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WHEN DO TWO RATIONAL FUNCTIONS HAVE THE SAME JULIA SET?

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ABSTRACT. It is proved that non-exceptional rational functions f and g on the Riemann sphere have the same measure of maximal entropy iff there exist iterates F of f and G of g and natural numbers M, N such that

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N. \tag{$*$}$$

If one assumes only that f, g have the same Julia set and no singular or parabolic domains of normality for the iterates, one also proves $(*)$.

INTRODUCTION

The most interesting dynamics of a rational function occurs on its Julia set and a very natural invariant measure whose support is the Julia set is the measure of maximal entropy (see [B], [CG], [ELyu]). We consider the following two problems: to describe all pairs of rational functions f and g such that:

(A) f and g have the same measure of maximal entropy,

or the apparently weaker requirement:

(B) f and g have the same Julia set.

In the present paper we solve the problem (A) for an arbitrary pair of non-exceptional (see the definition below) rational functions, and we solve the problem (B) in the class Ξ of rational functions with Julia sets not the whole Riemann sphere, a circle (or its arc), without parabolic periodic points and singular domains of the complement of Julia set (see also Proposition 1 and Remark 6 below). By solution of the problems we mean a functional equation between f and g, which is equivalent to having the same measure of maximal entropy (maximal measure), or the same Julia set. A corollary is that in the class Ξ the maximal measure is determined by the common Julia set (rigidity of maximal measure). An application to functional equations is done.

The problems (A) and (B) are closely related to the classical problem of commuting pairs of rational functions. In order to solve the latter problem, Fatou and Julia independently applied what is called now the Julia set of a rational function, introduced by them in [F1], [J1]. (Commuting rational functions have a common Julia set J and a common maximal measure.) Discovering fundamental properties

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of J, Fatou and Julia described [F2], [J2] all commuting rational functions under the restriction that the common Julia set J is not the whole Riemann sphere. Ritt [R1], [R2] gave an algebraic solution of the problem in general: except for explicitly described cases, if f and g commute, then they have a common iterate. These exceptions are exactly the *critically finite rational maps with parabolic orbifolds* in modern terminology [T], [DH], [E]. We call such functions *exceptional*. Recently Eremenko [E] has completed the method by Fatou and Julia studying the common maximal measure of commuting rational functions in the case $J = \overline{\mathbf{C}}$. Note that the problems (A)–(B) do not reduce to the commuting case (see Example below).

The problems (A) and (B) have been studied in [BE], [B1], [B2], [E], [L], [Fe]. In the class of polynomials the solution is known [BE], [B] (see also Remark 1).

Note that for polynomials (A) is equivalent to (B) because the maximal measures coincide with the harmonic measure for the basin of infinity. A similar idea will be used in section 2.

Let $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a rational function on the Riemann sphere $\overline{\mathbf{C}}$. Let $J(f)$ denote its Julia set, and $\mu(f)$ its unique probability measure of maximal entropy, [FLM], [Lyu], [M1]. Note that the support of $\mu(f)$ is $J(f)$, and that each iterate of f has the same Julia set and the same measure of maximal entropy. In what follows we always assume that all rational functions are *not critically finite with a parabolic orbifold*. The critically finite rational maps with parabolic orbifolds are completely classified in [DH]. For such functions the theorems of the paper are not true.

1. Rational functions with the same maximal measure

Theorem A. *Let f, g be two non-exceptional rational functions. The following conditions are equivalent:*

(A1) $\mu(f) = \mu(g)$;

(A2) *there exist iterates F of f and G of g, such that, for some natural numbers M and N the following equality holds:*

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N, \tag{$*$}$$

where $G^{-1} \circ G$ and $F^{-1} \circ F$ denote some single-, or multi-valued function obtained by the analytic continuation of a branch.

Moreover, in (A2) $(\deg G)^M = (\deg F)^N$.

Example. Let $\deg f = 2$. Consider the branch of the two-valued analytic function $h = f^{-1} \circ f$, which is different from the identity. Then h is a Möbius transformation, i.e. single-valued (singularities of h at the critical points of f are removable). Moreover, the function $g = h \circ f$ is rational and $\mu(f) = \mu(g)$. In this example $(g^{-1} \circ g) \circ g = (f^{-1} \circ f) \circ f$, where $g^{-1} \circ g =$id and $f^{-1} \circ f = h$. It is interesting to note that these functions do not commute: $g \circ f = h \circ f^2 \neq f^2 = f \circ g$. Moreover, all iterates of f and g are different: $g^n = h \circ f^n \neq f^n, n = 1, 2, \ldots$. Note also that here $h^2 = id$, by the definition.

Given a rational function f of degree 2, one can construct the map h explicitly as follows. Let c_1 and c_2 be different critical points of f, and let M be a Möbius transformation such that $M(0) = c_1$ and $M(\infty) = c_2$. Then $h = M \circ (-M^{-1})$. (In new coordinates it is just $z \to -z$.) Indeed, the critical points of the rational function $\widetilde{f} = M^{-1} \circ f \circ M$ are 0 and ∞. Since $\deg \widetilde{f} = 2$, it has to have the form $\widetilde{f}(z) = (az^2 + b)/(cz^2 + d)$, $ad - bc \neq 0$, i.e. $\widetilde{f}(-z) = \widetilde{f}(z)$. We conclude that $f \circ h(z) = M \circ \widetilde{f} \circ M^{-1} \circ M(-M^{-1}(z)) = M \circ \widetilde{f}(-M^{-1}(z)) = M \circ \widetilde{f} \circ M^{-1}(z) = f(z)$.

We should also point out that the group of all Möbius transformations, which are symmetries on the Julia set J of any rational function, is finite [L]. (A Möbius transformation L is defined to be a symmetry on J if $L(J) = J$ and, in the case J is the whole Riemann sphere, a circle, or an interval, if L preserves the measure μ. See also Remark 4.)

Remark 1. Let f and g be polynomials with the same Julia sets J. Such polynomials have been completely described in [BE], [B]. Let, for example, the degrees of f and g be equal (if not, we can consider $f \circ g$ and $g \circ f$ instead of f and g, as in [BE], or just pass to iterates of f and g). Then $g = L \circ f$, where L is a linear transformation (this immediately implies $L(J) = J$ because $x \in J$ iff $g \circ f^{-1}(x) \in J$). See [BE], [B2] for details; here is an outline of the proof as it follows from our paper: We can apply Theorem A (because $\mu(f) = \mu(g)$ =harmonic measure from ∞ in this case), or simply Proposition 1 of the next section. Since F and G are polynomials, $F^{-1} \circ F$ and $G^{-1} \circ G$ are asymptotically rotations around infinity, and we obtain from $(*)$ that the Bottcher coordinates for f and g must coincide: if $f(z) = az^p + \dots$, $g(z) = bz^q + \dots$, then there exists a univalent at infinity function B (the Bottcher coordinate function) such that $B(z)/z \to 1$ as $z \to \infty$ and

$$B \circ f(z) = a[B(z)]^p, B \circ g(z) = b[B(z)]^q.$$

If the degrees p and q are equal, we can proceed as in [BE]: $B \circ f = \gamma B \circ g$, $\gamma = b/a$, and, comparing positive powers of z in the expansions at ∞, we get: $g(z) + c = \gamma(f(z) + c)$, where c is some number (in fact, $B(z) = z + c + O(1/z)$ at infinity).

Remark 2. It is an interesting question whether $(*)$ can be simplified. For example, when (or whether) does the equation $(*)$ yield that $F^{-1} \circ F$ and $G^{-1} \circ G$ are Möbius transformations, as in the Example above? We should draw the reader's attention that it is not true in general that $F^N = G^M$ is equivalent to saying that both functions F and G are iterates of a common rational function (up to rotations). (See [R1] for an example, where Ritt writes: "...so that there exist permutable pairs of fractional functions which come neither from the multiplication theorems of the periodic functions, nor from the iteration of a function.") Let us make some more remarks. First, $(*)$ can be rewritten as:

$$G^M \circ (G \circ F^{-1}) = (G \circ F^{-1}) \circ F^N,$$

where the "conjugacy" $G \circ F^{-1}$ between G^M and F^N is, in general, multi-valued. Second, iterating $(*)$, we can replace there M and N by jM and jN respectively, $j = 1, 2, \dots$. This implies that any pre-periodic point for f is also pre-periodic for g.

The main ingredients of the proof of Theorem A will be given by Lemma 1 and Lemma 2 below.

Lemma 1. *Let ν be an ergodic invariant measure on the Julia set $J(f)$ with positive Lyapunov exponent $\chi = \int \log |f'| d\nu$. Then for every small positive σ there exists a set E of ν-measure $1 - \sigma$ and numbers $r > 0, K > 1$ and $T_0 > 0$ as follows:*

For every point $x \in E$ there exists $R(x)$, a subset of the set of real numbers, such that for every $T > T_0$ the set $R_T := R(x) \cap [0, T]$ occupies at least $5/8$ of the length of $[0, T]$, i.e. the Lebesgue measure $\mathrm{Leb}(R_T) > \frac{5}{8}T$ and for every $\tau \in R(x)$ there exists $n \in \mathbf{N}$ such that the following hold:

(a) *the map* $f^n : B(x, \exp(-\tau)) \to \overline{\mathbf{C}}$ *is injective and has bounded distortion:*

$$1/2 < |(f^n)'(x)/(f^n)'(y)| < 2,$$

for all $y \in B(x, \exp(-\tau))$,
(b) $B(f^n(x), r/K) \subset f^n(B(x, \exp(-\tau))) \subset B(f^n(x), r)$,
(c) $n \to \infty$ *as* $\tau \to \infty$.

Proof of Lemma 1. (A) Consider the inverse limit (natural extension in Rohlin terminology [Ro]) $(\tilde{J}, \tilde{f}, \tilde{\nu})$ of (J, f, ν).

Denote by $\pi : \tilde{J} \to J$ the projection on the 0 coordinate and by π_n the projection on n-th coordinate. Then for $\tilde{\nu}$-almost every $\tilde{x} \in \tilde{J}$ there exists $r = r(\tilde{x}) > 0$ such that there exist univalent branches F_n of f^{-n} on $B(\pi(\tilde{x}), r)$ for $n = 1, 2, \ldots$ for which $F_n(\pi(\tilde{x})) = \pi_{-n}(\tilde{x})$ and

$$\frac{1}{2} < \frac{|F_n'(\pi(\tilde{x}))|}{|F_n'(z)|} < 2$$

for every $z \in B(\pi(\tilde{x}), r)$, $n > 0$ (distances and derivatives in the Riemann metric on $\overline{\mathbf{C}}$).

Moreover r is measurable functions of $\tilde{x}$.

((A) follows easily from Pesin's theory [Pe]. It is stated explicitly in [PZ, Lemma 1] and a proof of its variant can be found in [Led1] or [P2, Sec.2]. See also [ELyu], [Led2], [M2], [P1, Sec.3]. The above distortion estimate can be deduced also from Koebe's Distortion Theorem.)

(B) Let us consider an arbitrary positive $\sigma < \frac{\chi}{8 \log L} (\leq 1/8)$ where $L := \sup |f'|$ and find two subsets $\tilde{E}$, $\tilde{E}_0$ of $\tilde{J}$ as follows:

(B1) $\tilde{E} \subset \tilde{E}_0$, $\tilde{\nu}(\tilde{E}_0) > 1 - \sigma/2$, $\tilde{\nu}(\tilde{E}) > 1 - \sigma$;

(B2) $\nu(E) > 1 - \sigma$, where $E = \pi(\tilde{E})$;

(B3) there exists $r > 0$ not depending on $\tilde{x} \in \tilde{E}_0$ such that univalent branches F_n of f^{-n} on $B(\pi(\tilde{x}), r)$ for $n = 1, 2, \ldots$ for which $F_n(\pi(\tilde{x})) = \pi_{-n}(\tilde{x})$ exist, and

$$\frac{1}{2} < \frac{|F_n'(\pi(\tilde{x}))|}{|F_n'(z)|} < 2$$

for every $z \in B(\pi(\tilde{x}), r)$, $n > 0$;

(B4) $\frac{1}{n} \log |(f^n)'(x)| \to \chi$ as $n \to \infty$ *uniformly* on $x \in E_0$ (Egoroff's Theorem);

(B5) Write $A_{\tilde{x},N} := \{n : 1 \leq n \leq N, \tilde{f}^n(\tilde{x}) \in \tilde{E}_0\}$ and write $D(\tilde{x}, \tilde{E}_0, N) := \sharp A_{\tilde{x},N}$. Then $D(\tilde{x}, \tilde{E}_0, N)/N \to \tilde{\nu}(\tilde{E}_0)$ as $N \to \infty$ *uniformly* on $\tilde{x} \in \tilde{E}$ (Birkhoff Ergodic Theorem and Egoroff's Theorem).

(C) Fix an arbitrary $\tilde{x} \in \tilde{E}$. Denote $\pi(\tilde{x})$ by x. For every $n = 1, 2, \ldots$ write $a(n) := \log |(f^n)'(x)|$. Observe that

(C1) $a(n+1) \leq a(n) + \log L$ and by (B4), (B5) and (B1) there exists N_0 such that for every $N \geq N_0$;

(C2) $a(N) \geq N\chi/2$;

(C3) $D(\tilde{x}, \tilde{E}_0, N)/N \geq 1 - \sigma$;

(C4) Note that the sequence $a(n)$ need not be monotone increasing. In the sequel we shall consider only such N that $a(n) \leq a(N)$ for every $n < N$.

(D) For every $n = 1, 2, \ldots$ write $I_n := [a(n), a(n) + \log L]$. Hence $\bigcup_{n=1}^{N} I_n \supset [\log L, a(N) + \log L]$.

Write $A'_{\tilde{x},N} := [1,N] \setminus A_{\tilde{x},N}$. Then by (C3) $\sum_{n\in A'_{\tilde{x},N}} |I_n| \le \sigma N \log L$. Hence, with the use of (C2), denoting $P_N := \bigcup_{n\in A_{\tilde{x},N}} I_n$

$$\mathrm{Leb}(P_N) \ge a(N) - \sigma N \log L \ge a(N) - 2\sigma(\log L)a(N)/\chi = a(N)(1 - \frac{2\sigma \log L}{\chi}).$$

(E) Observe that by our definitions the following holds:

(E1) $B(f^n(x), r/4L) \subset f^n(B(x, \frac{r}{2}\exp -t)) \subset B(f^n(x), r)$ for every $t \in P_N$; more precisely $t \in I_n$ where $n \in A_{\tilde{x},N}$.

This translates to

(E2) $B(f^n(x), r/4L) \subset f^n(B(x, \exp -\tau)) \subset B(f^n(x), r)$ for $n = n(\tau)$ for every $\tau \in P_N - \log\frac{r}{2}$.

(F) Define $R(x) := \bigcup_{N=1}^\infty P_N - \log\frac{r}{2}$; formally the construction depends on $\tilde{x}$ but in fact only on x.

We conclude that for every $T \ge a(N_0) - \log(r/2) + \log L$ and $R_T := R(x)\cap[0,T]$ for $a(N) - \log\frac{r}{2} + \log L \le T \le a(N+1) - \log\frac{r}{2} + \log L$, for every $\tau \in R_T$ (E2) holds and by (C4) $P_N - \log\frac{r}{2} \subset [0,T]$. Hence using (C1) we estimate

$$\mathrm{Leb}\, R_T \ge a(N)(1 - \frac{2\sigma\log L}{\chi}) \ge (a(N+1) - \log L)(1 - \frac{2\sigma \log L}{\chi})$$

$$\ge (T - 2\log L + \log\frac{r}{2})(1 - \frac{2\sigma\log L}{\chi}) \ge T(1 - \frac{2\sigma \log L}{\chi}) - Q \ge \frac{3}{4}T - Q$$

for the constant $Q = (2\log L - \log\frac{r}{2})(1 - \frac{2\sigma\log L}{\chi})$.

Thus, if T is big enough we obtain

$$\mathrm{Leb}\, R_T \ge \frac{5}{8}T.$$

The proof is over. □

Remark 3. In fact we have not used fully the μ-regularity of x, namely (B4) and (B5). We used only $\liminf \frac{1}{n}\log|(f^n)'(x)| > 0$ and (B3), more precisely the existence of the branch of f^{-n} from a big neighbourhood of $f^n(x)$ to a neighbourhood of x for a big proportion of n's. This might give a chance to get rid of measures in further considerations, i.e. to solve problem B without assuming there are no parabolic or singular domains in $\overline{\mathbf{C}}\setminus J$, under some non-uniform expanding assumptions. Compare [P3].

Lemma 2. *Let J be the Julia set of a non-exceptional rational function f. Fix a ball $B = B(x,r)$ centered at $x \in J$. Let H_n be a sequence of holomorphic functions in B such that:*

1. *The sequence H_n tends to a holomorphic function H in B.*
2. *For every n and z,*
$$z \in B\cap J \Longleftrightarrow H_n(z) \in H_n(B) \cap J.$$
3. *If J is the whole Riemann sphere, a circle, or an interval (in some holomorphic coordinates), then additionally, for every n, there is a constant $\alpha > 0$ so that $\mu(H_n(A)) = \alpha\mu(A)$, where $\mu = \mu(f)$ and A is any set such that $H_n : A \to \overline{\mathbf{C}}$ is injective.*

Then either the limit function H is constant, or $H_n = H$ for all big n.

Remark 4. A map with the properties 2. and 3. is called in [L] a *local symmetry* on J. Note that without the assumption 3 Lemma 2 is false; all appropriate translations or rotations are local symmetries for J Riemann sphere, interval or a circle.

Proof of Lemma 2 (see [L]). For the sake of completeness we reproduce the main steps of the proof here. An idea is to construct many shifts which leave the Julia set invariant. For this we consider a semi-group generated by the *local symmetries* H_n and f^{-n} in neighborhoods of repelling periodic points of f.

I. Let (Φ_n) be any sequence of holomorphic functions univalent on a ball $B(0,\varepsilon)$ such that $q_n = \Phi_n(0) \neq 0$, $n = 1, 2, \dots$, and $\Phi_n \to \mathrm{id}$ as $n \to \infty$. Given $|\lambda| > 1$, there are $a \neq 0$, $\delta > 0$ and positive integers sequences l_i, n_i such that for every $m \in \mathbf{N}$ and all big i the mappings

$$\Psi_i(z) = \lambda^{l_i - m}\Phi_{n_i}(\frac{\Phi_{n_i}^{-1}(z)}{\lambda^{l_i - m}})$$

are defined in $B(0,\delta)$ and

$$\Psi_i(z) \to z + \frac{a}{\lambda^m}$$

as $i \to \infty$.

Indeed, choose l_i and n_i such that $\lambda^{l_i} q_{n_i} \to a \neq 0$. Then we use the expansions

$$\lambda^l \Phi_n(\frac{\Phi_n^{-1}(z)}{\lambda^l}) = \lambda^l q_n + \alpha_1^{(n)}\Phi_n^{-1}(z) + \sum_{k=2}^{\infty} \alpha_k^{(n)} (\frac{\Phi_n^{-1}(z)}{\lambda^{l(1-1/k)}})^k,$$

where $\alpha_k^{(n)}$ are the coefficients of the power series expansion of Φ_n at 0. We have Cauchy's inequalities: $|\alpha_k^{(n)}| < C/(\varepsilon/2)^k$, for some $C > 0$ and all k. With the chosen $l_i \to \infty$ and n_i, it gives us the statement.

II. Let z belong to a half plane $\{\Re z > M_0\}$ and $\phi(z) = z + 1 + o(|z|^{-\gamma})$, $\gamma > 0$, as $z \to \infty$. Given $|\lambda| > 1$ and $c > 0$, there are sequences l_i, n_i and $M > M_0$ such that

$$\lambda^{-n_i}\phi^{l_i}(\lambda^{n_i} z) \to z + c,$$

$i \to \infty$, if $z \in \Pi = \{\Re z > M\}$.

To prove it, we choose a sequence n_i so that $\arg\lambda^{n_i} \to 0$ and then set $l_i = [c|\lambda|^{n_i}]$. Now the asymptotic $\phi^l(z) = z + l + o(l)$ if $z \to \infty$ and $l \to \infty$ leads to the conclusion.

III. There is no open domain U such that $U \bigcap J$ is diffeomorphic to the product of an interval and a Cantor set. A proof (due to A. Eremenko) can be found in [L].

IV. Assume that a limit function H of H_n is not a constant. We can set $H =$id. We can assume also that H_n are defined and univalent in a ball B centered at a repelling fixed point b of f (passing to an iterate) with multiplier $\lambda = f'(b)$. Let F be a branch of f^{-1} on B contracting to b. We let

$$F_n = H_n^{-1} \circ F \circ H_n.$$

Denote $b_n = H_n^{-1}(b)$. Then $F_n(b_n) = b_n$ and $F_n'(b_n) = 1/\lambda$.

Consider the case $b_n = b$ for some n. Let $R = f \circ F_n$. Then $R(b) = b$ and $R'(b) = 1$. If J coincides with $\overline{\mathbf{C}}$, a circle S, or an interval I, then R preserves the measure μ by the assumption 3. Looking at the corresponding Leau flower for R, we see that $R =$id. Now let J not be $\overline{\mathbf{C}}$, S, and I. Assume $R \neq$id. We make two changes of variable. First, we may assume that locally $f(z) = \lambda z$. Second, after a change $w = Az^{-p}$, with some $A > 0$ and $p \in \mathbf{N}$, the map R turns to a map of the form of p.II, and f turns to $w \mapsto \lambda^{-p} w$. Then applying p.II and returning to the original coordinate z, we see that for each point $x \in J$ close to b, J contains also an analytic arc joining x to b, which corresponds to a horizontal ray in the coordinate w. Then by III J is $\overline{\mathbf{C}}$, S, or I. A contradiction. (Another argument is that in the case of a Cantor set of rays in J to b, for a periodic point $b' \neq b$ close to b one has again a Cantor set of arcs to b' which implies $J = \overline{\mathbf{C}}$).

Thus the remaining case is $b_n \neq b$ for all n. We can linearize each F_n by a holomorphic Schroeder map h_n, $h_n(0) = b_n$, and F by h, $h(0) = b$ (so that $h_n = H_n^{-1} \circ h$). Then for passage maps $\Phi_n = h^{-1} \circ h_n$ we apply I. If λ is not real we can walk in J in arbitrarily small steps in two different directions which gives $J = \overline{\mathbf{C}}$. If λ is real we walk at least in the direction a. We conclude that J is either $\overline{\mathbf{C}}$ or an interval, or J is locally diffeomorphic to the product of a Cantor set and an interval. The latter case is ruled out by III. In the first two cases the measure μ is invariant under the shifts (by I). It is possible only if f is critically finite with parabolic orbifold (see [E]).

V. Thus $F = F_n$, i.e. F (a branch of f^{-1} in a neighborhood of the repelling periodic point of f) and all H_n commute. So each H_n is linear in some coordinates linearizing F in which b becomes 0. If we apply the result $F = F_n$ to another repelling periodic point of f close to b, we obtain $H_n =$id. (In [L] the reader can find a different argument.) $\square$

Proof of Theorem A. (A1) Let $\mu = \mu(f) = \mu(g)$, $J = J(f) = J(g)$. Since the Lyapunov exponents χ_f and χ_g are positive, we can apply Lemma 1. Take $\sigma < 1/2$, satisfying Lemma 1 for f and g and find the set E_1, and numbers $r_1 > 0$, $K_1 > 1$ and T_0^1 for f and $r_2 > 0$, $K_2 > 1$ and T_0^2 for g. There is a point $x \in E_1 \bigcap E_2$. For this point find the sets R_T^1 for f and R_T^2 for g, for all T big enough. Since each of these sets occupies more that half of the interval $[0, T]$ one can find a sequence of numbers $t_i \to \infty$, and two sequences of indexes $n_i^1 \to \infty$, $n_i^2 \to \infty$ such that the maps

$$f^{n_i^1} : B(x, \exp(-t_i)) \to \overline{\mathbf{C}},$$

$$g^{n_i^2} : B(x, \exp(-t_i)) \to \overline{\mathbf{C}}$$

are injective and

$$B(f^{n_i^1}(x), r_1/K_1) \subset f^{n_i^1}(B(x, \exp(-t_i))) \subset B(f^{n_i^1}(x), r_1),$$

$$B(g^{n_i^2}(x), r_2/K_2) \subset g^{n_i^2}(B(x, \exp(-t_i))) \subset B(g^{n_i^2}(x), r_2).$$

It is clear now that there exist a ball $B = B(a, r)$, with $a \in J$, and an infinite sequence of maps H_i, which are of the form $g^{l_i} \circ f^{-k_i}$, univalent on B and such that each $H_i(B)$ contains a ball of a fixed positive radius and is contained in another such ball (of a fixed radius). It means that $\{H_i\}$ is normal in B and the limit functions are not constants.

Now we use Lemma 2. Its assumption 3 holds because the Jacobians $\mathrm{Jac}_{\mu(f)} f$ and $\mathrm{Jac}_{\mu(g)} g$ are constant (see [M1]). (We say that for an arbitrary measure ν on J and a mapping $h : X \to J$ for $X \subset J$, a Jacobian $\mathrm{Jac}_\nu h$ exists and is equal to a function φ if for every Borel set $A \subset X$ on which h is injective, $\nu(h(X)) = \int_X \varphi d\nu$.)

As we assumed $\mu(f) = \mu(g) = \mu$, $\mathrm{Jac}_\mu H_n$ is constant. (So in Lemma 2 we could state the assumption 3 for every case, not only $J = \overline{\mathbf{C}}$ interval or a circle. This would simplify the proof. However in Section 2, Prop. 1, this is not so.)

Therefore, by Lemma 2, for some natural numbers m, n, k, and l, and for some branches f^{-n} and $f^{-(n+l)}$ defined in B,

$$g^m \circ f^{-n} = g^{m+k} \circ f^{-(n+l)}$$

identically in B. Rewrite it in the form

$$f^{-n} \circ f^{n+l} = g^{-m} \circ g^{m+k}$$

on $f^{-(n+l)}(B)$ and compose nm times.

Then we can set: $G = g^m$, $F = f^n$, $M = nk$, and $N = ml$.

(A2) Let, conversely,

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N.$$

Observe that this implies

$$(G^{-1} \circ G) \circ G^{iM} = (F^{-1} \circ F) \circ F^{iN}$$

with *the same functions* $G^{-1} \circ G$ *and* $F^{-1} \circ F$ *for all* $i = 1, 2, \ldots$. Because of the uniqueness of the measure of maximal entropy, it is enough to show that the measure $\mu = \mu(F)$ is the balanced measure for G^M too; i.e. the Jacobian exists and is constant (see [M1], [Lyu]). Denote by d_F and d_G the degrees of F and G. Let us fix any small open domain A. Let $B = G^M(A)$ and let A' be a component of $G^{-M}(A)$. Then

$$\mu(A') = d_F^{-2N} \mu((F^{-1} \circ F) \circ F^{2N}(A'))$$

$$= d_F^{-2N} \mu((G^{-1} \circ G) \circ G^{2M}(A')) = d_F^{-2N} \mu((G^{-1} \circ G)(B)). \tag{3}$$

Similarly

$$\mu(A) = d_F^{-N} \mu((G^{-1} \circ G)(B)). \tag{4}$$

Hence $\mu(A') = d_F^{-N} \mu(A)$, where $G^M : A' \to A$ is one-to-one (by the choice of A). It follows that $\deg G^M = d_F^N$ (see [M1]). The proof is completed. $\square$

2. Rational functions with the same Julia set

Theorem B (On rigidity of maximal measure)**.** *Let f,g be two rational functions without parabolic periodic points and singular domains* (*Siegel discs, Herman rings*), *Julia sets not* $\overline{\mathbf{C}}$, *a circle or an interval* (*in some holomorphic coordinates*). *Then the following conditions are equivalent:*

(B1) $J(f) = J(g)$;

(B2) $\mu(f) = \mu(g)$;

(B3) *there exist iterates F of f and G of g, such that, for some natural numbers M and N the following equality holds:*

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N,$$

where $G^{-1} \circ G$ and $F^{-1} \circ F$ denote some single-, or multi-valued function obtained by the analytic continuation of a branch.

Moreover, in (B3) $(degG)^M = (degF)^N$.

In fact, we prove a more general statement:

Proposition 1. *Let f, g be two arbitrary rational functions with the same Julia set $J = J(f) = J(g)$ not being a circle or interval. Suppose there exist periodic sinks p, q for f, g and components U, V of their basins such that $g^m(U) = V$, for some $m \geq 0$. Then the condition* (B3) *of Theorem* B *holds.*

Remark 5. This proposition yields Theorem B as follows. The assumptions of Theorem B imply there exists a periodic sink p of f. Its immediate f-basin W_f is eventually periodic under an iterate of g [S]; i.e. $g^m(W_f)$ is a basin of a periodic sink of g. So the assumptions of Proposition 1 are satisfied. (Note that the assumptions of Theorem B do not mean that f, g are hyperbolic; these assumptions do not exclude the presence of critical points in J.)

Proof of Proposition 1. Denote by U', V' some periodic components of the basins of p, q and $f^s(U) = U', g^t(V) = V'$. Let ν with index U, U', V, V' be harmonic measure on the appropriate boundary, viewed from p, q in the case of U', V'.

Then $\nu_{U'}$, $\nu_{V'}$ are ergodic invariant measures with positive Lyapunov exponents for f and g respectively. (By passing to iterations one can assume p, q are fixed points.)

Invariance (see for example [P2]): For every continuous $\varphi : U' \to \mathbf{R}$ we have

$$\int \varphi d\nu = \tilde{\varphi}(p) = \tilde{\varphi}(f(p)) = \widetilde{\varphi \circ f}(p) = \int \varphi \circ f d\nu,$$

where tilde denotes the harmonic extension to U' (solution of Dirichlet's problem) and $\nu = \nu_{U'}$.

Ergodicity: If $\varphi \circ f = \varphi$ (modν) on $\partial U'$ then $\varphi \circ f^n = \varphi$ for every $n \geq 0$. Hence $\tilde{\varphi} \circ f^n = \widetilde{\varphi \circ f^n} = \tilde{\varphi}$ on U'. Applying this for $n \to \infty$ we obtain $\tilde{\varphi}(p) = \tilde{\varphi}(z)$ for every $z \in U'$. So φ is constant.

Lyapunov exponent: It is not less than half of the entropy h_ν (by [Ru], cf. [M1], [P2]). Next recall that $\mathrm{h}_\nu(f) > 0$ iff f is not an automorphism (in ν), [Pa, Corollary 5.16].

Finally we prove that indeed f is not an automorphism (in ν). To that end it is sufficient to prove that for every Borel set $A \subset \partial U'$ with $\nu(A) > 0$, contained in a disc $B(z, r)$ so that the disc $B(z, 2r)$ does not contain critical values for f, there exist two different branches F_1, F_2 of f^{-1} so that $\nu(F_i(A)) > 0$ for $i = 1$ and $i = 2$. Let us use the notation $\nu(\Omega, w, K)$ for the harmonic measure of $K \subset \partial\Omega$ viewed from $w \in \Omega$ for a connected domain Ω. Going back to our situation we have $\nu(U', w, A) \leq C < 1$ for a constant C and every $w \in \partial(B(z, 2r) \cap U')$. Now $\nu(A) = \nu(U', p, A) > 0$ implies there exists $w_0 \in B(z, 2r) \cap U'$ such that $C < \nu(U', w_0, A)$. Denote by W the component of $B(z, 2r) \cap U'$ containing w_0. Then by the comparison of harmonic functions

$$\nu(W, w_0, A) \geq \frac{1}{1 - C}(\nu(U', w_0, A) - C) > 0$$

because the first inequality holds, by the definitions, on ∂W. Hence for $F_i, i = 1, 2$, being branches of f^{-1} mapping W in U' we obtain

$$\nu(U', F_i(w_0), F_i(A)) \geq \nu(F_i(W), F_i(w_0), F_i(A)) = \nu(W, w_0, A) > 0,$$

so $\nu(U', p, F_i(A)) > 0$. The proof that f is not an automorphism in ν is over.

In general since f, g are holomorphic, their compositions and inverse branches map sets of positive harmonic measure to the sets of positive harmonic measure. So we can use Lemma 1 to construct an infinite sequence of local symmetries H_i of J in a neighborhood of a point $a \in \partial U'$ of the form $H_i = g^{l_i+t+m} \circ f^{-k_i-s}$ (see Proof of Theorem A). We just find $x \in \partial U'$ such that x and $g^t(f^{-s})(x)$ satisfy the assertions of Lemma 1 for iteration of f and g respectively. Next we find $a \in \partial U'$ as a limit of $f^{k_i}(x)$, as in Proof of Theorem A. $\square$

Remark 6. Theorem B can be extended to rational functions with parabolic periodic points having simply connected immediate basins, by [PSV].

3. Functional equations

A classical result on commuting rational functions f and g states:

$$f \circ g = g \circ f \Longrightarrow f^m = g^n,$$

for some $m > 0, n > 0$ (if f and g are not critically finite with the same parabolic orbifold) (see Introduction).

Consider another functional equation:

$$f^2 \circ g = f \circ g \circ f,$$

i.e., f commutes with $f \circ g$. It yields

$$f^m = (f \circ g)^n$$

(if f and $f \circ g$ are not critically finite with the same parabolic orbifold). This gives no direct information about g.

On the other hand, Theorem A gives a way to separate the functions f and g in an appropriate functional equation. Indeed, if f commutes with $f \circ g$, then

$$\mu(f) = \mu(f \circ g) = \mu(g)$$

(because $\mathrm{Jac}_\mu\, g = \mathrm{Jac}_\mu(f \circ g)/(\mathrm{Jac}_\mu\, f) \circ g = \mathrm{Const}$, where $\mu = \mu(f) = \mu(f \circ g)$), and the equation $(*)$ holds (again, if f and g are not critically finite with the same parabolic orbifold).

The above is true of course for every functional equation between rational functions f and g whenever one can derive from the equation the coincidence of the maximal measures of f and g.

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