WHEN IS A MULTIPLICATIVE DERIVATION ADDITIVE?

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ABSTRACT. Our main objective in this note is to prove the following. Suppose R is a ring having an idempotent element $e(e\neq 0, e\neq 1)$ which satisfies:

- (M_1) xR=0 implies x=0.
- (M₂) eRx=0 implies x=0 (and hence Rx=0 implies x=0).
- (M_2) exeR(1-e)=0 implies exe=0.

If d is any multiplicative derivation of R, then d is additive.

KEY WORDS AND PHRASES. Ring, idempotent element, derivation, Peirce decomposition. 1980 AMS SUBJECT CLASSIFICATION CODES. 16A15. 16A70.

1. INTRODUCTION.

In [1], Martindale has asked the following question: When is a multiplicative mapping additive? He answered his question for a multiplicative isomorphism of a ring R under the existence of a family of idempotent elements in R which satisfies some conditions.

Over the past few years, many results concerning derivations of rings have been obtained. In this note, we introduce the definition of a multiplicative derivation of a ring R to be a mapping d of R into R such that d(ab) = d(a)b + ad(b), for all a,b in R. As Martindale did, we raise the following question: When is a multiplicative derivation additive? Fortunately, we can give a full answer for this question using Martindale's conditions when assumed for a single fixed idempotent in R.

In the ring R, let e be an idempotent element so that e \neq 0, e \neq 1 (R need not have an identity). As in [2], the two-sided Peirce decomposition of R relative to the idempotent e takes the form R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e). We will formally set e₁ = e and e₂ = 1-e. So letting R_{mn} = e_mRe_n; m,n = 1,2, we may write R = R₁₁ \oplus R₁₂ \oplus R₂₁ \oplus R₂₂. Moreover, an element of the subring R_{mn} will be denoted by x_{mn}.

From the definition of d we note that d(0) = d(00) = d(0) + 0d(0) = 0. Moreover, we have $d(e) = d(e^2) = d(e)e + ed(e)$. So we can express d(e) as $a_{11}^+ a_{12}^+ a_{21}^+ a_{22}^+$ and use the value of d(e) to get that $a_{11}^- a_{22}^-$, that is, $a_{11}^- a_{22}^-$. Consequently, we have $d(e) = a_{12}^- + a_{21}^-$.

Now let f be the inner derivation of R determined by the element $a_{12} - a_{21}$, that is $f(x) = [x, a_{12} - a_{21}]$ for all x in R. Therefore, $f(e) = [e, a_{12} - a_{21}] = a_{12} + a_{21}$.

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In the sequel, and without loss of generality, we can replace the multiplicative derivation d by the multiplicative derivation d-f, which we denote by D, that is, D = d - f. This yields D(e) = 0. This simplification is of great importance, for, as we will see, the subrings R_{mn} become invariant under the multiplicative derivation

2. A KEY LEMMA.

LEMMA 1. $D(R_{mn}) \subset R_{mn}$, m,n = 1,2.

PROOF. Let x_{11} be an arbitrary element of R_{11} . Then $D(x_{11}) = D(ex_{11}e) = eD(x_{11})e$ which is an element of R_{11} . For an element x_{12} in R_{12} , we have $D(x_{12}) = D(ex_{12}) = 0$ $eD(x_{12}) = b_{11} + b_{12}$. But $O = D(0) = D(x_{12}e) = D(x_{12})e = b_{11}$, hence $D(x_{12}) = b_{12}$ which belongs to R_{12} . In a similar fashion, for an element x_{21} in R_{21} , we have $\mathbb{D}(x_{21})$ belongs to R_{21} . Now take an element x_{22} in R_{22} . Write $D(x_{22}) = c_{11} + c_{12} + c_{21} + c_{22}$. So, $0 = D(ex_{22}) = eD(x_{22}) = c_{11} + c_{12}$, whence $c_{11} = c_{12} = 0$. Likewise $c_{21} = 0$, and thus $D(x_{22}) = c_{22}$ which is an element of R_{22} . This proves the lemma.

CONDITIONS OF MARTINDALE.

In his note [1], Martindale has given the following conditions which are imposed on a ring R having a family of idempotent elements $\{e_i: i \in I\}$.

- (1) xR = 0 implies x = 0.
- (2) If $e_x Rx = 0$ for each i in I, then x = 0 (and hence Rx = 0 implies x = 0).
- (3) For each i in I, $e_i \times e_i R(1-e_i) = 0$ implies $e_i \times e_i = 0$.

In our note, we find it appropriate to simply dispense with conditions (1), (2) and (3) altogether and instead substitute the following conditions:

- (M_1) xR = 0 implies x = 0.
- (M_2) eRx = 0 implies x = 0 (and hence Rx = 0 implies x = 0).
- (N_3) exeR(1-e) = 0 implies exe = 0.

4. AUXILIARY LEMMAS.

LEMMA 2. For any x_{mm} in R_{mm} and any x_{pq} in R_{pq} with $p \neq q$, we have

$$D(x_{mm} + x_{pq}) = D(x_{mm}) + D(x_{pq}).$$

PROOF. Assume m = p = 1 and q = 2.

Consider the sum $D(x_{11}) + D(x_{12})$. Let t_{1n} be an element of R_{1n} . Using Lemm 1, we have $[D(x_{11}) + D(x_{12})]t_{1n} = D(x_{11})t_{1n} = D(x_{11}t_{1n}) - x_{11}D(t_{1n}) = D[(x_{11} + x_{12})t_{1n}] - x_{11}D(t_{1n}) = D[(x_{11} + x_{12})t_{1n}]$ $x_{11}D(t_{1n}) = D(x_{11} + x_{12})t_{1n} + (x_{11} + x_{12})D(t_{1n}) - x_{11}D(t_{1n}) = D(x_{11} + x_{12})t_{1n}$. Thus,

$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]t_{1n} = 0.$$
Fashion for any t in R , we can set the following

In the same fashion, for any t_{2n} in R_{2n} , we can get the following

$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]t_{2n} = 0.$$
 Combining these results, we have
$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]R = 0.$$
 By condition

 (N_1) , we obtain

$$D(x_{11} + x_{12}) = D(x_{11}) + D(x_{12}).$$

In view of the symmetry resulting from condition (M₁) and the implication of condition (M_2) , we can find that the other three cases are easily shown in a similar fashion.

LEMMA 3. D is additive on R_{12} .

PROOF. Let x_{12} and y_{12} be two elements in the subring R_{12} , and consider the sum

 $D(x_{12}) + D(y_{12}).$

(A) For an element t_{1n} in R_{1n} , we have $[D(x_{12}) + D(y_{12})]t_{1n} = D(x_{12} + y_{12})t_{1n}$, since each side is zero by Lemma 1, so

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]t_{1n} = 0.$$

(B) Consider an element t_{2n} in R_{2n} . We have $(x_{12} + y_{12})t_{2n} = (e + x_{12})(t_{2n} + y_{12}t_{2n})$. Thus, $D[(x_{12} + y_{12})t_{2n}] = D(e + x_{12})(t_{2n} + y_{12}t_{2n}) + (e + x_{12})D(t_{2n} + y_{12}t_{2n})$ = $(D(e) + D(x_{12}))(t_{2n} + y_{12}t_{2n}) + (e + x_{12})(D(t_{2n}) + D(y_{12}t_{2n})) = D(x_{12})t_{2n} + x_{12}D(t_{2n})$ + $D(y_{12}t_{2n})$, by Lemmas 1 and 2. Thus, $D((x_{12} + y_{12})t_{2n}) = D(x_{12}t_{2n}) + D(y_{12}t_{2n})$. But $(D(x_{12}) + D(y_{12}))t_{2n} = D(x_{12})t_{2n} + D(y_{12})t_{2n} = D(x_{12}t_{2n}) + D(y_{12}t_{2n}) - (x_{12}+y_{12})D(t_{2n}) = D((x_{12} + y_{12})t_{2n}) - (x_{12} + y_{12})D(t_{2n}) = D(x_{12} + y_{12})t_{2n}$. Hence,

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]t_{2n} = 0.$$

Consequently, from (A) and (B) we have

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]R = 0.$$

By condition (M₁), we have

$$D(x_{12} + y_{12}) = D(x_{12}) + D(y_{12}).$$

LEMMA 4. D is additive on R_{11} .

PROOF. Let \mathbf{x}_{11} and \mathbf{y}_{11} be arbitrary elements in \mathbf{R}_{11} . For an element \mathbf{t}_{12} in \mathbf{R}_{12} , we have $(\mathbf{D}(\mathbf{x}_{11}) + \mathbf{D}(\mathbf{y}_{11}))\mathbf{t}_{12} = \mathbf{D}(\mathbf{x}_{11})\mathbf{t}_{12} + \mathbf{D}(\mathbf{y}_{11})\mathbf{t}_{12} = \mathbf{D}(\mathbf{x}_{11}\mathbf{t}_{12}) + \mathbf{D}(\mathbf{y}_{11}\mathbf{t}_{12}) - (\mathbf{x}_{11}\mathbf{t}_{12}) + \mathbf{D}(\mathbf{y}_{11}\mathbf{t}_{12}) = \mathbf{D}(\mathbf{x}_{11}\mathbf{t}_{12}) - (\mathbf{x}_{11}\mathbf{t}_{12}) + \mathbf{D}(\mathbf{y}_{11}\mathbf{t}_{12}) = \mathbf{D}(\mathbf{x}_{11}\mathbf{t}_{12} + \mathbf{y}_{11}\mathbf{t}_{12}) - (\mathbf{x}_{11}\mathbf{t}_{12}\mathbf{t}_{12}) = \mathbf{D}((\mathbf{x}_{11}\mathbf{t}_{12}\mathbf{t}_{12}) + \mathbf{D}(\mathbf{y}_{11}\mathbf{t}_{12}) = \mathbf{D}(\mathbf{x}_{11}\mathbf{t}_{12}\mathbf{t}_{12}) + \mathbf{D}(\mathbf{y}_{11}\mathbf{t}_{12}\mathbf{t}_{12}) = \mathbf{D}(\mathbf{x}_{11}\mathbf{t}_{12}\mathbf{t}_{12}\mathbf{t}_{12}\mathbf{t}_{12}) + \mathbf{D}(\mathbf{y}_{11}\mathbf{t}_{12}$

$$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]t_{12} = 0.$$

Therefore,

$$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]R_{12} = 0.$$

From Lemma 1, $D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})$ is an element in R_{11} , hence the above result with condition (M₃) give

$$D(x_{11} + y_{11}) = D(x_{11}) + D(y_{11}).$$

LEMMA 5. D is additive on $R_{11} + R_{12} = eR$.

PROOF. Consider the arbitrary elements \mathbf{x}_{11} , \mathbf{y}_{11} in \mathbf{R}_{11} and \mathbf{x}_{12} , \mathbf{y}_{12} in \mathbf{R}_{12} . So, Lemmas 2,3,4 give $\mathbf{D}((\mathbf{x}_{11}+\mathbf{x}_{12})+(\mathbf{y}_{11}+\mathbf{y}_{12}))=\mathbf{D}((\mathbf{x}_{11}+\mathbf{y}_{11})+(\mathbf{x}_{12}+\mathbf{y}_{12}))=\mathbf{D}(\mathbf{x}_{11}+\mathbf{y}_{11})+\mathbf{D}(\mathbf{x}_{12}+\mathbf{y}_{12})=\mathbf{D}(\mathbf{x}_{11}+\mathbf{D}(\mathbf{y}_{11})+\mathbf{D}(\mathbf{y}_{11})+\mathbf{D}(\mathbf{y}_{12})=(\mathbf{D}(\mathbf{x}_{11})+\mathbf{D}(\mathbf{x}_{12}))+(\mathbf{D}(\mathbf{y}_{11})+\mathbf{D}(\mathbf{y}_{12}))=\mathbf{D}(\mathbf{x}_{11}+\mathbf{x}_{12})+\mathbf{D}(\mathbf{y}_{11}+\mathbf{y}_{12})$. Thus D is additive on $\mathbf{R}_{11}+\mathbf{R}_{12}$. This proves the desired result.

MAIN THEOREM.

THEOREM. Let R be a ring containing an idempotent e which satisfies conditions (M_1) , (M_2) and (M_3) . If d is any multiplicative derivation of R, then d is additive.

PROOF. As we mentioned before, and without loss of generality, we can replace d by D. Let x and y be any elements of R. Consider D(x) + D(y). Take an element t in eR = $R_{11} + R_{12}$. Thus, tx and ty are elements of eR. According to Lemma 5, we can obtain t(D(x) + D(y)) = tD(x) + tD(y) = D(tx) + D(ty) - D(t)(x + y) = D(tx + ty) - D(t(x + y))

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+ tD(x + y). Thus, t(D(x) + D(y)) = tD(x + y). Since t is arbitrary in eR, we obtain eR(D(x) + D(y) - D(x + y)) = 0. By condition (M_2) , we get D(x + y) = D(x) + D(y),

which shows that the multiplicative derivation D is additive.

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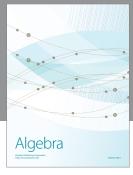
REFERENCES

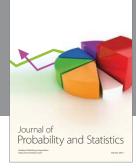
- MARTINDALE III, W.S. When are Multiplicative Mappings Additive ?, Proc. Amer. Math. Soc. 21 (1969), 695-698.
- 2. JACOBSON, N. Structure of Rings, Amer. Math. Soc. Collog. Publ. 37 (1964).











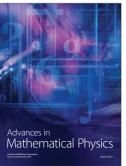






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