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WHEN IS A MULTIPLICATIVE DERIVATION ADDITIVE?

MOHAMAD NAGY DAIF

Department of Mathematics
Faculty of Education
Umm Al-Qura University
Taif, Saudi Arabia

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ABSTRACT. Our main objective in this note is to prove the following. Suppose R is a ring having an idempotent element e ($e \neq 0$, $e \neq 1$) which satisfies:

(M_1) $xR=0$ implies $x=0$.

(M_2) $eRx=0$ implies $x=0$ (and hence $Rx=0$ implies $x=0$).

(M_3) $exeR(1-e)=0$ implies $exe=0$.

If d is any multiplicative derivation of R, then d is additive.

KEY WORDS AND PHRASES. Ring, idempotent element, derivation, Peirce decomposition.
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1. INTRODUCTION.

In [1], Martindale has asked the following question : When is a multiplicative mapping additive ? He answered his question for a multiplicative isomorphism of a ring R under the existence of a family of idempotent elements in R which satisfies some conditions.

Over the past few years, many results concerning derivations of rings have been obtained. In this note, we introduce the definition of a multiplicative derivation of a ring R to be a mapping d of R into R such that $d(ab) = d(a)b + ad(b)$, for all a,b in R. As Martindale did, we raise the following question : When is a multiplicative derivation additive? Fortunately, we can give a full answer for this question using Martindale's conditions when assumed for a single fixed idempotent in R.

In the ring R, let e be an idempotent element so that $e \neq 0$, $e \neq 1$ (R need not have an identity). As in [2], the two-sided Peirce decomposition of R relative to the idempotent e takes the form $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. We will formally set $e_1 = e$ and $e_2 = 1-e$. So letting $R_{mn} = e_m R e_n$; $m,n = 1,2$, we may write $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$. Moreover, an element of the subring R_{mn} will be denoted by x_{mn}.

From the definition of d we note that $d(0) = d(00) = d(0)0 + 0d(0) = 0$. Moreover, we have $d(e) = d(e^2) = d(e)e + ed(e)$. So we can express d(e) as $a_{11} + a_{12} + a_{21} + a_{22}$ and use the value of d(e) to get that $a_{11} = a_{22}$, that is, $a_{11} = 0 = a_{22}$. Consequently, we have $d(e) = a_{12} + a_{21}$.

Now let f be the inner derivation of R determined by the element $a_{12} - a_{21}$, that is $f(x) = [x, a_{12} - a_{21}]$ for all x in R. Therefore, $f(e) = [e, a_{12} - a_{21}] = a_{12} + a_{21}$.

In the sequel, and without loss of generality, we can replace the multiplicative derivation d by the multiplicative derivation $d - f$, which we denote by D, that is, $D = d - f$. This yields $D(e) = 0$. This simplification is of great importance, for, as we will see, the subrings R_{mn} become invariant under the multiplicative derivation D.

2. A KEY LEMMA.

LEMMA 1. $D(R_{mn}) \subset R_{mn}$, $m,n = 1,2$.

PROOF. Let x_{11} be an arbitrary element of R_{11}. Then $D(x_{11}) = D(ex_{11}e) = eD(x_{11})e$ which is an element of R_{11}. For an element x_{12} in R_{12}, we have $D(x_{12}) = D(ex_{12}) = eD(x_{12}) = b_{11} + b_{12}$. But $0 = D(0) = D(x_{12}e) = D(x_{12})e = b_{11}$, hence $D(x_{12}) = b_{12}$ which belongs to R_{12}. In a similar fashion, for an element x_{21} in R_{21}, we have $D(x_{21})$ belongs to R_{21}. Now take an element x_{22} in R_{22}. Write $D(x_{22}) = c_{11}+c_{12}+c_{21}+c_{22}$. So, $0 = D(ex_{22}) = eD(x_{22}) = c_{11} + c_{12}$, whence $c_{11} = c_{12} = 0$. Likewise $c_{21} = 0$, and thus $D(x_{22}) = c_{22}$ which is an element of R_{22}. This proves the lemma.

3. CONDITIONS OF MARTINDALE.

In his note [1], Martindale has given the following conditions which are imposed on a ring R having a family of idempotent elements $\{e_i : i \in I\}$.

(1) $xR = 0$ implies $x = 0$.

(2) If $e_iRx = 0$ for each i in I, then $x = 0$ (and hence $Rx = 0$ implies $x = 0$).

(3) For each i in I, $e_ixe_iR(1-e_i) = 0$ implies $e_ixe_i = 0$.

In our note, we find it appropriate to simply dispense with conditions (1), (2) and (3) altogether and instead substitute the following conditions :

(M_1) $xR = 0$ implies $x = 0$.

(M_2) $eRx = 0$ implies $x = 0$ (and hence $Rx = 0$ implies $x = 0$).

(M_3) $exeR(1-e) = 0$ implies $exe = 0$.

4. AUXILIARY LEMMAS.

LEMMA 2. For any x_{mm} in R_{mm} and any x_{pq} in R_{pq} with $p \neq q$, we have

$$D(x_{mm} + x_{pq}) = D(x_{mm}) + D(x_{pq}).$$

PROOF. Assume $m = p = 1$ and $q = 2$.

Consider the sum $D(x_{11}) + D(x_{12})$. Let t_{1n} be an element of R_{1n}. Using Lemm 1, we have $[D(x_{11}) + D(x_{12})]t_{1n} = D(x_{11})t_{1n} = D(x_{11}t_{1n}) - x_{11}D(t_{1n}) = D[(x_{11} + x_{12})t_{1n}] - x_{11}D(t_{1n}) = D(x_{11} + x_{12})t_{1n} + (x_{11} + x_{12})D(t_{1n}) - x_{11}D(t_{1n}) = D(x_{11} + x_{12})t_{1n}$. Thus,

$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]t_{1n} = 0.$$

In the same fashion, for any t_{2n} in R_{2n}, we can get the following

$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]t_{2n} = 0.$$

Combining these results, we have $[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]R = 0$. By condition ($M_1$), we obtain

$$D(x_{11} + x_{12}) = D(x_{11}) + D(x_{12}).$$

In view of the symmetry resulting from condition (M_1) and the implication of condition (M_2), we can find that the other three cases are easily shown in a similar fashion.

LEMMA 3. D is additive on R_{12}.

PROOF. Let x_{12} and y_{12} be two elements in the subring R_{12}, and consider the sum

$D(x_{12}) + D(y_{12})$.

(A) For an element t_{1n} in R_{1n}, we have $[D(x_{12}) + D(y_{12})]t_{1n} = D(x_{12} + y_{12})t_{1n}$, since each side is zero by Lemma 1, so

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]t_{1n} = 0.$$

(B) Consider an element t_{2n} in R_{2n}. We have $(x_{12} + y_{12})t_{2n} = (e + x_{12})(t_{2n} + y_{12}t_{2n})$. Thus, $D[(x_{12} + y_{12})t_{2n}] = D(e + x_{12})(t_{2n} + y_{12}t_{2n}) + (e + x_{12})D(t_{2n}+ y_{12}t_{2n}) =(D(e) + D(x_{12}))(t_{2n}+ y_{12}t_{2n}) + (e + x_{12})(D(t_{2n}) + D(y_{12}t_{2n})) = D(x_{12})t_{2n} + x_{12}D(t_{2n}) + D(y_{12}t_{2n})$, by Lemmas 1 and 2. Thus, $D((x_{12} + y_{12})t_{2n}) = D(x_{12}t_{2n}) + D(y_{12}t_{2n})$. But $(D(x_{12}) + D(y_{12}))t_{2n} = D(x_{12})t_{2n} + D(y_{12})t_{2n} = D(x_{12}t_{2n}) + D(y_{12}t_{2n})-(x_{12}+y_{12})D(t_{2n})= D((x_{12} + y_{12})t_{2n}) - (x_{12} + y_{12})D(t_{2n}) = D(x_{12} + y_{12})t_{2n}$. Hence,

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]t_{2n} = 0.$$

Consequently, from (A) and (B) we have

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]R = 0.$$

By condition (M_1), we have

$$D(x_{12} + y_{12}) = D(x_{12}) + D(y_{12}).$$

LEMMA 4. D is additive on R_{11}.

PROOF. Let x_{11} and y_{11} be arbitrary elements in R_{11}. For an element t_{12} in R_{12}, we have $(D(x_{11}) + D(y_{11}))t_{12} = D(x_{11})t_{12} + D(y_{11})t_{12} = D(x_{11}t_{12}) + D(y_{11}t_{12}) - (x_{11}+ y_{11})D(t_{12})$. But $x_{11}t_{12}$ and $y_{11}t_{12}$ are in R_{12}, and D is additive on R_{12} by Lemma 3, hence $(D(x_{11}) + D(y_{11}))t_{12} = D(x_{11}t_{12} + y_{11}t_{12}) - (x_{11} + y_{11})D(t_{12}) = D((x_{11}+y_{11})t_{12}) - (x_{11} + y_{11})D(t_{12}) = D(x_{11} + y_{11})t_{12}$. thus we have

$$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]t_{12} = 0.$$

Therefore,

$$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]R_{12} = 0.$$

From Lemma 1, $D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})$ is an element in R_{11}, hence the above result with condition (M_3) give

$$D(x_{11} + y_{11}) = D(x_{11}) + D(y_{11}).$$

LEMMA 5. D is additive on $R_{11} + R_{12} = eR$.

PROOF. Consider the arbitrary elements x_{11}, y_{11} in R_{11} and x_{12}, y_{12} in R_{12}. So, Lemmas 2,3,4 give $D((x_{11} + x_{12}) + (y_{11} + y_{12})) = D((x_{11} + y_{11}) + (x_{12} + y_{12}))=D(x_{11}+ y_{11}) + D(x_{12} + y_{12}) = D(x_{11}) + D(y_{11}) + D(x_{12}) + D(y_{12}) = (D(x_{11}) + D(x_{12})) + (D(y_{11}) + D(y_{12})) = D(x_{11} + x_{12}) + D(y_{11} + y_{12})$. Thus D is additive on $R_{11} + R_{12}$. This proves the desired result.

5. MAIN THEOREM.

THEOREM. Let R be a ring containing an idempotent e which satisfies conditions (M_1), (M_2) and (M_3). If d is any multiplicative derivation of R, then d is additive.

PROOF. As we mentioned before, and without loss of generality, we can replace d by D. Let x and y be any elements of R. Consider $D(x) + D(y)$. Take an element t in $eR = R_{11} + R_{12}$. Thus, tx and ty are elements of eR. According to Lemma 5, we can obtain $t(D(x) + D(y)) = tD(x) + tD(y) = D(tx) + D(ty) - D(t)(x +y) = D(tx + ty)- D(t(x + y))$

+ tD(x + y). Thus, t(D(x) + D(y)) = tD(x + y). Since t is arbitrary in eR, we obtain eR(D(x) + D(y) - D(x + y)) = 0. By condition (M_2), we get

$$D(x + y) = D(x) + D(y),$$

which shows that the multiplicative derivation D is additive.

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