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Proton spin after 30 years: what we know and what we don't?

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ABSTRACT

More than three decades has passed since the European Muon Collaboration published the first surprising result on the spin structure of the proton. Much theoretical and experimental progress has been made in understanding the origins of the proton spin. In this review, we will discuss what we have learned so far, what are still missing, and what we shall expect to learn from the upcoming experiments including JLab 12 GeV and Electron-Ion Collider. In particular, we focus on first principles calculations and experimental measurements of the total gluon helicity ΔG , and quark and gluon orbital angular momenta.

Key points:

- There are two established approaches to look at the compositions of the proton spin: frame-independent spin structure, $\Delta\Sigma/2 + L_q + J_g = \hbar/2$ (“Ji sum rule”) and infinite-momentum-frame or parton spin structure, $\Delta\Sigma/2 + \Delta G + \ell_q + \ell_g = \hbar/2$ (“Jaffe-Manohar sum rule”).
- In the frame-independent approach, quark orbital L_q and gluon angular momentum contributions J_g can be extracted from moments of generalized parton distributions. Results from Jlab 6 GeV and HERMES experiments suggest a substantial quark orbital contribution L_q .
- In terms of partons, the quark and gluon helicity contributions, $\Delta\Sigma/2$ and ΔG , have a simple physical interpretation, and the result from RHIC-spin experiments has provided first important constraint on the total gluon helicity ΔG .
- Development of large-momentum effective theory along with lattice QCD simulations provides first-principles calculations of the spin structure. The recent results on $\Delta\Sigma$, L_q , J_g , and ΔG have provided intriguing theoretical pictures.
- JLab 12 GeV program will provide much improved information on quark orbital angular momentum L_q and ℓ_q . Future Electron-Ion Collider will provide high-precision measurements on the gluon helicity ΔG and gluon angular momentum J_g and ℓ_g .

Website summary: More than three decades have passed since the European Muon Collaboration published the first surprising result on the spin structure of the proton. Much theoretical and experimental progress has been made in understanding the origins of the proton spin. In this review, we will discuss what we have learned, what are still missing, and what we shall expect from the upcoming experiments including JLab 12 GeV and Electron-Ion Collider.

1 Introduction

The proton is a spin-1/2 particle, thought to be fundamental when discovered as a basic constituent of atomic nuclei by Rutherford in 1917¹. However, the subsequent measurement of its magnetic moment² showed a significant deviation from the Dirac value for a point-like object³. Ever since, the proton substructure along with the origin of its spin and magnetic moment has intrigued nuclear and particle physicists for nearly a century.

Every model of the proton ought to give an explanation for its spin: from the Skyrme model⁴, to Gell-Mann and Zweig's quark model^{5,6}, and to many other models proposed in the 70's and 80's^{7,8}. The simplest and most successful one has been the quark model which inspired, among others, the discovery of quantum chromodynamics (QCD)⁹, the fundamental theory of strong interactions. The non-relativistic quark model has an exceedingly simple explanation for the proton/neutron spin and the associate magnetic moments¹⁰, as well as their excitations¹¹: Three constituent quarks are all in the s -wave orbit, and their spins couple to $1/2$ in a way consistent with the $SU(2)_{\text{spin}} \times SU(3)_{\text{flavor}}$ combined spin-flavor symmetry¹².

The quark model picture was put under a straight test through polarized deep-inelastic scattering (DIS) on a polarized proton target¹³. In 1987, the European Muon Collaboration (EMC) delivered a measurement for the fraction of the proton spin carried by quarks^{14,15},

$$\Delta\Sigma(Q^2 = 10.7\text{GeV}^2) = 0.060 \pm 0.047 \pm 0.069, \quad (1)$$

which is consistent with zero! The EMC data also showed a significant deviation from the Ellis-Jaffe sum rule for the $g_1(x, Q^2)$ structure function based on the quark picture¹⁶. This result literally shocked the physics community, and created the so-called proton spin "crisis" or proton spin problem. The discrepancy has since inspired a large number of experimental and theoretical studies which have been reviewed in a number of papers¹⁷⁻²². The most important lesson we have learned is that the underlying theory for the proton structure, QCD, has a much more sophisticated way to build up the proton spin.

QCD is fundamental and beautiful on the one hand, and is sophisticated and defies simple ways to understand on the other. For example, it is no longer feasible, or we have failed so far, to come up with an entire quark and gluon wave function for the proton and to check the content of various components. Therefore, we will consider instead the so-called *sum rules* or decompositions of the spin into various physical parts. This has been the main approach to understand the origins of the proton spin so far.

This article is not a comprehensive review of hadron spin physics. In particular, it is not meant to be an update on the recent reviews^{19,22} which have done an excellent job. Rather, it focuses sharply on the questions related to the origins of the proton spin. We mainly discuss issues like: does it make sense to talk about different parts of the proton spin? What will be an interesting and physically meaningful decomposition for the spin? To what extent do we believe that we can measure each part experimentally? How can one calculate them in fundamental theory and put the results under experimental tests? We hope that, after 30 years of the EMC result, this article can help the physics community at large to understand what we know now, what we don't, and what we shall expect in the future. In particular, what the Electron Ion Collider (EIC) will help to answer the fundamental questions about the origins of the proton spin^{23,24}.

2 Spin Structure in Sum Rules

Without knowing the wave function, the angular momentum (AM) or spin structure of a composite system can be studied through various contributions to the total. Thus, to explore origins of the proton spin, we can start from QCD AM operator in terms of individual sources,

$$\vec{J}_{\text{QCD}} = \sum_{\alpha} \vec{J}_{\alpha}, \quad (2)$$

through which, the spin projection $\hbar/2$ can be expressed as a sum of different contributions.

One must be aware of some limitations in this approach. Since the proton is an eigenstate of the relativistic Pauli-Lubanski spin²⁵, the individual contributions can only be the quantum mechanical expectation values of the AM sources from the entire bound-state wave function. Moreover, they are "renormalization-scale dependent", because individual operators are not separately conserved, and the resulting ultra-violet (UV) divergences must be renormalized in the senses that the short distance physics is included in the effective AM operators²⁶. In non-relativistic systems, with the exception of particles moving in a magnetic field, the AM sources corresponding to different physical degrees of freedom obey the separate AM commutation relations. In quantum field theories, the simple commutation relations at the bare-field level are violated when dressed with interactions, and only the total AM commutation relations are protected by rotational symmetry²⁷. Finally, gauge symmetry imposes important constraints on what is physically measurable.

Still there exist more than one way to split the AM operator and derive spin sum rules for the proton. A physically-interesting spin sum rule shall have the following properties:

- *Experimental Measurability*. The overwhelming interest in the proton spin began with the EMC data. Much of the followup experiments, including HERMES and COMPASS, polarized RHIC²⁸, Jefferson Lab 12 GeV upgrade²⁹ and EIC^{23,24}, have been partially motivated to search a full understanding of the proton spin.

- *Frame Independence*: Since spin is an intrinsic property of a particle, one naturally searches for a description of its structure independent of its momentum. How the individual contributions depend on the reference frame requires understanding on the Lorentz transformation properties of \vec{J}_α . Since the proton structure probed in high-energy scattering is best described in the infinite momentum frame (IMF), a partonic picture of the spin is interesting in this special frame of reference.

According to these remarks, two sum rules have been well established in the literature (Table 1): the frame-independent one³⁰ and IMF one³¹, as we explain below.

2.1 QCD sources of angular momentum

To obtain a spin sum rule, we need an expression to the QCD AM operator. It can be derived through Noether's theorem³² based on space-time symmetry of QCD lagrangian density,

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu a} + \sum_f \bar{\psi}_f(i\not{D} - m_f)\psi_f, \quad (3)$$

where $F_a^{\mu\nu}$ is a gluon field strength tensor or simply gluon field with color indices $a = 1, \dots, 8$ and ψ_f a quark spinor field of flavor $f = u, d, s, \dots$. The relation between the gauge field and gauge potential A_a^μ is, $F_a^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - g f^{abc} A_b^\mu A_c^\nu$, and the covariant derivative is $D^\mu = \partial^\mu + igA^\mu$, with $A^\mu = A_a^\mu t^a$, and t^a are the generators of the SU(3) color group and f^{abc} are the structure constant. Straightforward calculation yields the canonical AM expression³¹

$$\vec{J}_{\text{QCD}} = \int d^3\vec{x} \left[\psi_f^\dagger \frac{\vec{\Sigma}}{2} \psi_f + \psi_f^\dagger \vec{x} \times (-i\vec{\partial}) \psi_f + \vec{E}_a \times \vec{A}_a + E_a^i (\vec{x} \times \vec{\partial}) A_a^i \right], \quad (4)$$

where $\vec{\Sigma} = \text{diag}(\vec{\sigma}, \vec{\sigma})$ with $\vec{\sigma}$ being the Pauli matrix, and the contraction of flavor (f) and color (a) indices, as well as the spatial Lorentz index “ i ”, is implied. The above expression contains four different terms, each of which has clear physical meaning in free-field theory. The first term corresponds to the quark spin, the second to the quark orbital AM (OAM), the third to the gluon spin, and the last one to the gluon OAM. Apart from the first term, the rest are not gauge-invariant under the general gauge transformation, $\psi \rightarrow U(x)\psi$ and $A^\mu \rightarrow U(x)(A^\mu + (i/g)\partial^\mu)U^\dagger(x)$, where $U(x)$ is an SU(3) matrix. However, the total is invariant under the gauge transformation up to a surface term at infinity which can be ignored in physical matrix elements.

Theoretically, the canonical form of the AM operator allows deriving an infinite number of spin sum rules with choices of gauges and/or frames of reference (hadron momentum)^{20,33}. In practice, only the infinite-momentum frame, relevant for interpreting high-energy scattering experiments, and physical gauge, such as Coulomb gauge, have shown related to experimental observables.

Using the Belinfante improvement procedure³⁴, one can obtain a gauge-invariant form from Eq. (4)³⁰,

$$\vec{J}_{\text{QCD}} = \int d^3x \left[\psi_f^\dagger \frac{\vec{\Sigma}}{2} \psi_f + \psi_f^\dagger \vec{x} \times (-i\vec{\partial} - g\vec{A}) \psi_f + \vec{x} \times (\vec{E} \times \vec{B}) \right], \quad (5)$$

All terms are manifestly gauge independent, with the second term as mechanical or kinetic OAM, and the third term gluon AM.

To evaluate the quark orbital and gluon contributions in a polarized proton state, we need the matrix elements of the QCD energy-momentum tensor (EMT), which can be split into the sum of the quark and gluon contributions, $T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}$, after Belinfante improvement. EMT defines the momentum density which is the source of AM density. The off-forward matrix elements of EMT have been parameterized as³⁰,

$$\langle P'S' | T_{q/g}^{\mu\nu} | PS \rangle = \bar{U}(P'S') \left[A_{q/g}(\Delta^2) \gamma^{(\mu} \bar{P}^{\nu)} + B_{q/g}(\Delta^2) \frac{\bar{P}^{(\mu} i\sigma^{\nu)\alpha} \Delta_\alpha}{2M} + \dots \right] U(PS), \quad (6)$$

where $\bar{P}^\mu = (P^\mu + P'^\mu)/2$, $\Delta^\mu = P'^\mu - P^\mu$. U and \bar{U} are Dirac spinors for the nucleon state, and A, B are form factors depending the momentum transfer squared, Δ^2 .

2.2 Helicity sum rules

Without loss of generality, one can assume the proton three-momentum to be $\vec{P} = (0, 0, P^z)$. In the case of longitudinal polarization, one has $\langle PS_z | J^z | PS_z \rangle = \hbar/2$ where S_z is spin polarization vector. The above equation is boost-invariant along the z -direction. This is a starting point to construct helicity (projection of the spin along the direction of motion) sum rules.

Frame-independent spin sum rule (Ji)	Infinite-momentum frame spin sum rule (Jaffe-Manohar)
$\frac{1}{2}\Delta\Sigma + L_q^z + J_g = \frac{\hbar}{2}$	$\frac{1}{2}\Delta\Sigma + \Delta G + \ell_q + \ell_g = \frac{\hbar}{2}$
<ul style="list-style-type: none"> • $\Delta\Sigma/2$ and L_q^z (sum to J_q) are the quark helicity and OAM, respectively; • Quark and gluon contributions J_q and J_g can be obtained from GPD moments; • The sum rule also works for the transverse angular momentum in the IMF. 	<ul style="list-style-type: none"> • ΔG is the gluon helicity, ℓ_q and ℓ_g are canonical OAM; • All terms have partonic interpretations, ℓ_q and ℓ_g are twist-three quantities; • ΔG is measurable from e.g. RHIC-spin and EIC; ℓ_q and ℓ_g can be extracted from GPDs.

Table 1. Two established proton spin sum rules: one is frame-independent³⁰ and the other is in the infinite-momentum frame³¹.

Using the gauge-invariant QCD AM in Eq.(5), one can write down the frame-independent sum rule^{30,35},

$$\frac{1}{2}\Delta\Sigma(\mu) + L_q^z(\mu) + J_g(\mu) = \frac{\hbar}{2}, \quad (7)$$

where $\Delta\Sigma/2$ is the quark helicity contribution measured in the EMC experiment, and L_q^z is total quark OAM contribution including all flavors of quarks. Together, they give the total quark AM contribution J_q . The last term, J_g , is the gluon contribution. All contributions depend on renormalization scheme and scale μ , which are usually taken to be dimensional regularization and (modified) minimal subtraction. It has been shown that both contributions are related to the form factors of the energy momentum tensor, $J_{q,g} = [A_{q,g}(0) + B_{q,g}(0)]/2$ ³⁰.

The frame-independence of the above sum rule means that the proton spin composition does not depend on its momentum so long its helicity is a good quantum number, be it in the finite momentum frame or infinite momentum frame. This is a nice feature because the wave function is clearly frame dependent.

Helicity sum rules can also be derived from the canonical expression of the QCD AM density in Eq. (4). Because of gauge dependence, one might outright dismiss the physical relevance of such sum rules. However, as we shall explain in the next subsection that the gluon helicity contribution in the IMF is actually physical. This prompts speculations that the quark and gluon canonical OAM might be measurable as well under such condition. Therefore, Jaffe and Manohar proposed a canonical spin sum rule in a nucleon state with $P^z = \infty$ ³¹,

$$\frac{1}{2}\Delta\Sigma(\mu) + \Delta G(\mu) + \ell_q(\mu) + \ell_g(\mu) = \frac{\hbar}{2}, \quad (8)$$

where ΔG is the gluon helicity and $\ell_{q,g}$ are the canonical quark and gluon OAM, respectively. Considerable studies have been made about this sum rule in the literature because of its relevance to parton physics of the proton. Recent precision studies of renormalization scale μ dependence of $\Delta\Sigma$ and ΔG have been reported in Ref.³⁶⁻³⁸, see also^{26,39} for the scale evolution of OAM contributions.

Recent developments⁴⁰⁻⁴³ have shown that the parton OAM can be closely connected to the quantum phase space Wigner function or distribution^{44,45}. Since the Wigner function describes the quantum distribution of quarks and gluons in both spatial and momentum spaces, we can construct the parton OAM by a properly-weighted integral. This leads to an intuitive explanation of the OAM contributions in the above two sum rules. The difference between two OAM's, the so-called potential angular momentum, comes from two different ways to define gauge links in the Wigner functions⁴³, one of which can be interpreted as final-state interaction effects in scattering experiments^{46,47}.

There has been other attempts in using Eq. (4) to write down sum rules in different frames and gauges, for example, the Coulomb gauge at finite hadron momentum⁴⁸. However, these sum rules have no known experimental measurements and remain a pure theoretical interest^{20,33}. Some of them are known to reduce to the Jaffe-Manohar sum rule in the IMF^{49,50}.

2.3 Why is gluon helicity in bound states a physical quantity?

In general, a gauge-dependent operator is not a physical observable and hence cannot be related to an experimental measurement. However, ΔG and OAM in IMF in Eq. (8) appear exceptional. This has been an interesting theoretical puzzle for many years, and has generated much debated in the literature^{33,48,51-53}.

Experimentally-measurable ΔG is the first moment of the gauge-invariant polarized gluon distribution⁵⁴,

$$\Delta G(Q^2) = \int_0^1 dx \Delta g(x, Q^2), \quad \Delta g(x) = \frac{i}{2x(P^+)^2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS | F^{+\alpha}(0) W(0, \lambda n) \tilde{F}_\alpha^+(\lambda n) | PS \rangle, \quad (9)$$

where $\tilde{F}^{\alpha\beta} = \varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu}/2$, and the light-cone gauge link $W(0, \lambda n)$ is defined in the adjoint representation of SU(3). [IMF quantities have been rewritten in the standard light-front notation $V^\pm = (V^0 \pm V^z)/\sqrt{2}$ after Lorentz transformation.] The above quantity clearly is gauge-invariant, but non-local. It does not seem to have a simple interpretation in a general gauge.

However, in the light-cone gauge $A^+ = 0$, the nonlocal operator in Eq. (9) reduces to the gluon ‘‘spin operator’’ in Eq. (4), and thus the experimental data on ΔG can be interpreted as the measurement of a contribution to the Jaffe-Manohar’s spin sum rule in this particular gauge. This suggests that a gauge-variant operator might correspond to an experimental observable in a specific gauge! The latter has inspired much discussions about the gauge symmetry and myriads of experimentally-unaccessible spin sum rules²⁰. The fundamental reason is, however, not about generalizing the concept of gauge invariance, but about the nature of the proton states in the IMF⁴⁹.

As realized by Weizsäcker and Williams in electromagnetism^{55,56}, the gauge field strength in a fast moving source is dominated by its transverse components. For a static charge, the electric field is purely longitudinal ($\vec{E} = \vec{E}_\parallel$ or without curl). As the charge moves with velocity $\beta = v/c$, where c is the speed of light, the field lines start to contract in the transverse direction due to Lorentz transformation. The moving charge forms an electric current that generates transverse magnetic fields,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_\perp, \quad (10)$$

and the gauge potential \vec{A} acquires a non-zero transverse component \vec{A}_\perp (divergence free). At large β , the field strength gets enhanced by a factor of $\beta\gamma$ ($\gamma = 1/\sqrt{1-\beta^2}$) in the transverse direction, whereas it is strongly suppressed in the longitudinal direction^{55,56}. In the limit of $\beta \rightarrow 1$ (or $\gamma \rightarrow \infty$), $\vec{E}_\perp \sim \vec{B}$, and $|\vec{E}_\perp| \gg |\vec{E}_\parallel|$, so the electromagnetic field can be approximated as free radiation!

The radiation fields have only two physical degrees of freedoms, and the longitudinal one in the gauge potential is just a pure gauge. Thus, for an on-shell photon, its helicity is physical and can be considered as gauge-invariant spin. One can superimpose such on-shell plane wave states with definite helicity to construct light modes with definite OAM or so-called twisted light^{57,58}. The gauge-invariant issue never arises because one deals with physical polarization at all time.

Analogously, the Weizsäcker-Williams approximation is also a valid picture for gluons in an ultra-relativistic proton⁵⁹. In the IMF, the gluons can also be approximated as free radiation, thus it only has two physical transverse polarizations. $A^+ = 0$ is a physical gauge which leaves the transverse polarizations of the radiation field intact. This justifies $\vec{E} \times \vec{A} = \vec{E}_\perp \times \vec{A}_\perp$ as the physical gluon spin (helicity) operator in the Jaffe-Manohar sum rule. The above consideration also applies to the associated canonical OAM, ℓ_q^z and ℓ_g^z , which implies partonic sum rules from them^{60–62}. However, the associated canonical OAMs involves transverse momentum integral by construction and their scale evolutions are much more complicated^{63,64}.

The situation is quite different, however, if one considers color fields inside a bound state which does not travel relativistically. The longitudinal gauge potential subjected to gauge transformation now contains a physical component whose effects cannot be separated from the transverse part. The gluons are now off-mass-shell, and the longitudinal polarization do has physical significance. Only gauge-invariant operators can pick up the correct physics from the longitudinal part of gluon potential.

Thus it is the physical states in IMF which ensure the total gluon helicity is measured through $\vec{E} \times \vec{A}$. The spin operator can have any longitudinal pure gauge potential which does not contribute to the physical matrix element. This situation is exactly opposite to the usual textbook formulation of gauge symmetry where the external states are gauge-dependent and the operators must be gauge invariant. When transforming the IMF states into ones with finite momentum through infinite Lorentz transformation, $\vec{E}_\perp \times \vec{A}_\perp$ becomes a non-local one in Eq. (9).

2.4 Sum rule for transverse angular momentum

For transverse polarization along, e.g., the x -direction, the transverse AM operator does not commute with the QCD Hamiltonian. However, according to the Lorentz transformation property of J^x , its expectation value in transversely-polarized state is well-defined⁶⁵,

$$\langle PS_x | J^x | PS_x \rangle = \gamma(\hbar/2). \quad (11)$$

where γ is the Lorentz boost factor. Therefore, the transverse AM J^x is a leading observable because it enhances under boost, a fact less appreciated in the literature. The potential contribution to the transverse AM from the non-intrinsic center-of-mass motion has led to incorrect results in the literature^{65–67}.

If we define, $J_{\perp}^{q,g} = \langle PS_{\perp} | J_{\perp q,g} | PS_{\perp} \rangle / (\gamma s_{\perp})$, then the quark and gluon contributions can again be related to the form factors in Eq. (6),

$$J_{\perp}^{q,g} = (A_{q,g} + B_{q,g})/2, \quad (12)$$

$$J_{\perp}^q + J_{\perp}^g = \hbar/2. \quad (13)$$

Both equations are the same as these in the helicity case. However, the separation of the quark spin and orbital contributions are frame dependent, with the former contribution going to zero in the infinite momentum limit⁶⁵.

A parton interpretation can be derived for the above result following an earlier suggestion in Ref.^{68,69}. The physical reason is that the transverse AM can be built from a longitudinal parton momentum with a transverse coordinate. One can define a parton AM partonic density^{43,63,70},

$$J_{\perp}^q(x) = x[q(x) + E_q(x)]/2, \quad J_{\perp}^g(x) = x[g(x) + E_g(x)]/2, \quad (14)$$

where $q(x)$ and $g(x)$ are the unpolarized quark/antiquark and gluon distributions, and $E_{q,g}(x)$ are a type of generalized parton distributions (GPDs)³⁰. GPD's are an extension of the well-known Feynman parton distribution and are defined as off-forward matrix element between nucleon states with different momenta, similar to form factors. They depend on three kinematic variables: x the longitudinal momentum fraction for the parton, ξ the skewness parameter represents the momentum transfer between the nucleon states along the longitudinal direction, and $t = \Delta^2$ the momentum transfer Δ^μ squared. They can be systematically studied through a new class of exclusive hadronic reactions³⁰. $J_{\perp}^{q,g}(x)$ are the AM densities carried by partons of momentum x in a transversely polarized nucleon in which partons are in general off the center of mass⁶⁹. Integrating the above over x give the total transverse AM carried by quarks and gluons, respectively.

3 Ab Initio Calculations of Spin in Lattice QCD

Because non-perturbative QCD is unusually challenging, a large number of models of the proton have been proposed in 70's and 80's, many of which use "effective" degrees of freedom. An introduction about these models can be found in the textbooks^{7,8}. A recent one is the holographic model in which the proton is pictured as a quark and di-quark bound state⁷¹. Since the connections between the model degrees of freedom and the fundamental ones are unknown, whereas high-energy experiments probe QCD quarks and gluons directly, we will discuss the theoretical calculations using QCD degrees of freedom only.

At present, the only systematic approach to solve the QCD proton structure is lattice field theory⁷², in which quark and gluon fields are put on four-dimensional Euclidean lattices with finite spacing a , and quantum correlation functions of fields are calculated using Feynman path integrals and Monte Carlo simulations. The physical limits are recovered when the lattice spacing a becomes sufficiently small compared to physical correlation length, the lattice volume much larger than hadron sizes, and the quark masses close to the physical ones⁷³. There are less systematic approaches such as Schwinger-Dyson (Bethe-Salpeter) equations⁷⁴ and instanton liquid models⁷⁵ in which a certain truncation is needed to find a solution. Although much progress has been made in these other directions, we focus on the lattice QCD method which can be systematically improved.

A complete physical calculation on lattice faces a number of obstacles. First, the total AM is a flavor-singlet quantity, and as such, one needs to compute the costly disconnected diagrams for the quarks. Since up and down quarks are light, computation demands at the physical pion mass are very high, as physical propagators becomes singular in the massless limit. Moreover, gluon observables need be calculated to complete the picture, which are known to be very noisy and large number of field configurations are needed for accuracy. At the same time, one needs to take the continuum and infinite volume limits. All of these add up to an extremely challenging task. However, a computation with all these issues taken into account has become feasible recently⁷⁶.

An additional challenge is present in quantities like ΔG , usually defined in terms of light-front correlations with real time dynamics. It is well-known that the real-time Monte carlo simulations demand exponentially-increasing resources. The recent development in large-momentum effective theory (LaMET) has opened the door for such time-dependent light-front correlations⁷⁷⁻⁷⁹.

3.1 Frame-independent helicity sum rule

The matrix elements of local operators, $\Delta\Sigma$, J_q and J_g , are relatively easier to calculate using the standard lattice QCD technique. Much progress has been made in understanding the content of manifestly gauge-invariant helicity sum rule (and hence the transverse AM sum rule as well by Eq. (13)).

The first calculation has been about the $\Delta\Sigma$ from different quark flavors⁸⁰. The relevant studies in the last two decades have been summarized in a recent review⁸¹. Important progress has been made in chiral-fermion calculations⁸² and at the

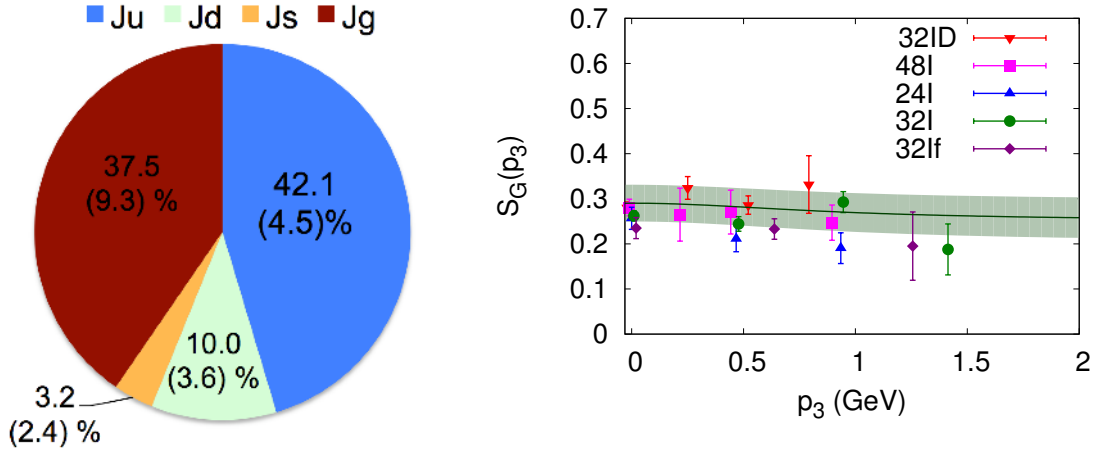


Figure 1. State-of-the-art lattice study on the proton spin. Left chart shows the spin decomposition in the frame-independent sum rule: total spin contributions from the up (J_u), down (J_d), strange (J_s) quarks and gluons (J_g). The numbers come from the ETMC collaboration⁷⁶ with = 38.2(3.1)% from total quark helicity contribution $\frac{1}{2}\Sigma$ and = 18.8(10.1)% from total quark orbital angular L_q . Right plot shows the gluon helicity contribution to the IMF sum rule, where the gluon spin is computed from different proton momenta (labeled as p_3 as x -axis) and lattice ensembles (different volumes and lattice spacing noted in the plot legend) by the χ QCD collaboration⁹⁵. The gluon spin reduces to ΔG when extrapolated to the IMF ($p_3 \rightarrow \infty$): $\Delta G = 0.251(50)$ or 50(9)% of the proton spin, which can be compared to the RHIC-spin determination, see, Fig. 2.

physical quark mass⁸³. The strange quark contribution has been calculated earlier in^{84,85} in consideration of the anomalous Ward identity. The total quark spin contribution to the proton helicity has been found consistently about 40%.

The calculation of the total quark and gluon angular momenta started in Ref.⁸⁶ where the quark part including the disconnected diagrams was calculated in the quenched approximation. The result of the total quark contribution is $J_q = 0.30 \pm 0.07$, i.e. 60%. Therefore about 40% of the proton spin is carried by gluon through simple sum rule deduction. Following the quenched studies^{87,88}, dynamical simulations have now become a standard⁸⁹⁻⁹³. A first complete study of the AM decomposition was made in Ref.⁸⁴, followed by a chiral dynamical simulation recently⁹⁴. A first study at the physical quark mass has appeared in Ref.⁸³.

A high-precision dynamical simulation at the physical pion mass has been finished recently⁷⁶. It was found that the total quark spin contribution is about 38.2%, and the orbital AM of the quarks about 18.8%, much reduced compared with quenched simulations. The total gluon contribution is 37.5%. The resulting pie chart is shown in Fig. 1. The total spin is 94.6% of $\hbar/2$ with an error bar of 14.2%. These results are largely consistent with the chiral fermion study in Ref.⁹⁴. All numbers are quoted in $\overline{\text{MS}}$ scheme at $\mu = 2$ GeV.

3.2 Gluon helicity in light-travelling proton

Calculation of the gluon helicity ΔG has not been possible for many years because it is intrinsically a time-dependent light-front quantity. However, a breakthrough in 2013 has finally been made by studying the frame dependence of non-local matrix elements⁴⁹. It was found that one can match the large-momentum matrix element of a static “gluon spin” operator calculable in lattice QCD to ΔG in the IMF⁴⁹. This idea was a prototype of LaMET, which was soon put forward as a general approach to calculate all parton physics^{77,78}.

The choice of the static “gluon spin” operator is not unique. There is a universality class of operators⁵⁰ whose IMF limit approach the free-field field operator in Eq. (4) in the light-front gauge. The simplest choice for the static “gluon spin” is the free-field operator $\vec{E} \times \vec{A}$ fixed in a time-independent gauge. For example, the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, axial gauges $A^z = 0$ and $A^0 = 0$ all maintain the transverse polarizations of the gluon field in the IMF limit, so they are viable options.

In the Coulomb gauge and $\overline{\text{MS}}$ scheme, the static “gluon spin” $\Delta \tilde{G}$ in a massive on-shell quark state at one-loop order is^{49,96}

$$\Delta \tilde{G}(P^z, \mu)(2S^z) = \langle PS | (\vec{E} \times \vec{A})^z | PS \rangle_q \Big|_{\vec{\nabla} \cdot \vec{A} = 0} = \frac{\alpha_s C_F}{4\pi} \left[\frac{5}{3} \ln \frac{\mu^2}{m^2} - \frac{1}{9} + \frac{4}{3} \ln \frac{(2P^z)^2}{m^2} \right] (2S^z), \quad (15)$$

where the subscript q denotes a quark. The collinear divergence is regulated by the finite quark mass m . The above result shows that the gluon state depends on the three-momentum P^z , as it should be. If we follow the procedure in⁹⁷ and take $P^z \rightarrow \infty$ limit

before UV regularization, which is the standard procedure to define partons⁴⁹,

$$\Delta G(\infty, \mu)(2S^z) = \langle PS | (\vec{E} \times \vec{A})^z | PS \rangle_q \Big|_{\vec{v} \cdot \vec{A} = 0} = \frac{\alpha_s C_F}{4\pi} \left(3 \ln \frac{\mu^2}{m^2} + 7 \right) (2S^z), \quad (16)$$

which is exactly the same as the light-front gluon helicity $\Delta G(\mu)$ appeared in Jaffe-Manohar spin sum rule⁵¹. Therefore, despite the difference in the UV divergence, the infrared-sensitive collinear divergences of $\Delta \tilde{G}(P^z, \mu)$ and $\Delta G(\mu)$ are exactly the same, which allows for a perturbative matching between them.

With the LaMET approach, ΔG was calculated in lattice QCD for the first time⁹⁵. In this calculation, the static gluon spin operator $\vec{E} \times \vec{A}$ in the Coulomb gauge was simulated on the lattice and converted to the continuum $\overline{\text{MS}}$ scheme with one-loop lattice perturbation theory, which is shown in Fig. 1. With leading-order matching and extrapolation to the IMF, the authors obtained $\Delta G(\mu = \sqrt{10} \text{ GeV}) = 0.251(47)(16)$, or 50(9)(3)% of the proton spin. A refined study on systematics and precise matching shall be made in the future.

3.3 Canonical OAM and Transverse AM density in light-travelling proton

To complete the Jaffe-Manohar picture of the proton spin, one needs to compute canonical OAM of the quarks and gluons in the IMF and light-front gauge. This can be done following the same approach above for ΔG . A study of calculating these in LaMET has been made in Ref.⁹⁸. One can start from the matrix elements, for example, in Coulomb gauge and at finite momentum P^z ,

$$\tilde{\ell}_q(\mu, P^z)(2S^z) = \langle PS | \int d^3 \vec{x} \psi_q^\dagger(\vec{x} \times (-i\vec{\nabla}))^z \psi_q | PS \rangle, \quad \tilde{\ell}_g(\mu, P^z)(2S^z) = \langle PS | \int d^3 \vec{x} E^{ia}(\vec{x} \times \vec{\nabla})^z A_a^i | PS \rangle, \quad (17)$$

which can be matched onto $\ell_{q,g}(\mu)$ in the Jaffe-Manohar sum rule. The matching expressions have been worked out in Coulomb gauge in Ref.⁹⁹. Mixings with potential AM contributions shall be taken into account⁴⁶. Because the matrix elements are spatial moments, one can either calculate them directly using \vec{x} -weighting on lattice^{100,101}, or by taking the zero-momentum-transfer limit of the momentum-density form factors. Computing the canonical quark OAM from lattice QCD has been carried out in Ref.^{102,103} using non-local operators, for which matching to the IMF quantities has yet to be studied.

Similar approach can be used to calculate the canonical OAM distributions $\ell_q(x, \mu)$ and $\ell_g(x, \mu)$ ^{60,62}. Since both distributions are sub-leading in high-energy experiments (the so-called twist-three), they may contain a zero-mode contribution at $x = 0$ ^{104,105}, which makes the experimental measurement of $\ell_{q,g}(\mu)$ through sum rules challenging. Mixings with other twist-three correlations with gluon fields must be considered.

Likewise, the transverse AM of the proton is a leading light-front observable, and has a partonic interpretation in terms of transverse AM density $J_\perp^{q,g}(x) = x(\{q, g\}(x) + E_{q,g}(x))/2$. While the singlet distributions $q(x), g(x)$ are well constrained and can be calculated on lattice with the standard LaMET method⁷⁹, little is known about GPD's $E_{q,g}(x)$. The moments of $E_{q,g}(x)$ can be calculated as a generalization of the form factors of the energy-momentum tensor. The x -distributions can also be obtained directly as the spatial moment of the gauge-invariant momentum-density correlation functions.

4 Experimental Progress and Electron-Ion Collider

We finally review the experimental progress in searching for the origins of the proton spin. Following the EMC, many experiments were launched to confirm the result. In the first subsection, we discuss efforts of nailing down quark helicity contribution Δq , particularly Δs , from SIDIS, and the gluon helicity ΔG from polarized proton-proton collisions at RHIC. In the second subsection, we review measuring the quark orbital AM contribution from a new class of experiments called deeply-virtual Compton scattering, first proposed and studied in Refs.^{30,106}. Following this, we consider the prospects of studying the proton spin structure at the EIC.

4.1 Nailing down the quark and gluon helicities

The majority of the experiment efforts followed the EMC experiment, measuring the polarized structure functions in DIS with polarized lepton on polarized target (proton, neutron, deuteron). Two important initiatives have also emerged: 1). The DIS experiment facilities extend their capabilities to measure the spin asymmetries in the semi-inclusive hadron production in DIS (SIDIS)^{109,110}, helping to identify the flavor dependence in the polarized quark distributions. 2). The Relativistic Heavy Ion Collider (RHIC) at the Brookhaven National Laboratory (BNL) started the polarized proton-proton scattering experiments, which opened new opportunities to explore the proton spin, in particular, for the helicity contributions from gluon and sea quarks. Most of these efforts have been covered in the recent reviews^{19,22}.

The total quark spin contribution has been well determined from DIS measurements: $\Delta \Sigma \approx 0.30$ with uncertainties around 0.10, see, e.g., recent global analyses from Refs.^{107,111,112}. However, for sea quark polarizations including \bar{u} , \bar{d} and s (\bar{s}), there

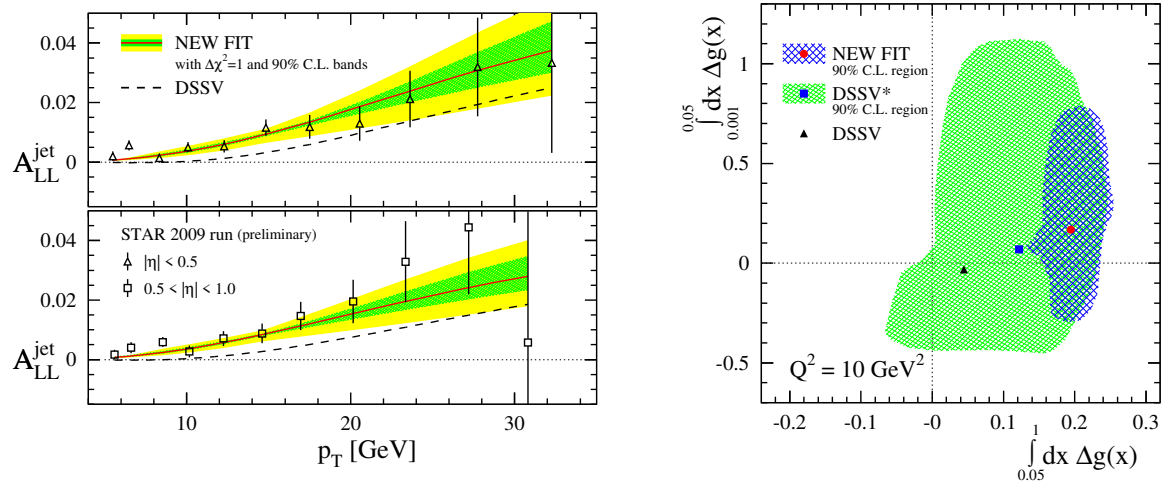


Figure 2. The relativistic heavy-ion collider (RHIC) at Brookhaven National Laboratory (BNL) provides a strong evidence of the gluon helicity contribution to the proton spin. (Left) Double spin asymmetry in inclusive jet production at RHIC compared to the the global analysis of DSSV-14¹⁰⁷, where the gluon helicity $\Delta g(x)$ plays an important role. (Right) Constraints on the gluon helicity contribution (labeled as “NEW FIT”) to the proton spin from the fit to the experimental data including that on the left plot. In the RHIC kinematics, i.e., $x > 0.05$, Δg was found positive and sizable: $\int_{0.05}^1 dx \Delta g(x) = 0.20^{+0.06}_{-0.07}$, as shown in abscissa. However, in the unexplored region of $x < 0.05$, the uncertainties are still significant as shown in ordinate. Source: experiment data from¹⁰⁸, DSSV-14 global analysis in Ref.¹⁰⁷.

exist larger uncertainties, in particular, in the strange quark polarization^{107,112,113}, where the constraints mainly come from SIDIS measurements by HERMES and COMPASS experiments. Recently, it was also found that the W boson spin asymmetries at center-of-mass energy $\sqrt{s} = 510$ GeV at RHIC have also improved the constraints on \bar{u} and \bar{d} polarization¹¹⁴. Very exciting results, in particular, for the double spin asymmetries in inclusive jet production from the RHIC experiments have provided stronger constraint on the gluon helicity¹⁰⁸, see Fig. 2. This promises great potential for future analysis from RHIC experiments to further reduce the uncertainties due to improved statistics^{115,116}. However, due to the kinematic limitations, the total gluon helicity contribution still has a significant uncertainty.

4.2 Quark OAM and GPD studies at JLab 12 GeV

It was found that the total quark (gluon) contribution to the proton spin (also the form factor of the QCD energy-momentum tensor) can be obtained from the moments of generalized parton distributions (GPD’s)³⁰,

$$J_q = \frac{1}{2} \Delta \Sigma_q + L_q = \lim_{t, \xi \rightarrow 0} \frac{1}{2} \int dx x [H^q(x, \xi, t) + E^q(x, \xi, t)] . \quad (18)$$

After subtracting the helicity contribution $\Delta \Sigma_q$ measured from inclusive and semi-inclusive DIS experiments, the above equation will provide the quark OAM contribution to the proton spin. The GPD’s can be measured in a new class of experiments called deep-exclusive processes, for example, deeply virtual Compton scattering (DVCS) and deeply virtual meson production (DVMP)^{30,106,118,119}. Both DVCS and DVMP processes belong to exclusive hard scattering processes in lepton-nucleon collisions. For example, in the DVCS process, as shown in Fig. 3, an incoming lepton scatters off the nucleon with momentum P and produces a high momentum real photon, and the recoiling nucleon with momentum P' . In this way, the quark spatial position and momentum can be sampled simultaneously. Review articles for GPDs and DVCS can be found in Ref.^{120–123}

Experimental efforts in these new processes have been made at various facilities, including HERMES at DESY¹²⁴, Jefferson Lab 6 GeV¹¹⁷, and COMPASS at CERN¹²⁵. In real photon exclusive production, the DVCS amplitude has interference with the Bethe-Heitler (BH) amplitude. This will, on the one hand, complicate the analysis of the cross section, and on the other hand, provide unique opportunities to direct access the DVCS amplitude through the interference. To obtain the constraints on the quark OAMs from these experiments, we need to find the observables which are sensitive to the GPD E ’s. Experiments on the DVCS from JLab Hall A¹¹⁷ and HERMES at DESY¹²⁴ have shown strong sensitivity to the quark OAMs in nucleon, see, e.g.,

Deeply Virtual Compton Scattering

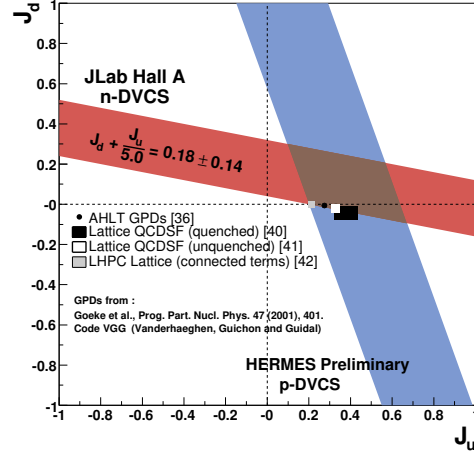
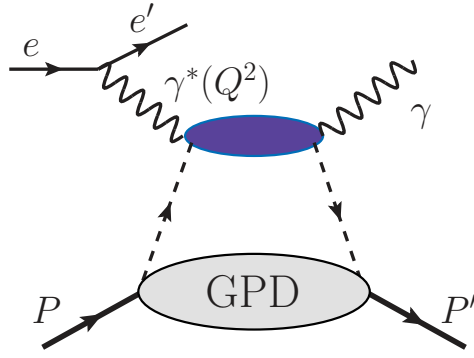


Figure 3. Extensive investigations of a new experimental process called deeply virtual Compton scattering (DVCS)¹⁰⁶, shown in the left, have provided a novel approach to study the quark orbital angular momentum in proton. In the DVCS process, an electron scatters off the nucleon with momentum P and produces a high momentum real photon and a recoiling proton with momentum P' . DVCS probes the generalized parton distributions which are sensitive to the orbital angular momenta of quarks and gluons. An example (right plot) is shown from JLab Hall A analysis of spin asymmetries in DVCS and the model-dependent constraints on the up and down quark total AM. Source: Ref. ¹¹⁷.

Fig. 3. In these experiments, the single spin asymmetries associated with beam or target in DVCS processes are measured, including the beam (lepton) single spin asymmetry and (target) nucleon single spin (transverse or longitudinal) asymmetries.

JLab 12 GeV facility has just started its experimental program. Multiple experiments on DVCS and DVMP have been planned in three experimental halls. One expects a new generation of precision data for extracting quark GPD's. From the phenomenology side, we need to construct more sophisticated parametrizations for the GPD's. In particular, in light of JLab experiments in next decade and future experiments at the EIC, a rigorous and collaborative approach has to be taken to perform the analysis of a large body of experimental data.

4.3 Prospects of the proton spin at EIC

In early 2020, the DOE announced that the next major facility for nuclear physics in the US will be a high-energy and high-luminosity polarized EIC to be built at BNL. EIC will be the first polarized electron-proton collider and the first electron-nucleus collider as well. The primary goal of the EIC is to precisely image gluon distributions in nucleons and nuclei, to reveal the origins of the nucleon mass and spin, and to explore the new QCD frontier of cold nuclear matter^{23,24}. EIC will impact our understanding of nucleon spin in many different ways. In the following, we highlight some of the most obvious ones:

- The quark and gluon helicity contributions to the proton spin are among the major emphases of the planned EIC. With the unique coverage in both x and Q^2 , it will provide the most stringent constraints on $\Delta\Sigma$ and ΔG ²³. Shown in Fig. 4 is the possible reduction in their uncertainties with the proposed EIC. Clearly, it will make a huge impact on our knowledge of these quantities, unparalleled by any other existing or anticipated facility.
- There will be a comprehensive research program on gluon GPD's at the EIC. Apart from providing the first hand constraints on the total quark/gluon AM contributions to the proton spin, the GPD's provide important information on the nucleon tomography, for example, the 3D imaging of partons inside the proton^{45,68}. With wide kinematic coverage at the EIC, a particular example shown in Fig. 4 is that the transverse imaging of the gluon can be precisely mapped out from the detailed measurement of hard exclusive J/ψ production. Together with the gravitational form factors extracted from the DVCS, this will provide unprecedented exploration of nucleon tomography and deepen our understanding of the nucleon spin structure in return. Pioneer experimental effort to constrain the gravitation form factor from DVCS experiment at JLab has been carried out in Ref. ¹²⁶.
- The EIC may shed light on the quark/gluon canonical OAM directly through various hard diffractive processes. A particular example has been studied recently in Refs. ^{127,128}. Here, by applying the connection between the parton Wigner distribution and OAM⁴⁰⁻⁴³, one can show that the single longitudinal target-spin asymmetry in the hard diffractive dijet production is sensitive to the canonical gluon OAM distribution. The associated spin asymmetry leads to a characteristic

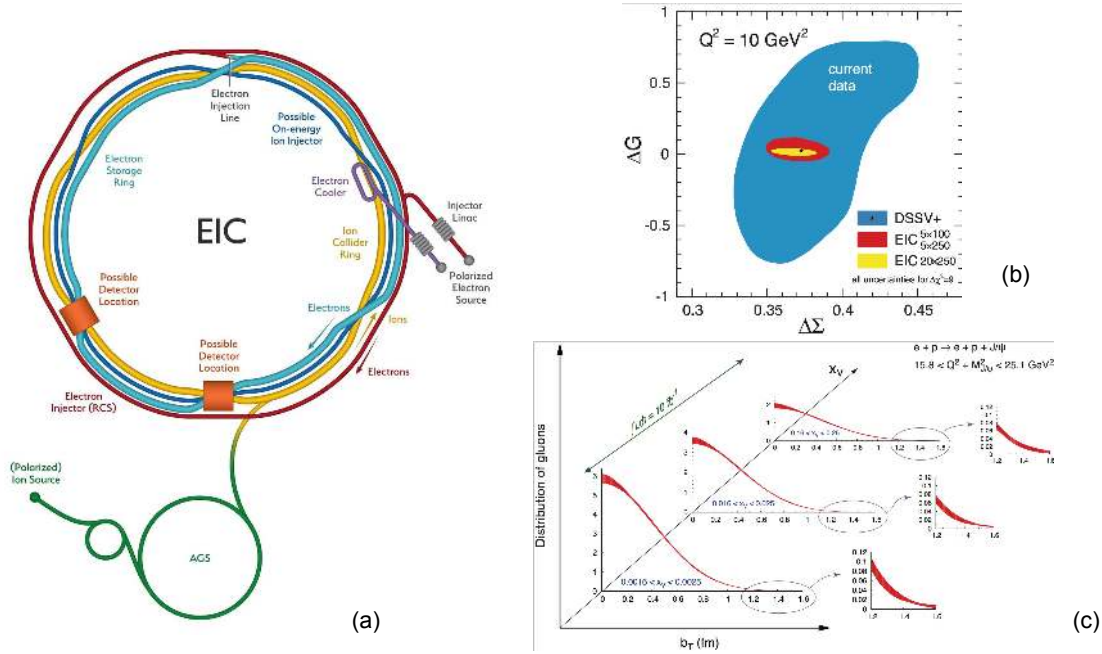


Figure 4. The planned Electron-Ion Collider (EIC) at BNL, New York, USA (Left, source: BNL). Highlights of the EIC impacts on our understanding of nucleon spin: precision on the total quark/gluon helicity contributions to the proton spin (upper right); gluon density in the transverse plane for different gluon parton x from deeply-virtual J/ψ production (lower right). This process can probe the generalized parton distribution for the gluon at small- x . The Fourier transform respect to the momentum transfer leads to the transverse spatial distribution of gluons in nucleon. High precision measurement of this process at the EIC will provide strong constraint on this tomography imaging. Source: Ref.²³.

azimuthal angular correlation between the proton momentum transfer and the relative transverse momentum between the quark-antiquark pair. With a hermetic detector designed for the EIC, this observable can be well studied in the future, and will help us obtain the final piece in the IMF helicity sum rule.

An important theoretical question concerns the asymptotic small- x behavior of the polarized PDFs and their contributions to the spin sum rule. There have been some progress to understand the proton spin structure at small- x from the associated small- x evolution equations^{129–134}. More theoretical efforts are needed to resolve the controversial issues raised in these derivations. The final answer to these questions will provide important guideline for the future EIC, where proton spin structure is one of the major focuses.

5 Conclusion

After 30 years since the EMC publication of the polarized DIS data, there has been much progress in understanding the spin structure. There are two well-established approaches to look at the composition of the proton spin: frame-independent approach (“Ji sum rule”) and infinite-momentum-frame parton approach (“Jaffe-Manohar sum rule”). In the frame-independent approach, the quark orbital and gluon contributions can be obtained from moments of generalized parton distributions. Results from Jlab 6 GeV experiments and HERMES data suggest a substantial quark orbital contribution. In the partonic picture of Jaffe and Manohar, the quark and gluon helicity have simple physics appeal, and the result from RHIC spin has provided important constraint on the total gluon helicity ΔG . Development of LaMET along with lattice QCD simulations provides the first-principles calculations of the spin structure, and the first results have provided an interesting overall picture. Jlab 12 GeV program will provide much improved data on the quark GPDs and OAM. EIC can provide high-precision measurements on the gluon helicity ΔG and total angular momentum contributions.

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References

1. Rutherford, E. Collision of α particles with light atoms. IV. An anomalous effect in nitrogen. *Phil. Mag. Ser.6* **37**, 581–587, DOI: [10.1080/14786431003659230](https://doi.org/10.1080/14786431003659230) (1919). [Phil. Mag.90,no.sup1,31(2010)].
2. Estermann, I., Frisch, R. & Stern, O. Magnetic Moment of the Proton. *Nature* **132**, 169–170, DOI: <https://doi.org/10.1038/132169a0> (1933).
3. Dirac, P. A. M. The quantum theory of the electron. *Proc. Roy. Soc. Lond.* **A117**, 610–624, DOI: [10.1098/rspa.1928.0023](https://doi.org/10.1098/rspa.1928.0023) (1928).
4. Skyrme, T. H. R. A Unified Field Theory of Mesons and Baryons. *Nucl. Phys.* **31**, 556–569, DOI: [10.1016/0029-5582\(62\)90775-7](https://doi.org/10.1016/0029-5582(62)90775-7) (1962).
5. Gell-Mann, M. A Schematic Model of Baryons and Mesons. *Phys. Lett.* **8**, 214–215, DOI: [10.1016/S0031-9163\(64\)92001-3](https://doi.org/10.1016/S0031-9163(64)92001-3) (1964).
6. Zweig, G. An SU(3) model for strong interaction symmetry and its breaking. Version 1. (1964).
7. Bhaduri, R. K. *Models of the Nucleon: From Quarks to Soliton* (1988).
8. Thomas, A. W. & Weise, W. *The Structure of the Nucleon* (Wiley, Germany, 2001).
9. Fritzsch, H., Gell-Mann, M. & Leutwyler, H. Advantages of the Color Octet Gluon Picture. *Phys. Lett.* **47B**, 365–368, DOI: [10.1016/0370-2693\(73\)90625-4](https://doi.org/10.1016/0370-2693(73)90625-4) (1973).
10. Greenberg, O. W. Spin and Unitary Spin Independence in a Paraquark Model of Baryons and Mesons. *Phys. Rev. Lett.* **13**, 598–602, DOI: [10.1103/PhysRevLett.13.598](https://doi.org/10.1103/PhysRevLett.13.598) (1964).
11. Isgur, N. & Karl, G. Positive Parity Excited Baryons in a Quark Model with Hyperfine Interactions. *Phys. Rev.* **D19**, 2653, DOI: [10.1103/PhysRevD.23.817.2](https://doi.org/10.1103/PhysRevD.23.817.2), [10.1103/PhysRevD.19.2653](https://doi.org/10.1103/PhysRevD.19.2653) (1979). [Erratum: Phys. Rev.D23,817(1981)].
12. Close, F. E. *An Introduction to Quarks and Partons* (1979).
13. Hughes, V. W. & Kuti, J. Internal Spin Structure of the Nucleon. *Ann. Rev. Nucl. Part. Sci.* **33**, 611–644, DOI: [10.1146/annurev.ns.33.120183.003143](https://doi.org/10.1146/annurev.ns.33.120183.003143) (1983).
14. Ashman, J. *et al.* A Measurement of the Spin Asymmetry and Determination of the Structure Function $g(1)$ in Deep Inelastic Muon-Proton Scattering. *Phys. Lett.* **B206**, 364, DOI: [10.1016/0370-2693\(88\)91523-7](https://doi.org/10.1016/0370-2693(88)91523-7) (1988).
15. Ashman, J. *et al.* An Investigation of the Spin Structure of the Proton in Deep Inelastic Scattering of Polarized Muons on Polarized Protons. *Nucl. Phys.* **B328**, 1, DOI: [10.1016/0550-3213\(89\)90089-8](https://doi.org/10.1016/0550-3213(89)90089-8) (1989).
16. Ellis, J. R. & Jaffe, R. L. A Sum Rule for Deep Inelastic Electroproduction from Polarized Protons. *Phys. Rev. D* **9**, 1444, DOI: [10.1103/PhysRevD.9.1444](https://doi.org/10.1103/PhysRevD.9.1444) (1974). [Erratum: Phys.Rev.D 10, 1669 (1974)].
17. Filippone, B. W. & Ji, X.-D. The Spin structure of the nucleon. *Adv. Nucl. Phys.* **26**, 1, DOI: [10.1007/0-306-47915-X_1](https://doi.org/10.1007/0-306-47915-X_1) (2001). [hep-ph/0101224](https://arxiv.org/abs/hep-ph/0101224).
18. Bass, S. D. The Spin structure of the proton. *Rev. Mod. Phys.* **77**, 1257–1302, DOI: [10.1103/RevModPhys.77.1257](https://doi.org/10.1103/RevModPhys.77.1257) (2005). [hep-ph/0411005](https://arxiv.org/abs/hep-ph/0411005).
19. Aidala, C. A., Bass, S. D., Hasch, D. & Mallot, G. K. The Spin Structure of the Nucleon. *Rev. Mod. Phys.* **85**, 655–691, DOI: [10.1103/RevModPhys.85.655](https://doi.org/10.1103/RevModPhys.85.655) (2013). [1209.2803](https://arxiv.org/abs/1209.2803).
20. Leader, E. & Lorcé, C. The angular momentum controversy: What’s it all about and does it matter? *Phys. Rept.* **541**, 163–248, DOI: [10.1016/j.physrep.2014.02.010](https://doi.org/10.1016/j.physrep.2014.02.010) (2014). [1309.4235](https://arxiv.org/abs/1309.4235).
21. Ji, X. Proton Tomography Through Deeply Virtual Compton Scattering. *Natl. Sci. Rev.* **4**, 213–223, DOI: [10.1093/nsr/nwx024](https://doi.org/10.1093/nsr/nwx024) (2017). [1605.01114](https://arxiv.org/abs/1605.01114).
22. Deur, A., Brodsky, S. J. & De Téramond, G. F. The Spin Structure of the Nucleon. *Rept. Prog. Phys.* **82**, DOI: [10.1088/1361-6633/ab0b8f](https://doi.org/10.1088/1361-6633/ab0b8f) (2019). [1807.05250](https://arxiv.org/abs/1807.05250).
23. Accardi, A. *et al.* Electron Ion Collider: The Next QCD Frontier. *Eur. Phys. J.* **A52**, 268, DOI: [10.1140/epja/i2016-16268-9](https://doi.org/10.1140/epja/i2016-16268-9) (2016). [1212.1701](https://arxiv.org/abs/1212.1701).
24. Boer, D. *et al.* Gluons and the quark sea at high energies: Distributions, polarization, tomography. (2011). [1108.1713](https://arxiv.org/abs/1108.1713).
25. Tung, W. K. *Group Theory In Physics* (1985).
26. Ji, X.-D., Tang, J. & Hoodbhoy, P. The spin structure of the nucleon in the asymptotic limit. *Phys. Rev. Lett.* **76**, 740–743, DOI: [10.1103/PhysRevLett.76.740](https://doi.org/10.1103/PhysRevLett.76.740) (1996). [hep-ph/9510304](https://arxiv.org/abs/hep-ph/9510304).

27. Ji, X. Comment on ‘Spin and Orbital Angular Momentum in Gauge Theories: Nucleon Spin Structure and Multipole Radiation Revisited’. *Phys. Rev. Lett.* **104**, 039101, DOI: [10.1103/PhysRevLett.104.039101](https://doi.org/10.1103/PhysRevLett.104.039101) (2010).
28. Bunce, G., Saito, N., Soffer, J. & Vogelsang, W. Prospects for spin physics at RHIC. *Ann. Rev. Nucl. Part. Sci.* **50**, 525–575, DOI: [10.1146/annurev.nucl.50.1.525](https://doi.org/10.1146/annurev.nucl.50.1.525) (2000). [hep-ph/0007218](https://arxiv.org/abs/hep-ph/0007218).
29. Dudek, J. *et al.* Physics Opportunities with the 12 GeV Upgrade at Jefferson Lab. *Eur. Phys. J.* **A48**, 187, DOI: [10.1140/epja/i2012-12187-1](https://doi.org/10.1140/epja/i2012-12187-1) (2012). [1208.1244](https://arxiv.org/abs/1208.1244).
30. Ji, X.-D. Gauge-Invariant Decomposition of Nucleon Spin. *Phys. Rev. Lett.* **78**, 610–613, DOI: [10.1103/PhysRevLett.78.610](https://doi.org/10.1103/PhysRevLett.78.610) (1997). [hep-ph/9603249](https://arxiv.org/abs/hep-ph/9603249).
31. Jaffe, R. L. & Manohar, A. The G(1) Problem: Fact and Fantasy on the Spin of the Proton. *Nucl. Phys.* **B337**, 509–546, DOI: [10.1016/0550-3213\(90\)90506-9](https://doi.org/10.1016/0550-3213(90)90506-9) (1990).
32. Noether, E. Invariant Variation Problems. *Gott. Nachr.* **1918**, 235–257, DOI: [10.1080/00411457108231446](https://doi.org/10.1080/00411457108231446) (1918). [physics/0503066](https://arxiv.org/abs/physics/0503066).
33. Ji, X., Xu, Y. & Zhao, Y. Gluon Spin, Canonical Momentum, and Gauge Symmetry. *JHEP* **08**, 082, DOI: [10.1007/JHEP08\(2012\)082](https://doi.org/10.1007/JHEP08(2012)082) (2012). [1205.0156](https://arxiv.org/abs/1205.0156).
34. Belinfante, F. J. On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields. *Physica* **7**, 449 – 474, DOI: [https://doi.org/10.1016/S0031-8914\(40\)90091-X](https://doi.org/10.1016/S0031-8914(40)90091-X) (1940).
35. Ji, X.-D. Lorentz symmetry and the internal structure of the nucleon. *Phys. Rev.* **D58**, 056003, DOI: [10.1103/PhysRevD.58.056003](https://doi.org/10.1103/PhysRevD.58.056003) (1998). [hep-ph/9710290](https://arxiv.org/abs/hep-ph/9710290).
36. Moch, S., Vermaseren, J. & Vogt, A. The Three-Loop Splitting Functions in QCD: The Helicity-Dependent Case. *Nucl. Phys. B* **889**, 351–400, DOI: [10.1016/j.nuclphysb.2014.10.016](https://doi.org/10.1016/j.nuclphysb.2014.10.016) (2014). [1409.5131](https://arxiv.org/abs/1409.5131).
37. Behring, A. *et al.* The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements. *Nucl. Phys. B* **948**, 114753, DOI: [10.1016/j.nuclphysb.2019.114753](https://doi.org/10.1016/j.nuclphysb.2019.114753) (2019). [1908.03779](https://arxiv.org/abs/1908.03779).
38. de Florian, D. & Vogelsang, W. Spin budget of the proton at NNLO and beyond. *Phys. Rev. D* **99**, 054001, DOI: [10.1103/PhysRevD.99.054001](https://doi.org/10.1103/PhysRevD.99.054001) (2019). [1902.04636](https://arxiv.org/abs/1902.04636).
39. Thomas, A. W. Interplay of Spin and Orbital Angular Momentum in the Proton. *Phys. Rev. Lett.* **101**, 102003, DOI: [10.1103/PhysRevLett.101.102003](https://doi.org/10.1103/PhysRevLett.101.102003) (2008). [0803.2775](https://arxiv.org/abs/0803.2775).
40. Hatta, Y. Notes on the orbital angular momentum of quarks in the nucleon. *Phys. Lett. B* **708**, 186–190, DOI: [10.1016/j.physletb.2012.01.024](https://doi.org/10.1016/j.physletb.2012.01.024) (2012). [1111.3547](https://arxiv.org/abs/1111.3547).
41. Lorce, C. & Pasquini, B. Quark Wigner Distributions and Orbital Angular Momentum. *Phys. Rev. D* **84**, 014015, DOI: [10.1103/PhysRevD.84.014015](https://doi.org/10.1103/PhysRevD.84.014015) (2011). [1106.0139](https://arxiv.org/abs/1106.0139).
42. Lorce, C., Pasquini, B., Xiong, X. & Yuan, F. The quark orbital angular momentum from Wigner distributions and light-cone wave functions. *Phys. Rev.* **D85**, 114006, DOI: [10.1103/PhysRevD.85.114006](https://doi.org/10.1103/PhysRevD.85.114006) (2012). [1111.4827](https://arxiv.org/abs/1111.4827).
43. Ji, X., Xiong, X. & Yuan, F. Proton Spin Structure from Measurable Parton Distributions. *Phys. Rev. Lett.* **109**, 152005, DOI: [10.1103/PhysRevLett.109.152005](https://doi.org/10.1103/PhysRevLett.109.152005) (2012). [1202.2843](https://arxiv.org/abs/1202.2843).
44. Ji, X.-d. Viewing the proton through ‘color’ filters. *Phys. Rev. Lett.* **91**, 062001, DOI: [10.1103/PhysRevLett.91.062001](https://doi.org/10.1103/PhysRevLett.91.062001) (2003). [hep-ph/0304037](https://arxiv.org/abs/hep-ph/0304037).
45. Belitsky, A. V., Ji, X.-d. & Yuan, F. Quark imaging in the proton via quantum phase space distributions. *Phys. Rev. D* **69**, 074014, DOI: [10.1103/PhysRevD.69.074014](https://doi.org/10.1103/PhysRevD.69.074014) (2004). [hep-ph/0307383](https://arxiv.org/abs/hep-ph/0307383).
46. Wakamatsu, M. More on the relation between the two physically inequivalent decompositions of the nucleon spin and momentum. *Phys. Rev. D* **85**, 114039, DOI: [10.1103/PhysRevD.85.114039](https://doi.org/10.1103/PhysRevD.85.114039) (2012). [1204.2860](https://arxiv.org/abs/1204.2860).
47. Burkardt, M. Parton Orbital Angular Momentum and Final State Interactions. *Phys. Rev. D* **88**, 014014, DOI: [10.1103/PhysRevD.88.014014](https://doi.org/10.1103/PhysRevD.88.014014) (2013). [1205.2916](https://arxiv.org/abs/1205.2916).
48. Chen, X.-S., Lu, X.-F., Sun, W.-M., Wang, F. & Goldman, T. Spin and orbital angular momentum in gauge theories: Nucleon spin structure and multipole radiation revisited. *Phys. Rev. Lett.* **100**, 232002, DOI: [10.1103/PhysRevLett.100.232002](https://doi.org/10.1103/PhysRevLett.100.232002) (2008). [0806.3166](https://arxiv.org/abs/0806.3166).
49. Ji, X., Zhang, J.-H. & Zhao, Y. Physics of the Gluon-Helicity Contribution to Proton Spin. *Phys. Rev. Lett.* **111**, 112002, DOI: [10.1103/PhysRevLett.111.112002](https://doi.org/10.1103/PhysRevLett.111.112002) (2013). [1304.6708](https://arxiv.org/abs/1304.6708).

50. Hatta, Y., Ji, X. & Zhao, Y. Gluon helicity ΔG from a universality class of operators on a lattice. *Phys. Rev.* **D89**, 085030, DOI: [10.1103/PhysRevD.89.085030](https://doi.org/10.1103/PhysRevD.89.085030) (2014). [1310.4263](https://arxiv.org/abs/1310.4263).
51. Hoodbhoy, P., Ji, X.-D. & Lu, W. Implications of color gauge symmetry for nucleon spin structure. *Phys. Rev. D* **59**, 074010, DOI: [10.1103/PhysRevD.59.074010](https://doi.org/10.1103/PhysRevD.59.074010) (1999). [hep-ph/9808305](https://arxiv.org/abs/hep-ph/9808305).
52. Hoodbhoy, P. & Ji, X.-D. Does the gluon spin contribute in a gauge invariant way to nucleon spin? *Phys. Rev. D* **60**, 114042, DOI: [10.1103/PhysRevD.60.114042](https://doi.org/10.1103/PhysRevD.60.114042) (1999). [hep-ph/9908275](https://arxiv.org/abs/hep-ph/9908275).
53. Hatta, Y. Gluon polarization in the nucleon demystified. *Phys. Rev.* **D84**, 041701, DOI: [10.1103/PhysRevD.84.041701](https://doi.org/10.1103/PhysRevD.84.041701) (2011). [1101.5989](https://arxiv.org/abs/1101.5989).
54. Manohar, A. V. Polarized parton distribution functions. *Phys. Rev. Lett.* **66**, 289–292, DOI: [10.1103/PhysRevLett.66.289](https://doi.org/10.1103/PhysRevLett.66.289) (1991).
55. von Weizsacker, C. F. Radiation emitted in collisions of very fast electrons. *Z. Phys.* **88**, 612–625, DOI: [10.1007/BF01333110](https://doi.org/10.1007/BF01333110) (1934).
56. Williams, E. J. Correlation of certain collision problems with radiation theory. *Kong. Dan. Vid. Sel. Mat. Fys. Med.* **13N4**, 1–50 (1935).
57. Allen, L., Beijersbergen, M. W., Spreeuw, R. J. C. & Woerdman, J. P. Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes. *Phys. Rev.* **A45**, 8185–8189, DOI: [10.1103/PhysRevA.45.8185](https://doi.org/10.1103/PhysRevA.45.8185) (1992).
58. Bliokh, K. Y. & Nori, F. Transverse and longitudinal angular momenta of light. *Phys. Rept.* **592**, 1–38, DOI: [10.1016/j.physrep.2015.06.003](https://doi.org/10.1016/j.physrep.2015.06.003) (2015). [1504.03113](https://arxiv.org/abs/1504.03113).
59. Altarelli, G. & Parisi, G. Asymptotic Freedom in Parton Language. *Nucl. Phys.* **B126**, 298–318, DOI: [10.1016/0550-3213\(77\)90384-4](https://doi.org/10.1016/0550-3213(77)90384-4) (1977).
60. Hagler, P. & Schafer, A. Evolution equations for higher moments of angular momentum distributions. *Phys. Lett.* **B430**, 179–185, DOI: [10.1016/S0370-2693\(98\)00414-6](https://doi.org/10.1016/S0370-2693(98)00414-6) (1998). [hep-ph/9802362](https://arxiv.org/abs/hep-ph/9802362).
61. Harindranath, A. & Kundu, R. On Orbital angular momentum in deep inelastic scattering. *Phys. Rev. D* **59**, 116013, DOI: [10.1103/PhysRevD.59.116013](https://doi.org/10.1103/PhysRevD.59.116013) (1999). [hep-ph/9802406](https://arxiv.org/abs/hep-ph/9802406).
62. Bashinsky, S. & Jaffe, R. L. Quark and gluon orbital angular momentum and spin in hard processes. *Nucl. Phys.* **B536**, 303–317, DOI: [10.1016/S0550-3213\(98\)00559-8](https://doi.org/10.1016/S0550-3213(98)00559-8) (1998). [hep-ph/9804397](https://arxiv.org/abs/hep-ph/9804397).
63. Hoodbhoy, P., Ji, X.-D. & Lu, W. Quark orbital - angular - momentum distribution in the nucleon. *Phys. Rev.* **D59**, 014013, DOI: [10.1103/PhysRevD.59.014013](https://doi.org/10.1103/PhysRevD.59.014013) (1999). [hep-ph/9804337](https://arxiv.org/abs/hep-ph/9804337).
64. Hatta, Y. & Yao, X. QCD evolution of the orbital angular momentum of quarks and gluons: Genuine twist-three part. *Phys. Lett. B* **798**, 134941, DOI: [10.1016/j.physletb.2019.134941](https://doi.org/10.1016/j.physletb.2019.134941) (2019). [1906.07744](https://arxiv.org/abs/1906.07744).
65. Ji, X. & Yuan, F. Transverse spin sum rule of the proton. (2020). [2008.04349](https://arxiv.org/abs/2008.04349).
66. Bakker, B. L. G., Leader, E. & Trueman, T. L. A Critique of the angular momentum sum rules and a new angular momentum sum rule. *Phys. Rev.* **D70**, 114001, DOI: [10.1103/PhysRevD.70.114001](https://doi.org/10.1103/PhysRevD.70.114001) (2004). [hep-ph/0406139](https://arxiv.org/abs/hep-ph/0406139).
67. Leader, E. New relation between transverse angular momentum and generalized parton distributions. *Phys. Rev.* **D85**, 051501, DOI: [10.1103/PhysRevD.85.051501](https://doi.org/10.1103/PhysRevD.85.051501) (2012). [1109.1230](https://arxiv.org/abs/1109.1230).
68. Burkardt, M. Impact parameter space interpretation for generalized parton distributions. *Int. J. Mod. Phys. A* **18**, 173–208, DOI: [10.1142/S0217751X03012370](https://doi.org/10.1142/S0217751X03012370) (2003). [hep-ph/0207047](https://arxiv.org/abs/hep-ph/0207047).
69. Burkardt, M. Transverse deformation of parton distributions and transversity decomposition of angular momentum. *Phys. Rev. D* **72**, 094020, DOI: [10.1103/PhysRevD.72.094020](https://doi.org/10.1103/PhysRevD.72.094020) (2005). [hep-ph/0505189](https://arxiv.org/abs/hep-ph/0505189).
70. Ji, X., Xiong, X. & Yuan, F. Transverse Polarization of the Nucleon in Parton Picture. *Phys. Lett.* **B717**, 214–218, DOI: [10.1016/j.physletb.2012.09.027](https://doi.org/10.1016/j.physletb.2012.09.027) (2012). [1209.3246](https://arxiv.org/abs/1209.3246).
71. Brodsky, S. J., de Teramond, G. F., Dosch, H. G. & Erlich, J. Light-Front Holographic QCD and Emerging Confinement. *Phys. Rept.* **584**, 1–105, DOI: [10.1016/j.physrep.2015.05.001](https://doi.org/10.1016/j.physrep.2015.05.001) (2015). [1407.8131](https://arxiv.org/abs/1407.8131).
72. Wilson, K. G. Confinement of Quarks. *Phys. Rev.* **D10**, 2445–2459, DOI: [10.1103/PhysRevD.10.2445](https://doi.org/10.1103/PhysRevD.10.2445) (1974).
73. Rothe, H. J. Lattice gauge theories: An Introduction. *World Sci. Lect. Notes Phys.* **43**, 1–381 (1992). [World Sci. Lect. Notes Phys.59,1(1997); World Sci. Lect. Notes Phys.74,1(2005); World Sci. Lect. Notes Phys.82,1(2012)].
74. Maris, P. & Roberts, C. D. Dyson-Schwinger equations: A Tool for hadron physics. *Int. J. Mod. Phys. E* **12**, 297–365, DOI: [10.1142/S0218301303001326](https://doi.org/10.1142/S0218301303001326) (2003). [nucl-th/0301049](https://arxiv.org/abs/nucl-th/0301049).

75. Schäfer, T. & Shuryak, E. V. Instantons in QCD. *Rev. Mod. Phys.* **70**, 323–426, DOI: [10.1103/RevModPhys.70.323](https://doi.org/10.1103/RevModPhys.70.323) (1998). [hep-ph/9610451](https://arxiv.org/abs/hep-ph/9610451).
76. Alexandrou, C. *et al.* Complete flavor decomposition of the spin and momentum fraction of the proton using lattice QCD simulations at physical pion mass. *Phys. Rev.* **D101**, 094513, DOI: [10.1103/PhysRevD.101.094513](https://doi.org/10.1103/PhysRevD.101.094513) (2020). [2003.08486](https://arxiv.org/abs/2003.08486).
77. Ji, X. Parton Physics on a Euclidean Lattice. *Phys. Rev. Lett.* **110**, 262002, DOI: [10.1103/PhysRevLett.110.262002](https://doi.org/10.1103/PhysRevLett.110.262002) (2013). [1305.1539](https://arxiv.org/abs/1305.1539).
78. Ji, X. Parton Physics from Large-Momentum Effective Field Theory. *Sci. China Phys. Mech. Astron.* **57**, 1407–1412, DOI: [10.1007/s11433-014-5492-3](https://doi.org/10.1007/s11433-014-5492-3) (2014). [1404.6680](https://arxiv.org/abs/1404.6680).
79. Ji, X., Liu, Y.-S., Liu, Y., Zhang, J.-H. & Zhao, Y. Large-Momentum Effective Theory. (2020). [2004.03543](https://arxiv.org/abs/2004.03543).
80. Dong, S., Lagae, J.-F. & Liu, K. Flavor singlet $g(A)$ from lattice QCD. *Phys. Rev. Lett.* **75**, 2096–2099, DOI: [10.1103/PhysRevLett.75.2096](https://doi.org/10.1103/PhysRevLett.75.2096) (1995). [hep-ph/9502334](https://arxiv.org/abs/hep-ph/9502334).
81. Lin, H.-W. *et al.* Parton distributions and lattice QCD calculations: a community white paper. *Prog. Part. Nucl. Phys.* **100**, 107–160, DOI: [10.1016/j.pnpnp.2018.01.007](https://doi.org/10.1016/j.pnpnp.2018.01.007) (2018). [1711.07916](https://arxiv.org/abs/1711.07916).
82. Liang, J., Yang, Y.-B., Draper, T., Gong, M. & Liu, K.-F. Quark spins and Anomalous Ward Identity. *Phys. Rev. D* **98**, 074505, DOI: [10.1103/PhysRevD.98.074505](https://doi.org/10.1103/PhysRevD.98.074505) (2018). [1806.08366](https://arxiv.org/abs/1806.08366).
83. Alexandrou, C. *et al.* Nucleon Spin and Momentum Decomposition Using Lattice QCD Simulations. *Phys. Rev. Lett.* **119**, 142002, DOI: [10.1103/PhysRevLett.119.142002](https://doi.org/10.1103/PhysRevLett.119.142002) (2017). [1706.02973](https://arxiv.org/abs/1706.02973).
84. Deka, M. *et al.* Lattice study of quark and glue momenta and angular momenta in the nucleon. *Phys. Rev.* **D91**, 014505, DOI: [10.1103/PhysRevD.91.014505](https://doi.org/10.1103/PhysRevD.91.014505) (2015). [1312.4816](https://arxiv.org/abs/1312.4816).
85. Gong, M. *et al.* Strange and charm quark spins from the anomalous Ward identity. *Phys. Rev.* **D95**, 114509, DOI: [10.1103/PhysRevD.95.114509](https://doi.org/10.1103/PhysRevD.95.114509) (2017). [1511.03671](https://arxiv.org/abs/1511.03671).
86. Mathur, N., Dong, S. J., Liu, K. F., Mankiewicz, L. & Mukhopadhyay, N. C. Quark orbital angular momentum from lattice QCD. *Phys. Rev.* **D62**, 114504, DOI: [10.1103/PhysRevD.62.114504](https://doi.org/10.1103/PhysRevD.62.114504) (2000). [hep-ph/9912289](https://arxiv.org/abs/hep-ph/9912289).
87. Hagler, P. *et al.* Moments of nucleon generalized parton distributions in lattice QCD. *Phys. Rev.* **D68**, 034505, DOI: [10.1103/PhysRevD.68.034505](https://doi.org/10.1103/PhysRevD.68.034505) (2003). [hep-lat/0304018](https://arxiv.org/abs/hep-lat/0304018).
88. Gockeler, M. *et al.* Generalized parton distributions from lattice QCD. *Phys. Rev. Lett.* **92**, 042002, DOI: [10.1103/PhysRevLett.92.042002](https://doi.org/10.1103/PhysRevLett.92.042002) (2004). [hep-ph/0304249](https://arxiv.org/abs/hep-ph/0304249).
89. Brommel, D. *et al.* Moments of generalized parton distributions and quark angular momentum of the nucleon. *PoS LATTICE2007*, 158, DOI: [10.22323/1.042.0158](https://doi.org/10.22323/1.042.0158) (2007). [0710.1534](https://arxiv.org/abs/0710.1534).
90. Bratt, J. D. *et al.* Nucleon structure from mixed action calculations using 2+1 flavors of asqtad sea and domain wall valence fermions. *Phys. Rev.* **D82**, 094502, DOI: [10.1103/PhysRevD.82.094502](https://doi.org/10.1103/PhysRevD.82.094502) (2010). [1001.3620](https://arxiv.org/abs/1001.3620).
91. Syritsyn, S. N. *et al.* Quark Contributions to Nucleon Momentum and Spin from Domain Wall fermion calculations. *PoS LATTICE2011*, 178, DOI: [10.22323/1.139.0178](https://doi.org/10.22323/1.139.0178) (2011). [1111.0718](https://arxiv.org/abs/1111.0718).
92. Alexandrou, C. *et al.* Moments of nucleon generalized parton distributions from lattice QCD. *Phys. Rev.* **D83**, 114513, DOI: [10.1103/PhysRevD.83.114513](https://doi.org/10.1103/PhysRevD.83.114513) (2011). [1104.1600](https://arxiv.org/abs/1104.1600).
93. Alexandrou, C. *et al.* Nucleon form factors and moments of generalized parton distributions using $N_f = 2 + 1 + 1$ twisted mass fermions. *Phys. Rev.* **D88**, 014509, DOI: [10.1103/PhysRevD.88.014509](https://doi.org/10.1103/PhysRevD.88.014509) (2013). [1303.5979](https://arxiv.org/abs/1303.5979).
94. Yang, Y.-B. A Lattice Story of Proton Spin. *PoS LATTICE2018*, 017, DOI: [10.22323/1.334.0017](https://doi.org/10.22323/1.334.0017) (2019). [1904.04138](https://arxiv.org/abs/1904.04138).
95. Yang, Y.-B. *et al.* Glue Spin and Helicity in the Proton from Lattice QCD. *Phys. Rev. Lett.* **118**, 102001, DOI: [10.1103/PhysRevLett.118.102001](https://doi.org/10.1103/PhysRevLett.118.102001) (2017). [1609.05937](https://arxiv.org/abs/1609.05937).
96. Chen, X.-S., Sun, W.-M., Wang, F. & Goldman, T. Proper identification of the gluon spin. *Phys. Lett.* **B700**, 21–24, DOI: [10.1016/j.physletb.2011.04.045](https://doi.org/10.1016/j.physletb.2011.04.045) (2011). [1101.5358](https://arxiv.org/abs/1101.5358).
97. Weinberg, S. Dynamics at infinite momentum. *Phys. Rev.* **150**, 1313–1318, DOI: [10.1103/PhysRev.150.1313](https://doi.org/10.1103/PhysRev.150.1313) (1966).
98. Zhao, Y., Liu, K.-F. & Yang, Y. Orbital Angular Momentum and Generalized Transverse Momentum Distribution. *Phys. Rev.* **D93**, 054006, DOI: [10.1103/PhysRevD.93.054006](https://doi.org/10.1103/PhysRevD.93.054006) (2016). [1506.08832](https://arxiv.org/abs/1506.08832).
99. Ji, X., Zhang, J.-H. & Zhao, Y. Justifying the Naive Partonic Sum Rule for Proton Spin. *Phys. Lett.* **B743**, 180–183, DOI: [10.1016/j.physletb.2015.02.054](https://doi.org/10.1016/j.physletb.2015.02.054) (2015). [1409.6329](https://arxiv.org/abs/1409.6329).

100. Gadiyak, V., Ji, X.-d. & Jung, C.-w. A Lattice study of the magnetic moment and the spin structure of the nucleon. *Phys. Rev. D* **65**, 094510, DOI: [10.1103/PhysRevD.65.094510](https://doi.org/10.1103/PhysRevD.65.094510) (2002). [hep-lat/0112040](https://arxiv.org/abs/hep-lat/0112040).
101. Blum, T. *et al.* Lattice Calculation of Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment. *Phys. Rev. D* **93**, 014503, DOI: [10.1103/PhysRevD.93.014503](https://doi.org/10.1103/PhysRevD.93.014503) (2016). [1510.07100](https://arxiv.org/abs/1510.07100).
102. Engelhardt, M. Quark orbital dynamics in the proton from Lattice QCD – from Ji to Jaffe-Manohar orbital angular momentum. *Phys. Rev. D* **95**, 094505, DOI: [10.1103/PhysRevD.95.094505](https://doi.org/10.1103/PhysRevD.95.094505) (2017). [1701.01536](https://arxiv.org/abs/1701.01536).
103. Engelhardt, M. *et al.* From Ji to Jaffe-Manohar orbital angular momentum in Lattice QCD using a direct derivative method. (2020). [2008.03660](https://arxiv.org/abs/2008.03660).
104. Aslan, F. & Burkardt, M. Singularities in Twist-3 Quark Distributions. *Phys. Rev. D* **101**, 016010, DOI: [10.1103/PhysRevD.101.016010](https://doi.org/10.1103/PhysRevD.101.016010) (2020). [1811.00938](https://arxiv.org/abs/1811.00938).
105. Ji, X. Fundamental Properties of the Proton in Light-Front Zero Modes. (2020). [2003.04478](https://arxiv.org/abs/2003.04478).
106. Ji, X.-D. Deeply virtual Compton scattering. *Phys. Rev. D* **55**, 7114–7125, DOI: [10.1103/PhysRevD.55.7114](https://doi.org/10.1103/PhysRevD.55.7114) (1997). [hep-ph/9609381](https://arxiv.org/abs/hep-ph/9609381).
107. de Florian, D., Sassot, R., Stratmann, M. & Vogelsang, W. Evidence for polarization of gluons in the proton. *Phys. Rev. Lett.* **113**, 012001, DOI: [10.1103/PhysRevLett.113.012001](https://doi.org/10.1103/PhysRevLett.113.012001) (2014). [1404.4293](https://arxiv.org/abs/1404.4293).
108. Adamczyk, L. *et al.* Precision Measurement of the Longitudinal Double-spin Asymmetry for Inclusive Jet Production in Polarized Proton Collisions at $\sqrt{s} = 200$ GeV. *Phys. Rev. Lett.* **115**, 092002, DOI: [10.1103/PhysRevLett.115.092002](https://doi.org/10.1103/PhysRevLett.115.092002) (2015). [1405.5134](https://arxiv.org/abs/1405.5134).
109. Airapetian, A. *et al.* Quark helicity distributions in the nucleon for up, down, and strange quarks from semi-inclusive deep-inelastic scattering. *Phys. Rev. D* **71**, 012003, DOI: [10.1103/PhysRevD.71.012003](https://doi.org/10.1103/PhysRevD.71.012003) (2005). [hep-ex/0407032](https://arxiv.org/abs/hep-ex/0407032).
110. Alekseev, M. *et al.* Flavour Separation of Helicity Distributions from Deep Inelastic Muon-Deuteron Scattering. *Phys. Lett. B* **680**, 217–224, DOI: [10.1016/j.physletb.2009.08.065](https://doi.org/10.1016/j.physletb.2009.08.065) (2009). [0905.2828](https://arxiv.org/abs/0905.2828).
111. de Florian, D., Sassot, R., Stratmann, M. & Vogelsang, W. Extraction of Spin-Dependent Parton Densities and Their Uncertainties. *Phys. Rev. D* **80**, 034030, DOI: [10.1103/PhysRevD.80.034030](https://doi.org/10.1103/PhysRevD.80.034030) (2009). [0904.3821](https://arxiv.org/abs/0904.3821).
112. Nocera, E. R., Ball, R. D., Forte, S., Ridolfi, G. & Rojo, J. A first unbiased global determination of polarized PDFs and their uncertainties. *Nucl. Phys. B* **887**, 276–308, DOI: [10.1016/j.nuclphysb.2014.08.008](https://doi.org/10.1016/j.nuclphysb.2014.08.008) (2014). [1406.5539](https://arxiv.org/abs/1406.5539).
113. Ethier, J. J., Sato, N. & Melnitchouk, W. First simultaneous extraction of spin-dependent parton distributions and fragmentation functions from a global QCD analysis. *Phys. Rev. Lett.* **119**, 132001, DOI: [10.1103/PhysRevLett.119.132001](https://doi.org/10.1103/PhysRevLett.119.132001) (2017). [1705.05889](https://arxiv.org/abs/1705.05889).
114. Adam, J. *et al.* Measurement of the longitudinal spin asymmetries for weak boson production in proton-proton collisions at $\sqrt{s} = 510$ GeV. *Phys. Rev. D* **99**, 051102, DOI: [10.1103/PhysRevD.99.051102](https://doi.org/10.1103/PhysRevD.99.051102) (2019). [1812.04817](https://arxiv.org/abs/1812.04817).
115. Aschenauer, E.-C. *et al.* The RHIC SPIN Program: Achievements and Future Opportunities. (2015). [1501.01220](https://arxiv.org/abs/1501.01220).
116. Adam, J. *et al.* Longitudinal double-spin asymmetry for inclusive jet and dijet production in pp collisions at $\sqrt{s} = 510$ GeV. *Phys. Rev. D* **100**, 052005, DOI: [10.1103/PhysRevD.100.052005](https://doi.org/10.1103/PhysRevD.100.052005) (2019). [1906.02740](https://arxiv.org/abs/1906.02740).
117. Mazouz, M. *et al.* Deeply virtual compton scattering off the neutron. *Phys. Rev. Lett.* **99**, 242501, DOI: [10.1103/PhysRevLett.99.242501](https://doi.org/10.1103/PhysRevLett.99.242501) (2007). [0709.0450](https://arxiv.org/abs/0709.0450).
118. Collins, J. C., Frankfurt, L. & Strikman, M. Factorization for hard exclusive electroproduction of mesons in QCD. *Phys. Rev. D* **56**, 2982–3006, DOI: [10.1103/PhysRevD.56.2982](https://doi.org/10.1103/PhysRevD.56.2982) (1997). [hep-ph/9611433](https://arxiv.org/abs/hep-ph/9611433).
119. Mankiewicz, L., Piller, G., Stein, E., Vantinen, M. & Weigl, T. NLO corrections to deeply virtual Compton scattering. *Phys. Lett. B* **425**, 186–192, DOI: [10.1016/S0370-2693\(98\)00190-7](https://doi.org/10.1016/S0370-2693(98)00190-7) (1998). [Erratum: *Phys.Lett.B* 461, 423–423 (1999)], [hep-ph/9712251](https://arxiv.org/abs/hep-ph/9712251).
120. Ji, X.-D. Off forward parton distributions. *J. Phys. G* **24**, 1181–1205, DOI: [10.1088/0954-3899/24/7/002](https://doi.org/10.1088/0954-3899/24/7/002) (1998). [hep-ph/9807358](https://arxiv.org/abs/hep-ph/9807358).
121. Diehl, M. *Generalized parton distributions*. Ph.D. thesis (2003). DOI: [10.1016/j.physrep.2003.08.002](https://doi.org/10.1016/j.physrep.2003.08.002). [hep-ph/0307382](https://arxiv.org/abs/hep-ph/0307382).
122. Ji, X. Generalized parton distributions. *Ann. Rev. Nucl. Part. Sci.* **54**, 413–450, DOI: [10.1146/annurev.nucl.54.070103.181302](https://doi.org/10.1146/annurev.nucl.54.070103.181302) (2004).
123. Belitsky, A. & Radyushkin, A. Unraveling hadron structure with generalized parton distributions. *Phys. Rept.* **418**, 1–387, DOI: [10.1016/j.physrep.2005.06.002](https://doi.org/10.1016/j.physrep.2005.06.002) (2005). [hep-ph/0504030](https://arxiv.org/abs/hep-ph/0504030).

124. Airapetian, A. *et al.* Measurement of Azimuthal Asymmetries With Respect To Both Beam Charge and Transverse Target Polarization in Exclusive Electroproduction of Real Photons. *JHEP* **06**, 066, DOI: [10.1088/1126-6708/2008/06/066](https://doi.org/10.1088/1126-6708/2008/06/066) (2008). [0802.2499](https://arxiv.org/abs/0802.2499).
125. Akhunzyanov, R. *et al.* Transverse extension of partons in the proton probed in the sea-quark range by measuring the DVCS cross section. *Phys. Lett. B* **793**, 188–194, DOI: [10.1016/j.physletb.2019.04.038](https://doi.org/10.1016/j.physletb.2019.04.038) (2019). [1802.02739](https://arxiv.org/abs/1802.02739).
126. Burkert, V., Elouadrhiri, L. & Girod, F. The pressure distribution inside the proton. *Nature* **557**, 396–399, DOI: [10.1038/s41586-018-0060-z](https://doi.org/10.1038/s41586-018-0060-z) (2018).
127. Ji, X., Yuan, F. & Zhao, Y. Hunting the Gluon Orbital Angular Momentum at the Electron-Ion Collider. *Phys. Rev. Lett.* **118**, 192004, DOI: [10.1103/PhysRevLett.118.192004](https://doi.org/10.1103/PhysRevLett.118.192004) (2017). [1612.02438](https://arxiv.org/abs/1612.02438).
128. Hatta, Y., Nakagawa, Y., Yuan, F., Zhao, Y. & Xiao, B. Gluon orbital angular momentum at small- x . *Phys. Rev.* **D95**, 114032, DOI: [10.1103/PhysRevD.95.114032](https://doi.org/10.1103/PhysRevD.95.114032) (2017). [1612.02445](https://arxiv.org/abs/1612.02445).
129. Blumlein, J. & Vogt, A. The Singlet contribution to the structure function $g_1(x, Q^2)$ at small x . *Phys. Lett. B* **386**, 350–358, DOI: [10.1016/0370-2693\(96\)00958-6](https://doi.org/10.1016/0370-2693(96)00958-6) (1996). [hep-ph/9606254](https://arxiv.org/abs/hep-ph/9606254).
130. Kovchegov, Y. V., Pitonyak, D. & Sievert, M. D. Helicity Evolution at Small- x . *JHEP* **01**, 072, DOI: [10.1007/JHEP01\(2016\)072](https://doi.org/10.1007/JHEP01(2016)072) (2016). [Erratum: *JHEP* 10, 148 (2016)], [1511.06737](https://arxiv.org/abs/1511.06737).
131. Kovchegov, Y. V., Pitonyak, D. & Sievert, M. D. Small- x asymptotics of the quark helicity distribution. *Phys. Rev. Lett.* **118**, 052001, DOI: [10.1103/PhysRevLett.118.052001](https://doi.org/10.1103/PhysRevLett.118.052001) (2017). [1610.06188](https://arxiv.org/abs/1610.06188).
132. Kovchegov, Y. V. & Sievert, M. D. Small- x Helicity Evolution: an Operator Treatment. *Phys. Rev. D* **99**, 054032, DOI: [10.1103/PhysRevD.99.054032](https://doi.org/10.1103/PhysRevD.99.054032) (2019). [1808.09010](https://arxiv.org/abs/1808.09010).
133. Boussarie, R., Hatta, Y. & Yuan, F. Proton Spin Structure at Small- x . *Phys. Lett. B* **797**, 134817, DOI: [10.1016/j.physletb.2019.134817](https://doi.org/10.1016/j.physletb.2019.134817) (2019). [1904.02693](https://arxiv.org/abs/1904.02693).
134. Tarasov, A. & Venugopalan, R. The role of the chiral anomaly in polarized deeply inelastic scattering I: Finding the triangle graph inside the box diagram in Bjorken and Regge asymptotics. (2020). [2008.08104](https://arxiv.org/abs/2008.08104).