When the Identity Theorem "seems" to fail

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Abstract

The Identity Theorem states that an analytic function (real or complex) on a connected domain is uniquely determined by its values on a sequence of distinct points that converge to a point of its domain. This result is not true in general in the real setting if we relax the analyticity hypothesis on the function to infinitely many times differentiability. In fact, we construct an algebra of functions $\mathcal{A}$ enjoying the following properties: (i) $\mathcal{A}$ is uncountably infinitely generated (that is, the cardinality of a minimal system of generators of $\mathcal{A}$ is $\mathfrak{c}$), (ii) every nonzero element of $\mathcal{A}$ is nowhere analytic, (iii) $\mathcal{A} \subset \mathcal{C}^\infty(\mathbb{R})$, (iv) every element of $\mathcal{A}$ has infinitely many zeroes in $\mathbb{R}$, and (v) for every $f \in \mathcal{A}$ and $n \in \mathbb{N}$, $f^{(n)}$ (the n-th derivative of f) enjoys the same properties as the elements in $\mathcal{A} \setminus \{0\}$. This construction complements those made by Cater and Kim & Kwon, and published in the *American Mathematical Monthly* in 1984 and 2000, respectively.

1 The Identity Theorem. Examples and Counterexamples

In Complex Analysis, the *Identity Theorem* (see, e.g. [6, 8]) states that, if two holomorphic functions f and g defined on a domain (a connected open subset) $D \subset \mathbb{C}$ agree on a set A which has an accumulation point in D, then $f = g$ all over D. Of course, one surprising consequence of this fact is that any analytic function is completely determined by its values on any neighborhood V in D, no matter *how small* V is.

In a totally different framework, a real function is said to be real analytic if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point. Of course, the *Identity Theorem* also holds for real analytic functions, but one needs to be careful when applying it, since (in $\mathbb{R}$) one can have $\mathcal{C}^\infty$ functions that are not analytic, as the following *well-known* function shows (see Figure 2(a)):

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

As some simple calculations would entail, the above function only agrees with its Taylor series expansion at $x = 0$. As a standard application of the *Baire*

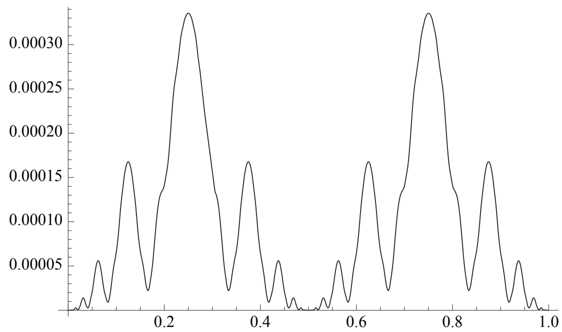


Figure 1: The function $\psi(x)$ provided by Kim and Kwon in [17].

category theorem one can obtain that most infinitely differentiable functions are nowhere analytic (see, e.g., [9]). In particular, the set of all continuous but nowhere differentiable functions on $\mathbb{R}$ is of the second category in $\mathcal{C}(\mathbb{R})$ (for a recent account of this class of continuous nowhere differentiable functions, we refer the interested reader to the recent work [16]).

In this series of examples, it is important to mention the work of Cater [7] who showed that if $F \subset \mathcal{C}^\infty[0,1]$ consists of the functions not expressible as a power series on any nondegenerate interval in $[0,1]$, then there exists a vector space in $F \cup \{0\}$ of dimension $\mathfrak{c}$ (the continuum). Later, García, Palmberg, and the fourth named author [12] showed that there actually exists an uncountably infinitely generated algebra, every nonzero element of which is $\mathcal{C}^\infty$ and nonanalytic. Recently Bernal-González [5] showed (among other results) that, although the set of nowhere analytic functions on $[0,1]$ is clearly not a linear space, there exists a dense linear submanifold, every nonzero element of which belongs to the space of $\mathcal{C}^\infty$-smooth functions.

Also Kim and Kwon [14, 17] constructed examples of nowhere analytic increasing smooth functions in $\mathbb{R}$. An example of this kind is, for instance, the function $\int \psi(x)dx$, where $\psi : \mathbb{R} \to \mathbb{R}$ (see Figure 1) is given by

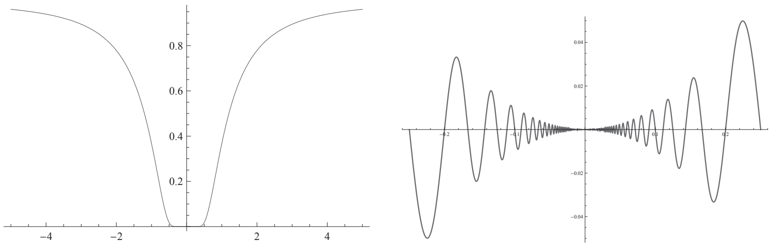
$$\psi(x) = \sum_{j=1}^{\infty} \frac{1}{j!} \phi\left(2^j x - [2^j x]\right), \tag{1.1}$$

where $[\cdot]$ denotes the *greatest integer function* and $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(x) = \begin{cases} e^{\frac{-1}{x^2}} \cdot e^{\frac{-1}{(x-1)^2}} & \text{if } 0 < x < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

We would also like to refer the reader to the very interesting work of Bastin et al. [4], where the authors study the genericity of functions which are nowhere analytic in a measure-theoretic sense.

Of course, if an entire function has infinitely many zeroes with an accumulation point, then (by the Identity Theorem) it must be the zero function (see,



(a) Graph of f. (b) Graph of g.

Figure 2: Graphs of the functions f and g.

also, [6], for a very accessible work on uniqueness theorems for analytic functions). For real functions this does not hold. For instance, the differentiable function $g : \mathbb{R} \to \mathbb{R}$ given by (see Figure 2(b))

$$g(x) = \begin{cases} x^2 \sin(\frac{\pi}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

has the infinite set $Z = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$ as its set of zeroes, Z has an accumulation point (0) but, obviously, $g \neq 0$.

Of course, and after all the previous battery of examples, we can't help but notice how rich the structure of certain sets of functions can get. Take a function with some special or unexpected property (for example, any of the above). Coming up with a concrete example of such a function can be difficult. Actually, it may seem so difficult that if one succeeds, one could then think that there cannot be too many functions of that kind. Moreover, probably one cannot find infinite dimensional vector spaces or infinitely generated algebras of such functions. However, and as we just saw, this is exactly what has occurred.

The search for large algebraic structures of functions with *pathological properties* has lately become somewhat of a new trend in mathematics. In fact, even new mathematical words have been introduced. We say that a set of functions $M \subset \mathcal{C}(\mathbb{R})$ is *lineable* if there exists an infinite dimensional linear space $Y \subset M \cup \{0\}$ (see [1, 18], or [3, 11, 15] for very recent references). Analogously, a set of functions $M \subset \mathcal{C}(\mathbb{R})$ is said to be *algebrable* if there exists an infinitely generated algebra of functions $Y \subset M \cup \{0\}$ (see [2]).

After all of the above, the following question comes naturally.

> *Are there nonzero real valued differentiable functions with infinitely many zeroes, possessing derivatives of all orders, and also nowhere analytic? And, how big is this set of functions? What algebraic/linear structure does this set possess?*

In this note, we shall provide answers to the above questions. Moreover, we shall construct an algebra $\mathcal{A}$ of real valued functions enjoying, simultaneously, each of the following properties.

(i) $\mathcal{A}$ is uncountably infinitely generated. That is, the cardinality of a minimal system of generators of $\mathcal{A}$ is $\mathfrak{c}$.

(ii) Every nonzero element of $\mathcal{A}$ is nowhere analytic.

(iii) $\mathcal{A} \subset \mathcal{C}^\infty(\mathbb{R})$.

(iv) Every element of $\mathcal{A}$ has infinitely many zeroes in $\mathbb{R}$.

(v) For every $f \in \mathcal{A}$ and $n \in \mathbb{N}$, $f^{(n)}$ (the n-th derivative of f) enjoys the same properties as the elements in $\mathcal{A} \setminus \{0\}$.

Functions with infinitely many zeros in a closed finite interval are known as *annulling functions* (see [10, Definition 2.1]). Very recently Enflo, Gurariy, and the fourth named author proved that for every infinite dimensional closed subspace X of $\mathcal{C}[0,1]$, the subset of its annulling functions contains an infinite dimensional closed subspace [10, Corollary 3.8]. The question of the existence of an algebra of such functions inside of $\mathcal{C}[0,1]$ is what shall also be solved in this note.

2 The algebra of functions

Let $\mathcal{H}$ be a Hamel basis of $\mathbb{R}$. That is, a basis of the real numbers $\mathbb{R}$, considered as a $\mathbb{Q}$-vector space. Furthermore, without loss of generality, we can assume that $\mathcal{H}$ consists only of positive real numbers.

Let us now consider the minimum algebra of $\mathcal{C}(\mathbb{R})$ that contains the family of functions $\{\rho_\alpha\}_{\alpha\in\mathcal{H}}$ with $\rho_\alpha : \mathbb{R} \to \mathbb{R}$ defined as follows.

$$\rho_\alpha(x) = \sum_{j=1}^{\infty} \frac{\lambda_j(x)}{\mu_j} \phi\left(2^j x - [2^j x]\right) \alpha^j.$$

As before, $[\cdot]$ denotes the *greatest integer function*, for $j \in \mathbb{N}$

$$\lambda_j(x) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{2^j}, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu_j = s_k!$ if $s_{k-1} < j \leq s_k$ where s_k is the sum of the first k positive integers.

These ρ_α's are quite similar to the function ψ introduced by Kim and Kwon in [17]. A sketch of what these ρ_α's look like can be seen in Figure 3.

Remark 2.1. Let us recall that if $\{f_j(x)\}_{j=1}^\infty$ is a sequence of continuously differentiable functions on $\mathbb{R}$ such that $\sum_{j=1}^\infty f_j(x)$ converges pointwise to $f(x)$ and $\sum_{j=1}^\infty f_j'(x)$ converges uniformly on $\mathbb{R}$, then $f(x)$ is differentiable and $f'(x) = \sum_{j=1}^\infty f_j'(x)$. Also, if there is a sequence $\{M_j\}_{j\in\mathbb{N}}$ of nonnegative real numbers such that $|f_j(x)| \leq M_j$ for all $x \in \mathbb{R}$ and $\sum_{j=1}^\infty M_j < \infty$, then $\sum_{j=1}^\infty f_j(x)$ converges uniformly on $\mathbb{R}$ (this is Weierstrass' M-test).

The following proposition shall be crucial in the proof of our main theorem.

Proposition 2.2. *All functions $\{\rho_\alpha\}_{\alpha\in\mathcal{H}}$ are $\mathcal{C}^\infty$ and nowhere analytic. Moreover, all the derivatives and the function itself vanish at the points*

$$\left\{\frac{1}{2^j} : j \in \mathbb{N} \cup \{0\}\right\}.$$

Proof. The function $\phi(x)$ is smooth everywhere and analytic except at $x = 0$ and $x = 1$. Moreover, it is flat at both of these points; that is, all its derivatives and the function $\phi(x)$ itself evaluated at those points are also 0. If we replace x by $2^j x - [2^j x]$, then the behavior of $\phi(x)$ over the interval $[0, 1]$ is replicated by $\phi(2^j x - [2^j x])$ on any dyadic interval of the form $\left[\frac{m-1}{2^j}, \frac{m}{2^j}\right]$ for all $m \in \mathbb{Z}$.

Let $\phi_j(x) := \frac{\lambda_j(x)}{\mu_j}\phi(2^j x - [2^j x])\alpha^j$. Notice that, due to the flatness of ϕ at 0 and 1, ϕ_j is smooth everywhere (and analytic everywhere but at $x = \frac{m}{2^l}$ for all $m \in \mathbb{Z}$ and $0 \leq l \leq j$). Now, $\sum_{j=1}^\infty \phi_j^{(k)}$ is uniformly bounded for all $k \in \mathbb{N}$, from which (by Remark 2.1) it follows that ρ_α is smooth for all α's considered.

In order to prove that ρ_α is nowhere analytic, we follow the same procedure used by Kim and Kwon in [17, Theorem 1]. Assume indeed that ρ_α is analytic at a point, so it is also analytic on an interval. Since the dyadics are dense, ρ_α is analytic at some $x_0 = \frac{m}{2^n}$, with m odd. If $1 \leq j \leq n-1$, then ϕ_j is analytic at x_0 and hence

$$\widehat{\rho}_\alpha(x) := \sum_{j=n}^\infty \phi_j(x)$$

is also analytic at x_0. However, $\widehat{\rho}_\alpha^{(k)}(x_0) = 0$ for all integers $k \geq 0$. This contradicts the fact that $\widehat{\rho}_\alpha(x)$ is positive in some punctured neighborhood of x_0. □

Now, it is time to state and prove the result that complements those from [7, 17].

Theorem 2.3. *If $\mathcal{A}$ is the algebra generated by $\{\rho_\alpha\}_{\alpha\in\mathcal{H}}$, then*

(i) $\mathcal{A}$ is uncountably infinitely generated,

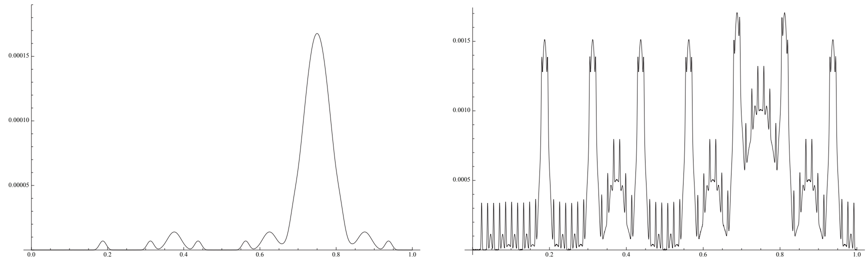
(ii) every non-zero element of $\mathcal{A}$ is nowhere analytic,

(iii) $\mathcal{A} \subset \mathcal{C}^\infty(\mathbb{R})$,

(iv) every non-zero element of $\mathcal{A}$ is an annulling function on $\mathbb{R}$, and

(v) for every $f \in \mathcal{A}$ and $n \in \mathbb{N}$, $f^{(n)}$ enjoys the same properties as the elements in $\mathcal{A} \setminus \{0\}$.

Proof. Any element $h \in \mathcal{A}$ can be written as $h(x) = \sum_{k=1}^n \beta_k \rho_{\alpha_k}^{m_k}(x)$ with $\alpha_k \in \mathcal{H}$, and $m_k \in \mathbb{N}$, for $k = 1, \ldots, n$. Let us suppose that $h \equiv 0$ and we evaluate $h(x)$ at the points $x_j = \frac{3}{2^{j+1}}$, for $j = s_{n-1} + 1, \ldots, s_n$. Evaluating the function $\rho_{\alpha_k}^{m_k}$ at the points x_j, the infinite sum is reduced to



Graph of $\rho_{1/2}$. Graph of ρ_π.

Figure 3: Graphs of ρ_α for some choices of α.

$$\rho_{\alpha_k}^{m_k}\left(\frac{3}{2^{j+1}}\right) = \left(\frac{\phi(1/2)\alpha_k^j}{s_n!}\right)^{m_k} = \frac{e^{-8m_k}\alpha_k^{jm_k}}{(s_n!)^{m_k}}.$$

Therefore, if we consider the system of equations obtained from the conditions $h\left(\frac{3}{2^{j+1}}\right) = 0$ for $j = s_{n-1}+1, \ldots, s_n$, we obtain the following:

$$\begin{pmatrix} \frac{e^{-8m_1}\alpha_1^{m_1(s_{n-1}+1)}}{s_n!^{m_1}} & \cdots & \frac{e^{-8m_n}\alpha_n^{m_n(s_{n-1}+1)}}{s_n!^{m_n}} \\ \vdots & \ddots & \vdots \\ \frac{e^{-8m_1}\alpha_1^{m_1 s_n}}{s_n!^{m_1}} & \cdots & \frac{e^{-8m_n}\alpha_n^{m_n s_n}}{s_n!^{m_n}} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

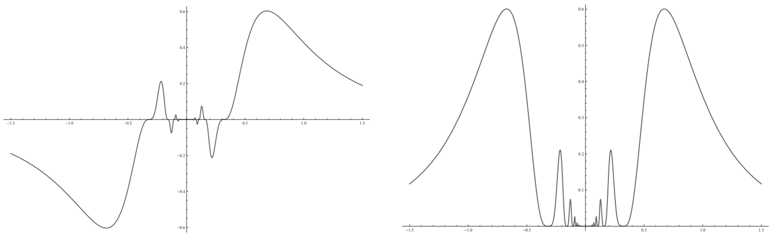
If (for all $j = 1, \ldots, n$) we multiply the jth-column of the above matrix by $\frac{(s_n!^{m_j})e^{8m_j}}{\alpha_j^{m_j(s_{n-1}+1}}$, we have that the former system is equivalent to a system with the following matrix,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix},$$

which is non-singular since it is a Vandermonde-type matrix (and also because the α_k's are different elements of the Hamel basis $\mathcal{H}$). Therefore, $\beta_i = 0$ for $i = 0, \ldots, n$. The rest of the statements follow directly from Proposition 2.2. □

Remark 2.4. Although the previous result gives us what we were aiming for, we would also like to provide a less sophisticated way that might be more accessible at an undergraduate level. Most Math majors are familiar with examples of $\mathcal{C}^\infty$ nonanalytic functions, such as

$$F(x) = \begin{cases} e^{-\frac{1}{|x|}}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



$\lambda = 1/3$ and $m = 3$ $\qquad$ $\lambda = 1/3$ and $m = 4$

Figure 4: Graphs of $f_{\lambda,m}$ for some choices of m and λ.

It can be very easily verified that $F \in \mathcal{C}^\infty(\mathbb{R})$ and its derivatives (of all orders) at 0 are all null. We can slightly modify F and define a family of functions $\{f_{\lambda,m}\}_{\lambda>0,m\in\mathbb{N}}$ with $f_{\lambda,m} : \mathbb{R} \to \mathbb{R}$ defined by (see Figure 4)

$$f_{\lambda,m}(x) = \begin{cases} e^{-\frac{\lambda}{|x|}} \sin^m\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We next consider the minimum algebra of $\mathcal{C}(\mathbb{R})$, call it $\mathcal{B}$, generated by the previous family. We shall spare the details of the calculations as an exercise to the interested reader. Most of the details involve intricate (but simple) calculations, although we also refer the reader to [7, p. 620] for other techniques that are helpful to this purpose. In any case, let us give a sketch of the steps that one can follow to show that the functions in $\mathcal{B}$ enjoy "almost" all the required properties. We shall also provide hints on each step to help the reader throughout the proof.

Let $\mathcal{B}$ denote the algebra of $\mathcal{C}(\mathbb{R})$ generated by the family $\{f_{\lambda,m}\}_{\lambda>0,m\in\mathbb{N}}$. Then, we have the following.

1. All functions $\{f_{\lambda,m}\}_{\lambda>0,m\in\mathbb{N}}$ are in $\mathcal{C}^\infty(\mathbb{R})$ and their derivatives (of all orders) at 0 are all null. [Hint: This can be seen by restricting our attention to the interval $[0,\infty)$ (due to the symmetry) and by proving that the k-th derivative of $f_{\lambda,m}$ is of the form

$$f^{(k)}_{\lambda,m}(x) = \begin{cases} \frac{e^{-\frac{\lambda}{x}}}{x^{2k}} \sin^{r(k)}\left(\frac{1}{x}\right) P^{\lambda,m}_k\left(x, \sin\left(\frac{1}{x}\right), \cos\left(\frac{1}{x}\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $r : \mathbb{N} \to \mathbb{N}$ and the $P^{\lambda,m}_k$'s are polynomials that can be inductively obtained.]

2. $\mathcal{B}$ is uncountably infinitely generated. [Hint: Notice that any element $h \in \mathcal{B}$ can be written as

$$h(x) = \alpha_1 e^{-\frac{\lambda_1}{|x|}} + \cdots + \alpha_p e^{-\frac{\lambda_p}{|x|}} + \alpha_{p+1} \sin^{k_1} x^{-1} e^{-\frac{\lambda_{p+1}}{|x|}} + \cdots + \alpha_{p+q} \sin^{k_q} x^{-1} e^{-\frac{\lambda_{p+q}}{|x|}},$$

and, next, assume that $h \equiv 0$ and show that all the α_k's are null.]

3. Every nonzero element of $\mathcal{B}$ is nonanalytic. [This is a consequence of (1).] If $h \in \mathcal{B}$, then its derivatives at zero are always zero. Hence, h cannot be analytic. Of course in Theorem 2.3 we obtained an algebra of nowhere analytic functions whereas here we only obtain nonanalyticity (at $x = 0$).

4. $\mathcal{B} \subset \mathcal{C}^\infty(\mathbb{R})$ and every element of $\mathcal{B}$ is an annulling function on $\mathbb{R}$.

5. For every $f \in \mathcal{B}$ and $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{B}$. [Hint: This follows from an iterative application of Rolle's Theorem to the derivatives of f. It can be seen that $f^{(k)}$ also vanishes at an infinite set of points and that $f^{(k)} \in \mathcal{B}$ for every $k \in \mathbb{N}$.]

Of course, this previous technique, although more accessible to an undergraduate student, only provides nonanalyticity, instead of an algebra of nowhere analytic functions. Clearly, as we showed earlier, all the other remaining properties we looked for are fulfilled by the functions from both algebras.

Remark 2.5. We would like to finish this note by mentioning that the result in Theorem 2.3 is the best possible in the following sense.

(a) The dimension of $\mathcal{A}$ (as a vector space) is the *largest* possible, $\mathfrak{c}$, since the dimension of the space of continuous functions is also $\mathfrak{c}$. Also, the cardinality of the system of generators of $\mathcal{A}$ is the *biggest* possible for the same reason.

(b) If we restrict ourselves to the interval $[0, 1]$ (or to any compact interval for that matter), the corresponding algebra $\mathcal{A}$ cannot be constructed being close in $\mathcal{C}[0, 1]$. This is due to the fact that Gurariy showed in [13] that the set of differentiable functions on $[0, 1]$ does not contain an infinite dimensional closed subspace.

To summarize, there is no way to improve the "size" of $\mathcal{A}$ or its topological structure by making it closed.

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