When the Cartesian Product of Two Directed Cycles is Hyperhamiltonian

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ABSTRACT

We say a digraph G is hyperhamiltonian if there is a spanning closed walk in G which passes through one vertex exactly twice and all others exactly once. We show the cartesian product $Z_a \times Z_b$ of two directed cycles is hyperhamiltonian if and only if there are positive integers m and n with $ma + nb = ab + 1$ and $gcd(m, n) = 1$ or 2. We obtain a similar result for the vertex-deleted subdigraphs of $Z_a \times Z_b$.

S. Curran [4, Theorem 4.3] observed that by using the theory of torus knots it is easy to prove that the cartesian product $Z_a \times Z_b$ of two directed cycles is hamiltonian if and only if there are positive integers m and n with $ma + nb = ab$ and $gcd(m, n) = 1$. Using Curran's ideas, Penn and Witte [2] proved that $Z_a \times Z_b$ is hypohamiltonian if and only if there are positive integers m and n with $ma + nb = ab - 1$ and $gcd(m, n) = 1$. (A digraph is said to be *hypohamiltonian* if it is not hamiltonian but every vertex-deleted subdigraph is hamiltonian.) Motivated by these results we define a digraph to be *hyperhamiltonian* if there is a spanning closed walk which passes through one vertex exactly twice and all others exactly once and determine when $Z_a \times Z_b$ is hyperhamiltonian and when a vertex-deleted subdigraph of $Z_a \times Z_b$ is hyperhamiltonian. We assume the reader is familiar with [2] and with the background on torus knots given in [1, Section 4].

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Theorem 1. The cartesian product $Z_a \times Z_b$ of two directed cycles is hyperhamiltonian if and only if there are positive integers m and n with $ma + nb = ab + 1$ and $gcd(m, n) = 1$ or 2.

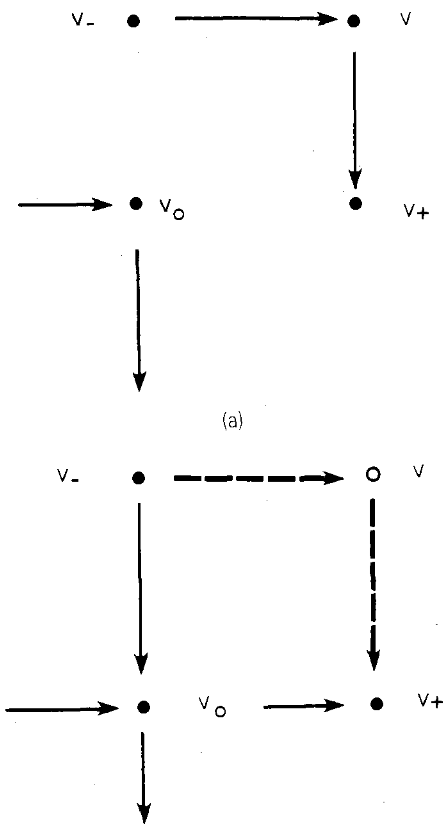
Proof. Let C be a hyperhamiltonian closed walk in $Z_a \times Z_b$. Then C decomposes uniquely into a pair of edge-disjoint circuits C_1 and C_2 with a common vertex. Let (m_1, n_1) and (m_2, n_2) be the knot classes of C_1 and C_2, respectively. Then $(m_1 + m_2)a + (n_1 + n_2)b = ab + 1$. If C_1 crosses C_2, then the algebraic intersection number is ± 1, so $\pm 1 = m_1 n_2 - n_1 m_2 = (m_1 + m_2)n_2 - (n_1 + n_2)m_2$ and it follows that $gcd(m_1 + m_2, n_1 + n_2) = 1$. If C_1 does not cross C_2, then the intersection number is $m_1 n_2 - n_1 m_2 = 0$. From this and the fact that m_1, n_1, m_2, n_2 are non-negative and $gcd(m_1, n_1) = 1 = gcd(m_2, n_2)$, we obtain $m_1 = m_2$ and $n_1 = n_2$. Thus $2m_1 a + 2n_1 b = ab + 1$ and $gcd(2m_1, 2n_1) = 2$.

Now assume there are positive integers m and n such that $ma + nb = ab + 1$ and $gcd(m, n) = 1$ or 2. Clearly a and b are relatively prime, so the group $Z_a \times Z_b$ is generated by $(-1, 1)$. Let H be the spanning subdigraph of $Z_a \times Z_b$ in which a vertex $d(-1, 1)$ travels by $(0, 1)$ if $0 \leq d < nb$ and it travels by $(1, 0)$ if $nb \leq d \leq ab$. Since the in-degree and the out-degree of each vertex are equal, H is the union of edge disjoint circuits $C_1, C_2, \ldots, C_N$. All vertices of H have out-degree 1, except $(0, 0)$ which has out-degree 2, so, by renumbering if necessary, we may assume that C_1 and C_2 are the only circuits in the decomposition of H which intersect. We wish to show that C_1 and C_2 are, in fact, the only circuits in the decomposition of H, so that H is the union of two circuits with a single point of intersection, for then H can obviously be realized as a closed walk with one repeated vertex. To this end, suppose $N \geq 3$. Then, for $i \geq 3$, C_i is disjoint from C_1 and C_2, so $\text{knot}(C_1) = \text{knot}(C_i) = \text{knot}(C_2)$. Set $(r, s) = N \cdot \text{knot}(C_1) = \Sigma_{i=1}^{N} \text{knot}(C_i)$. Then $ra + sb = ab + 1$. Since a and b are relatively prime, only one pair of positive integers u and v may satisfy $ua + vb = ab + 1$. Thus $(r, s) = (m, n)$. Hence $gcd(m, n) = gcd(r, s) = N > 2$, a contradiction. ∎

We next consider vertex-deleted subdigraphs $Z_a \times Z_b - \{v\}$. Since $Z_a \times Z_b$ is vertex-transitive, the particular vertex which is deleted is unimportant.

Theorem 2. Let $v \in Z_a \times Z_b$. Then $Z_a \times Z_b - \{v\}$ is hyperhamiltonian if and only if there are positive integers m and n with $ma + nb = ab$ and $gcd(m, n) = 1$ or 2.

Proof. The proof of necessity is similar to that in Theorem 1. To prove sufficiency, let H_0 be a spanning subdigraph of $Z_a \times Z_b$ with $\text{knot}(H_0) = (m, n)$. (Namely: Put $m_0 = ma/lcm(a, b)$ and $n_0 = nb/lcm(a, b)$. Let m_0 cosets of $(-1, 1)$ travel by $(1, 0)$ in H_0; and let the other n_0 cosets travel by $(0, 1)$.) Since m_0 and n_0 are positive, there must be some vertex w, such that w travels by $(0, 1)$, and $w - (1, 0)$ travels by $(1, 0)$. Replacing H_0 by a translate if necessary, we assume $w = v$. Set $v_- = v - (1, 0)$, $v_+ = v + (0, 1)$, and $v_0 = v + (-1, 1)$. In H_0, there is an arc from v_- to v, and from v to v_+. Create a new digraph H by removing the vertex v from H_0 (and removing the two arcs inci-



(b)

FIGURE 1. (a) The digraph H_0. (b) The digraph H. Dotted arcs and the vertex v have been deleted.

dent with v), and by inserting two arcs: from v_- to v_0, and from v_0 to v_+ (see Figure 1). Then H is a spanning subdigraph of $Z_a \times Z_b - \{v\}$. An argument similar to that in Theorem 1 shows H is the union of two circuits with a single point of intersection, namely v_0. Thus $Z_a \times Z_b - \{v\}$ is hyperhamiltonian. ∎

We remark that $Z_5 \times Z_7$ is a cartesian product of two cycles which is neither hamiltonian, hypohamiltonian nor hyperhamiltonian. Likewise, $Z_3 \times Z_7 - \{(0,0)\}$ is a vertex-deleted subdigraph of a cartesian product which is not hamiltonian, hypohamiltonian, or hyperhamiltonian.

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