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When Convex Analysis Meets Mathematical Morphology on Graphs

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Abstract. In recent years, variational methods, *i.e.*, the formulation of problems under optimization forms, have had a great deal of success in image processing. This may be accounted for by their good performance and versatility. Conversely, mathematical morphology (MM) is a widely recognized methodology for solving a wide array of image processing-related tasks. It thus appears useful and timely to build bridges between these two fields. In this article, we propose a variational approach to implement the four basic, structuring element-based operators of MM: dilation, erosion, opening, and closing. We rely on discrete calculus and convex analysis for our formulation. We show that we are able to propose a variety of continuously varying operators in between the dual extremes, *i.e.*, between erosions and dilation; and perhaps more interestingly between openings and closings. This paves the way to the use of morphological operators in a number of new applications.

Keywords: Optimization, convex analysis, discrete calculus, graphs.

1 Introduction

In recent years, variational methods have received a great deal of attention in the image processing community, mainly due to their success in addressing a wide range of signal and image processing tasks [24] (*e.g.*, denoising, restoration, reconstruction, impainting, segmentation,...). Concurrently, mathematical morphology (MM) is widely recognized as a fundamental methodology for solving image processing problems [23]. It thus appears useful to build bridges between these two fields. A first step in this direction may consist of looking for variational formulations of the essential structuring-element based operators of morphological processing, *i.e*, the dilation, erosion, opening, and closing operators.

Several formulations can be found in the literature. Since the dilation, for instance, is in some way similar to a propagation, it seems natural to express it as a propagation or a diffusion PDE. The first proposal for a PDE-based formulation of morphology was independently proposed by Alvarez *et al.* [1,2], Brockett and Maragos [4], and Boomgaard and Smeulders [27]. In [19], this approach is studied from a more geometrical point of view. In [10], standard derivatives are replaced by metric-based differential calculus. In [17], Maragos proposes a PDE-based formalism based on the slope transform. The slope transform is studied in details, including its relationship with convex analysis in [14]. We also mention [3,25], where binary dilations and erosions are modeled as curve evolution with constant normal speed. An overview of PDE-based, contrast-invariant image processing, including morphology operators, is given in [13]. Many references could be cited here, not limited to dilation and erosions; for example, the equivalents of connected operators are proposed in PDE form in [21]. Obviously all these approaches need to be discretized for practical implementations, typically with finite differences. All of these share a similar goal: they seek for implementing the notion of morphological operator in the continuous domain. The discrete structure used to specifying the operators is not seen as important.

In contrast, some formulations exploit the discrete nature of image data. For instance, an algebraic analysis of the theory of differential operators [5] leads, in the discrete case, to the discrete exterior calculus [9,8]. In [12], Grady proposes a graph-based, variational implementation, with applications to imaging. The variational discrete calculus is expanded in [6]. A similar approach is followed in [11]. An overview of the various graph-based variational approaches is given in [16]. In all of these, the graph is not a discretization artifact, it remains a core feature, allowing for more flexibility and arguably better data fidelity.

In the discrete setting, a natural framework for mathematical morphology is precisely that of graphs. Pioneering papers are in [28,15]. Many works have been proposed since, for instance [20,7]. The framework has been extended far beyond the basic operators, the interested reader can refer to the recent survey given in [22].

To the best of our knowledge, a variational framework for implementing the morphological operators on graphs has not been proposed. Maragos [18] was the first to propose a variational approach to study the basic and the connected operators of MM, but in a continuous setting. A graph-based PDE approach to MM is proposed in [26], however the approach is not variational. In this work, we follow an alternative, conceptually simpler way, merging variational formulation and graphs, based on discrete calculus and convex analysis.

The graph-based variational framework for MM developped in this paper, allows us to overcome some of the main difficulties of the continuous setting. In particular, in the continuous domain, it has been not possible so far to propose a proper differential definition of the combined operators: openings and closings; a major difficulty being the idempotence of these operators. In contrast, we show in this paper that the discrete nature of the graph allows a complete definition of these operators. In future work, such a variational formulation of morphological operators could be used as asymmetric, non self-dual, regularisers, something difficult (and thus unusual) in standard applications of the variational framework.

The organization of the paper is the following. By designing objective functions using judiciously chosen support functions, we show that a fully convex formulation of dilation, erosion, dilation, and opening is possible. More precisely, we establish that these morphological operators correspond to asymptotic forms of the solutions to classes of convex optimization problems. As a side result, by considering a non-asymptotic regime, a broader set of solutions can be obtained which allow us to generate, in a continuous manner, intermediate behaviours between dilation/erosion and opening/closing. Our proposal for a graph-based variational formulation of the erosion and dilation operators is exposed in section 2, while the one for opening and closing operators is developed in section 3. Due to space constraints, the proofs of the results in this paper will be provided in an extended version.

Notation. Let $G = (V, E)$ be a valued directed reflexive graph. The cardinality of the vertex set V is assumed to be equal to $n \in \mathbb{N}^*$. In the case of images, the vertices reduce to pixels. An edge joining vertex $v_i \in V$ to vertex $v_j \in V$ with $(i, j) \in \{1, \dots, n\}^2$, $i \neq j$, is denoted by $e_{i,j} \in E$. For every $i \in \{1, \dots, n\}$, a weight $y_i \in \mathbb{R}$ is associated with each vertex v_i, and we introduce the set of neighboring indices of i, $N_i = \{j \in \{1, \dots, n\} \mid e_{i,j} \in E\}$, which will be assumed to be nonempty. The reciprocal neighborhood of N_i is $\check{N}_i = \{j \in \{1, \dots, n\} \mid i \in N_j\}$. Since G is reflexive, $(\forall i \in \{1, \dots, n\})\ i \in N_i \cap \check{N}_i$.

2 A unifying variational formulation of erosion, dilation and median

For every $i \in \{1, \dots, n\}$, let U_i be a nonempty subset of V (although it may be any subset, we will be mainly interested in the case when it corresponds to a neighborhood or reciprocal neighborhood of node i). We will first consider the following convex minimization problem:

$$\underset{x=(x_i)_{1\leqslant i\leqslant n}\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^{n} \sum_{j\in U_i} \varphi_i\big(\sigma_{\Omega_i}(x_i - y_j - w_{i,j})\big) \tag{2.1}$$

where $(w_{i,j})_{j\in U_i} \in \mathbb{R}^{|U_i|}$ is a vector of shift parameters, $\varphi_i\colon\ [0, +\infty[\ \to \mathbb{R}$ is a (strictly) increasing lower-semicontinuous convex function, Ω_i is a closed real interval, and σ_{Ω_i} denotes the support function of Ω_i, which is defined as

$$\begin{aligned}(\forall \upsilon \in \mathbb{R}) \qquad \sigma_{\Omega_i}(\upsilon) &= \sup_{\xi\in\Omega_i} \upsilon\xi \\ &= \begin{cases} \alpha_i^-\upsilon & \text{if } \upsilon < 0 \\ 0 & \text{if } \upsilon = 0 \\ \alpha_i^+\upsilon & \text{if } \upsilon > 0 \end{cases} \end{aligned} \tag{2.2}$$

with $\alpha_i^- = \inf \Omega_i$ and $\alpha_i^+ = \sup \Omega_i$.

Any solution to the multivariate optimization problem (2.1) is a vector of node weights which can be viewed as the result of a generally nonlinear processing applied to the original vector of node weights $y = (y_i)_{1\leqslant i\leqslant n}$. The result is dependent on the choice of the intervals $(\Omega_i)_{1\leqslant i\leqslant n}$ and our goal will be to better understand this dependence. In order to guarantee the existence of a solution to problem (2.1), it will be assumed that:

Assumption 1 For every $i \in \{1, \dots, n\}$, $-\infty \leqslant \alpha_i^- < 0 < \alpha_i^+ \leqslant +\infty$ with $(\alpha_i^-, \alpha_i^+) \neq (-\infty, +\infty)$.

Note that α_i^- (resp. α_i^+) is allowed to be equal to $-\infty$ (resp. $+\infty$), which will turn out to be useful in the rest of our discussion.

A first result concerning the solution to this optimization problem is as follows:

Proposition 1. *Suppose that Assumption 1 holds. Then Problem (2.1) admits a solution.*
Let $i \in \{1, \dots, n\}$ and let $\widehat{x}_i$ be the i-th component of such a solution $\widehat{x}$. We have the following properties:

(i) *If $\alpha_i^+ = +\infty$, then*

$$\widehat{x}_i = \min \left\{ y_j + w_{i,j} \;\middle|\; j \in U_i \right\}. \tag{2.3}$$

(ii) *If $\alpha_i^- = -\infty$, then*

$$\widehat{x}_i = \max \left\{ y_j + w_{i,j} \;\middle|\; j \in U_i \right\}. \tag{2.4}$$

(iii) *If φ_i is strictly convex, then $\widehat{x}_i$ also is uniquely defined.*

Remark 1. We see that standard morphological mathematical operators are recovered as specific solutions to Problem (2.1):

– If $\alpha_i^+ = +\infty$ and $U_i = N_i$ with $i \in \{1, \dots, n\}$, then (2.3) corresponds to an erosion with (spatially variant) structuring element:

$$\big(\forall (i,j) \in \{1, \dots, n\}\big) \qquad \widetilde{\omega}_{i,j} = \begin{cases} -\omega_{i,j} & \text{if } j \in N_i \\ -\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

– If $\alpha_i^- = -\infty$ and $U_i = N_i$ with $i \in \{1, \dots, n\}$, then (2.4) corresponds to a dilation with (spatially variant) structuring element:

$$\big(\forall (i,j) \in \{1, \dots, n\}\big) \qquad \widetilde{\omega}_{i,j} = \begin{cases} \omega_{i,j} & \text{if } j \in N_i \\ -\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

By making $-\alpha_i^+/\alpha_i^-$ vary, intermediate behaviours between an erosion and a dilation can be obtained. In general however, one must resort to numerical methods to find a solution to Problem (2.1).

We now mention a specific choice of functions $(\varphi_i)_{1 \leqslant i \leqslant n}$, for which minimizers can be expressed in closed forms.

Proposition 2. *Suppose that Assumption 1 holds. Let $i \in \{1, \dots, n\}$ and let $\widehat{x}_i$ be the i-th component of the optimal solution to Problem (2.1) where $\varphi_i \colon [0, +\infty[\mapsto \mathbb{R} \colon \xi \mapsto \xi^2$. Let us denote by $(z_{i,j})_{1 \leqslant j \leqslant |U_i|}$ the coefficients $(y_j + w_{i,j})_{j \in U_i}$ which have been re-indexed in increasing order, and let us set $z_{i,0} = -\infty$ and $z_{i,|U_i|+1} = +\infty$. If $(\alpha_i^-, \alpha_i^+) \in \mathbb{R}^2$, then $\widehat{x}_i$ is the unique value in $[z_{i,j_i-1}, z_{i,j_i}]$ with $j_i \in \{1, \dots, |U_i| + 1\}$ such that*

$$\widehat{x}_i = \frac{(\alpha_i^+)^2 \sum_{j=1}^{j_i-1} z_{i,j} + (\alpha_i^-)^2 \sum_{j=j_i}^{|U_i|} z_{i,j}}{(j_i - 1)(\alpha_i^+)^2 + (|U_i| - j_i + 1)(\alpha_i^-)^2} \tag{2.7}$$

(with the conventions $\sum_{j=1}^{0} \cdot = \sum_{j=|U_i|+1}^{|U_i|} \cdot = 0$).

Remark 2. When $\alpha_i^- = -\alpha_i^+$ with $i \in \{1, \dots, n\}$ and $\alpha_i^+ \in]0, +\infty[$, then the expression of $\widehat{x}_i$ given in the above proposition reduces to the standard averaged value of $(y_j + w_{i,j})_{j \in U_i}$.

An image processing illustration of Proposition 2 is provided in the first two columns of Fig. 2. The original 8-bit images are noisy versions of a synthetic one of size 100×104 called Pyramid and of a natural scene of size 512×512 called Goldhill, displayed in 1. The graph consists here of a regular 4-connected grid. As expected, nonlinear filters ranging from a dilation (on the left) to an erosion (on the right) are generated, with the local averaging (third image) as a special case.

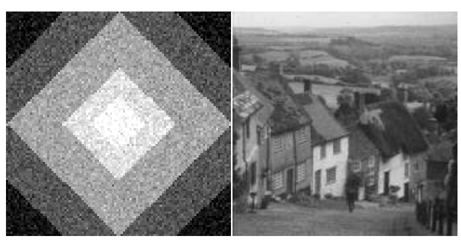

Fig. 1. Pyramid and Goldhill images.

Another simple choice of functions $(\varphi_i)_{1 \leqslant i \leqslant n}$ is addressed next:

Proposition 3. *Suppose that Assumption 1 holds. Let $i \in \{1, \dots, n\}$ and let $\widehat{x}_i$ be the i-th component of an optimal solution to Problem (2.1) where $\varphi_i \colon [0, +\infty[\mapsto \mathbb{R} \colon \xi \mapsto \xi$. Let us denote by $(z_{i,j})_{1 \leqslant j \leqslant |U_i|}$ the coefficients $(y_j + w_{i,j})_{j \in U_i}$ which have been re-indexed in increasing order. Assume that $(\alpha_i^-, \alpha_i^+) \in \mathbb{R}^2$ and set $j_i = \left\lfloor \frac{|U_i|}{1 - \alpha_i^+/\alpha_i^-} \right\rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the lower rounding operation.*

Then, if $\frac{|U_i|}{1 - \alpha_i^+/\alpha_i^-} \notin \mathbb{N}$, $\widehat{x}_i = z_{i,j_i}$.

If $\frac{|U_i|}{1 - \alpha_i^+/\alpha_i^-} \in \mathbb{N}$, then $\widehat{x}_i$ can be chosen equal to any value in $[z_{i,j_i-1}, z_{i,j_i}]$.

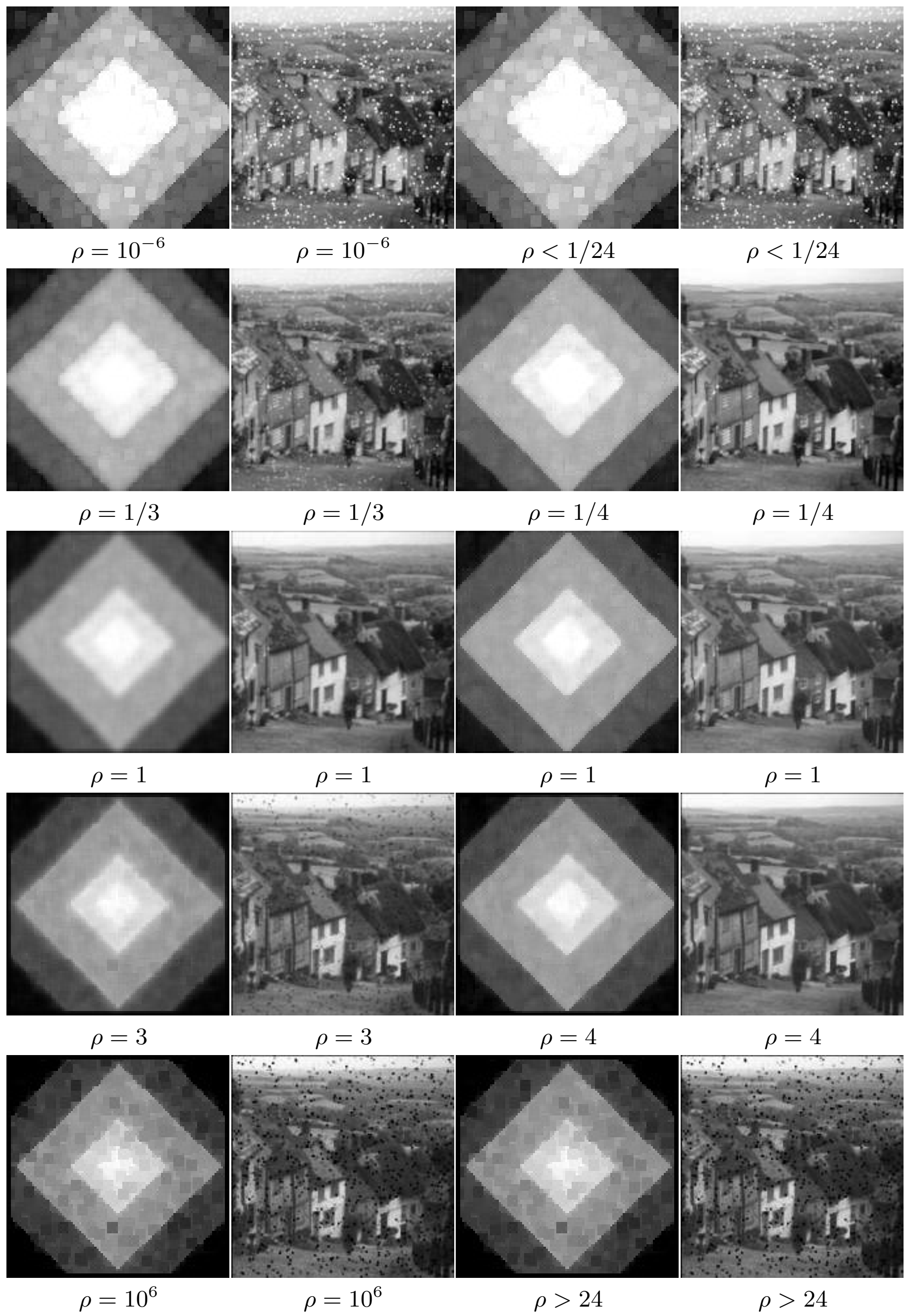

Fig. 2. First two columns: solutions to Problem (2.1) for Pyramid and Goldhill images when, for every pixel i, $\varphi_i \colon [0, +\infty[\mapsto \mathbb{R} \colon \xi \mapsto \xi^2$.
Last two columns: solutions to Problem (2.1) for Pyramid and Goldhill images when, for every pixel i, $\varphi_i \colon [0, +\infty[\mapsto \mathbb{R} \colon \xi \mapsto \xi$.
In both cases, $-\alpha_i^+/\alpha_i^- = \rho$, $(w_{i,j})_{j \in U_i} = 0$, and U_i is a 5×5 spatial neighborhood.

An illustration of Proposition 3 is provided in the last two columns of Fig. 2. The first (resp. last) row exacly corresponds to a dilation (resp. an erosion), as explained in the remark below. Note that we do not observe on the intermediate images the blur effect which is noticeable on those displayed in the first two columns of Fig. 2.

Remark 3. Under the assumptions of the above proposition, the following specific solutions can be obtained:

– median filtering
  If $\alpha_i^- = -\alpha_i^+$, $\alpha_i^+ \in]0,+\infty[$ and $U_i = N_i$ with $i \in \{1,\ldots,n\}$, then the i-th component $\widehat{x}_i$ of an optimal solution to Problem (2.1) is

$$\widehat{x}_i = \mathrm{median}\{y_j + w_{i,j} \mid j \in N_i\}. \tag{2.8}$$

– erosion
  If $-\alpha_i^+/\alpha_i^- > |N_i| - 1$ and $U_i = N_i$, then $\widehat{x}_i$ is given by (2.3).
– dilation
  If $-\alpha_i^+/\alpha_i^-(|N_i| - 1) < 1$ and $U_i = N_i$, then $\widehat{x}_i$ is given by (2.4).

More generally, by varying $-\alpha_i^+/\alpha_i^-$, any rank-order filtering operator between a dilation and an erosion can be generated, so putting more or less emphasis on large/small intensity values for vertices in the neighborhood N_i.

3 Towards a variational formulation of opening and closing

Let us now consider the more sophisticated convex minimization problem:

$$\underset{\substack{x=(x_i)_{1\leqslant i\leqslant n}\in\mathbb{R}^n\\ t=(t_i)_{1\leqslant i\leqslant n}\in\mathbb{R}^n}}{\mathrm{minimize}} \quad \sum_{i=1}^{n}\Big(\sum_{j\in N_i}\varphi_i\big(\sigma_{\Omega_i}(t_i - y_j - w_{i,j})\big) + \sum_{j\in\check{N}_i}\psi_i\big(\sigma_{\Lambda_i}(x_i - t_j - \nu_{i,j})\big)\Big) \tag{3.1}$$

where, for every $i \in \{1,\ldots,n\}$, $(w_{i,j})_{j\in N_i} \in \mathbb{R}^{|N_i|}$, $(\nu_{i,j})_{j\in N_i} \in \mathbb{R}^{|\check{N}_i|}$, $\varphi_i\colon [0,+\infty[\to \mathbb{R}$ and $\psi_i\colon [0,+\infty[\to \mathbb{R}$ are (strictly) increasing lower-semicontinuous convex functions, Ω_i and Λ_i are closed real intervals with $\alpha_i^- = \inf\Omega_i$, $\alpha_i^+ = \sup\Omega_i$, $\beta_i^- = \inf\Lambda_i$, and $\beta_i^+ = \sup\Lambda_i$.

Subsequently, we will suppose that:

Assumption 2 For every $i \in \{1,\ldots,n\}$, $-\infty \leqslant \alpha_i^- < 0 < \alpha_i^+ \leqslant +\infty$ with $(\alpha_i^-,\alpha_i^+) \neq (-\infty,+\infty)$, and $-\infty \leqslant \beta_i^- < 0 < \beta_i^+ \leqslant +\infty$ with $(\beta_i^-,\beta_i^+) \neq (-\infty,+\infty)$.

A first result on the general solutions to Problem (3.1) is as follows:

Proposition 4. *Suppose that Assumption 2 holds. Problem (3.1) admits a solution. Let $(\widehat{x}, \widehat{t})$ be such a solution. Then, $\widehat{t}$ is a solution to the convex optimization problem:*

$$\underset{t=(t_i)_{1\leqslant i\leqslant n}\in\mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^{n}\Big(\sum_{j\in N_i}\varphi_i\big(\sigma_{\Omega_i}(t_i-y_j-w_{i,j})\big)+\Psi_i(t)\Big) \tag{3.2}$$

where, for every $i\in\{1,\ldots,i\}$, Ψ_i is the finite convex function given by

$$\Psi_i\colon t\mapsto \inf_{x_i\in\mathbb{R}}\sum_{j\in\check{N}_i}\psi_i\big(\sigma_{\Lambda_i}(x_i-t_j-\nu_{i,j})\big). \tag{3.3}$$

In addition, if $(\widehat{t}_j)_{1\leqslant j\leqslant n}$ are the components of $\widehat{t}$, then the components $(\widehat{x}_i)_{1\leqslant i\leqslant n}$ of $\widehat{x}$ are such that

$$(\forall i\in\{1,\ldots,n\})\qquad \widehat{x}_i\in\underset{x_i\in\mathbb{R}}{\text{Argmin}}\sum_{j\in\check{N}_i}\psi_i\big(\sigma_{\Lambda_i}(x_i-\widehat{t}_j-\nu_{i,j})\big). \tag{3.4}$$

Moreover, if, for every $i\in\{1,\ldots,n\}$, φ_i and ψ_i are strictly convex, $(\widehat{x},\widehat{t})$ is uniquely defined.

Let us now focus on the case when, for every $i\in\{1,\ldots,n\}$, $\psi_i\colon [0,+\infty[\,\to\mathbb{R}\colon \xi\mapsto\xi^{\tau_i}$, where $\tau_i\in[1,+\infty[$. The next result shows that, under some asymptotic conditions, Problem (3.1) can basically be decoupled into two simpler optimization tasks:

Proposition 5. *Suppose that Assumption 2 holds. Let $(\mu^{(k)})_{k\in\mathbb{N}}$ be a sequence of positive reals such that*

$$\lim_{k\to+\infty}\mu^{(k)}=0. \tag{3.5}$$

For every $i\in\{1,\ldots,n\}$, let $(\Lambda_i^{(k)})_{k\in\mathbb{N}}$ be sequences of closed intervals such that, for every $k\in\mathbb{N}$,

$$\inf\Lambda_i^{(k)}=\mu^{(k)}\beta_i^-,\qquad \sup\Lambda_i^{(k)}=\mu^{(k)}\beta_i^+. \tag{3.6}$$

For every $k\in\mathbb{N}$, let

$$(\widehat{x}^{(k)},\widehat{t}^{(k)})\in\underset{\substack{x=(x_i)_{1\leqslant i\leqslant n}\in\mathbb{R}^n\\ t=(t_i)_{1\leqslant i\leqslant n}\in\mathbb{R}^n}}{\text{Argmin}}\;\sum_{i=1}^{n}\Big(\sum_{j\in N_i}\varphi_i\big(\sigma_{\Omega_i}(t_i-y_j-w_{i,j})\big) + \sum_{j\in\check{N}_i}\big(\sigma_{\Lambda_i^{(k)}}(x_i-t_j-\nu_{i,j})\big)^{\tau_i}\Big). \tag{3.7}$$

Then, $(\widehat{x}^{(k)},\widehat{t}^{(k)})_{k\in\mathbb{N}}$ is a bounded sequence. In addition, if $(\widehat{x},\widehat{t})$ is a cluster point of $(\widehat{x}^{(k)},\widehat{t}^{(k)})_{k\in\mathbb{N}}$, then $\widehat{x}=(\widehat{x}_i)_{1\leqslant i\leqslant n}$ and $\widehat{t}=(\widehat{t}_i)_{1\leqslant i\leqslant n}$ where

$$(\forall i\in\{1,\ldots,n\})\qquad \widehat{t}_i\in\underset{t_i\in\mathbb{R}}{\text{Argmin}}\sum_{j\in N_i}\varphi_i\big(\sigma_{\Omega_i}(t_i-y_j-w_{i,j})\big) \tag{3.8}$$

$$\widehat{x}_i\in\underset{x_i\in\mathbb{R}}{\text{Argmin}}\sum_{j\in\check{N}_i}\big(\sigma_{\Lambda_i}(x_i-\widehat{t}_j-\nu_{i,j})\big)^{\tau_i}. \tag{3.9}$$

Remark 4. Under the assumptions of Proposition 5, for every $i \in \{1,\dots,N\}$, if $t_i \mapsto \sum_{j\in N_i} \varphi_i\big(\sigma_{\Omega_i}(t_i - y_j - w_{i,j}\big)$ has a unique minimizer $\widehat{t}_i$, then the i-th component $(\widehat{t}_i^{(k)})_{k\in\mathbb{N}}$ of $(\widehat{t}^{(k)})_{k\in\mathbb{N}}$ converges to $\widehat{t}_i$. In addition, if $(\widehat{t}_j)_{j\in\check{N}_i}$ is uniquely defined and $x_i \mapsto \sum_{j\in\check{N}_i}\big(\sigma_{\Lambda_i}(x_i - \widehat{t}_j - \nu_{i,j})\big)^{\tau_i}$ has a unique minimizer $\widehat{x}_i$, then the i-th component $(\widehat{x}_i^{(k)})_{k\in\mathbb{N}}$ of $(\widehat{x}^{(k)})_{k\in\mathbb{N}}$ converges to $\widehat{x}_i$. The latter condition holds, in particular, when, for every $j \in \check{N}_i$, φ_j is a strictly convex function and $\tau_i > 1$.

Combining Propositions 1 and 5 with the above remark yields the following result:

Corollary 1. *Suppose that, for every $i \in \{1,\dots,n\}$, $\alpha_i^- \in]-\infty,0[$, $\alpha_i^+ = +\infty$, $\beta_i^- = -\infty$, and $\beta_i^+ \in]0,+\infty[$ (resp. $\alpha_i^- = -\infty$, $\alpha_i^+ \in]0,+\infty[$, $\beta_i^- \in]-\infty,0[$, and $\beta_i^+ = +\infty$). Let $(\mu^{(k)})_{k\in\mathbb{N}}$ be a sequence of positive reals satisfying (3.5). For every $i \in \{1,\dots,n\}$, let $(\Lambda_i^{(k)})_{k\in\mathbb{N}}$ be sequences of closed intervals such that, for every $k \in \mathbb{N}$, (3.6) holds. For every $k \in \mathbb{N}$, let $(\widehat{x}^{(k)},\widehat{t}^{(k)})$ be given by (3.7). Then, $\lim_{k\to+\infty}\widehat{x}^{(k)} = (\widehat{x}_i)_{1\leqslant i\leqslant n}$ and $\lim_{k\to+\infty}\widehat{t}^{(k)} = (\widehat{t}_i)_{1\leqslant i\leqslant n}$, where*

$$
\begin{aligned}
(\forall i \in \{1,\dots,n\}) \qquad & \widehat{t}_i = \min\big\{y_j + w_{i,j} \;\big|\; j \in N_i\big\} && (3.10)\\
& (\textit{resp. } \widehat{t}_i = \max\big\{y_j + w_{i,j} \;\big|\; j \in N_i\big\}) \\
& \widehat{x}_i = \max\big\{\widehat{t}_j + \nu_{i,j} \;\big|\; j \in \check{N}_i\big\} && (3.11)\\
& (\textit{resp. } \widehat{x}_i = \min\big\{\widehat{t}_j + \nu_{i,j} \;\big|\; j \in \check{N}_i\big\}).
\end{aligned}
$$

Remark 5.

(i) If

$$(\forall i \in \{1,\dots,n\})(\forall j \in N_i) \qquad \omega_{i,j} = -\nu_{j,i} \tag{3.12}$$

then, under the assumptions of Corollary 1, we asymptotically obtain an opening (resp. a closing) operator with structuring element (2.5) (resp. (2.6)).

(ii) When, for every $i \in \{1,\dots,n\}$, $\varphi_i\colon [0,+\infty[\mapsto \mathbb{R}\colon \xi \mapsto \xi$ and $\tau_i = 1$, it follows from Proposition 3 (see also Remark 3) that, if $(\forall i \in \{1,\dots,n\})$ $(\alpha_i^-,\beta_i^-) \in]-\infty,0[^2$ and $(\alpha_i^+,\beta_i^+) \in]0,+\infty[^2$ are such that

$$
\begin{aligned}
& -\frac{\alpha_i^+}{\alpha_i^-} > |N_i| - 1 \quad \text{and} \quad -\frac{\beta_i^+}{\beta_i^-}(|\check{N}_i| - 1) < 1 \\
(\text{resp.} \quad & -\frac{\alpha_i^+}{\alpha_i^-}(|N_i| - 1) < 1 \quad \text{and} \quad -\frac{\beta_i^+}{\beta_i^-} > |\check{N}_i| - 1), \qquad (3.13)
\end{aligned}
$$

then the conclusions of Corollary 1 also hold.

Illustrations of the effect of the proposed operators on the two image examples we already considered are shown in Fig. 3.

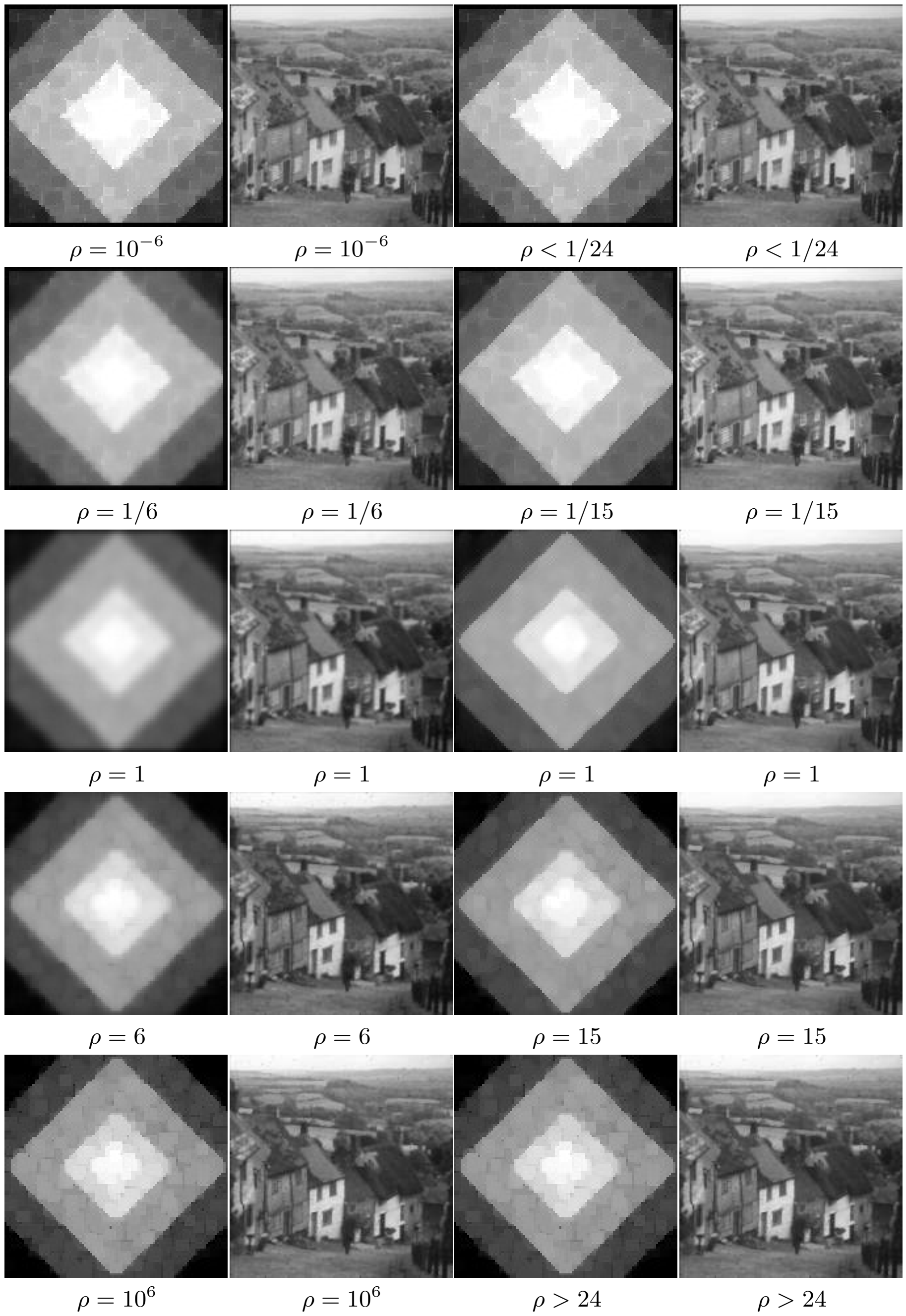

Fig. 3. Solutions to (3.8)-(3.9) for Pyramid and Goldhill images when $-\alpha_i^+/\alpha_i^- = -\beta_i^-/\beta_i^+ = \rho$, $(w_{i,j})_{j\in N_i} = (\nu_{i,j})_{j\in \check{N}_i} = 0$, and N_i (and $\check{N}_i$) is a 5×5 spatial neighborhood. For every pixel i, we have on the first two columns $\varphi_i\colon [0,+\infty[\mapsto \mathbb{R}\colon \xi \mapsto \xi^2$ and $\tau_i = 2$, whereas on the last two columns, $\varphi_i\colon [0,+\infty[\mapsto \mathbb{R}\colon \xi\mapsto \xi$ and $\tau_i = 1$.

4 Conclusion

In this paper, we have introduced what we may call *variational dilation, erosion, opening, and closing*. We have seen that these operators can be applied in graph processing and that they are defined thanks to support functions of closed real intervals and increasing convex functions. By varying these parameters, a wide class of operators can be defined, potentially leading to much flexibility in their use. In particular, we have proved that standard morphological dilation, erosion, opening, and closing are recovered as limit cases. For application purposes that we plan to investigate soon, this means that we are now able to use those operators for example as asymmetric regularizers, *e.g.* to favor bright contrast over dark contrast or conversely.

Building on the present paper, two main theoretical avenues can now be explored. The first one consists of more deeply analyzing the properties of these new variational operators and to see whether other classical morphological operators have their variational counterparts. The second direction is to look for extensions of the energy functions which have been set to define these operators in order to address more general problems of interest, *e.g.*, those involving local adaptive decision processes or some statistical knowledge on the target signal or the noise. Then, more attention should be paid to optimization algorithms allowing us to efficiently solve such problems.

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