different distances instead of the norm for the Pasch-Hausdorff envelope, or half the norm square for the Moreau envelope. For example, Bregman distances [43] $D(x, y) = f(x) - f(y) - \langle\nabla f(x), x - y\rangle$ associated with some functions f, and divergence measures e.g. based on the Shannon entropy could be considered. Generalizations to quasi-convex or γ-convex functions fit also our framework.

2. Computer-Aided Convex Analysis

Introduction. While optimization algorithms avoid explicitly computing the conjugate, motivated by the study of some Hamilton-Jacobi partial differential equations, computational algorithms have been developed to compute it on grids. A log-linear algorithm named the Fast Legendre Transform (FLT for short, by analogy with the Fast Fourier Transform) was first introduced [45, 62, 160, 206, 236] to be subsequently improved by a linear-time algorithm: The Linear-time Legendre transform (LLT) [161]. Another linear-time algorithm, motivated by applications in image processing, was obtained by computing the Moreau envelope [70, 69, 81].

While fast algorithms have been the main strategy to compute convex transforms, different frameworks have also been investigated. A parametric framework was introduced in [118] and further expanded in [165]. It relies on the parametrization of the Fenchel conjugate to recover its graph up to affine parts. However, its restrictions led to the introduction of hybrid symbolic-numeric algorithms by considering the class of piecewise linear-quadratic (PLQ) functions [165]. (PLQ functions are lower-semicontinuous (lsc) proper extended-valued functions with piecewise linear domain for which the function is either linear or quadratic on each piece of its domain; See Section 2.3.)

Lately, a new strategy using graph-matrix calculus to compute only the graph of the transforms was introduced in [96] and further developed in [22]. For example, one can recover the graph of M_λ by quadrature from

$$\operatorname{gph} \nabla M_\lambda f = \begin{bmatrix} I & \lambda I \\ 0 & I \end{bmatrix} \operatorname{gph} \partial f = \{(x + \lambda y, y) : (x, y) \in \operatorname{gph} \partial f\},$$

where I is the $n \times n$ identity matrix, $\operatorname{gph} \nabla M_\lambda f = \{(x, \nabla M_\lambda f(x)) : x \in \mathbb{R}^n\}$, $\operatorname{gph} \partial f = \{(x, y) : y \in \partial f(x)\}$, and

$$\partial f(x) = \{y \in \mathbb{R}^n : \forall x' \in \mathbb{R}^n, f(x') \geq f(x) + \langle y, x' - x\rangle\}$$

is the subdifferential of Convex Analysis. Whether graph-matrix calculus will provide efficient and competing algorithm is the subject of ongoing research.

We now recall what we consider the three main approaches to compute convex transforms: symbolic computation, fast algorithms, and PLQ-based algorithms.

2.1. Symbolic Computation.

The natural strategy to compute the Fenchel conjugate is to differentiate the function under the supremum to obtain an equation satisfied by all the critical points. The difficulty resides in solving such an equation, which amounts to inverting the gradient of the function. For commonly used functions, symbolic computation software allows to perform some computation. Maple implementations were presented in [26] for the one-dimensional case, and in [40] for the multi-dimensional case. Large classes of functions can now be considered and some explicit formulas for the conjugate have been found using these packages. The packages offer a very efficient method to build some intuition, and to check one's computation.

However, the symbolic computation approach suffers from an intrinsic limitation: there may not be any closed form solution for the conjugate. Indeed, consider computing the conjugate of an even degree polynomial. If the degree is greater or equal to six, computing the conjugate involves finding the zeros of a polynomial of degree at least five, which may not admit a closed form. Moreover, in some cases, the explicit formula for the original function is not available e.g. the function is only available through a black box. So when the symbolic packages fail or are not applicable, ones turns to numerical computation, which is the subject of the next two subsections.

2.2. Fast Algorithms.

The idea of a fast algorithm to compute the Fenchel conjugate was first formulated in [45], and independently in [236]. It was then investigated in [62, 206, 160] under the name Fast Legendre Transform. Its log-linear worst-case time complexity $O(n \log n)$ was later improved in [161] to a linear time complexity $O(n)$. All subsequently developed algorithms focus on either computing the conjugate or the Moreau envelope. As we mentioned, both computations are equivalent.

The first step in any fast algorithm is to reduce computations to functions of one variable by noting that

$$M_\lambda(s_1, \ldots, s_d) = \inf_{x_1} [\frac{|s_1 - x_1|^2}{2\lambda} + \cdots + \inf_{x_d} [\frac{|s_d - x_d|^2}{2\lambda} + f(x)] \ldots]. \tag{7}$$

A similar formula holds for the conjugate. Hence, all computations for functions in $\mathbb{R}^d$ can be reduced to computing transforms in $\mathbb{R}$ several times.

The above "factorization" formula has been extended as a generalized distributive law to encompass various transforms beyond convex analysis [3]. In fact, by considering semi-rings instead of the usual $(\mathbb{R}, +, .)$ algebra, a common framework exists that encompasses the Fast Fourier Transform on any finite Abelian group, the fast Hadamard transform, Viterbi's algorithm, and Belief propagation algorithms [2, 139]. Among the many applications of such transforms, factor graphs have been applied to protein function [155] and, using the sum-product algorithm, to wireless communication [53].

2.2.1. *The Linear-time Legendre Transform (LLT) Algorithm.* The main idea behind the LLT algorithm is to note that computing the Fenchel conjugate is equivalent to computing the convex envelope (the convex envelope of a function f is the largest convex function that lies below f). While it is well-known that, for proper lsc convex functions, computing the conjugate of the conjugate gives the closed convex envelope, the LLT reverses the order: It first computes the convex envelope as a pre-processing step, and then computes the conjugate. More precisely, we first consider a discrete version of the transform

$$f_X^*(s) = \max_{x_i \in X}[s_j x_i - f(x_i)],$$

where the maximum is taken over $X = \{x_1, \dots, x_n\}$, and f_X^* is to be computed at all the slopes $s_j \in S = \{s_1, \dots, s_m\}$. The goal of the algorithm is to reduce the brute force computation of $O(nm)$ to $O(n+m)$. Since in practice we take $m = n$ to obtain a good numerical precision, the goal is to reduce the complexity from quadratic to linear.

Computing the lower convex envelope of the set of points $(x_i, f(x_i))$ in the plane can be achieved in linear time using the Beneath-Beyond algorithm [75, 214], since the sequence x_i can be assumed sorted without any loss of generality: $x_i < x_{i+1}$. Now any point which is not a vertex of the convex hull, can be safely discarded since the maximum can never be attained at a point strictly in the interior of the epigraph, and vertices allow us to recover all points on the boundary of the epigraph. So it is sufficient to focus on vertices of the convex hull.

After precomputation, we can assume the points $(x_i, f(x_i))$ are vertices of the convex hull. Hence the finite difference slopes $c_i := \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$ form an increasing sequence. Now computing the Fenchel conjugate amounts to merging the finite difference slopes c_i with the slopes s_j since $c_{i-1} < s_j < c_i$ implies that the maximum in the definition of the conjugate is attained at x_i. More details on the LLT algorithm, including its proof of correctness, can be found in [161].

Note that no convexity assumption is made on the input data. (If the data is convex, the precomputation step can be skipped.) Convexity is explicitly introduced to speed up the computation, but the algorithm applies to nonconvex data.

2.2.2. *The Parabolic Envelope (PE) Algorithm.* The PE algorithm was introduced in [70] and later independently in [81]. It focuses on computing the discrete Moreau envelope

$$M_{\lambda,X}(s_j) = \min_{x_i \in X}[f(x_i) + \frac{\|x_i - s_j\|^2}{2\lambda}],$$

where as above $i = 1, \dots n$ and $j = 1, \dots m$. Assume $m = n$. The goal is again to reduce the quadratic brute force computation to linear. The PE algorithm shares the same efficiency as the LLT algorithm, and is an alternative algorithm to compute the Moreau envelope or the Fenchel conjugate.

The key step is to note that the computation amounts to finding the lower envelope of the family of parabola $s \mapsto f(x_i) + \frac{\|x_i - s\|^2}{2\lambda}$. Such envelope can be computed in linear time by adding parabola one at a time, since computing the intersection between two parabola can be done in constant time. See [163] for more details, comparison with other algorithms, and a Scilab [233] implementation.

2.3. **Piecewise Linear-Quadratic (PLQ) algorithms.** As mentioned above, PLQ functions are defined as the set of lower-semicontinuous (lsc) proper extended-valued functions with piecewise linear domain for which the function is either linear or quadratic on each piece of its domain. Convex PLQ functions are known in Convex Analysis for being closed under the Moreau envelope and the conjugate transformations [230, 11.14 p. 484]. The class of PLQ functions of one variable corresponds to lsc proper extended-valued piecewise quadratic functions. In that setting, PLQ algorithms refer to algorithms to compute fundamental convex transforms applied to the class of PLQ functions. Such algorithms run in linear time [165]

The PLQ algorithms were introduced specifically to compute composition of convex transforms such as the proximal average [165]. Such computation becomes very technical using fast transform algorithms since one has to keep track of the dual domain explicitly. Moreover, to obtain a reasonable numerical approximation of the result, one needs considerable knowledge of the dual domain of any intermediate transform. Such requirements make the fast algorithms cumbersome beyond a few compositions.

The key idea of PLQ algorithms is to explicitly represent convex functions. Fast algorithms manipulate points, so the underlying model is either a sample function, or a piecewise linear approximation. One reason the class of piecewise linear functions is not rich enough for our purpose, is the Moreau envelope of a piecewise linear function is no longer piecewise linear even for simple functions like the indicator of a single point. On the contrary, the class of piecewise linear-quadratic functions (functions whose domain can be expressed as the union of finitely many convex polyhedra, relative to each of which the function is at most quadratic) is closed under all major convex operations: addition, scalar multiplication, Fenchel conjugacy, and Moreau envelope. Hence, the computation of such transforms, or of compositions of such transforms, can be done symbolically. Moreover, there is no need to track the dual (or primal) domain of the function.

The PLQ algorithm to compute the conjugate amounts to matching each primal domain part with its dual counterpart, then computation is done symbolically. (See [165] for more details.) The price to pay for such simplicity is that we can no longer use the factorization formula, so computations beyond functions of one variable are the subject of active research.

2.4. **Nonconvex Extensions.** Several previously mentioned algorithms can handle nonconvex functions. The LLT and PE fast algorithms can be used to compute the conjugate and the Moreau envelope of nonconvex functions. In fact, considering that the conjugate is always a convex function that depends only on the convex envelope and using Formula (5), algorithms restricted to convex functions can be readily extended to nonconvex functions by first convexifying the function, then computing its conjugate (this is the principle of the LLT algorithm), and if needed its Moreau envelope. Hence, the PLQ algorithms can be extended to nonconvex functions as soon as one can compute the convex envelope of a PLQ function (which is a PLQ function) [248].

Finally we note that a completely different approach was taken in [44] to compute the inf-convolution of nonconvex data in subquadratic time. While breaking the quadratic barrier, this algorithm is of course not optimal when the data is convex.

We now consider application areas benefiting from the previous framework.

3. Antiderivatives, Network Flow, Phase Transition, Electrical Networks, and Robot Navigation

3.1. **Finite Convex Integration.** Consider the following problem: given a finite set x_i^* of subgradients at points x_i, find a convex function f such that $x_i^* \in \partial f(x_i)$. The problem has been tackled in [143] under the name finite convex integration with links to linear programming. It can also be interpreted as a feasibility problem induced by a system of difference constraints [1, Section 4.5], which can be solved using shortest path algorithms.

Tools from monotone operator theory and the mid-point proximal average operator allowed to build a method that gives the same solution for (x_i, x_i^*) in the primal space as for (x_i^*, x_i) in the dual space: it is symmetric with respect to convex duality [24]. This "compatibility" with convex duality, which was a key requirement of the original problem, is naturally obtained using the proximal average. We summarize the results below to emphasize the role played by the PLQ algorithms in the numerical examples. (The availability of efficient algorithms also played a critical role in conjecturing the results.)

Assume x_i, x_i^* are given for $i = 1, \ldots, n$. We say a function f is an antiderivative if $x_i^* \in \partial f(x_i)$ for $i = 1, \ldots n$. The derivative is said *intrinsic* if in addition the function f does not depend on the order of the points x_i. A method m, which given a set $A = \{(x_i, x_i^*)\}$ produces an intrinsic antiderivative m_A is said to be *primal-dual symmetric* if m applied to the set $A^{-1} = \{(x_i^*, x_i)\}$ gives the conjugate of m applied to the set A:

$$m_{A^{-1}} = m_A^*. \tag{8}$$

The key idea behind primal-dual symmetric anti-derivative is that the input data is symmetric with respect to convex duality i.e. any antiderivative f satisfies $x_i^* \in \partial f(x_i)$ and $x_i \in \partial f^*(x_i^*)$. We would like the method to preserve that symmetry, which is the meaning of Formula (8).

While there are many antiderivatives, there is a priori no reason for primal-dual symmetric antiderivative methods to exist. However, it turns out that using the midpoint proximal average operator of two functions f_0 and f_1

$$\mathcal{P}(f_0, f_1) := \Big(\tfrac{1}{2}\big(f_0 + \tfrac{1}{2}\|\cdot\|^2\big)^* + \tfrac{1}{2}\big(f_1 + \tfrac{1}{2}\|\cdot\|^2\big)^*\Big)^* - \tfrac{1}{2}\|\cdot\|^2,$$

one creates primal-dual symmetric antiderivatives from any antiderivative using the fact that the midpoint proximal average of two antiderivatives is also an antiderivative. Given a method m producing intrinsic antiderivatives m_A for the set A, define the new method $\mathfrak{m}$ by

$$\mathfrak{m}_{\mathcal{A}} = \mathcal{P}\big(m_A, m_{A^{-1}}^*\big).$$

Then $\mathfrak{m}$ produces primal-dual symmetric antiderivatives [24].

The numerical computation of primal-dual symmetric antiderivative amounts to computing proximal averages. The PLQ algorithms are ideally suited for that task: they achieve the optimal linear time worst-case complexity as well as provide robust numerical results when applied several times to compute compositions of convex transforms as is needed to calculate the proximal average.

3.2. **Network Flow.** Computing Linear Cost Network Flow on Series-Parallel Networks is another problem related to graph theory [250]. (We refer to references in [250] for its importance in combinatorial optimization.) Assume $\mathcal{G}$ is a strongly connected directed graph with vertex set $\mathcal{V}$, and edge set $\mathcal{E}$. Each edge $(i,j) \in \mathcal{E}$ is associated with a flow $x_{i,j}$, which is lower- and upper-bounded $-\infty < l_{i,j} \le x_{i,j} \le u_{i,j} < +\infty$, and a flow cost per unit $c_{i,j}$. The linear cost network flow problem is to minimize the total cost of the arc flows, subject to capacity and conservation constraints, in other words to solve

$$\begin{array}{lll}
\text{minimize} & \sum_{(i,j)\in\mathcal{E}} c_{i,j} x_{i,j} & \\
\text{subject to} \;\; \forall i \in \mathcal{V} & \sum\limits_{\{j|(i,j)\in\mathcal{E}\}} x_{j,i} = \sum\limits_{\{j|(i,j)\in\mathcal{E}\}} x_{i,j}, & \text{(Conservation condition)} \\
\qquad\quad \forall (i,j) \in \mathcal{E} & l_{i,j} \le x_{i,j} \le u_{i,j}, & \text{(Capacity condition).}
\end{array}$$

To solve the problem efficiently, nested sums and nested infimal convolutions are computed [250]. Series-parallel networks give naturally rise to such nested operators since the network can be decomposed as a sequence of serial (resp. parallel) joins corresponding to sum (resp. inf-convolution) operations. The solution to the above problem can be expressed as computing a function $f = C(f_1, \dots, f_m)$ with f_i convex piecewise linear functions, and $C(f_1, \dots, f_m) = C(f_1, \dots, f_k) \odot C(f_{k+1}, \dots, f_m)$ where $\odot$ is either the addition or the inf-convolution operator: $\odot \in \{+, \oplus\}$.

The key idea to obtain an efficient algorithm is to sort grid nodes to compute the sum, and to sort the slopes to compute the inf-convolution. (A similar sorting strategy was used for the LLT algorithm except instead of inserting slopes in a sorted list, two sorted lists were merged; see Section 2.2.1.) The algorithm amounts to computing nested inf-convolutions and sums of convex piecewise-linear functions, and is an alternative approach to the Fast Algorithms of Section 2.2. The resulting worst-case computation cost is $O(m \log^2 m)$ where m is the number of arcs in the graph.

3.3. **Thermodynamics: Phase Transition.** In numerical simulation of multiphasic flows [109], we consider two phases of a same pure body. The second principle of thermodynamics states that the system will evolve until the entropy reaches a maximum. Let S (resp. S_1, S_2) denote the entropy of the mixture (resp. of the first phase, the second phase), and let $W = (M, V, E)$ (resp. W_1, W_2) be the vector of mass, volume, and energy for the mixture (resp. the first and second phase). If the two phases are perfectly immiscible we have $M_1 + M_2 = M$ (conservation of mass), $E_1 + E_2 = E$ (conservation of energy), and $V_1 + V_2 \le V$

(immiscible phases). Moreover, if the pressures of the two fluids are always positive, the volume constraint is saturated: $V_1 + V_2 = V$. Therefore, the optimization problem becomes a sup-convolution

$$S(W) = \max_{W_1+W_2=W} S_1(W_1) + S_2(W_2).$$

Assuming the entropies of the two phases are known, the mixture entropy can be computed numerically using either a fast algorithm or the PLQ algorithms through Formula (2) since the functions S_1, and S_2 are concave. As mentioned in [109], such numerical computation is especially useful in the absence of closed form solutions.

Thermodynamics links to Convex Analysis run deeper than the above instance. The study of thermodynamic equilibrium is closely linked to the operation of convexification [217]. Consider the phase equilibrium problem at a point d (phase vector) at constant volume. It corresponds to

$$\min\{\sum_{i=1}^{q} \lambda_i E(d^i) \ : \ \sum_{i=1}^{q} \lambda_i d^i = d, \sum_{i=1}^{q} \lambda_i = 1, \lambda_i > 0\}$$

where E is a function associated with the Helmholtz free energy, $d = m/V$, m is the mole vector: $m^i > 0$ is the number of moles of the i^{th} fluid, and V is the volume. To recover the physical phases from the phase vector d, use $m^i = V_i d^i$, and $V_i = \lambda_i V$. The solution to the optimization problem is the convex envelope of E. (The convex envelope is the largest convex function upper bounded by E.) When the function E is smooth (at least C^1), at a point d, the optimal solution satisfies $\nabla E(d^i) = \nabla E(d^j)$ for all i and j, which represents the equality of the chemical potentials, and $E(d^i) - \langle \nabla E(d^i), d^i\rangle = E(d^j) - \langle \nabla E(d^j), d^j\rangle$, which expresses the equality of pressure in each phase. The phase equilibrium at constant pressure problem consists in minimizing the Gibbs free energy instead of the Helmholtz free energy. Since the former is obtained as the Legendre transform of the later, Computational Convex Analysis algorithms allow to compute it efficiently as soon as the convex envelope E is calculated. Global minimization of the Gibbs free energy to solve the chemical and phase transition problem was studied in [180].

Another application explored the analysis of the distribution of chemical compounds in the atmosphere. In [147], a measure of roughness is defined, and is further applied in [148]. It consists in smoothing noisy data by rolling a parabola from above, then rolling another parabola from below, and considering the area between the parabolas as the measure of roughness. The efficient computation of the measure is performed with the LLT algorithm. Intuitively, smoothing with a parabola corresponds to computing a Moreau envelope, which is equivalent to computing the Legendre conjugate by Formulas (5)–(6).

3.4. **Electrical Networks.** The study of a *mechanical system* consisting of two springs in series can be performed by computing the total potential energy of the system, which is the inf-convolution of the potential energy of each spring. Such systems with series and/or parallel strings are similar to electrical networks. In fact, the study of electrical circuits motivated the definition of the parallel addition and parallel subtraction operators, which corresponds to the inf-convolution and deconvolution of quadratic functions. Anderson [6, 7, 8] defined the parallel addition operator, and Mazure [174, 177, 175, 178, 176, 119] studied its properties from a Convex Analysis perspective (some of her results also apply to nonconvex functions). Consider [117, Example IV.2.3.8 p. 165]: an electrical circuit is made up of two generalized resistors A_1 and A_2 connected in parallel, and we want to find the equivalent resistor. By Maxwell's variational principle, a given current-vector $i \in \mathbb{R}^n$ is distributed among the two branches such that the dissipated power $\langle A_1 i_1, i_1\rangle + \langle A_2 i_2, i_2\rangle$ is minimal. So the real current distribution $i = \bar{i_1} + \bar{i_2}$ satisfies

$$\langle A_1 \bar{i_1}, \bar{i_1}\rangle + \langle A_2 \bar{i_2}, \bar{i_2}\rangle = \inf_{i_1+i_2=i} \{\langle A_1 i_1, i_1\rangle + \langle A_1 i_2, i_2\rangle\}.$$

When the matrices A_1 and A_2 are positive definite, the solution corresponds to the inf-convolution of two quadratic forms $f_j(x) = \langle A_j x, x\rangle/2$ for $j = 1, 2$. The result $(f_1 \oplus f_2)$ is the quadratic form associated with $A_{1,2} := (A_1^{-1} + A_2^{-1})^{-1}$. All such inf-convolutions can be evaluated numerically efficiently using Computational Convex Analysis algorithms. Similarly, the parallel subtraction, which corresponds to replacing a resistor with an equivalent circuit using two resistors in parallel, can be computed using for example PLQ algorithms.

While a short history of parallel sum related to electrical networks is provided in the introduction to [10], applications to network connections are explored in [9, 196]. Parallel sum have also found applications in quantum effects [90]. Extensions of the parallel sum have also been considered e.g. the quasi-projection operator [72] defined as $!(A,B) = 2A(A+B)^+B$, where $+$ denotes the Moore-Penrose inverse, reduces to the harmonic mean $!(A,B) = 2(A^{-1}+B^{-1})^{-1}$ when A and B are invertible. The parallel addition was also defined as the limit of the sequence $(x^{-1}+b_n^{-1})^{-1}$ when $b_n \to b$ in [18], and of the sequence $((A+\varepsilon I)^{-1}+(B+\varepsilon I)^{-1})^{-1}$ when $\varepsilon \downarrow 0$ in [140]. A generalization to connections through an axiomatic approach is given in [141] (including an interpretation of series-parallel networks). The relation between the Moore-Penrose generalized inverse of the sum of two matrices and their parallel sum can be found in [85]. The variational characterization using the inf-convolution was investigated in [203] while the parallel sum of k matrices was studies in [245]. (See also [209] for further studies of the parallel sum.) A new regularization process based on parallel addition was studied in [211]. See also [222] for a generalization to monotone operators, and [77, 78] for another generalization. All such generalizations fit into the general framework of Computational Convex Analysis, and as such can benefit from its fast algorithms.

3.5. **Robot Navigation.** Building from related work on the slope transform [71, 107, 108], the Legendre-Fenchel transform has been investigated to navigate a robot in a 2D space [137]. The LLT algorithm was adapted to handle discrete convex, concave, and nonconvex functions. It was then extended to polygons by splitting a function in several such pieces, and computing its conjugate on each piece. In that context, the key property of the Legendre-Fenchel transform is its ability to detect contact between bodies using slopes. Another important property used in [137] is Formula (2) to reduce inf-convolution of convex functions to Legendre-Fenchel transforms.

While the framework of [137] focuses on piecewise linear functions and polygons and as such relies on results first established for the LLT algorithm, it could be extended to PLQ functions, which would make the addition operator trivial instead of explicitly generating the domain of the conjugate of the sum as the union of the domain of each conjugate. Such extension requires generalizing the PLQ framework to nonconvex functions [248], and considering piecewise quadratic approximation of objects instead of polygon approximations.

Robot navigation has long been performed using distance transforms whose computation is a special case of computing the Moreau envelope (see Section 4.1 below). We refer to [243] for a fast distance transform based heuristic path planning algorithm, and to [110, 111] for robot manipulator path planning. Both build from the work in [129] further developed in [126, 127]. See also the Jarvis' previous work on collision free path planning [128, 125].

Extensions of the original robot navigation problem include covert robotic [172] (move a robot while escaping sentinels' notice), real-time detection and navigation [257], outdoor robot navigation using vision [56], robot exploration with industrial applications [260, 261, 262], and multidimensional alignment [138]. Since all such extensions rely on the computing the distance transform, they can be performed using the Computational Convex Analysis algorithms.

An interesting link between distance transform and Hamilton-Jacobi equations is illustrated in [242], in which the robot path planning problem is solved by considering an Hamilton-Jacobi-Bellman equation instead of computing distance transforms. It is another example of the links between Section 4 and Section 5.

We now turn our attention to applications arising from image science.

4. Image Processing, Computer Vision, and Mathematical Morphology

4.1. **Image Processing: Distance Transforms.** In image processing, distance transforms have been investigated for decades [232, 65] due to their diverse applications (see for example [39, 64, 247, 215] and references therein). For a binary image B defined as an application from $\{1,\dots,n\}\times\{1,\dots,m\}$ to $\{0,1\}$, the distance transform is the mapping that associates B with an $\{1,\dots,n\}\times\{1,\dots,m\}$ array D defined as follow. Assume B has at least one pixel p with $B[p]=0$. For each pixel p in B, $D[p]$ contains the Euclidean distance to the closest pixel in B containing the value 0. In practice, to restrict computation to integer arithmetic the distance transform D^2 is computed, for example for

$$\mathrm{B}=\begin{bmatrix}1&1&1&1&1\\1&0&1&1&1\\1&1&1&1&1\\1&1&1&0&1\\1&1&1&1&1\end{bmatrix},\ \mathrm{D}^2=\begin{bmatrix}2&1&2&5&8\\1&0&1&4&5\\2&1&2&1&2\\5&4&1&0&1\\8&5&2&1&2\end{bmatrix}.$$

Several algorithms [39, 46, 54, 64, 173, 238, 112] were introduced to compute the Euclidean Distance Transform (EDT), and the performance of some of them were recently compared [79]. Recent research focuses on simplifying the algorithms while still achieving linear-time complexity. For example, the fact that the Euclidean distance transform computation is equivalent to computing the lower envelope of quadratic functions was exploited in [70, 69, 81] to achieve a simple linear-time algorithm. Other algorithms based on monotonicity or neighborhood properties also managed to achieve linear complexity [89, 238].

The relationship between the Moreau envelope and the Legendre conjugate was exploited in [162] to reduce the core of the distance transform computation to the LLT algorithm as follow. The squared distance transform is the application D^2 from $\{1,\dots,n\}\times\{1,\dots,m\}$ to the set of non-negative integers defined by

$$D^2(p)=\min_{q\in O}\|p-q\|^2,$$

where $O=\{q;B(q)=0\}$ is the set of pixels with value 0 in B. Using the indicator function $I(p)=0$ if $B(p)=0$ and $+\infty$ otherwise, we find that the square distance transform is the Moreau envelope of I:

$$D^2(p)=\min_q[\|p-q\|^2+I(q)].$$

Hence, distance transform algorithms are particular cases of discrete Moreau envelope algorithms.

Examples of applications based on distance transforms include the morphometry of nerve cross-sections, the registration of Magnetic Resonance images, camera path-planning in virtual endoscopy, and tissue classification in Magnetic Resonance images; see [64].

4.2. **Image Processing: Generalized Distance Transforms.** We detail two contributions in computer vision and object recognition that rely on efficient algorithms for the generalized distance transform.

4.2.1. *Efficient Belief Propagation for Early Vision* [83]. Early vision problems such as stereo and image restoration have been solved using Markov Random Field (MRF) models. Since the resulting problems are NP hard, approximation techniques based on graph cuts and belief propagation have been used with high accuracy results in practice. However, both approaches are still computationally expensive especially compared with local methods that are faster but produce poorer results.

A general framework consists of finding a labeling function $f:p\mapsto f_p$ from the set of pixels $\mathcal{P}$ to the set of labels $\mathcal{L}$ (labels may correspond to disparities or intensities) by minimizing an energy function

$$E(f)=\sum_{p\in\mathcal{P}}D_p(f_p)+\sum_{(p,q)\in\mathcal{N}}V(f_p-f_q),$$

where $\mathcal{N}$ is the set of edges in the four-connected image grid graph, $D_p(f_p)$ is the cost of assigning label f_p to pixel p (data cost), and $V(f_p-f_q)$ measures the cost of assigning labels f_p and f_q to two neighboring pixels (discontinuity cost).

The max-product belief propagation (BP) algorithm can be used to find a labeling. It is an iterative algorithm that works by passing messages in parallel around the graph. Denoting $m^t_{p\to q}$ the message that node p sends to a neighboring node q at iteration t, it can be summarized as follow:

(1) Initialize $m^0_{p\to q}$ to 0.
(2) At each iteration t ($t=1$ to T) compute

$$m^t_{p\to q}(f_q)=\min_{f_p}\left(V(f_p-f_q)+D_p(f_p)+\sum_{s\in\mathcal{N}(p)\backslash q}m^{t-1}_{s\to p}(f_p)\right).$$

(3) After T iterations, compute the belief vector

$$b_q(f_q) = D_q(f_q) + \sum_{p \in \mathcal{N}(q)} m^T_{p \to q}(f_q).$$

(4) Finally compute

$$f^*_q = \operatorname*{Argmin}_{f_q} b_q(f_q).$$

The key step in the algorithm for our purposes is Step (2). It can be rewritten as

$$m^t_{p \to q}(f_q) = \min_{f_p} \left(F(f_p) + V(f_p - f_q) \right),$$

with $F(f_p)$ gathering all the data for the label f_p. As such, we are required to compute a min convolution at each iteration. Using a fast algorithm, the quadratic computation cost is reduced to linear thereby reducing the BP algorithm cost of $O(nk^2T)$ to $O(nkT)$, where $n = |\mathcal{P}|$ is the number of pixels in the image, $k = |\mathcal{L}|$ is the number of possible labels for each pixel, and T is the number of iterations.

The above improvement is only valid for specific functions V corresponding to different models. The Potts model consists of a piecewise constant function $V(x) = 0$ when $x = 0$ and d otherwise. A direct approach leads to a linear time algorithm. For the linear model $V(x) = c|x|$, and the truncated linear model $V(x) = \min(c|x|, d)$, the computation is similar to computing a distance transform since it amounts to computing the min convolution with a linear cost. In that case, the min convolution corresponds to the computation of a Pasch-Hausdorff regularization (see Section 1.2). Finally a quadratic model and a truncated quadratic model are equivalent to the computation of Moreau envelope (or Euclidean distance transform) and so be computed in linear time.

More general distances could be used while still keeping a linear cost, e.g., any function V for which the intersection between two translations of its graph can be computed in constant time results in a linear time algorithm without making any convexity assumption. When convexity is present Formula (2) gives a linear-time algorithm.

4.2.2. *Pictorial Structures for Object Recognition* [82]. To recognize generic objects in an image, and to learn from example images, an energy function is minimized. It measures the match cost for each part of a pictorial model, and a deformation cost for each pair of connected parts. More precisely, a pictorial model is considered as an undirected graph $G = (V, E)$ with n parts $V = \{v_1, \dots, v_n\}$ defining the vertices, and the links between connected parts defining the edges E. A configuration $L = (l_1, \dots, l_n)$, where each l_i specifies the location of v_i, defines an instance of the object. The optimal match is obtained by solving

$$L^* = \operatorname*{Argmin}_{L} \left(\sum_{i=1}^{n} m_i(l_i) + \sum_{(v_i, v_j) \in E} d_{ij}(l_i, l_j) \right)$$

where the function $m_i(l_i)$ measures the degree of mismatch when part v_i is placed at location l_i in the image, and $d_{i,j}(l_i, l_j)$ measures the degree of deformation of the model when part v_i is placed at location l_i and part v_j is placed at location l_j.

While the minimization for arbitrary graphs $G = (V, E)$ and arbitrary functions m_i, d_{ij} is NP-hard, special cases can be solved efficiently. In particular, when the graph is a chain, a dynamic programming solution runs in $O(h^2 n)$, where n is the number of parts of the model and h the number of possible locations of each part. This complexity can be improved when the d_{ij} are restricted to the Mahalanobis distance between transformed locations

$$d_{ij}(l_i, l_j) = (T_{ij}(l_i) - T_{ji}(l_j))^T M_{ij}^{-1} T_{ij}(l_i) - T_{ji}(l_j)).$$

Then a minimization algorithm can be obtained that runs in $O(h'n)$, where h' is the number of grid locations in a discretization of the space of transformed locations given by T_{ij} and T_{ji}.

More precisely for an acyclic graph $G = (V, E)$, pick v_r an arbitrary node as the root of a tree. Denote by d_i the depth level of node v_i (the depth level of v_r is 0). For any vertex $v_j \neq v_r$, the best location given a location for its parent v_i is

$$B_j(l_i) = \min_{l_j} \left(m_j(l_j) + d_{ij}(l_i, l_j) + \sum_{v_c \in C_j} B_c(l_j) \right),$$

where C_j is the set of children of node v_j. Consequently a dynamic programming approach (computing B_j from the bottom up and then tracing the solution to get the argmin) gives a $O(nh^2)$ algorithm. However, using generalized distance transforms from Computational Convex Analysis [81, 162, 164], the computation is reduced to a $O(nh)$ cost. That improvement makes the difference between an algorithm too costly in practice and one fast enough for applications. (See also [63] for the application of fast algorithms in that context, and [52, 61] for a Convex Analysis point of view.)

4.3. **Differential Morphology.** Image processing has long been using morphological operators to compute various transformations. The core operators are the dilation and the erosion operators

$$\begin{aligned} (f \oplus g)(x) &= \sup_{y \in B}[f(y) + g(x-y)], \\ (f \ominus g)(x) &= \inf_{y \in B}[f(y) - g(x-y)], \end{aligned}$$

which correspond to the inf-convolution and deconvolution operators of convex analysis. See for instance [251] on the Minkowski addition operators for sets. Composition of these give smoothing filters like the opening $f \mapsto ((f \ominus g) \oplus g)$ and the closing $f \mapsto ((f \oplus g) \ominus g)$ operators. From dilation, one can define the morphological gradient, which is important in edge detection for image segmentation. The link between mathematical morphology in image processing, and Convex Analysis was noticed by Maragos in [168] who also noted the connection with partial differential equations like the Hamilton-Jacobi and the Eikonal equation. Moreover, Maragos made the connection with the Legendre-Fenchel transform through the slope transform [167, 107]. He further investigated the relation with PDEs and made the link between distance transforms and PDEs using level set methods [170, 171]; while Lucet [162, 164] took the reverse view of using fast algorithms from Section 2.2 to compute distance transforms. The connection to Hamilton-Jacobi equations was also made in [11, 239], and to the Eikonal equation in [130].

More traditional algorithms to compute dilation and erosion were presented in [255] for binary images, and in [256] in a broader context. Other efficient algorithms for morphological operators were presented in [94] (see also [73, 204]). A technique to compute the erosion using the FFT was presented in [251]. More connection between morphology and Convex Analysis were used in [38] with an explanation of the relationship using the max-plus algebra in [48]. In our context, Computational Convex Analysis provides a common framework for mathematical morphology algorithms.

5. Partial Differential Equations

While links between Convex Analysis and Partial Differential Equations (PDE) are well-known, recent work focused on using efficient numerical methods in one field to solve a problem in the other. In this section, we first explain how Convex Analysis helps finding solution to an Hamilton-Jacobi PDE. Conversely, we then explain how efficient PDE solvers help computing a fundamental Convex Analysis transform: the convex envelope.

5.1. **Lax-Hopf Formula.** The Lax and the Hopf functions are explicit solutions of

$$\begin{cases} \frac{\partial u}{\partial t} + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = g(\cdot) & \text{in } \mathbb{R}^n, \end{cases}$$

when either H or g is convex (where Du stands for the derivative of u with respect to the space variable x). They are defined as follow.

$$\begin{aligned} u_{\text{Lax}}(x,t) &= \inf_{y \in \mathbb{R}^n} \sup_{q \in \mathbb{R}^n} [g(x-y) + \langle y, q \rangle - tH(q)] = (g \oplus (tH)^*)(x), \\ u_{\text{Hopf}}(x,t) &= \sup_{q \in \mathbb{R}^n} \inf_{y \in \mathbb{R}^n} [g(x-y) + \langle y, q \rangle - tH(q)] = (g^* + tH)^*(x). \end{aligned}$$

The study of their properties using tools from Convex Analysis was performed in [122] (see also [123]). The formulas were extended further in [231]. The extension to quasiconvex functions was performed in [20] while Bardi et al. [19] considered the nonconvex nonconcave case. The use of fast algorithms to compute the solutions numerically was investigated in [62]. Considering there are numerous results on Hamilton-Jacobi-Bellman equations, we refer to [122] for an introduction from the point of view of Convex Analysis.

While the Hopf function can be computed in linear time as several conjugates, none of the current algorithms allows the computation of the inf-convolution in linear time. (We cannot use Formula (2) since the function u_0 is not assumed convex.) A nonlinear-time algorithm was proposed in [62] but it does not scale well with the dimension. Another better than brute force (subquadratic) general inf-convolution algorithm was investigated in [44].

The FLT and the faster LLT algorithms have also been used in efficient numerical simulations of the Burgers equation. For example, in [252] an adhesion model is investigated and numerical simulations (using the FLT) are performed to compare theories on mass distribution in the universe. The tools used are the Fenchel conjugate, the convex envelope, and other Convex Analysis arguments. The same algorithm is key to numerous numerical simulations for the Burgers' equation [15, 28, 87, 88, 103, 104, 205].

5.2. **Convexification.** For a locally bounded function $u_0 : \mathbb{R}^N \to \mathbb{R}$, the system

$$\begin{cases} \frac{\partial u}{\partial t} = \sqrt{1 + \|Du\|^2} F(Du, D^2 u) & \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

models the motion of the graph of the solution $u(t, \cdot)$ in the normal direction at each point, with speed $F(Du, D^2u)$. Using $F(Du, D^2u) = \min(0, \lambda_{\min}(D^2u))$, where $\lambda_{\min}$ denotes the smallest eigenvalue of D^2u, and assuming that u_0 defined on the closure of a convex open bounded set Ω is lsc on the closure of Ω and continuous on the boundary of Ω, the (unique viscosity) solution $u(t, \cdot)$ converges to the convex envelope of u_0 when $t \to \infty$ (see [253]). While finite difference methods were used to compute the convex hull, the reverse could also be done: using computational geometry algorithms to compute the solution to the partial differential equation above.

More recently, the convex envelope was found to be the solution of a nonlinear obstacle problem. The convex envelope u of the function $g : \mathbb{R}^n \to \mathbb{R}$ is a viscosity solution of

$$\max(u(x) - g(x), -\lambda_1[u](x)) = 0,$$

where $\lambda_1[u](x)$ is the smallest eigenvalue of the Hessian $D^2u(x)$ [208]. That formulation was further studied in [207] to obtain a PDE-based numerical algorithm to compute the convex envelope. The above equation offers the following advantage over a direct computation of the convex envelope: it provides a local characterization in contrast to the global nature of the convex envelope, it can provide a certificate that a given function is the convex envelope, and it allows the definition of approximate solutions.

The convex envelope of f is also the solution of

$$\min \int_{[a,b]} \sqrt{1 + \dot{u}^2(s)} ds$$

under the constraints $u \in W^{1,1}[a, b]$ (the Sobolev space), $u \le f$ on $[a, b]$, $u(a) = f(a)$, and $u(b) = f(b)$. (Minimizing that equation is the same as minimizing the curve length (length of the epigraph boundary) of u.) The problem can then be discretized and, assuming the initial function u is upper semicontinuous (usc) on $[a, b]$, its solution converges uniformly to the convex envelope [131].

In [47], the convex envelope of a function φ is computed as the solution to the problem

$$\varphi^{**}(\alpha) = \inf_{v \in W_0^{1,\infty}(\Omega)} \frac{1}{|\Omega|} \int_\Omega \varphi(\alpha + \nabla v(x)) dx,$$

while the convex envelope of a function f is approximated in [113] as the solution of

$$\min \frac{1}{2} \int_\Omega (u - f)^2$$

with the constraints $u \in BV^2(\Omega)$, $u = \hat{f}$ on $\partial\Omega$, and $u \leq f$ on Ω. (BV^2 is the space of bounded second variation, and $\hat{f}$ is the set of Dirichlet data on $\partial\Omega$, which is assumed known a priori.)

Other recent work on computing the convex envelope has focused on polynomials for which the computation of the convex envelope can be transformed into a minimization problem on a set of probability measures [184]. The later can be reduced to a semidefinite programming problem corresponding to the Hamiltonian of a convex formulation of the problem. See also [185] for another application of the method of moments and its relation with the convex envelope.

PDE have also been used for global optimization through a smoothing method linked with the convex envelope. More precisely, a cost function f of an unconstrained global optimization problem is smoothed with a function $u : [0, T) \times \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) - \max(0, \Delta u(t, x)),$$

with $0 < t < T$ and $u(0, \cdot) = f$ (Δu denotes the Laplacian of u with respect to x). It can be shown that the convex envelope of a solution $u(t, \cdot)$ is the convex envelope of f. Using the fact that global minima of the convex envelope are the same as the original function, an algorithm is devised to compute the global minimum [145].

Finally, we mention an area where Convex Analysis techniques have benefited PDEs. N. Ghoussoub recently introduced the theory of anti-selfdual Lagrangians to prove new variational principles of dynamical systems [91, 92, 93]. The foundation of anti-selfdual Lagrangian is Convex Analysis: Fenchel conjugate, Fenchel inequality, and the convex subdifferential. It will be interesting to see if Computational Convex Analysis algorithms contribute to solving numerically these dynamical systems.

5.3. **Interface Propagation.** The search for numerical methods to solve Hamilton-Jacobi equations has given rise to very efficient numerical schemes to compute curve evolutions and interface propagations. The Fast Marching method, the Level Set method, and the Fast Sweeping method are examples of such methods with a wide range of applications [235] (see also [249] for the application of level set methods to image science). In our context, these methods have been considered to compute distance transforms in image processing, which are a particular case of Moreau envelope. They could also be used to compute the convex envelope. Note that the main advantage of such methods is not the speed of computation, since they are outperformed by computational geometry and fast algorithms, but their potential ability to build a nonuniform grid on which the convex envelope is approximated.

Moreover, recent investigation into the fast sweeping methods for static Hamilton-Jacobi equations require the computation of the Legendre transform [132], which is performed symbolically or numerically using the fast transform algorithms. The Fenchel conjugate allows the transformation of the evolutive Hamilton-Jacobi equation of the first order into a Bellman equation with a finite horizon control problem [80]. The Hamiltonian can then be computed using the fast algorithms. Both articles refer to the original Fast Legendre Transform algorithm, which has since been superceded by the Linear-time Legendre Transform algorithm.

6. Multifractal Analysis, Network Communication, and Extremal Algebra

6.1. **Multifractal Analysis.** Fractal processes have allowed significant advances in a variety of fields, e.g. turbulence theory [237], stock market modeling, image processing, medical data, geophysics, network modelings [37, 225] (and in particular TCP traffic [156]), computer worms in network [55], analysis of paleoclimatic records [133], Blast furnace [121], etc. (see also references in [223]). Multifractal analysis describes the singularities (points where the function is nonsmooth) of a signal locally via the Hölder exponent, and globally via the large deviation multifractal spectrum $f(a)$ which estimates the exponential speed of decay of the probability to encounter a singularity equal to a at resolution n, when n tends to infinity.

To provide a global analysis of a univariate function $Y(t)$, called a fractal process, we use the concept of box-dimension (see [223] for details). Introduce

$$h_k^{(n)} := -\frac{1}{n} \log_2 \sup\{|Y(s) - Y(t)| \ : \ (k-1)2^{-n} \leq s \leq t \leq (k+2)2^{-n}\}$$

to define the *grain multifractal spectrum* as

$$f(a) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log N^{(n)}(a,\epsilon)}{n\log 2}, \text{ where } N^{(n)}(a,\epsilon) = \#\{k \ : \ |h_k^{(n)} - a| < \epsilon\}.$$

Using Large Deviation Principles, we can interpret the coarse spectrum f by studying the partition function

$$\tau_h(q) := \liminf_{n\to\infty} \frac{\log S_n^{(n)}(q)}{-n\log 2} \text{ where } S_n^{(n)}(q) := \sum_{k=0}^{2^n-1} 2^{-nqh_k^{(n)}},$$

defined for all $q \in \mathbb{R}$. Under the appropriate assumptions, one can prove that τ_h is the (concave) Legendre-Fenchel transform of f:

$$\tau_h(q) = f^*(q) := \inf_a (qa - f(a)).$$

(Properties of the Legendre conjugate of interest to multifractal analysis are introduced in [224].)

In practice, to compute f it is simpler to evaluate τ_h and then compute its conjugate using one of the Computational Convex Analysis algorithms. To facilitate such computation, a fast Legendre-Fenchel transform algorithm is included in the multifractal analysis toolbox FracLab [86].

Using multifractal analysis, the detection of artificial objects within natural environments was studied by noting that artificial objects have a wider Legendre spectrum than natural ones. Hence, a given image is subdivided into several subareas for which the Legendre spectrum is computed. The results are then compared to locate any artificial object [50]. (While fast algorithms can be used to compute the Legendre spectrum, the authors of [50] used a combination of numerical summations and limits coupled with the explicit Legendre conjugate formula for the function considered.)

6.2. **Network Communication.** The single output function $y(k)$ of a simple communication network with a single input function $x(k)$ is computed as the infimal convolution $y = (h \oplus x)$ of the input with the response characteristic function h, which is network and protocol dependent [120]. The goal is to recover the response characteristic by deconvolution: $h = (y \ominus x)$. Since the functions may not be convex, the Legendre transform is extended to handle nonconvex/nonconcave data as the set-valued slope transform

$$\mathcal{L}[x(u^*)](s) = \{x(u^*) - su^* \mid s = \frac{dx}{du}(u^*)\},$$

Then specialized computations are performed to evaluate the deconvolution operation associated with the slope transform. Note that the formulation of the extended Legendre transform is close to the parametric Legendre transform algorithm introduced and studied in [118]. Moreover, the recently extended PLQ algorithms allow computing with nonconvex functions [248] thereby opening the door to the efficient numerical computation of the Legendre transform. Another possibility is to use a general inf-convolution algorithm for nonconvex functions [44].

In a more general setting, the TCP protocol is covered by a theory of deterministic queuing systems found in computer networks called Network calculus [16]. It is the equivalent of system theory for which the usual $(\mathbb{R}, +.\cdot)$ algebra has been replaced with the commutative dioid $(\mathbb{R} \cup \{+\infty\}, \min, +)$. The usual convolution operation now becomes the inf-convolution (called min-plus convolution), its dual the deconvolution (called min-plus deconvolution), while the equivalent of the Fourier transform is the Fenchel conjugate [146]. While some authors have noted the connection between Network calculus and Convex Analysis, it does not appear that the full power of Convex Analysis, e.g. support functions and subadditive functions [117], has been fully exploited yet.

The foundation of network calculus are the min-plus convolution (inf-convolution) and the min-plus deconvolution [84]

$$(f \oplus g)(t) = \inf_u f(t-u) + g(u), \ \ (f \ominus g)(t) = \sup_u f(t+u) - g(u), \text{ with } t \geq u \geq 0, t \geq 0, u \geq 0.$$

Then relevant network entities e.g. arrival curves $\alpha(t)$, and service curves $\beta(t)$ can be computed. For example, the service curve $\beta(t)$ of the *concatenation* of n service elements with service curves $\beta_i(t)$ is $\beta(t) = \oplus_{i=1}^n \beta_i(t)$; a service element $\beta(t)$ with input bounded by $\alpha(t)$ admits a bound on its output $\alpha'(t)$ given by $\alpha' = (\alpha \ominus \beta)$. Of course, all the numerical computation of the inf-convolution, and deconvolution can be achieved with Computational Convex Analysis algorithms.

Diving deeper in min-plus algebra, one finds that eigenfunctions with respect to min-plus deconvolution are the affine functions admitting the Fenchel conjugate as eigenvalue. In fact, the Fenchel conjugate transforms inf-convolutions in additions and deconvolutions in subtractions. As such it becomes very advantageous to work in the Legendre domain. In addition, Fenchel's duality Theorem gives upper bounds on network entities [84].

6.3. **Max-Plus, Tropical, Idempotent, and Extremal Algebras.** A further generalization of Network Calculus is provided by replacing the usual arithmetic operations with new operations satisfying the idempotent property. Important semi-rings are the max-plus $\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +)$, and the min-plus $\mathbb{R}_{\min} := (\mathbb{R} \cup \{+\infty\}, \min, +)$ algebras, although many other semi-rings have been studied, which have strong relation to convex operators e.g. $(\mathcal{C}, \oplus, \odot)$ where $\mathcal{C}$ is the set of all convex compact subsets of $\mathbb{R}^d$ equiped with the Minkowski operations: $A \oplus B = \operatorname{co}(A \cup B)$ is the convex hull of the union, and $A \odot B = \{x \;:\; x = a + b, \text{ where } a \in A, b \in B\}$. The new addition $\oplus$ is idempotent: for all x, $x \oplus x = x$. Note that the terminology varies: idempotent semi-rings are sometimes called tropical semi-rings, idempotent semi-fields, minimax algebras, or extremal algebras.

The max-plus and min-plus idempotent semi-rings can be seen as the limit of the usual algebra under various transforms, e.g. $u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h))$ gives $u \oplus_h v \to \max(u, v)$ as $h \to 0$. The passage from $\mathbb{R}$ to $\mathbb{R}_{\max}$ (or min) is sometimes called the Maslov dequantization or Cole-Hopf transformation [157]. In the max-plus (and min-plus) algebra, the role of the Fourier transform and the convolution operators are played by the Legendre-Fenchel transform and the inf-convolution operator.

More result on idempotent calculus, in particular with links to Hamilton-Jacobi equations, can be found in [68, 67]. We refer to the recent survey [157] for more information and further references, and to [17] (especially Section 9.4) for an introduction in the context of discrete event systems (see also the introduction to a nonlinear theory for discrete event systems based on Max-plus algebra in [101, 102]).

The area of application of idempotent semi-rings is wide ranging: discrete mathematics, computer science, computer languages, linguistic problems, finite automata, optimization problems on graphs, discrete event systems and Petri nets, stochastic systems, evaluation of computer performance, computational problems, mathematical economics, etc. See references in [157]. Specific research has also focused on car-traffic laws [158, 159].

Many equivalent results from the usual algebra have been obtained, sometimes simplified due to the idempotent property. Numerous concepts have been investigated: the equivalent of the Riemann sum allows one to define a corresponding measure theory, while abstract convex sets have given rise to global algorithms for Lipschitz functions by extending the cutting plane algorithm [29]. A strong motivation to study such semi-rings comes from the fact that some nonlinear equations in the regular algebra become linear in the idempotent semi-ring, e.g. the Hamilton-Jacobi equation is linear over $\mathbb{R}_{\max}$. An abstract linear algebra theory has been studied within which the properties of the Legendre-Fenchel transform allow to hugely reduce the cost of computation for Hamilton-Jacobi equations [179]. More recent work has focused on geometry properties like convexity [60]. Algorithms also have their counterpart, and generic implementations over abstract semi-rings have been studied, while generalized linear algebra methods like the Jacobi Gauss-Seidel, and Gauss-Jordan method correspond to path-finding problems [51]. More details on semi-rings can be found in the monographs [95, 97, 98] while [99] gives an historical perspective with precise naming of the various semi-rings.

In the context of Computational Convex Analysis, the fact that the counterpart to the Fourier transform is the Legendre-Fenchel transform strongly highlights the importance of the fast algorithms of Section 2.2, especially the LLT which can be seen as a counterpart to the FFT. The huge importance of the Fourier transform in signal analysis is directly translated into the critical importance of the Legendre-Fenchel transform in Convex Analysis and Optimization. The link between the Fourier, Legendre, and Cramer transform was studied in [48] making a connection between linear and morphological system theory, the later been seen as linear system theory in the max-plus algebra. Connection between the Fourier Transform and the Fenchel conjugate were also made in [166].

One original motivation to introduce the maximum transform was to transform inf-convolution into addition [31] with a view toward applications in resource allocation [134]. It was later applied to nonlinear knapsack problems for which Formula (2) mitigated the curse of dimensionality [202]. Computational Convex Analysis provides the algorithmic framework to compute efficiently in these algebras.

7. Additional Applications

While additional applications can be found in the literature [241], let us emphasize a few more fields. Economics has long been using convex analysis tools. Consider the following simple economic example: a person buys quantities of a product from two manufacturers, with prices depending on the quantities bought. Minimizing the total cost amounts to computing the inf-convolution of the costs. The general dynamic programming (DP) model for linear transition benefits from conjugate duality, which reduces the curse of dimensionality by reformulating the problem as a recursive sequence of inf-convolutions, and computing its dual in which inf-convolutions are reduced to additions [135]. Similarly a general discrete resource allocation problem reduces to a sequence of inf-convolutions, which in the dual becomes a sequence of additions [136]. Note that the reduction of inf-convolution to addition was already noted in [32].

7.1. **Morphology Neural Networks.** Morphology Neural Networks is another field benefiting from non-standard algebra. In a classical neural net, each node combines information by multiplying output values x_i and corresponding weights λ_i, and summing: $y = \sum_i \lambda_i x_i$. However, in a morphology neural net, values and corresponding weights are added, and then the maximum value is taken: $y = \max_i \lambda_i + x_i$. The result can be computed using inf-convolution algorithms. Morphology neural networks arose from applications in image processing, which is not surprising considering the relation between classical dilation and erosion operators in image processing and the $\mathbb{R}_{\max}$ algebra. Of course, erosion and dilation corresponds to deconvolution and infconvolution in convex analysis, see Section 4.3.

We refer to [66] for an introduction to morphology neural networks with some initial applications, and to [227] for another introduction with a computation of the capabilities of such neural network. While Computational Convex Analysis algorithms provide efficient numerical tools to accelerate calculations at each node of the neural network, their systematic application in that context has not been studied yet.

7.2. **Medical imaging.** Many image reconstruction methods rely on the Radon transform to reconstruct an image. Then the problem of detecting singularities (discontinuities between adjacent pixels e.g. an edge in a picture) in the image becomes important since those correspond to a crack in a solid e.g. an aircraft wing or an engine, or a rupture in a tissue in medical diagnosis. It turns out that the singularities of the radon transform of a function f are related to the singularities of the function f through the Legendre transform [219]: if a curve S is the graph of a smooth function $y = g(x)$, then the dual curve S^* in the appropriate coordinates (β, q) is the graph of the function $q = h(\beta)$, where $h = L(g)$ is the Legendre transform of g, whichis defined when the gradient is invertible by

$$L(f)(s) := \langle s, \nabla^{-1} f(s) \rangle - f(\nabla^{-1} f(s)).$$

It coincides with the Legendre-Fenchel transform when in addition the function f is convex. When the gradient is not invertible, the Legendre transform may be multi-valued, and it has been generalized accordingly (see [219, Definition 1], the slope transform [71, 108, 167] and references therein).

While the computation of the Legendre transform may be ill-posed, this is not the case for the Legendre-Fenchel transform, see [219, Section 4.3] which also lists various methods to compute the Legendre transform numerically (at a single point contrary to the fast algorithms of Section 2.2). The stable computation of the generalized Legendre transform is investigated in [218] (see also [220, 259] for further results on that topic).

The computational convex analysis algorithms allow the visualization of the Legendre transform. More work is needed to compare their efficiency with current approaches in medical imaging. Especially relevant is the extension of PLQ algorithms to two variable functions, which would allow the fast detection of singular points.

The problem is generalized in [221], which considers the X-ray transform of a function f as the function which associates to each straight line l in $\mathbb{R}^3$, the integral of f over l with respect to the Lebesgue measure on l. The Radon transform uses planes in $\mathbb{R}^3$ instead of straight lines. The general case involves considering linear subspaces of arbitrary dimensions. As already mentioned, the main application of such investigation is computerized tomography when one looks for boundary of bones, or for holes in solids.

8. Conclusion

We presented core convex transforms, Computational Convex Analysis algorithms to compute them, and a wide range of application areas using them. We note the following directions for future research.

New Convex Analysis transforms, like the kernel average [27], have been recently introduced that require the extension of current algorithms. While some transforms like the Moreau envelope and the Fenchel conjugate can be computed efficiently for convex and nonconvex functions, the efficient computation for others is limited to convex functions e.g. the inf-convolution. There are also important applications that require the computation of the closest closed convex function of a nonconvex function i.e. to project on the cone of closed convex functions [142], which is closely linked to the problem of shape-preserving interpolation and approximation.

From the application perspective, researchers should use the most efficient algorithms e.g. the LLT algorithm instead of the FLT algorithm, and use the power of the Convex Analysis machinery. Rifkin and Lippert's contribution to Machine Learning [226] is one such example. It finds new results by using Fenchel duality coupled with Tikhonov regularization, instead of the classical Lagrangian duality.

Acknowledgments

The author would like to thank Pr. J.-B. Hiriart-Urruty for a careful reading of the manuscript which greatly improved the paper, Pr. W. Hare for pointing out several typos, and Pr. H. H. Bauschke for his continuous support and encouragements.

The paper was initially announced as a companion paper of [163] under the title *Fast Moreau Envelope Computation II: Applications*.

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows*, Prentice Hall Inc., Englewood Cliffs, NJ, 1993. Theory, algorithms, and applications.

[2] S. M. Aji and R. J. Mceliece, *The generalized distributive law*, Information Theory, IEEE Transactions on, 46 (2000), pp. 325–343.

[3] S. M. Aji and R. J. McEliece, *The generalized distributive law and free energy minimization*, in Proc. 39th Annu. Allerton Conf. Communication, Control, and Computing, Oct. 2001, pp. 672–681. Monticello, IL.

[4] M. Akian, J.-P. Quadrat, and M. Viot, *Bellman processes*, in 11th International Conference on Analysis and Optimization of Systems : Discrete Event Systems, S. Verlag, ed., vol. 199 of Lecture notes in control and information sciences, 1994.

[5] M. Akian, J.-P. Quadrat, and M. Viot, *Duality between probability and optimization*, in Idempotency (Bristol, 1994), vol. 11 of Publ. Newton Inst., Cambridge Univ. Press, Cambridge, 1998, pp. 331–353.

[6] W. N. Anderson, Jr. and R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl., 26 (1969), pp. 576–594.

[7] W. N. Anderson, Jr., R. J. Duffin, and G. E. Trapp, *Parallel subtraction of matrices (Hermitian semidefinite)*, Proc. Nat. Acad. Sci. U.S.A., 69 (1972), pp. 2530–2531.

[8] W. N. Anderson, Jr., T. D. Morley, and G. E. Trapp, *Characterization of parallel subtraction*, Proc. Nat. Acad. Sci. U.S.A., 76 (1979), pp. 3599–3601.

[9] W. N. Anderson, Jr. and G. E. Trapp, *Inequalities for the parallel connection of resistive n-port networks*, J. Franklin Inst., 209 (1975), pp. 305–313.

[10] J. Antezana, G. Corach, and D. Stojanoff, *Bilateral shorted operators and parallel sums*, Linear Algebra Appl., 414 (2006), pp. 570–588.

[11] A. B. Arehart, L. Vincent, and B. B. Kimia, *Mathematical morphology: The Hamilton-Jacobi connection*, in Computer Vision, 1993. Proceedings., Fourth International Conference on, 1993, pp. 215–219.

[12] M. Atteia and M. Raïssouli, *Self dual operators on convex functionals; geometric mean and square root of convex functionals*, J. Convex Anal., 8 (2001), pp. 223–240.

[13] H. Attouch and D. Azé, *Approximation and regularization of arbitrary functions in Hilbert spaces by the Lasry-Lions method*, Ann. Inst. H. Poincaré. Anal. Non Linéaire, 10 (1993), pp. 289–312.

[14] H. Attouch and R. J.-B. Wets, *A convergence theory for saddle functions*, Trans. Amer. Math. Soc., 280 (1983), pp. 1–41.

[15] E. Aurell, S.-g. N. Gurbatov, and S.-g. I. Simdyankin, *Numerical proof of self-similarity in Burgers' turbulence*, Tech. Rep. http://arxiv.org/abs/patt-sol/9602005, arXiv.org, 1996.

[16] F. Baccelli and D. Hong, *TCP is max-plus linear and what it tells us on its throughput*, in SIGCOMM '00: Proceedings of the conference on Applications, Technologies, Architectures, and Protocols for Computer Communication, New York, NY, USA, 2000, ACM Press, pp. 219–230.

[17] F. L. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity: An Algebra for Discrete Event Systems*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Ltd., Chichester, 1992.

[18] H. Bae and Y. Lim, *Parallel addition and fixed points of compressions on symmetric cones*, Math. Nachr., 246/247 (2002), pp. 20–30.

[19] M. Bardi and S. Faggian, *Hopf-type estimates and formulas for nonconvex nonconcave Hamilton-Jacobi equations*, SIAM J. Math. Anal., 29 (1998), pp. 1067–1086 (electronic).

[20] E. N. Barron, R. Jensen, and W. Liu, *Explicit solution of some first-order PDE's*, J. Dynam. Control Systems, 3 (1997), pp. 149–164.

[21] H. H. Bauschke, P. L. Combettes, and D. Noll, *Joint minimization with alternating Bregman proximity operators*, Pac. J. Optim., 2 (2006), pp. 401–424.

[22] H. H. Bauschke, R. Goebel, Y. Lucet, and X. Wang, *The proximal average: Applications, extensions, and computation*, tech. rep., University of British Columbia, 2008.

[23] H. H. Bauschke, Y. Lucet, and M. Trienis, *How to transform one convex function continuously into another*, SIAM Rev., 50 (2008), pp. 115–132.

[24] H. H. Bauschke, Y. Lucet, and X. Wang, *Primal-dual symmetric intrinsic methods for finding antiderivatives of cyclically monotone operators*, SIAM J. Control Optim., 46 (2007), pp. 2031–2051.

[25] H. H. Bauschke, E. Matoušková, and S. Reich, *Projection and proximal point methods: Convergence results and counterexamples*, Nonlinear Anal., 56 (2004), pp. 715–738.

[26] H. H. Bauschke and M. von Mohrenschildt, *Symbolic computation of Fenchel conjugates*, SIGSAM Bull., 40 (2006).

[27] H. H. Bauschke and X. Wang, *The kernel average of two convex functions and its application to the extension and representation of monotone operators*, Trans. Amer. Math. Soc., (2007).

[28] J. Bec, Frisch, U., and K. Khanin, *Kicked Burgers turbulence*, J. Fluid Mech., 416 (2000), pp. 239–267.

[29] G. Beliakov, *A review of applications of the cutting angle methods*, in Continuous optimization, vol. 99 of Appl. Optim., Springer, New York, 2005, pp. 209–248.

[30] R. Bellman and W. Karush, *On a new functional transform in analysis: the maximal transform*, Bull. Amer. Math. Soc., 67 (1961), pp. 501–503.

[31] R. Bellman and W. Karush, *Mathematical programming and the maximum transform*, Journal of the Society for Industrial and Applied Mathematics, 10 (1962), pp. 550–567.

[32] R. Bellman and W. Karush, *Functional equations in the theory of dynamic programming. XII. An application of the maximum transform*, J. Math. Anal. Appl., 6 (1963), pp. 155–157.

[33] ———, *On the maximum transform*, J. Math. Anal. Appl., 6 (1963), pp. 67–74.

[34] F. Bernard and L. Thibault, *Prox-regularity of functions and sets in Banach spaces*, Set-Valued Anal., 12 (2004), pp. 25–47.

[35] ———, *Prox-regular functions in Hilbert spaces*, J. Math. Anal. Appl., 303 (2005), pp. 1–14.

[36] ———, *Uniform prox-regularity of functions and epigraphs in Hilbert spaces*, Nonlinear Anal., 60 (2005), pp. 187–207.

[37] G. Bianchi, F. Vieira, and L. L. Ling, *A novel network traffic predictor based on multifractal traffic characteristic*, in Global Telecommunications Conference, 2004. GLOBECOM '04. IEEE, vol. 2, Dec. 2004, pp. 680–684.

[38] R. V. D. Boomgaard and H. J. A. M. Heijmans, *Mathematical Morphology and its Applications to Image and Signal Processing*, vol. 18 of Computational Imaging and Vision, Springer Netherlands, 2002, ch. Morphological Scale-Space Operators: An Algebraic Framework, pp. 282–290.

[39] G. Borgefors, *Distance transformations on hexagonal grids*, PRL, 9 (1989), pp. 97–105.

[40] J. M. Borwein and C. H. Hamilton, *Symbolic Fenchel conjugation*, Math. Programming, (2007). Online.

[41] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 3, Springer-Verlag, New York, 2000. Theory and examples.

[42] M. Bougeard, J.-P. Penot, and A. Pommellet, *Towards minimal assumptions for the infimal convolution regularization*, J. Approx. Theory, 64 (1991), pp. 245–270.

[43] L. M. Bregman, *A relaxation method of finding a common point of convex sets and its application to the solution of problems in convex programming*, Ž. Vyčisl. Mat. i Mat. Fiz., 7 (1967), pp. 620–631.

[44] D. Bremner, T. M. Chan, E. D. Demaine, J. Erickson, F. Hurtado, J. Iacono, S. Langerman, and P. Taslakian, *Necklaces, convolutions, and $X + Y$*, in Algorithms—ESA 2006, vol. 4168 of Lecture Notes in Comput. Sci., Springer, Berlin, 2006, pp. 160–171.

[45] Y. Brenier, *Un algorithme rapide pour le calcul de transformées de Legendre–Fenchel discrètes*, C. R. Acad. Sci. Paris Sér. I Math., 308 (1989), pp. 587–589.

[46] H. Breu, J. Gil, D. Kirkpatrick, and M. Werman, *Linear time Euclidean distance transform algorithms*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 17 (1995), pp. 529–533.

[47] B. Brighi and M. Chipot, *Approximated convex envelope of a function*, SIAM J. Numer. Anal., 31 (1994), pp. 128–148.

[48] B. Burgeth and J. Weickert, *An explanation for the logarithmic connection between linear and morphological system theory*, International Journal of Computer Vision, 64 (2005), pp. 157–169(13).

[49] J. V. Burke and M. Qian, *On the superlinear convergence of the variable metric proximal point algorithm using Broyden and BFGS matrix secant updating*, Math. Program., 88 (2000), pp. 157–181.

[50] Y. Caron, P. Makris, and N. Vincent, *A method for detecting objects using Legendre transform*, in RFAI team publication, Maghrebian Conference on Computer Science MCSEAI, Annaba (Algeria), May 2002, pp. 219–225.
[51] B. A. Carré, *An algebra for network routing problems*, J. Inst. Math. Appl., 7 (1971), pp. 273–294.
[52] C. Chaux, P. L. Combettes, J.-C. Pesquet, and V. R. Wajs, *A forward-backward algorithm for image restoration with sparse representations*, in Signal Processing with Adaptative Sparse Structured Representations (SPARS'05), Nov. 2005, pp. 49–52.
[53] J.-C. Chen, Y.-C. Wang, and J.-T. Chen, *A novel broadcast scheduling strategy using factor graphs and the sum-product algorithm*, Wireless Communications, IEEE Transactions on, 5 (2006), pp. 1241–1249.
[54] L. Chen and H. Y. H. Chuang, *A fast algorithm for Euclidean distance maps of a 2-d binary image*, Inf. Process. Lett., 51 (1994), pp. 25–29.
[55] Y. Chen, Y. Dong, D. Lu, and Y. Pan, *Intelligence and Security Informatics*, vol. 3495 of Lecture Notes in Computer Science, Springer Berlin / Heidelberg, 2005, ch. The Multi-fractal Nature of Worm and Normal Traffic at Individual Source Level, pp. 505–510.
[56] Y. T. Chin, H. Wang, L. P. Tay, H. Wang, and W. Y. C. Soh, *Vision guided agv using distance transform*, in Proceedings of the 32nd ISR (International Symposium on Robotics) 19-21 April 2001, 2001.
[57] F. H. Clarke, R. J. Stern, and P. R. Wolenski, *Proximal smoothness and the lower-C^2 property*, J. Convex Anal., 2 (1995), pp. 117–144.
[58] G. Cohen, S. Gaubert, and J.-P. Quadrat, *Duality and separation theorems in idempotent semimodules*, Linear Algebra Appl., 379 (2004), pp. 395–422. Tenth Conference of the International Linear Algebra Society.
[59] ———, *Projection and aggregation in maxplus algebra*, in Current trends in nonlinear systems and control, Systems Control Found. Appl., Birkhäuser Boston, Boston, MA, 2006, pp. 443–454.
[60] G. Cohen, S. Gaubert, J.-P. Quadrat, and I. Singer, *Max-plus convex sets and functions*, in Idempotent mathematics and mathematical physics, vol. 377 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2005, pp. 105–129.
[61] P. L. Combettes and V. R. Wajs, *Signal recovery by proximal forward-backward splitting*, SIAM Multiscale Model. Simul., 4 (2005), pp. 1168–1200.
[62] L. Corrias, *Fast Legendre–Fenchel transform and applications to Hamilton–Jacobi equations and conservation laws*, SIAM J. Numer. Anal., 33 (1996), pp. 1534–1558.
[63] D. Crandall, P. F. Felzenszwalb, and D. P. Huttenlocher, *Object recognition by combining appearance and geometry.*, in Toward Category-Level Object Recognition, 2006, pp. 462–482.
[64] O. Cuisenaire, *Distance Transformations: Fast Algorithms and Applications to Medical Image Processing*, PhD thesis, Université Catholique de Louvain, Louvain-la-Neuve, Belgium, Oct. 1999.
[65] P.-E. Danielsson, *Euclidean distance mapping*, Computer Graphics and Image Processing, 14 (1980), pp. 227–248. CGIP, Academic Press.
[66] J. L. Davidson and F. Hummer, *Morphology neural networks: an introduction with applications*, Circuits Syst. Signal Process., 12 (1993), pp. 177–210.
[67] P. Del Moral and M. Doisy, *Maslov idempotent probability calculus. I*, Teor. Veroyatnost. i Primenen., 43 (1998), pp. 735–751.
[68] ———, *Maslov idempotent probability calculus. II*, Teor. Veroyatnost. i Primenen., 44 (1999), pp. 384–400.
[69] L. Deniau, *Proposition d'un opérateur géométrique pour l'analyse et l'identification de signaux et images*, PhD thesis, Université de Paris-Sud, Centre d'Orsay, Dec. 1997.
[70] L. Deniau and J. Blanc-Talon, *Fractal analysis with Hausdorff distance under affine transformations*, tech. rep., ETCA-CREA-SP, 1995.
[71] L. Dorst and R. van den Boomgaard, *Morphological signal processing and the slope transform*, Signal Processing, 38 (1994), pp. 79–98.
[72] K. R. Driessel and A. Gerisch, *Zero-preserving iso-spectral flows based on parallel sums*, Linear Algebra Appl., 421 (2007), pp. 69–84.
[73] M. V. Droogenbroeckb and H. Talbot, *Fast computation of morphological operations with arbitrary structuring elements*, Pattern Recognition Letters, 17 (1996), pp. 1451–1460.
[74] J. Eckstein, *Nonlinear proximal point algorithms using Bregman functions, with applications to convex programming*, Math. Oper. Res., 18 (1993), pp. 202–226.
[75] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, EATC Monographs on Theoretical Computer Science, Springer-Verlag, 1987.
[76] R. Ellaia and J.-B. Hiriart-Urruty, *The conjugate of the difference of convex functions*, J. Optim. Theory Appl., 49 (1986), pp. 493–498.
[77] S.-L. Eriksson-Bique and H. Leutwiler, *A generalization of parallel addition*, Aequationes Math., 38 (1989), pp. 99–110.
[78] ———, *Minimal operators from a potential-theoretic viewpoint [MR1266217 (95a:47012)]*, in ICPT '91 (Amersfoort, 1991), Kluwer Acad. Publ., Dordrecht, 1994, pp. 47–59.
[79] R. Fabbri, O. M. Bruno, J. C. Torelli, and L. da F. Costa, *2d euclidean distance transform algorithms: A comparative survey*, ACM Computing Surveys, 40 (2007), pp. 2:1–2:44.
[80] M. Falcone and R. Ferretti, *Semi-Lagrangian schemes for Hamilton-Jacobi equations, discrete representation formulae and Godunov methods*, J. Comput. Phys., 175 (2002), pp. 559–575.

[81] P. F. Felzenszwalb and D. P. Huttenlocher, *Distance transforms of sampled functions*, Tech. Rep. TR2004-1963, Cornell Computing and Information Science, Sept. 2004.

[82] ———, *Pictorial structures for object recognition*, Int. J. Comp. Vision, 61 (2005), pp. 55–79.

[83] ———, *Efficient belief propagation for early vision*, Int. J. Comp. Vision, 70 (2006), pp. 41–54.

[84] M. Fidler and S. Recker, *Conjugate network calculus: a dual approach applying the Legendre transform*, Comput. Networks, 50 (2006), pp. 1026–1039.

[85] J. A. Fill and D. E. Fishkind, *The Moore-Penrose generalized inverse for sums of matrices*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 629–635 (electronic).

[86] *Fraclab: A fractal analysis toolbox for signal and image processing*. http://complex.inria.fr/, 2001.

[87] U. Frisch and J. Bec, *Burgulence*, in Les Houches 2000: New Trends in Turbulence, A. M. Lesieur and e. F. David, eds., Springer EDP-Sciences, 2001, pp. 341–383.

[88] U. Frisch, J. Bec, and B. Villone, *Singularities and the distribution of density in the Burgers/adhesion model*, Physica D, 152-153 (2001), pp. 620–635.

[89] M. Gavrilova and M. H. Alsuwaiyel, *Two algorithms for computing the Euclidean distance transform*, Tech. Rep. 2000-661-13, Computer Science Technical Reports, University of Calgary, 2000.

[90] A. Gheondea and S. Gudder, *Sequential product of quantum effects*, Proc. Amer. Math. Soc., 132 (2004), pp. 503–512 (electronic).

[91] N. Ghoussoub, *A theory of anti-selfdual Lagrangians: stationary case*, C. R. Math. Acad. Sci. Paris, 340 (2005), pp. 245–250.

[92] N. Ghoussoub and A. Moameni, *Selfdual variational principles for periodic solutions of hamiltonian and other dynamical systems*, tech. rep., Department of Mathematics, University of British Columbia, Sept. 2006.

[93] ———, *Anti-symmetric hamiltonians (ii): Variational resolutions for navier-stokes and other nonlinear evolutions*, tech. rep., Department of Mathematics, University of British Columbia, 2007.

[94] J. Gil and R. Kimmel, *Efficient dilation, erosion, opening, and closing algorithms*, IEEE Transactions on Pattern Analysis and Machine Intelligence, 24 (2002).

[95] K. Głazek, *A guide to the literature on semirings and their applications in mathematics and information sciences*, Kluwer Academic Publishers, Dordrecht, 2002. With complete bibliography.

[96] R. Goebel, *Self-dual smoothing of convex and saddle functions*, J. Convex Anal., 15 (2008), pp. 179–190.

[97] J. S. Golan, *Power algebras over semirings*, vol. 488 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1999. With applications in mathematics and computer science.

[98] ———, *Semirings and their applications*, Kluwer Academic Publishers, Dordrecht, 1999. Updated and expanded version of *The theory of semirings, with applications to mathematics and theoretical computer science* [Longman Sci. Tech., Harlow, 1992; MR1163371 (93b:16085)].

[99] ———, *Some recent applications of semiring theory*, in International Conference on Algebra in Memory of Kostia Beidar, National Cheng Kung University, Tainan, Mar. 2005.

[100] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim., 29 (1991), pp. 403–419.

[101] J. Gunawardena, *An introduction to idempotency*, in Idempotency (Bristol, 1994), vol. 11 of Publ. Newton Inst., Cambridge Univ. Press, Cambridge, 1998, pp. 1–49.

[102] ———, *From max-plus algebra to nonexpansive mappings: a nonlinear theory for discrete event systems*, Theoret. Comput. Sci., 293 (2003), pp. 141–167. Max-plus algebras.

[103] S. N. Gurbatov, S. I. Simdyankin, E. Aurell, U. Frisch, and G. Tth, *On the decay of Burgers turbulence*, J. Fluid Mech., 344 (1997), pp. 339–374.

[104] S. N. Gurbatov and A. V. Troussov, *The decay of multiscale signals - deterministic model of the Burgers turbulence*, tech. rep., arxiv.org, 2000.

[105] A. Hamdi, *A Moreau-Yosida regularization of a difference of two convex functions*, Appl. Math. E-Notes, 5 (2005), pp. 164–170 (electronic).

[106] W. Hare and C. Sagastizábal, *Benchmark of some nonsmooth optimization solvers for computing nonconvex proximal points*, Pac. J. Optim., 2 (2006), pp. 545–573.

[107] H. J. Heijmans and P. Maragos, *Lattice calculus of the morphological slope transform*, Signal Processing, 59 (May 1997), pp. 17–42(26).

[108] H. J. Heijmans and I. S. Molchanov, *Morphology on convolution lattices with applications to the slope transform and random set theory*, Journal of Mathematical Imaging and Vision, 8 (1998), pp. 199–214.

[109] P. Helluy and N. Seguin, *Relaxation models of phase transition flows*, M2AN Math. Model. Numer. Anal., 40 (2006), pp. 331–352.

[110] B. J. Hendrey and R. A. Jarvis, *Robot manipulator path planning*, in TENCON '92. Technology Enabling Tomorrow : Computers, Communications and Automation towards the 21st Century. 1992 IEEE Region 10 International Conference, 11-13 Nov 1992, Melbourne, Australia, vol. 2, 1992, pp. 865–870.

[111] ———, *A path planning heuristic for robotic manipulators*, in Intelligent Robots and Systems '93, IROS '93. Proceedings of the 1993 IEEE/RSJ International Conference on, 26-30 Jul 1993, Yokohama, Japan, vol. 2, 1993, pp. 1417–1423.

[112] W. H. Hesselink, *A linear-time algorithm for Euclidean feature transform sets*, Inf. Process. Lett., 102 (2007), pp. 181–186.

[113] W. Hinterberger and O. Scherzer, *Variational methods on the space of functions of bounded Hessian for convexification and denoising*, Computing, 76 (2006), pp. 109–133.

[114] J.-B. Hiriart-Urruty, *Extension of Lipschitz functions*, J. Math. Anal. Appl., 77 (1980), pp. 539–554.

[115] ———, *Lipschitz r-continuity of the approximate subdifferential of a convex function*, Math. Scand., 47 (1980), pp. 123–134.

[116] ———, *The deconvolution operation in convex analysis: an introduction*, Kibernet. Sistem. Anal., (1994), pp. 97–104, 190.

[117] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms*, vol. 305–306 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1993. Vol I: Fundamentals, Vol II: Advanced theory and bundle methods.

[118] J.-B. Hiriart-Urruty and Y. Lucet, *Parametric computation of the Legendre–Fenchel conjugate*, J. Convex Anal., 14 (2007), pp. 657–666.

[119] J.-B. Hiriart-Urruty and M.-L. Mazure, *Formulations variationnelles de l'addition parallèle et de la soustraction parallèlle d'opérateurs semi-définis positifs*, C. R. Acad. Sci. Paris Sér. I Math., 302 (1986), pp. 527–530.

[120] T. Hisakado, K. Okumura, V. Vukadinovic, and L. Trajkovic, *Characterization of a simple communication network using Legendre transform*, in Proc. IEEE Int. Symp. Circuits and Systems, vol. 3, May 2003, pp. 738–741.

[121] S. hua Luo, X. guan Liu, and J. sun Zeng, *Identification of multi-fractal characteristics of silicon content in blast furnace hot metal*, ISIJ Int., 8 (2007), pp. 1102–1107.

[122] C. Imbert, *Analyse non lisse : - Fonction d'appui de la Jacobienne généralisée de Clarke et de son enveloppe plénière - Quelques applications aux équations de Hamilton-Jacobi du premier ordre (fonctions de Hopf-Lax, Hamiltoniens diff. convexes, solutions sci)*, PhD thesis, Laboratoire de Mathématiques pour l'Industrie et la Physique, Université Paul Sabatier - Toulouse III, May 2000.

[123] ———, *Convex analysis techniques for Hopf-Lax formulae in Hamilton-Jacobi equations*, J. Nonlinear Convex Anal., 2 (2001), pp. 333–343.

[124] A. N. Iusem, B. F. Svaiter, and M. Teboulle, *Entropy-like proximal methods in convex programming*, Math. Oper. Res., 19 (1994), pp. 790–814.

[125] R. Jarvis, *Distance transform based path planning for robot navigation*, in Recent Trends in Mobile Robots, Y. Zheng, ed., vol. 11 of Robotics and Automated Systems, World Scientific, 1993, ch. 1, pp. 3–31.

[126] R. Jarvis, *An articulated six wheel drive robot for very rough terrain navigation*, in Proc. 2002 Australian Conference on Robotics and Automation, 27-29 November 2002, Auckland, 2002, pp. 133–139.

[127] ———, *Distance transform based visibility measures for covert path planning in known but dynamic environments*, in 2nd International Conference on Autonomous Robots and Agents December 13-15, 2004 Palmerston North, New Zealand, Dec. 2004, pp. 396–400.

[128] R. A. Jarvis, *Collision-free trajectory planning using distance transforms*, Trans. Inst. Eng. Aust. Mech. Eng., ME10 (1985), pp. 187–191.

[129] R. A. Jarvis, *Collision-free path planning in time-varying environments*, in Intelligent Robots and Systems '89. (IROS '89) 'The Autonomous Mobile Robots and Its Applications'. , Proceedings. IEEE/RSJ International Workshop on, 4-6 Sep 1989, Tsukuba, Japan, 1989, pp. 99–106.

[130] S. K., B. S., T. A., and Z. S.W., *Hamilton-jacobi skeletons*, International Journal of Computer Vision, 48 (July 2002), pp. 215–231(17).

[131] F. Kadhi and A. Trad, *Characterization and approximation of the convex envelope of a function*, J. Optim. Theory Appl., 110 (2001), pp. 457–466.

[132] C.-Y. Kao, S. Osher, and Y.-H. Tsai, *Fast sweeping methods for static Hamilton-Jacobi equations*, SIAM J. Numer. Anal., 42 (2005), pp. 2612–2632 (electronic).

[133] L. Karimovaa, Y. Kuandykova, N. Makarenkoa, M. Novakb, and S. Helamac, *Fractal and topological dynamics for the analysis of paleoclimatic records*, Physica A: Statistical and Theoretical Physics, 373 (2007).

[134] W. Karush, *A general algorithm for the optimal distribution of effort*, Management Science, 9 (1962), pp. 50–72.

[135] C. M. Klein, *Conjugate duality and its implications in dynamic programming*, Math. Comput. Modelling, 14 (1990), pp. 151–154.

[136] C. M. Klein and T. L. Morin, *Conjugate duality and the curse of dimensionality*, European Journal of Operational Research, 50 (1991), pp. 220–228.

[137] B. Koopen, *Contact of bodies in 2D-space: Implementing the Discrete Legendre Transform*, AI Master's thesis, Intelligent Autonomous Systems Group, University of Amsterdam, Feb. 2002.

[138] D. Kozinska, O. J. Tretiak, J. Nissanov, and C. Ozturk, *Multidimensional alignment using the Euclidean distance transform*, Graphical Models and Image Processing, 59 (1997), pp. 373–387.

[139] F. R. Kschischang, B. Frey, and H.-A. Loeliger, *Factor graphs and the sum-product algorithm*, IEEE Trans. Inform. Theory, 47 (2001), pp. 498–519.

[140] F. Kubo, *Conditional expectations and operations derived from network connections*, J. Math. Anal. Appl., 80 (1981), pp. 477–489.

[141] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., 246 (1979/80), pp. 205–224.

[142] T. Lachand-Robert and É. Oudet, *Minimizing within convex bodies using a convex hull method*, SIAM J. Optim., 16 (2005), pp. 368–379 (electronic).

[143] D. Lambert, J.-P. Crouzeix, V. H. Nguyen, and J.-J. Strodiot, *Finite convex integration*, J. Convex Anal., 11 (2004), pp. 131–146.

[144] J.-M. Lasry and P.-L. Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math., 55 (1986), pp. 257–266.

[145] M. S. K. Lau and C. P. Kwong, *A smoothing method of global optimization that preserves global minima*, J. Global Optim., 34 (2006), pp. 369–398.

[146] J.-Y. Le Boudec and P. Thiran, *Network calculus*, vol. 2050 of Lecture Notes in Computer Science, Springer-Verlag, Berlin, 2001. A theory of deterministic queuing systems for the internet.

[147] B. Legras, B. Joseph, and F. Lefèvre, *Vertical diffusivity in the lower stratosphere from lagrangian back-trajectory reconstructions of ozone profiles*, Journal of Geophysical Research-Atmospheres, 108 (2003), pp. 148–227.

[148] B. Legras, I. Pisso, G. Berthet, and F. Lefèvre, *Variability of the Lagrangian turbulent diffusion in the lower stratosphere*, Atmospheric Chemistry and Physics, 5 (2005), pp. 1605–1622.

[149] B. Lemaire, *The proximal algorithm*, in New methods in Optimization and their industrial uses (Pau/Paris, 1987), vol. 87 of Internat. Schriftenreihe Numer. Math., Birkhäuser, Basel, 1989, pp. 73–87.

[150] C. Lemaréchal and F. Oustry, *Growth conditions and $\mathcal{U}$-Lagrangians*, Set-Valued Anal., 9 (2001), pp. 123–129. Wellposedness in optimization and related topics (Gargnano, 1999).

[151] C. Lemaréchal, F. Oustry, and C. Sagastizábal, *The $\mathcal{U}$-Lagrangian of a convex function*, Trans. Amer. Math. Soc., 352 (2000), pp. 711–729.

[152] C. Lemaréchal and C. Sagastizábal, *More than first-order developments of convex functions: primal-dual relations*, J. Convex Anal., 3 (1996), pp. 255–268.

[153] ———, *Practical aspects of the Moreau-Yosida regularization: theoretical preliminaries*, SIAM J. Optim., 7 (1997), pp. 367–385.

[154] ———, *Variable metric bundle methods: from conceptual to implementable forms*, Math. Programming, 76 (1997), pp. 393–410.

[155] S. Letovsky and S. Kasif, *Predicting protein function from protein/protein interaction data: a probabilistic approach.*, Bioinformatics, 19 Suppl 1 (2003).

[156] J. Lévy Véhel and B. Sikdar, *A multiplicative multifractal model for tcp traffic*, iscc, 00 (2001), p. 0714.

[157] G. L. Litvinov, *Maslov dequantization, idempotent and tropical mathematics: A brief introduction*, Journal of Mathematical Sciences, 140 (2007), pp. 426–444.

[158] P. Lotito, E. Mancinelli, and J.-P. Quadrat, *Traffic assignment & Gibbs-Maslov semirings*, in Idempotent mathematics and mathematical physics, vol. 377 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2005, pp. 209–219.

[159] P. A. Lotito, E. M. Mancinelli, and J.-P. Quadrat, *A min-plus derivation of the fundamental car-traffic law*, IEEE Trans. Automat. Control, 50 (2005), pp. 699–705.

[160] Y. Lucet, *A fast computational algorithm for the Legendre–Fenchel transform*, Comput. Optim. Appl., 6 (1996), pp. 27–57.

[161] ———, *Faster than the Fast Legendre Transform, the Linear-time Legendre Transform*, Numer. Algorithms, 16 (1997), pp. 171–185.

[162] ———, *A linear Euclidean distance transform algorithm based on the Linear-time Legendre Transform*, in Proceedings of the Second Canadian Conference on Computer and Robot Vision (CRV 2005), Victoria BC, May 2005, IEEE Computer Society Press, pp. 262–267.

[163] ———, *Fast Moreau envelope computation I: Numerical algorithms*, Numer. Algorithms, 43 (2006), pp. 235–249.

[164] ———, *New sequential exact Euclidean distance transform algorithms based on convex analysis*, Image and Vision Computing, (2006). online.

[165] Y. Lucet, H. H. Bauschke, and M. Trienis, *The piecewise linear-quadratic model for computational convex analysis*, Comput. Optim. Appl., (2006). Online.

[166] P. Maragos, *Morphological systems: Slope transforms and max-min difference and differential equations*, Signal Processing, 38 (1994), pp. 57–77.

[167] P. Maragos, *Slope transforms: theory and application to nonlinear signal processing*, Signal Processing, IEEE Transactions on [see also Acoustics, Speech, and Signal Processing, IEEE Transactions on], 43 (1995), pp. 864–877. 1053-587X.

[168] P. Maragos, *Differential morphology and image processing*, IEEE Transactions on Image Processing, 5 (1996), pp. 922–927.

[169] ———, *Differential morphology*, in Nonlinear Image Processing, S. K. Mitra and G. L. Sicuranza, eds., Academic Press Series in Communications, Networking, and Multimedia, Academic Press, 2001, ch. 10, pp. 289–329.

[170] ———, *Handbook of Image and Video Processing*, Elsevier, second ed., 2005, ch. PDEs for Morphological Scale-Spaces and Eikonal Applications, pp. 587–612.

[171] P. Maragos and M. A. Butt, *Curve evolution, differential morphology, and distance transforms applied to multiscale and Eikonal problems*, Fundamenta Informaticae, 41 (2000), pp. 91–129.

[172] M. Marzouqi and R. A. Jarvis, *Covert robotics: hiding in known environments*, in Robotics, Automation and Mechatronics, 2004 IEEE Conference on, 1-3 Dec. 2004, Singapore, vol. 2, Dec. 2004, pp. 804–809.

[173] C. R. Maurer Jr, R. Qi, and V. V. Raghavan, *A linear time algorithm for computing exact Euclidean distance transforms of binary images in arbitrary dimensions*, IEEE Trans. Pattern Anal. Mach. Intell., 25 (2003), pp. 265–270.

[174] M.-L. Mazure, *L'addition parallèle d'opérateurs interprétée comme inf-convolution de formes quadratiques convexes*, RAIRO Modél. Math. Anal. Numér., 20 (1986), pp. 497–515.

[175] ———, *La soustraction parallèle d'opérateurs interprétée comme déconvolution de formes quadratiques convexes*, Optimization, 18 (1987), pp. 465–484.

[176] ———, *Shorted operators through convex analysis*, in Trends in mathematical optimization (Irsee, 1986), vol. 84 of Internat. Schriftenreihe Numer. Math., Birkhäuser, Basel, 1988, pp. 139–152.

[177] ———, *Équations de convolution et formes quadratiques*, Ann. Mat. Pura Appl. (4), 158 (1991), pp. 75–97.

[178] M.-L. Mazure and M. Volle, *Équations inf-convolutives et conjugaison de Moreau-Fenchel*, Ann. Fac. Sci. Toulouse Math. (5), 12 (1991), pp. 103–126.

[179] W. M. McEneaney, *Combining Legendre/Fenchel transformed operators on max-plus spaces for nonlinear control solution*, in Decision and Control, 2004. CDC. 43rd IEEE Conference on, vol. 2, Dec. 2004, pp. 1152–1157.

[180] K. McKinnon and M. Mongeau, *A generic global optimization algorithm for the chemical and phase equilibrium problem*, J. Global Optim., 12 (1998), pp. 325–351.

[181] F. Meng, D. Sun, and G. Zhao, *Semismoothness of solutions to generalized equations and the Moreau-Yosida regularization*, Math. Program., 104 (2005), pp. 561–581.

[182] F. Meng and G. Zhao, *On second-order properties of the Moreau-Yosida regularization for constrained nonsmooth convex programs*, Numer. Funct. Anal. Optim., 25 (2004), pp. 515–529.

[183] F. W. Meng and Y. Hao, *Piecewise smoothness for Moreau-Yosida approximation to a piecewise C^2 convex function*, Adv. Math. (China), 30 (2001), pp. 354–358.

[184] R. J. Meziat, D. P. no, and P. Pedregal, *An alternative approach for non-linear optimal control problems based on the method of moments*, Computational Optimization and Applications, 38 (2007), pp. 147–171.

[185] R. J. Meziat and J. Villalobos, *Analysis of microstructures and phase transistion phenomena in one-dimensional, non-linear elasticity by convex optimization*, Struct. Multidisc. Optim., 32 (2006), pp. 507–519.

[186] R. Mifflin, *A quasi-second-order proximal bundle algorithm*, Math. Programming, 73 (1996), pp. 51–72.

[187] R. Mifflin, L. Qi, and D. Sun, *Properties of the Moreau-Yosida regularization of a piecewise C^2 convex function*, Math. Program., 84 (1999), pp. 269–281.

[188] R. Mifflin and C. Sagastizábal, *$\mathcal{VU}$-decomposition derivatives for convex max-functions*, in Ill-posed variational problems and regularization techniques (Trier, 1998), vol. 477 of Lecture Notes in Econom. and Math. Systems, Springer, Berlin, 1999, pp. 167–186.

[189] ———, *Functions with primal-dual gradient structure and $\mathcal{U}$-Hessians*, in Nonlinear optimization and related topics (Erice, 1998), vol. 36 of Appl. Optim., Kluwer Acad. Publ., Dordrecht, 2000, pp. 219–233.

[190] ———, *On $\mathcal{VU}$-theory for functions with primal-dual gradient structure*, SIAM J. Optim., 11 (2000), pp. 547–571 (electronic).

[191] ———, *Primal-dual gradient structured functions: second-order results; links to epi-derivatives and partly smooth functions*, SIAM J. Optim., 13 (2003), pp. 1174–1194 (electronic).

[192] ———, *On the relation between $\mathcal{U}$-Hessians and second-order epi-derivatives*, European J. Oper. Res., 157 (2004), pp. 28–38.

[193] ———, *$\mathcal{VU}$-smoothness and proximal point results for some nonconvex functions*, Optim. Methods Softw., 19 (2004), pp. 463–478.

[194] ———, *Relating $\mathcal{U}$-Lagrangians to second-order epi-derivatives and proximal-tracks*, J. Convex Anal., 12 (2005), pp. 81–93.

[195] R. Mifflin and C. Sagastizábal, *A $\mathcal{VU}$-algorithm for convex minimization*, Mathematical Programming, (2005).

[196] S. K. Mitra and K. Manjunatha Prasad, *The regular shorted matrix and the hybrid sum*, Adv. in Appl. Math., 18 (1997), pp. 403–422.

[197] J.-J. Moreau, *Inf-convolution des fonctions numériques sur un espace vectoriel*, C. R. Acad. Sci. Paris, 256 (1963), pp. 5047–5049.

[198] ———, *Propriétés des applications "prox"*, C. R. Acad. Sci. Paris, 256 (1963), pp. 1069–1071.

[199] ———, *Proximité et dualité dans un espace Hilbertien*, Bull. Soc. Math. France, 93 (1965), pp. 273–299.

[200] ———, *Convexity and duality*, in Functional Analysis and Optimization, Academic Press, New York, 1966, pp. 145–169.

[201] ———, *Inf-convolution, sous-additivité, convexité des fonctions numériques*, J. Math. Pures Appl. (9), 49 (1970), pp. 109–154.

[202] T. L. Morin and R. E. Marsten, *An algorithm for nonlinear knapsack problems*, Management Science, 22 (1976), pp. 1147–1158.

[203] T. D. Morley, *Parallel summation, Maxwell's principle and the infimum of projections*, J. Math. Anal. Appl., 70 (1979), pp. 33–41.

[204] N. Nikopoulos and I. Pitas, *A fast implementation of 3-D binary morphological transformations*, IEEE Transactions on Image Processing, 9 (2000).

[205] A. Noullez, S. N. Gurbatov, E. Aurell, and S. I. Simdyankin, *The global picture of self-similar and not self-similar decay in Burgers turbulence*, Tech. Rep. nlin.CD/0409022, arXiv.org eprint archive, Sept. 2004.

[206] A. Noullez and M. Vergassola, *A fast Legendre transform algorithm and applications to the adhesion model*, J. Sci. Comput., 9 (1994), pp. 259–281.

[207] A. M. Oberman, *Computing the convex envelope using a nonlinear partial differential equation*, Mathematical Models and Methods in Applied Sciences, (2007). Accepted.

[208] ———, *The convex envelope is the solution of a nonlinear obstacle problem*, Proc. Amer. Math. Soc., 135 (2007), pp. 1689–1694 (electronic).

[209] È. L. Pekarev and J. L. Šmul′jan, *Parallel addition and parallel subtraction of operators*, Izv. Akad. Nauk SSSR Ser. Mat., 40 (1976), pp. 366–387, 470.

[210] J.-P. Penot, *Proximal mappings*, J. Approx. Theory, 94 (1998), pp. 203–221.

[211] J.-P. Penot and R. Ratsimahalo, *On the Yosida approximation of operators*, Proc. Roy. Soc. Edinburgh Sect. A, 131 (2001), pp. 945–966.

[212] R. Poliquin and R. T. Rockafellar, *Generalized Hessian properties of regularized nonsmooth functions*, SIAM J. Optim., 6 (1996), pp. 1121–1137.

[213] ———, *Prox-regular functions in variational analysis*, Trans. Amer. Math. Soc., 348 (1996), pp. 1805–1838.

[214] F. P. Preparata and M. I. Shamos, *Computational Geometry*, Texts and monographs in computer science, Springer-Verlag, third ed., 1990.

[215] S. Prohaska and H. C. Hege, *Fast visualization of plane-like structures in voxel data*, in VIS '02: Proceedings of the conference on Visualization '02, IEEE Computer Society, 2002, pp. 29–36. Boston, Massachusetts.

[216] L. Qi, *Second-order analysis of the Moreau-Yosida regularization*, in Nonlinear analysis and convex analysis (Niigata, 1998), World Sci. Publishing, River Edge, NJ, 1999, pp. 16–25.

[217] P. J. Rabier and A. Griewank, *Generic aspects of convexification with applications to thermodynamic equilibrium*, Arch. Rational Mech. Anal., 118 (1992), pp. 349–397.

[218] A. G. Ramm, A. M. STeinberg, and A. I. Zaslavsky, *Stable calculation of the Legendre transform*, Journal of Mathematical Analysis and Applications, 178 (1993), pp. 592–602.

[219] A. G. Ramm and A. I. Zaslavsky, *Reconstructing singularities of a function from its Radon transform*, Math. Comput. Modelling, 18 (1993), pp. 109–138.

[220] ———, *Singularities of the Radon transform*, Bull. Amer. Math. Soc. (N.S.), 28 (1993), pp. 109–115.

[221] ———, *X-ray transform, the Legendre transform, and envelopes*, J. Math. Anal. Appl., 183 (1994), pp. 528–546.

[222] J. P. Revalski and M. Théra, *Enlargements and sums of monotone operators*, Nonlinear Anal., 48 (2002), pp. 505–519.

[223] R. H. Riedi, *Multifractal processes*, in Theory and applications of long-range dependence, Birkhäuser Boston, Boston, MA, 2003, pp. 625–716.

[224] R. H. Riedi, M. S. Crouse, V. J. Ribeiro, and R. G. Baraniuk, *A multifractal wavelet model with application to network traffic*, IEEE Transactions on Information Theory, 45 (1999), pp. 992–1018.

[225] R. H. Riedi and J. L. Vhel, *Multifractal properties of TCP traffic: a numerical study*, Tech. Rep. 3129, INRIA, Rocquencourt, Mar. 1997.

[226] R. M. Rifkin and R. A. Lippert, *Value regularization and Fenchel duality*, J. Mach. Learn. Res., 8 (2007), pp. 441–479.

[227] G. Ritter and P. Sussner, *An introduction to morphological neural networks*, in ICPR '96: Proceedings of the International Conference on Pattern Recognition (ICPR '96) Volume IV-Volume 7472, Washington, DC, USA, 1996, IEEE Computer Society, pp. 709–717.

[228] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New york, 1970.

[229] ———, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optimization, 14 (1976), pp. 877–898.

[230] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[231] R. T. Rockafellar and P. R. Wolenski, *Convexity in Hamilton-Jacobi theory. II. Envelope representations*, SIAM J. Control Optim., 39 (2000), pp. 1351–1372 (electronic).

[232] A. Rosenfeld and J. L. Pfaltz, *Sequential operations in digital picture processing*, J. ACM, 13 (1966), pp. 471–494.

[233] *Scilab*. http://www.scilab.org, 1994.

[234] J. A. Sethian, *Fast marching methods*, SIAM Review, 41 (1999), pp. 199–235.

[235] ———, *Level Set Methods and Fast Marching Methods*, vol. 3 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, second ed., 1999. Evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science.

[236] Z.-S. She, E. Aurell, and U. Frisch, *The inviscid Burgers equation with initial data of brownian type*, Comm. Math. Phys., 148 (1992), pp. 623–641.

[237] B. Shi, B. Vidakovic, and G. G. Katul, *Quantifying the effects of atmospheric stability on the multifractal spectrum of turbulence*, tech. rep., Georgia Institute of Techology, School of Industrial and Systems Engineering, Statistics Group, 2004.

[238] F. Y. Shih and Y.-T. Wu, *Fast Euclidean distance transformation in two scans using a 3×3 neighborhood*, Comput. Vis. Image Underst., 93 (2004), pp. 195–205.

[239] K. Siddiqi, S. Bouix, A. Tannenbaum, and S. W. Zucker, *The Hamilton-Jacobi skeleton*, Seventh International Conference on Computer Vision (ICCV'99), 02 (1999), p. 828.

[240] T. Strömberg, *Representation formulae for infimal convolution with applications*, in Analysis, algebra, and computers in mathematical research (Luleå, 1992), vol. 156 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1994, pp. 319–334.

[241] ———, *The operation of infimal convolution*, Dissertationes Math. (Rozprawy Mat.), 352 (1996).

[242] S. Sundar and Z. Shiller, *Optimal obstacle avoidance based on the Hamilton-Jacobi-Bellman equation*, IEEE Transactions on Robotics and Automation, 13 (1997), pp. 305–310.

[243] K. W. Tang and R. A. Jarvis, *A fast algorithm to plan a collision-free path in cluttered 2d environments*, in Robotics, Automation and Mechatronics, 2004 IEEE Conference on, 1-3 Dec. 2004, Singapore, vol. 2, 2004, pp. 786–791.

[244] M. Teboulle, *Entropic proximal mappings with applications to nonlinear programming*, Math. Oper. Res., 17 (1992), pp. 670–690.

[245] Y. Tian, *How to express a parallel sum of k matrices*, J. Math. Anal. Appl., 266 (2002), pp. 333–341.

[246] A. N. Tikhonov and V. A. Morozov, *Methods for the regularization of ill-posed problems*, Vychisl. Metody i Programmirovanie, (1981), pp. 3–34.

[247] J. Toriwaki and K. Mori, *Distance transformation and skeletonization of 3d pictures and their applications to medical images*, in Digital and Image Geometry: Advanced Lectures, R. K. E. G. Bertrand, A. Imiya, ed., vol. 2243 / 2001 of Lecture Notes in Computer Science, Springer-Verlag Heidelberg, 2001, pp. 412–429.

[248] M. Trienis, *Computational convex analysis: From continuous deformation to finite convex integration*, Master's thesis, University of British Columbia, 2007.

[249] Y.-H. R. Tsai and S. Osher, *Total variation and level set methods in image science*, Acta Numer., 14 (2005), pp. 509–573.

[250] P. Tseng and Z.-Q. Luo, *On computing the nested sums and infimal convolutions of convex piecewise-linear functions*, Journal of Algorithms, 21 (1996), pp. 240–266.

[251] A. V. Tuzikov, G. L. Margolin, and A. I. Grenov, *Convex set symmetry measurement via Minkowski addition*, J. Math. Imaging Vision, 7 (1997), pp. 53–68.

[252] M. Vergassola, B. Dubrulle, U. Frisch, and N. A., *Burgers's equation, devil's staircases and the mass distribution function for large-scale structures*, Astronomy and Astrophysics, 289 (1994), pp. 325–356.

[253] L. A. Vese, *A method to convexify functions via curve evolution*, Comm. Partial Differential Equations, 24 (1999), pp. 1573–1591.

[254] C. Villani, *Optimal transport, old and new*, 2008. http://www.umpa.ens-lyon.fr/˜cvillani/Cedrif /B07C.StFlour.pdf.

[255] L. Vincent, *Morphological transformations of binary images with arbitrary structuring elements*, Signal Processing, 22 (1991), pp. 3–23.

[256] ———, *Mathematical Morphology in Image Processing*, Marcel-Dekker, New York, Sept. 1992, ch. Morphological Algorithms, pp. 255–288.

[257] E. T. P. Wong and R. Jarvis, *Real time obstacle detection and navigation planning for a humanoid robot in an indoor environment*, in Proceedings of the 2004 IEEE Conference on Robotics, Automation and Mechatronics, 1-3 December 2004, Singapore, Dec. 2004, pp. 693–699.

[258] K. Yosida, *Functional analysis*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

[259] A. I. Zaslavsky, *Legendre transforms and dual varieties*, J. Math. Anal. Appl., 183 (1994), pp. 670–684.

[260] A. Zelinsky, *A mobile robot exploration algorithm*, IEEE Transactions on Robotics and Automation, 8 (1992), pp. 707–717.

[261] A. Zelinsky, *A navigation algorithm for industrial mobile robots*, in Emerging Technologies and Factory Automation, 1992. IEEE International Workshop on, 11-14 Aug 1992, 1992, pp. 525–530.

[262] A. Zelinsky, *Using path transforms to guide the search for findpath in 2d*, International Journal of Robotics Research, 13 (1994), pp. 315–325.

Computer Science, I. K. Barber School of Arts & Sciences Unit # 4, University of British Columbia Okanagan, 3333 University Way, Kelowna BC V1V 1V7, Canada

E-mail address: yves.lucet@ubc.ca

URL: http://people.ok.ubc.ca/ylucet/