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When Does Eddy Viscosity Damp Subfilter Scales Sufficiently?

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Abstract Large eddy simulation (LES) seeks to predict the dynamics of spatially filtered turbulent flows. The very essence is that the LES-solution contains only scales of size $\geq\Delta$, where Δ denotes some user-chosen length scale. This property enables us to perform a LES when it is not feasible to compute the full, turbulent solution of the Navier-Stokes equations. Therefore, in case the large eddy simulation is based on an eddy viscosity model we determine the eddy viscosity such that any scales of size $<\Delta$ are dynamically insignificant. In this paper, we address the following two questions: how much eddy diffusion is needed to (a) balance the production of scales of size smaller than Δ; and (b) damp any disturbances having a scale of size smaller than Δ initially. From this we deduce that the eddy viscosity ν_e has to depend on the invariants $q = \frac{1}{2}\mathrm{tr}(S^2)$ and $r = -\frac{1}{3}\mathrm{tr}(S^3)$ of the (filtered) strain rate tensor S. The simplest model is then given by $\nu_e = \frac{3}{2}(\Delta/\pi)^2|r|/q$. This model is successfully tested for a turbulent channel flow ($\mathrm{Re}_\tau = 590$).

Keywords Large eddy simulation · Eddy viscosity · Confinement of turbulent dynamics · Invariants of strain rate tensor

1 Introduction

1.1 Navier-Stokes Equations and Turbulence

The Navier-Stokes (NS) equations provide an appropriate model for turbulent flow. In the absence of compressibility ($\nabla \cdot u = 0$), the equations are

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 2\nu\nabla \cdot S(u) \tag{1}$$

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where u is the fluid velocity field, p stands for the pressure, ν denotes the viscosity, and $S(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the velocity gradient. Turbulent flow is generally visualized as a cascade of kinetic energy, where the large scales of motion are driven. Since the large scales cannot reach a near equilibrium between the rate at which energy is supplied and the rate at which energy is dissipated (by the action of viscosity), they break up, transferring their energy to somewhat smaller scales; this transport process is governed by the nonlinearity in (1). The smaller scales undergo a similar break-up process, and transfer their energy to yet smaller scales. The energy cascade continues until the scale becomes so small that dissipation is getting predominant. The entire spectrum—ranging from the scales where the flow is driven to the smallest, dissipative scales—is to be resolved numerically when turbulence is computed directly from the NS-equations. This, however, is not feasible in many cases, i.e., the NS-equations (1) do not provide a tractable model (see e.g. Ref. [1]). Hence, finding a coarse-grained description is one of the main challenges to turbulence research. A promising methodology for that is large eddy simulation (LES). Put simply, the aim of LES is to compute only the scales of motion (eddies) which are larger than some user-chosen length scale.

1.2 Large Eddy Simulation

Large eddy simulation seeks to predict the dynamics of spatially filtered turbulent flows. Therefore a spatial filter, with filter length Δ, is applied to (1),

$$\partial_t \overline{u} + (\overline{u} \cdot \nabla)\overline{u} + \nabla \overline{p} - 2\nu\nabla \cdot S(\overline{u}) = \nabla \cdot \left(\overline{u}\,\overline{u}^T - \overline{uu^T}\right) \tag{2}$$

where it is assumed that the filter $u \mapsto \overline{u}$ commutes with differentiation. The right-hand side represents the effects of the residual scales on the 'large eddies' (the part of the fluid motion with velocity $\overline{u}$). It depends on both u and $\overline{u}$, due to the nonlinearity. The dependence on u is removed by introducing a closure model $\tau(\overline{u})$ that approximates $\overline{uu^T} - \overline{u}\,\overline{u}^T$ somehow. The dynamics of the large eddies is then governed by

$$\partial_t v + (v \cdot \nabla)v + \nabla \tilde{p} - 2\nu\nabla \cdot S(v) = -\nabla \cdot \tau(v) \tag{3}$$

where the variable name is changed from $\overline{u}$ to v to stress that the solution of (3) differs from that of (2), because the closure model (represented by τ) is not exact. The inequality $\tau(v) \neq \overline{uu^T} - \overline{u}\,\overline{u}^T$ is crucial, since information is to be lost: the solution v of (3) must possess less scales of motion (degrees of freedom) than the Navier-Stokes solution u. Especially in case the filter is an isomorphism, it is theoretically possible to express the subfilter contribution $\overline{uu^T} - \overline{u}\,\overline{u}^T$ exactly in terms of the filtered field $\overline{u}$ and thus to construct an exact model. This, however, does not reduce the complexity of the problem; see also [2] for a more detailed discussion. Finding a closure model that is both inexact (to reduce the complexity of the flow) and accurate (to approximate the dynamics of the larger eddies) represents the main difficulty to LES.

The LES equations (3) need be discretized in space and time to find a numerical approximation of v. In principle, any discretization method that is suited for the Navier-Stokes equations (1) can be applied to the LES equations (3) too. Here, the grid is to be taken such that the smallest scale in the flow field can be represented. In many applications, however, the LES-grid is so coarse that the discretization may be viewed as a second filter that truncates the solution at the size of the grid. In that case subgrid-scale effects are to be taken into account, see [3, 4], e.g.

We take the closure model τ symmetric, because $\overline{u}\,\overline{u}^T - \overline{uu^T}$ is symmetric. Additionally, we can take the trace of τ equal to zero without loss of generality, because the trace can be included as part of the scalar $\tilde{p}$. Since turbulence is so far from being completely understood, there is a wide range of closure models, mostly based on heuristic, *ad hoc* arguments that cannot be derived from the NS-equations, see for example [5] and the references therein. The most commonly used closure model is given by

$$\tau(v) = -2\nu_e S(v) \tag{4}$$

where ν_e denotes the eddy viscosity. Notice that this closure model is not time reversible (for $\nu_e > 0$); forward in time it provides dissipation, i.e., the complexity of the flow can be reduced, depending on the amount eddy viscosity.

The classical Smagorinsky model reads

$$\nu_e = C_S^2 \Delta^2 \sqrt{4q} \tag{5}$$

where $q(v)$ is an invariant of the strain rate tensor $S(v)$:

$$q(v) = \frac{1}{2}\mathrm{tr}(S^2(v))$$

Here it may be remarked that (5) is commonly formulated in terms of $|S(v)| = \sqrt{2\,\mathrm{tr}(S(v)^2)} = \sqrt{4q}$. Further, it may be noticed that the model (5) depends only on the length Δ of the filter, and not on the details of the map $u \mapsto \overline{u}$. Various value for the Smagorinsky constant C_S have been proposed, mainly ranging from $C_S = 0.1$ to $C_S = 0.17$ [1]. Instead of adhering to a constant value one can also take $C_S = C_S(v)$. In the well-known dynamical procedure, for instance, the coefficient C_S is computed with the help of the Jacobi identity (in least-square sense) [6].

1.3 Problem Setting

The solution v of (3) is composed of eddies of different size. The very essence of large eddy simulation is that v contains only eddies of size $\geq \Delta$, where Δ is the user-chosen length of the filter $u \mapsto \overline{u}$. This property enables us to solve (3) numerically when it is not feasible to compute the solution of (1), i.e., the full turbulent flow field u. Equations (3)–(4) is formally equivalent to the NS-equations (1) with a modified diffusion coefficient; hence, the desired effect is to eliminate all scales of size $< \Delta$. Therefore we view the eddy viscosity as a function of v that is to be determined such that the dynamically significant scales of motion in the solution v of (3)–(4) are greater than (or equal to) Δ. Strictly speaking, the eddy viscosity is to be determined such that the corresponding solution v of (3)–(4) forms the 'best' approximation of $\overline{u}$. To that end, however, we have to rely on phenomenological arguments, because we do not know how to derive the 'best' eddy viscosity from the Navier-Stokes equations. In the present approach we try not to make any specific assumptions (about the spectrum, e.g.). Rather, we focus on the question: "when does eddy viscosity damp subfilter scales sufficiently?"

2 When Does Eddy Viscosity Stop the Production of Smaller Scales of Motion from Continuing at the Filter Scale?

In this section, a lower bound for the eddy viscosity is determined from the requirement that the production of any eddies of size smaller than Δ by the nonlinear mechanism in the left-hand side of (3) is counteracted by the dissipation in the right-hand side of (3), where the closure model is given by (4).

To that end, we consider an arbitrary part Ω_Δ with diameter Δ of the flow domain and define the filtering operator $u \mapsto \overline{u}$ by

$$\overline{u} = \frac{1}{|\Omega_\Delta|} \int_{\Omega_\Delta} u(x,t)\, dx$$

In other words, the filtered velocity $\overline{u}$ is equal to the average of u over Ω_Δ. This filter is known as a box or top-hat filter.

Furthermore, we suppose that the solution v of (3) is periodic on Ω_Δ, so that boundary terms resulting from integration by parts (in the computations to come) vanish. It may be emphasized here that a solution u of (1) is generally not Δ-periodic. However, if Δ tends towards the smallest scale of motion homogeneity may be assumed, i.e., then the sub Δ-scale flow tends to become Δ-periodic. In the computations to come the periodicity conditions are applied to v, not to u nor $\overline{u}$, and Δ is supposed to be the smallest scale in v. This partially justifies our choice of the boundary conditions for v.

Poincaré's inequality states that there exists a constant C_Δ, depending only on Ω_Δ, such that for every function v in the Sobolev space $W^{1,2}(\Omega_\Delta)$,

$$\int_{\Omega_\Delta} \|v - \overline{v}\|^2\, dx \le C_\Delta \int_{\Omega_\Delta} \|\nabla v\|^2\, dx \tag{6}$$

The optimal constant C_Δ, the Poincaré constant for the domain Ω_Δ, is the inverse of the smallest (non-zero) eigenvalue of the dissipative operator $-\nabla^2$ on Ω_Δ [7]. It is given by

$$C_\Delta = (\Delta/\pi)^2$$

for convex domains Ω_Δ [8].

The residual field $v' = v - \overline{v}$ contains eddies of size smaller than Δ. These eddies are produced by the nonlinear, convective term in (3). The eddy viscosity must keep them from becoming dynamically significant. Poincaré's inequality (6) shows that the $L^2(\Omega_\Delta)$ norm of the residual field v' is bounded by a constant (independent of v) times the $L^2(\Omega_\Delta)$ norm of ∇v. Consequently, we can confine the dynamically significant part of the motion to scales $\ge \Delta$ by damping the velocity gradient with the help of an eddy viscosity. To see how the evolution of the $L^2(\Omega_\Delta)$ norm of ∇v is to be damped, we consider the residual field v' first:

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \frac{1}{2}\|v'\|^2\, dx = -\nu \int_{\Omega_\Delta} \|\nabla v'\|^2\, dx + \int_{\Omega_\Delta} T(\overline{v}, v')\, dx - \nu_e \int_{\Omega_\Delta} \|\nabla v'\|^2\, dx$$

Here, $\int_{\Omega_\Delta} T(\overline{v}, v')\, dx$ represents the energy transfer from $\overline{v}$ to v'. Equation (3) should not produce subfilter scales, i.e., the eddy diffusion has to balance the energy transfer at the scale set by the filter. Now suppose that the eddy viscosity is taken such that the last two terms in the right-hand side above cancel each other out. Then we have

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \frac{1}{2}\|v'\|^2\, dx = -\nu \int_{\Omega_\Delta} \|\nabla v'\|^2\, dx \tag{7}$$

This equation shows that the evolution of the energy of v' is not depending on $\overline{v}$. Stated otherwise, the energy of subfilter scales dissipates at a natural rate, without any forcing mechanism involving scales larger than Δ. With the help of the Poincaré inequality (6), we obtain from (7) that

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega_\Delta}\|v'\|^2(x,t)\,dx \leq (-2\nu/C_\Delta)\int_{\Omega_\Delta}\|v'\|^2(x,t)\,dx$$

The Gronwall lemma leads then to

$$\int_{\Omega_\Delta}\|v'\|^2(x,t)\,dx \;\leq\; \exp(-2\nu t/C_\Delta)\int_{\Omega_\Delta}\|v'\|^2(x,0)\,dx$$

In other words, the energy of the subfilter scales decays at least as fast as $\exp(-2\nu t/C_\Delta)$, for *any* filter length Δ. Applying Poincaré's inequality and Gronwalls lemma to

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega_\Delta}\frac{1}{2}\|\nabla v\|^2\,dx = -\nu\int_{\Omega_\Delta}\|\nabla^2 v\|^2\,dx \tag{8}$$

results into the same rate of decay:

$$\begin{aligned}\int_{\Omega_\Delta}\|v'\|^2(x,t)\,dx &\overset{(6)}{\leq} C_\Delta\int_{\Omega_\Delta}\|\nabla v\|^2(x,t)\,dx\\ &\overset{(8)}{\leq} C_\Delta\exp(-2\nu t/C_\Delta)\int_{\Omega_\Delta}\|\nabla v\|^2(x,0)\,dx\end{aligned}$$

To give (8) also a somewhat different interpretation, we take an arbitrary part Ω_ℓ with diameter ℓ of the flow domain, with $\ell \geq \Delta$ and consider both the incompressible NS equations (1) and the LES-model given by (3)–(4) on Ω_ℓ with periodic boundary conditions. Furthermore we supply energy to the flow at a given rate. This energy cannot escape from the box Ω_ℓ, since we have applied periodic conditions. Hence, the energy has to dissipate within Ω_ℓ at the rate at which it is supplied. In case of the NS equations the dissipation rate is given by $\epsilon = \nu\int_{\Omega_\ell}\|\nabla u\|^2 dx$. In the LES-model, the dissipation rate becomes $\int_{\Omega_\ell}(\nu+\nu_e)\|\nabla v\|^2 dx$. Without eddy viscosity, i.e., $\nu_e = 0$, the dissipation rate of the LES is much smaller than ϵ if $v \approx \overline{u}$. Indeed, the mapping $u \mapsto \overline{u}$ smoothes the velocity field, i.e., reduces the gradient. Now suppose that the amount of eddy viscosity is taken too little. Then, $\|\nabla v\|^2$ will have (a tendency) to increase, because the energy that is supplied to the flow is to be dissipated at the given rate. Since the norm of the velocity gradient $\|\nabla v\|$ provides a consistent characterization of the reciprocal of the time scale, an increase of $\|\nabla v\|$ implies that smaller time scales are produced. That is, eddies of size ℓ in the field v become unstable and break up, transferring their energy to smaller eddies. So, an increase of $\int_{\Omega_\ell}\|\nabla v\|^2\,dx$ indicates that scales with a length smaller than ℓ are produced. A LES can deal with this only if $\ell > \Delta$. In other words, the eddy viscosity has to stop the production of smaller scales from continuing at the scale $\ell = \Delta$ set by the filter. Therefore, $\int_{\Omega_\Delta}\|\nabla v\|^2\,dx$ should decrease. Equation (8) states that this integral decreases at the rate given by the fluid viscosity.

The (minimum) amount of eddy viscosity needed to satisfy the dissipative condition (8) can be derived by taking the L^2 inner product of (3)–(4) with $\nabla^2 v$. Integration by parts yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega_\Delta}\frac{1}{2}\|\nabla v\|^2\,dx &= -\nu\int_{\Omega_\Delta}\|\nabla^2 v\|^2\,dx\\ &\quad+\int_{\Omega_\Delta}((v\cdot\nabla)v\cdot\nabla^2 v - \nu_e\|\nabla^2 v\|^2)\,dx\end{aligned} \tag{9}$$

where ν_e is assumed to be constant in Ω_Δ. As remarked before, the boundary terms that result from the integration by parts vanish because Ω_Δ is a periodic box. Thus we see that (8) holds if

$$\nu_e \int_{\Omega_\Delta} \|\nabla^2 v\|^2 \, dx = \int_{\Omega_\Delta} (v \cdot \nabla) v \cdot \nabla^2 v \, dx \tag{10}$$

Chae [9] (pp. 791–792) showed that, for a periodic box, the convective term in the right-hand side of (10) is equal to $4\int_{\Omega_\Delta} r(v)\, dx$, where

$$r(v) = -\frac{1}{3}\mathrm{tr}(S^3(v)) = -\det S(v)$$

is an invariant of the strain rate tensor $S(v)$. It may be remarked here that the calculations by Chae are done for the 3D Euler equations; yet one can add the viscous term to each step of the calculations in Ref. [9]. The other nonzero invariant of $S(v)$, $q(v) = \frac{1}{2}\mathrm{tr}(S^2(v))$, has the property that $4\int_{\Omega_\Delta} q(v)\, dx = \int_{\Omega_\Delta} \|\nabla \times v\|^2 \, dx$. See again Ref. [9] for instance. Since $\nabla^2 v = -\nabla \times \omega$ with $\omega = \nabla \times v$, it follows that (10) is equivalent to

$$\nu_e \int_{\Omega_\Delta} q(\omega)\, dx = \int_{\Omega_\Delta} r(v)\, dx \tag{11}$$

In conclusion, the eddy viscous damping in (9) counteracts the nonlinear production of scales $< \Delta$ if the eddy viscosity is taken according to (11). Here, the left-hand side represents the dissipation, and the right-hand side stands for the production, i.e., the invariant $q(\omega)$ provides a measure for the dissipation, and the production can be quantified with the help of the invariant $r(v)$.

2.1 Vortex Stretching

A noticeable difference between (11) and the standard Smagorinsky model (5) with C_S constant is that the standard model depends only on the invariant q—i.e., not on r. The role of the invariant $r(v)$ can be explained with the help of the vorticity $\omega = \nabla \times v$. By taking the curl of (3)–(4) we find the vorticity equation and from that we obtain that the enstrophy is governed by

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \frac{1}{2}\|\omega\|^2 \, dx = \int_{\Omega_\Delta} \omega \cdot S\omega \, dx - (\nu + \nu_e) \int_{\Omega_\Delta} \|\nabla \omega\|^2 \, dx$$

In the right-hand side we recognize the vortex stretching term that can produce smaller scales of motion and the eddy diffusive term that should counteract the production of smaller scales at the scale Δ. It can be shown that (see [9], e.g.)

$$\nu_e \int_{\Omega_\Delta} \|\nabla \omega\|^2 \, dx = \nu_e \int_{\Omega_\Delta} 4q(\omega)\, dx \quad \text{and} \quad \int_{\Omega_\Delta} 4\, r(v)\, dx = \int_{\Omega_\Delta} \omega \cdot S\omega \, dx$$

Notice that the latter equality shows that $r(v)$ is a measure for the vortex stretching. Thus, (11) can also be interpreted as follows: the eddy viscosity is taken such that the corresponding damping of the enstrophy equals the production by means of the vortex stretching mechanism:

$$\nu_e \int_{\Omega_\Delta} \|\nabla \omega\|^2 \, dx = \int_{\Omega_\Delta} \omega \cdot S(v)\omega \, dx \tag{12}$$

In other words, the eddy viscosity prevents the intensification of vorticity at the scale Δ set by the map $u \mapsto \overline{u}$. Finally, it may be emphasized that (11) and (12) are equivalent.

2.2 Modeling Consistency

The dissipative term in (11) can be bounded from below with the help of Poincaré's inequality:

$$\int_{\Omega_\Delta} q(\omega)\,dx = \int_{\Omega_\Delta} \frac{1}{4}\|\nabla\omega\|^2\,dx \geq \frac{1}{C_\Delta}\int_{\Omega_\Delta} \frac{1}{4}\|\omega\|^2\,dx = \frac{1}{C_\Delta}\int_{\Omega_\Delta} q(v)\,dx$$

where the equality-sign holds if ω is fully aligned with the eigenfunction of the dissipative operator $-\nabla^2$ on Ω_Δ associated with the smallest non-zero eigenvalue. Consequently, the eddy viscous term in (11) dominates the nonlinear, convective term if

$$\frac{\nu_e}{C_\Delta}\int_{\Omega_\Delta} q(v)\,dx \geq \int_{\Omega_\Delta} r(v)\,dx \tag{13}$$

Condition (13) ensures that subfilter scales are dynamically insignificant, meaning that the energy of the scales of size $\leq \Delta$ decays at least as fast as $\exp(-2\nu t/C_\Delta)$, for any filter length Δ, if the eddy viscosity is taken such that (13) holds.

It has not been established, thus far, that the choice of the minimal eddy viscosity satisfying (13), i.e.,

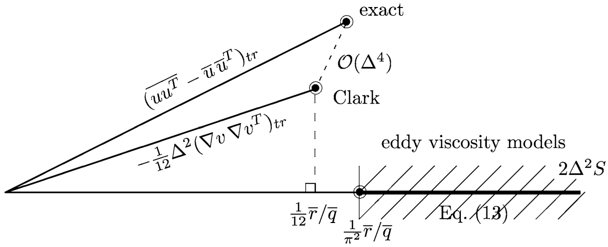
$$\nu_e = C_\Delta\, \overline{r(v)}/\overline{q(v)} \tag{14}$$

will adequately model the subfilter contributions to the evolution of the filtered velocity. From the filtered Navier-Stokes solution $\overline{u}$, we can analyze the consistency of the eddy viscosity model $(\overline{u}\overline{u}^T - \overline{uu^T})_{tr} \approx 2\nu_e S(\overline{u})$ by *a priori* testing. Here we consider the traceless part (defined by $A_{tr} = A - \frac{1}{3}\mathrm{tr}(A)I$), because the trace of $\overline{u}\,\overline{u}^T - \overline{uu^T}$ can be incorporated into the pressure. A series expansion gives $(\overline{u}\,\overline{u}^T - \overline{uu^T})_{tr} = -\frac{\Delta^2}{12}(\nabla\overline{u}\,\nabla\overline{u}^T)_{tr} + \mathcal{O}(\Delta^4)$. The leading term is known as the Clark model [10]. Unfortunately, the Clark model cannot be used as a stand-alone LES model, since it produces a finite time blow-up of the kinetic energy [11]. In other words, the Clark model can produce length-scales smaller than Δ. Projecting both (14) and the Clark model onto $S(v)$ leads to the following consistency question

$$2C_\Delta \frac{\overline{r}}{\overline{q}}\int_{\Omega_\Delta} S(v):S(v)\,dx \stackrel{?}{=} -\frac{\Delta^2}{12}\int_{\Omega_\Delta} (\nabla v \nabla v^T)_{tr} : S(v)\,dx \tag{15}$$

The integral in the right-hand side equals $-4\int_{\Omega_\Delta} r(v)\,dx$ [9]. This shows that r provides a measure of the alignment of the Clark model and S. By definition we have $S:S = 2q$. Consequently, (15) shows that the order of the modeling error is optimal if $C_\Delta = \Delta^2/12$. This value is in fair agreement with the Poincaré constant, $C_\Delta = \Delta^2/\pi^2$; yet, it is slightly lower. The overall situation is sketched in Fig. 1. The horizontal axis in this figure represents all possible eddy viscosity models; the axis is parameterized by the eddy viscosity. The shaded part of the horizontal axis in Fig. 1 depicts the subset of eddy viscosities that satisfy (13). The projection of the Clark model onto the horizontal axis falls outside the shaded area; hence it cannot be guaranteed that this projection damps subfilter scales adequately. This reflects that the Clark model can produce subfilter scales [11]. Equation (14) forms the best approximation of the projection of the Clark model provided the eddy viscosity model is restricted by (13). The projection onto $S(v)$ is chosen to evaluate the dissipation provided by these models.

Fig. 1 Some LES-models in the space of symmetric 3 × 3 tensors



Furthermore, (14) yields $\nu_e = 0$ in any (part of the) flow where $r = 0$. That is, the eddy viscosity vanishes if the nonlinear transport to scales $< \Delta$ is absent. At a no-slip wall $r = 0$ too; hence $\nu_e = 0$ at the wall. In homogeneous, isotropic turbulence, we have $r/q \propto \mathrm{Re}^{1/2}$. Therefore $\nu_e/\nu \propto \mathrm{Re}^{3/2}$ for fixed Δ. Additionally, we obtain that $\nu_e + \nu \to \nu$ if $\nu \propto \mathrm{Re}^{-1} \propto \Delta^2 r/q \propto \Delta^2\,\mathrm{Re}^{1/2}$, that is if $\Delta \propto \mathrm{Re}^{-3/4}$. This shows that the eddy viscosity given by (14) vanishes as Δ is of the order of $\mathrm{Re}^{-3/4}$, i.e., if Δ approaches the Kolmogorov scale.

3 When Does Eddy Viscosity Damp Subfilter-scale Disturbances Properly?

In Sect. 2, we have determined the eddy viscosity such that the energy that is transferred from the large eddies (scales of size $\geq \Delta$) to the subfilter scales is dissipated so fast that the production of subfilter scales by the nonlinear mechanism in the left-hand side of (3) becomes dynamically irrelevant. Condition (12) ensures that the transfer of energy *from* the large eddies to the subfiler scales is balanced properly by the eddy dissipation. This condition is necessary, but not sufficient, to limit the dynamics governed by (3) to scales of size $\geq \Delta$. To that end, the energy that is transferred from the subfilter scales *to* the large eddies, should be dynamically insignificant too. Here it may be noticed that according to (13) we can simply take $\nu_e = 0$ if $\overline{r} < 0$. This is because (13) states only that the eddy dissipation balances (or dominates) the transfer of energy from the larger eddies to the subfilter scales; hence (13) does not prescribe the eddy viscosity if energy is transferred from scales of size smaller than Δ to larger scales (backscatter). Also in the case $\overline{r} < 0$ the solution v of (3) should solely depend on scales of size $\geq \Delta$. To see if the backward cascade of energy is properly closed, we suppose that the velocity field v does not contain subfilter scales initially, i.e., at time $t = 0$ we have $v = \overline{v}$ in an arbitrary part Ω_Δ with diameter Δ of the flow domain. Now, we superimpose an instantaneous, solenoidal, subfilter-scale perturbation δv to v on Ω_Δ. Initially one may conceive the unperturbed field $v = \overline{v}$ as being constant on Ω_Δ, whereas the perturbation δv is any (non-constant) periodic function on Ω_Δ. The evolution of the perturbed velocity $v + \delta v$ is governed by (3)–(4) with v replaced by $v + \delta v$. As before, we take the eddy viscosity constant in Ω_Δ. The dynamics of the disturbance is then given by

$$\partial_t \delta v + (v \cdot \nabla)\delta v + (\delta v \cdot \nabla) v + (\delta v \cdot \nabla)\delta v + \nabla \delta p = 2(\nu + \nu_e)\nabla \cdot S(\delta v) \qquad (16)$$

where δv stands for a arbitrary, space-periodic, vector field having divergence equal to zero, and $\overline{\delta v} = 0$ at $t = 0$. Scales of size smaller than Δ do not become dynamically significant if the difference between the two solutions v and $v + \delta v$ vanishes sufficiently fast. The evolution of the $L^2(\Omega_\Delta)$ norm of δv is given by

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \frac{1}{2}\|\delta v\|^2\, dx \overset{(16)}{=} -\int_{\Omega_\Delta} \delta v \cdot S(v)\delta v\, dx - (\nu_e + \nu)\int_{\Omega_\Delta} \|\nabla \delta v\|^2\, dx$$

Notice that a number of terms vanish here, because the convective operator is skew symmetric, i.e., $(v\cdot\nabla)^* = -(v\cdot\nabla)$ [12]. The term $\int_{\Omega_\Delta} \delta v\cdot S(v)\delta v\,dx$ represents the energy transfer from v to δv. In the absence of this term we can simply take $\nu_e = 0$. As in Sect. 2 the eddy viscosity is taken such that it neutralizes the production term,

$$\nu_e \int_{\Omega_\Delta} \|\nabla\delta v\|^2\,dx = -\int_{\Omega_\Delta} \delta v\cdot S(v)\delta v\,dx \tag{17}$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega_\Delta} \frac{1}{2}\|\delta v\|^2\,dx = -\nu\int_{\Omega_\Delta} \|\nabla\delta v\|^2\,dx$$

This equation shows that the evolution of the energy of the perturbation δv is not depending on the unperturbed, base flow v. Compare (7). As a result, the energy of subfilter scales dissipates at a natural rate (without any nonlinear mechanism involving scales larger than Δ). Again with the help of the Poincaré inequality (6) and Gronwalls lemma, we get

$$\int_{\Omega_\Delta} \|\delta v\|^2(x,t)\,dx \le \exp(-2\nu t/C_\Delta)\int_{\Omega_\Delta} \|\delta v\|^2(x,0)\,dx$$

This shows that the energy of subfilter disturbances decays at least as fast as $\exp(-2\nu t/C_\Delta)$, for *any* filter length Δ. So, in conclusion, the LES-model given by (3)–(4) is stable with respect to subfilter disturbances—i.e., the backward cascade of energy is properly closed—if (17) holds.

4 A Condition for the Eddy Dissipation

The solution v of (3) should contain only eddies of size $\ge \Delta$, where Δ is the user-chosen length of the filter. This condition is worked out for an eddy viscosity model in Sects. 2–3. From that we conclude that the eddy viscosity is to be taken such that (12) and (17) are satisfied. The essential difference between these two equalities is the sign of the transport term: if we would simply disregard the difference between δv with ω and name them both ϕ, then (12) and (17) become

$$\nu_e \int_{\Omega_\Delta} \|\nabla\phi\|^2\,dx = \pm\int_{\Omega_\Delta} \phi\cdot S(v)\phi\,dx$$

where the plus sign corresponds to (12) and the minus sign to (17). From a physical point of view the plus sign represents the requirement that the forward cascade of energy stops at the scale Δ set by the filter (that is, the production of smaller scales stops at the filter scale); the minus sign expresses that there is no backward cascade too (that is, scales of size $< \Delta$ cannot become dynamically relevant, since any subfilter perturbations decay exponentially fast).

The dissipative term in the expression above can be bounded from below with the help of Poincaré's inequality:

$$\int_{\Omega_\Delta} \|\nabla\phi\|^2\,dx \ge \frac{1}{C_\Delta}\int_{\Omega_\Delta} \|\phi\|^2\,dx$$

Notice again that the Poincaré constant C_Δ is the inverse of the smallest (non-zero) eigenvalue of the dissipative operator $-\nabla^2$ on Ω_Δ. The equality holds if ϕ is aligned with the associated eigenfunction. Therefore we may take the eddy viscosity such that

$$\nu_e \int_{\Omega_\Delta} \|\phi\|^2 \, dx \geq \pm C_\Delta \int_{\Omega_\Delta} \phi \cdot S(v)\phi \, dx \tag{18}$$

Consequently, the eddy viscosity can be expressed in terms of the Poincaré constant C_Δ and the spectrum of the strain rate tensor S. The eigenvalues of the symmetric matrix S can be ordered as follows: $\lambda_1 \leq \lambda_2 \leq \lambda_3$. The sum of the eigenvalues is zero, because $\mathrm{tr}(S) = 0$. Hence, $\lambda_1 \leq 0$ and $\lambda_3 \geq 0$. The characteristic equation reads $\lambda^3 - q\lambda + r = 0$. The three roots of this cubic equation can be computed analytically:

$$\lambda_1 = -|S|\sqrt{\frac{1}{3}} \cos\left(\frac{\theta}{3}\right)$$

$$\lambda_2 = -|S|\sqrt{\frac{1}{3}} \cos\left(\frac{\theta}{3} - \frac{2\pi}{3}\right)$$

$$\lambda_3 = -|S|\sqrt{\frac{1}{3}} \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)$$

with

$$\theta = \arccos\left(\frac{1}{2}r \Big/ \sqrt{\left(\frac{1}{3}q\right)^3}\right)$$

The eigenvalues must be real-valued, because S is symmetric. The characteristic equation has three real roots if and only if the invariants $q = \frac{1}{2}\mathrm{tr}(S^2)$ and $r = -\frac{1}{3}\mathrm{tr}(S^3)$ satisfy

$$27r^2 - 4q^3 \leq 0 \tag{19}$$

Hence, $\theta \in [0, \pi]$. Now suppose that ϕ in (18) is fully aligned with the eigenvector associated with λ_i, with $i = 1, 2, 3$. Then, the mean value theorem for integration states that there exists a point ξ_i in Ω_Δ such that

$$\int_{\Omega_\Delta} \phi \cdot S(v)\phi \, dx = \int_{\Omega_\Delta} \lambda_i \phi \cdot \phi \, dx = \lambda_i(\hat{v}_i) \int_{\Omega_\Delta} \phi \cdot \phi \, dx$$

where $\lambda_i(\hat{v}_i) = \lambda_i(v(\xi_i, t))$. So, with the help of Rayleigh's principle we obtain from (18) that $\nu_e = C_\Delta \max\{|\lambda_1(\hat{v}_1)|, \lambda_3(\hat{v}_3)\}$. This eddy viscosity guarantees that v does not contain dynamically significant scales of size smaller than Δ. Therefore, we may assume that $\hat{v}_i \approx \overline{v} \approx v$ in Ω_Δ. Hence, we may replace $\hat{v}_i$ by v in the above expression for ν_e. Notice that this formally means that higher-order terms are neglected; for instance, we make use of $\overline{v} = v + \mathcal{O}(\Delta^2)$. In this way we get an eddy viscosity depending upon the solution v of (3),

$$\nu_e(v) = C_\Delta \max\{|\lambda_1(v)|, \lambda_3(v)\} = C_S^2(\theta)\Delta^2|S(v)| \tag{20}$$

where the Smagorinsky coefficient C_S becomes a function of θ:

$$C_S^2(\theta) = \frac{1}{\pi^2\sqrt{3}} \max\left\{\cos\left(\frac{\theta}{3}\right), -\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)\right\}$$

It may be emphasized here that the right-hand side is non-negative. The maximum value of the Smagorinsky coefficient C_S becomes $C_S = \frac{1}{\pi 3^{1/4}} \approx 0.24$, which is relatively high in comparison to both empirical fits to DNS results and Lilly's result, $C_S = 0.17$ [13]. See also Ref. [14]. This overprediction is caused by the assumed alignment of ϕ with the eigenfunction associated with the extreme eigenvalue (the maximum of $|\lambda_1|$ and λ_3). The underlying problem has a likeness with the dynamical behavior of the $L^2(\Omega_\Delta)$ norm of the vorticity. In the absence of viscosity (Euler equation.), we have

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega_\Delta} \frac{1}{2} \|\omega\|^2 \, dx = \int_{\Omega_\Delta} \omega \cdot S(v) \omega \, dx$$

This immediately implies that growth rate of the $L^2(\Omega_\Delta)$ norm of the vorticity can be bounded with the help of the extreme eigenvalue $\max\{|\lambda_1|, \lambda_3\}$, like in (20). If, however, ω is not taken arbitrary, but constrained to solutions of the incompressible Euler equations it can be proven rigorously that the middle eigenvalue λ_2, and not the extreme eigenvalues $|\lambda_1|$ and λ_3, governs the growth rate of the $L^2(\Omega_\Delta)$ norm of the vorticity, see Theorem 2.2 in Ref. [9], for example. In case of the Navier-Stokes equations the theory is not developed that far, but our current knowledge states that the vorticity has a strong tendency to align with the eigenfunction associated with the middle eigenvalue λ_2, see for instance [15]. It may be noted that we have only considered the case $\phi = \omega$ here; yet also for $\phi = \delta v$ it can be concluded that ϕ cannot be fully aligned with the extreme eigenfunctions of $S(v)$. So, in theory, the estimate (20) can be improved by restricting ϕ in (18) to solutions of the governing equations (that is, the vorticity equation if $\phi = \omega$ and (16) in case $\phi = \delta v$). The necessary mathematics is, however, not at hand. Moreover, it is not very likely that theorems about the alignment of ϕ and the eigenvectors of S will be proven soon, because that requires a substantial advancement in the mathematical theory of turbulence.

The discussion above suggests to enlarge our view by letting the eddy viscosity depend on the directions of the eigenvectors of the strain rate tensor S. Accordingly, the eddy viscosity changes from a scalar into a second-order tensor. Extending our analysis in that way leads to the following second-order tensor $\pm C_\Delta E \Lambda E^T$, where Λ is a diagonal matrix with the eigenvalues λ_i as the entries on the main diagonal, and E is the matrix with the associated (normalized) eigenvectors as column vectors. That is, the eddy viscosity tensor is given by $\pm C_\Delta S(v)$. The closure model is then given by $\tau = \pm 2 C_\Delta S^2$, and the dissipation due to the closure model becomes $\pm 2 C_\Delta S^2 : S = \mp 6 C_\Delta r$. Since we have to take the sign that leads to the largest amount of dissipation, we arrive at the following condition for the eddy dissipation:

$$\tau(v) : S(v) = -6 C_\Delta |r(v)| \tag{21}$$

This condition states that the dissipation caused by the closure model (the left-hand side) has to balance the nonlinear transport (the right-hand side) at the scale Δ. It has been derived without making any assumptions about the alignment of ϕ and the eigenvectors of the strain rate tensor. Rather it is assumed that the eddy viscosity depends on the directions of the eigenvectors.

Finally it may be noted that the Clark model (that is, the leading-order term in the approximation of $\overline{uu^T} - \overline{u}\,\overline{u}^T$) does not satisfy (21). Figure 1 suggests that modifying the constant (1/12) in the Clark model suffices to provide enough dissipation. This is indeed true if $r > 0$, i.e., for the case shown in Fig. 1. Yet, the Clark model does not provide sufficient dissipation in case $r < 0$, cf. [4]. In particularly, it does not satisfy the stability condition that is considered in Sect. 3. In other words, the Clark model is unstable with respect to perturbations having a length scale smaller than Δ, see also [16], e.g. Therefore, in practice the Clark model is always supplemented by an eddy viscosity model, resulting in a so-called mixed model, see [4] e.g.

5 Towards an Eddy Viscosity Model

Condition (21) is formally derived for a tensorial eddy viscosity model. Nevertheless, we interpret (21) as an appropriate approximation of the eddy dissipation of a scalar eddy viscosity model too. The dissipation resulting from a scalar eddy viscosity model is given by $\tau(v):S(v) = -2\nu_e S(v):S(v) = -4\nu_e q(v)$. Thus, with the help of (21) we get

$$\nu_e(v) = \frac{3}{2} C_\Delta \frac{|r(v)|}{q(v)} \tag{22}$$

Here it may be noted that the middle eigenvalue of S can be bounded by $|r|/q \le |\lambda_2| \le \frac{3}{2}|r|/q$; the equality in the upper bound holds if $27r^2 = 4q^3$.

In the derivation of (22) we have assumed that (21) holds (approximately) for a scalar eddy viscosity model. To test this hypothesis, we will derive the model (22) also differently. That is, we compute the eddy viscosity ν_e directly from (14) in case $r > 0$. For $r < 0$, we take

$$\nu_e(v) = C_\Delta \frac{\overline{|r(v)|}}{\overline{q(v)}} \tag{23}$$

Now the problem is that we need know how r and q vary within Ω_Δ to compute $\overline{r(v)}$ and $\overline{q(v)}$. Therefore, we decompose v with the help of the filter into $v = \overline{v} + v'$. The residual v' represents the behavior of the large-eddy solution v within the filter-box Ω_Δ. This part of v is needed to compute the eddy viscosity directly from (23), i.e., to compute the ratio of $\overline{|r(v)|}$ and $\overline{q(v)}$. Here, we cannot simply take $\overline{q(v)} = q(\overline{v})$, because the relation between q and v is nonlinear (similarly for r). This problem is similar to the closure problem in LES, except that the original closure problem concerns the residual of the Navier-Stokes solution u, whereas here it is about the residual of the large-eddy solution v. It may be noted that we can also compute the eddy viscosity directly from (23) provided that the grid is taken such that the residual v' is fully resolved numerically. Obviously, this implies that the grid size is to be taken smaller than the filter width Δ. Therefore the computational costs will be higher than usual. This approach is successfully tested for decaying isotropic turbulence (the Comte-Bellot & Corrsin experiment at $\mathrm{Re}_\lambda = 71.6$ [17]), see Ref. [18] for more details.

Here we apply an approximate deconvolution method that recovers some of the information lost in the filtering process, see [19], e.g. To recover an approximation for v' we consider the series expansion of v around $\overline{v}$. Ignoring terms that are of the order Δ^4, we get the approximation $v' \approx -\frac{1}{24}\Delta^2\nabla^2\overline{v}$. Notice that the deconvolution method is commonly applied to approximate the subfilter part of the Navier-Stokes solution u, whereas it is used to approximate the subfilter part of the LES-velocity v here. In homogeneous, isotropic turbulence we have $r \propto \mathrm{Re}^{3/2}$ and $q \propto \mathrm{Re}^{1/2}$; hence the ratio of r and $q^{3/2}$ scales like Re^0. This scaling law suggests to take $\overline{r(v)}/\overline{q(v)}^{3/2} \approx r(\overline{v})/q(\overline{v})^{3/2}$. Thus (23) leads to

$$\nu_e \approx C_\Delta \frac{|r(\overline{v})|}{q(\overline{v})^{3/2}} \left(\frac{\overline{q(v)}}{q(\overline{v})}\right)^{1/2} \sqrt{q(\overline{v})}$$

Furthermore with the help of the approximate deconvolution method and the Poincaré inequality (6) it can be shown that

$$\overline{q(v)} = \frac{1}{4}\overline{\|\nabla v\|^2} \approx \frac{1}{4}\overline{\left\|\nabla\left(\overline{v} - \frac{1}{24}\Delta^2\nabla^2\overline{v}\right)\right\|^2}$$
$$\stackrel{(6)}{\leq} \frac{1}{4}\left(1 + \frac{1}{24}\Delta^2/C_\Delta\right)^2\overline{\|\nabla\overline{v}\|^2} = c^2\overline{q(\overline{v})}$$

with $c = 1 + \pi^2/24$, where the equality-sign holds (once again) if v is fully aligned with eigenfunction of $-\nabla^2$ on Ω_Δ associated with the eigenvalue $1/C_\Delta$. Since $\overline{q(\overline{v})} \approx q(\overline{v}) = q(v) + \mathcal{O}(\Delta^2)$ we obtain (in lowest order) the eddy viscosity model given by (22) with the constant $\frac{3}{2}$ replaced by c. Because $c \approx 1.4$, we may conclude that the two ways of deriving (22) yield approximately the same result (i.e., just a slight difference in the constant), which partially justifies the application of the dissipative condition (21) to a scalar eddy viscosity.

Equation (22) is invariant under rotation of coordinate axis, since it depends on the invariants of $S(v)$. The eddy viscosity model (22) can be put into the standard notation (5) by introducing the relation

$$C_S^2(v) = \frac{3 \quad |r(v)|}{4\pi^2\sqrt{q^3(v)}} \tag{24}$$

In homogeneous, isotropic turbulence we have $C_S^2 \propto r/\sqrt{q^3} \propto \mathrm{Re}^0$, i.e., the Smagorinsky coefficient is (in lowest order) independent of the Reynolds number Re. So, if we average (24) over the homogeneous directions we obtain an approximately constant coefficient C_S^2 that is valid for a wide range of Reynolds numbers (in case of homogeneous, isotropic turbulence). This partially agrees with Smagorinsky's reasoning, in which C_S^2 is taken constant (once again: provided that $r \propto \mathrm{Re}^{3/2}$ and $q \propto \mathrm{Re}^{1/2}$).

The invariants of S are constrained by (19). Consequently, (22) yields an eddy viscosity in the range

$$0 \leq \nu_e \leq \frac{1}{2\pi^2\sqrt{3}}\Delta^2|S|$$

This shows that the largest value of the Smagorinsky coefficient C_S is equal to $1/\sqrt{2\pi^2\sqrt{3}} \approx 0.17$. Remarkably this maximum value is identical to Lilly's value, $C_S = 0.17$ [13], which implies that the standard Smargorinsky model (5) with $C_S = 0.17$ has (more than) sufficient eddy dissipation. This upper bound was also found in Ref. [14] by means of other reasoning. Interestingly, the value $C_S = 0.17$ has been found too large in many numerical experiments. In turbulent shear flow, for instance, the value of the coefficient C_S is often reduced to the relatively low value $C_S = 0.1$ to give the standard model a fair change for success.

6 First Results

In summary, the eddy viscosity model given by (22) has the following properties: (a) $\nu_e = 0$ in any (part of the) flow where $r = 0$, i.e., the eddy viscosity vanishes if the nonlinear transport to scales $< \Delta$ is absent; hence (b) $\nu_e = 0$ in any 2D flow; (c) $\nu_e = 0$ at a wall; (d) $\nu_e \to 0$ if $\Delta \propto \mathrm{Re}^{-3/4}$; (e) $C_S \leq 0.17$. It goes without saying that the performance of the eddy viscosity model (22) has to be investigated for many cases. As a first step it was tested for turbulent channel flow by means of a comparison with direct numerical simulations. This flow forms a prototype for near-wall turbulence: virtually every LES has been tested for it. The results are compared to the DNS data of Moser et al. [20] at $\mathrm{Re}_\tau = 590$. In fact, we should compare the LES-solution v to the filtered DNS-solution $\overline{u}$. Yet, since the filtered DNS-solution is not presented in Ref. [20] we will compare v directly to u. The dimensions of the channel are taken identical to those of the DNS of Moser et al., i.e., $2\pi \times 2 \times \pi$. The computational grid used for the large-eddy simulation consists of 64^3 points. The DNS was performed on a $384 \times 257 \times 384$ grid, i.e., the DNS uses about 144 times more grid points than the present LES. The LES-results were obtained with an incompressible code that uses a fourth-order, symmetry-preserving, finite-volume discretization. Details about the numerics can be found in Ref. [21].

The eddy viscosity model given by (22) has been derived for continuous variables. To start, we imposed Condition (8) on solutions of (3)–(4). This led to (11), see Sect. 2. A discrete representation of (11) may be derived along similar lines. To that end both the PDE's (3)–(4) and Condition (8) are to be discretized. The resulting discrete condition may then be worked out in the manner of the continuous condition, but now in the discrete setting, yielding a discrete representation of (11). In Sect. 3, we superimposed subfilter-scale perturbations to a solution v of (3)–(4) and required that these perturbations dissipate at a natural rate. Also, this analysis may be performed for the discrete set of equations. In this way we obtain a discrete representation of the eddy viscosity model (22). Obviously, the result will depend on the details of the discretization method that is applied to (3)–(4). Generically, it will again be of the form given by (22) with q and r replaced by the invariants of the discrete rate-of-strain tensor, and the Poincaré constant C_Δ replaced by the inverse of the eigenvalue, corresponding to the scale Δ, of the discrete approximation of $-\nabla^2$. To explain our approximation of the Poincaré constant, we consider a second-order central discretization on a uniform grid with spacing dx, dy and dz. The largest eigenvalue of the discrete approximation of $-\nabla^2$ is then given by

$$\mu_{max} = \frac{4}{dx^2} + \frac{4}{dy^2} + \frac{4}{dz^2}$$

This eigenvalue describes the greatest possible damping in the numerical simulation, i.e., it provides a measure for the dissipation at the scale of the grid cell. Hence if we take Ω_Δ equal to the grid cell and approximate the smallest eigenvalue of $-\nabla^2$ on Ω_Δ by μ_{max}, we arrive at the following approximate relation:

$$C_\Delta \approx \frac{1}{\mu_{max}} \tag{25}$$

Thus in case $dx = dy = dz = h$, we get $\Delta \approx h$. In case the grid is nonuniform, μ_{max} can be approximate locally by multiplying the discrete dissipative operator with the mode associated with the highest frequency that fits on the grid (i.e., the $+1, -1, +1$ mode). With the help of (25) we can compute Δ for a given grid (and discretization of the dissipative operator $-\nabla^2$). It may be noted that the resulting relation between Δ and the grid width differs from the usual expression $\Delta = (dx\, dy\, dz)^{1/3}$ if the grid is (strongly) nonuniform.

The eddy viscosity model (22) is essentially not more complicated to implement in a LES-code than the standard Smagorinsky model (with C_S constant). Indeed, the model (22) is expressed in terms of the invariants of the strain rate tensor and does not involve explicit filtering. The invariant $q = \frac{1}{4}|S|^2$ is to be computed in any case; the computation of r is just as difficult. Unlike the standard Smagorinsky model (even with the relatively low value

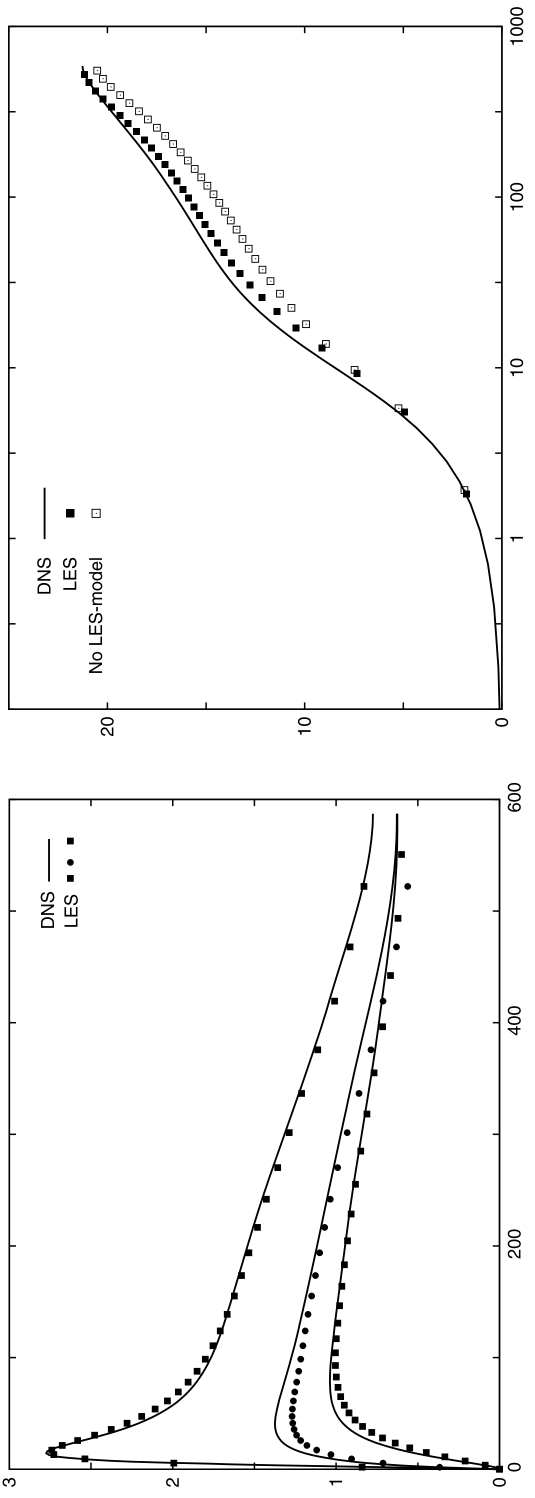


Fig. 2 The *right* figure shows the mean velocity (in wall coordinates) obtained with the help of the 64^3 LES and the DNS by Moser et al. [20]. Results obtained on the 64^3 LES-grid without closure model (i.e., $\nu_e = 0$) are also shown for reference (*open symbols*). The *left* figure displays the root-mean-square of the fluctuating velocities. The *boxes* and *circles* represent LES data; every symbol corresponds to data in a grid point

$C_S = 0.1$), the present model showed an appropriate behavior. As can be seen in Fig. 2 both the mean velocity and the root-mean-square of the fluctuating velocity are in good agreement with the DNS. To illustrate how much the eddy viscosity model contributes to the quality of the solution, the mean velocity profile obtained on the 64^3 LES-grid without closure model (i.e., $\nu_e = 0$) is also shown in Fig. 2.

The results shown in Fig. 2 were obtained with Ω_Δ equal to the grid cell, that is the filter length Δ is of roughly the same size as spacing h of the grid. As a result, the eddy viscosity has been computed such that (1) the corresponding eddy dissipation stops the production of subgrid (= subfilter) scales; and (2) any subgrid perturbations simply dissipate (decay at their natural rate); see Sects. 2 and 3, respectively. So, the eddy viscosity ensures that the subgrid part is dynamically irrelevant, which motivates the choice $\Delta \approx h$. On the other hand, however, it is obvious that with this choice the Δ-scale in v may not be resolved accurately. Consequently, the tail of the spectrum of the numerical approximation of v may contain large discretization errors. In the present case, the tail decays monotonously and there are no gridpoint-to-gridpoint oscillations (wiggles) observable, i.e., there are no strong signs of underresolution. In addition the rms-profiles are approximated well.

6.1 Towards a Dynamic Eddy Viscosity Model

Finally, it may be remarked that the model given by (22) can be further simplified with the help of the constraint (19). With the help of this inequality, we obtain the following upper bound from (22):

$$\nu_e \leq C_\Delta \left| \frac{1}{2} r \right|^{1/3} \tag{26}$$

Thus, (the upper bound for) the eddy viscosity is proportional to the geometric mean of the eigenvalues of the strain rate tensor. For that reason a new dynamic eddy viscosity model based on

$$\nu_e \sim \Delta^2 |r|^{1/3} = \Delta^2 |\lambda_1 \lambda_2 \lambda_3|^{1/3}$$

was proposed in Ref. [18]. The model coefficient is computed dynamically using the Germano identity. This dynamic eddy viscosity model correctly predicts the decay rate for decaying isotropic turbulence and the predicted energy spectra are in good agreement with filtered DNS results. Also, it is tested successfully for the $\mathrm{Re}_\tau = 590$ channel flow. Details can be found in Ref. [18].

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