Which Cayley graphs are integral?

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Abstract

Let G be a non-trivial group, $S \subseteq G \setminus \{1\}$ and $S = S^{-1} := \{s^{-1} \mid s \in S\}$. The Cayley graph of G denoted by $\Gamma(S : G)$ is a graph with vertex set G and two vertices a and b are adjacent if $ab^{-1} \in S$. A graph is called integral, if its adjacency eigenvalues are integers. In this paper we determine all connected cubic integral Cayley graphs. We also introduce some infinite families of connected integral Cayley graphs.

1 Introduction and Results

We say that a graph is integral if all the eigenvalues of its adjacency matrix are integers. The notion of integral graphs was first introduced by Harary and Schwenk in 1974 [12].

In 1976 Bussemaker and Cvetković [7], proved that there are exactly 13 connected cubic integral graphs. The same result was independently proved by Schwenk [16] who unlike the effort in [7] avoids the use of computer search to examine all the possibilities. However the work of Schwenk [16] was inspired and stimulated by Cvetković attempt [9] to find the connected cubic integral graphs where he had displayed twelve such graphs, and had restricted the remaining possibilities to ninety-five potential spectra, Schwenk has produced a complete and self-contained solution.

It is known that the size of a connected k-regular graph with diameter d is bounded above by $\frac{k(k-1)^d-2}{k-2}$ (see, for example [10]). In [9], it is noted that if we know the graph is integral then $d \leqslant 2k$ because there are at most $2k+1$ distinct eigenvalues. Consequently, the upper bound of the size of a connected k-regular integral graph is

$$n \leqslant \frac{k(k-1)^{2k}-2}{k-2}.$$

Using Brendan McKay's program **geng** for generating graphs, nowadays it is easy to see that there are exactly 263 connected integral graphs on up to 11 vertices (see [3, 4]). In 2009 Alon et al. [1] show that the total number of adjacency matrices of integral graphs with n vertices is less than or equal to $2^{\frac{n(n-1)}{2}-\frac{n}{400}}$ for a sufficiently large n. For the background and some known results about integral graphs, we refer the reader to the survey [5].

The problem of characterizing integral graphs seems to be very difficult and so it is wise to restrict ourselves to certain families of graphs. Here we are interested to study Cayley graphs. Let G be a non-trivial group with the identity element 1, $S \subseteq G \setminus \{1\}$ and $S = S^{-1} := \{s^{-1} | s \in S\}$. The Cayley graph of G denoted by $\Gamma(S : G)$ is the graph with vertex set G and two vertices a and b are adjacent if $ab^{-1} \in S$. If S generates G then $\Gamma(S : G)$ is connected. A Cayley graph is simple and vertex transitive.
We denote the symmetric group and the alternating group on n letters by S_n and A_n, respectively. Also C_m and D_{2n} are used for the cyclic group of order m and dihedral group of order $2n$ ($n > 2$).
The main question that we are concerned here is the following:
Which Cayley graphs are integral?
It is clear that if $S = G \setminus \{1\}$, then $\Gamma(S : G)$ is the complete graph with $|G|$ vertices and so it is integral. Klotz and Sander [14] showed that all nonzero eigenvalues of $\Gamma(U_n : \mathbb{Z}_n)$ are integers dividing the value $\varphi(n)$ of the Euler totient function, where $\mathbb{Z}_n$ is the cyclic group of order n and U_n is the subset of all elements of $\mathbb{Z}_n$ of order n. W. So [17] characterize integral graphs among circulant graphs. By using a result of Babai [2] which presents the spectrum of a Cayley graph in terms of irreducible characters of the underlying group, we give some infinite families of integral Cayley graphs.
The study of Cayley graphs of the symmetric group generated by transpositions is interesting (See [11]). In this paper we show $\Gamma(S : S_n)$ is integral, where $S = \{(12), (13), \ldots, (1n)\}$ and $n \in \{3, 4, 5, 6\}$. We also characterize all connected cubic integral Cayley graphs and introduce some infinite family of connected integral Cayley graphs.
The main results are the following.

Theorem 1.1 *There are exactly seven connected cubic integral Cayley graphs. In particular, for a finite group G and a subset $S = S^{-1} \not\ni 1$ with three elements, $\Gamma(S : G)$ is integral if and only if G is isomorphic to one the following groups: C_2^2, C_4, C_6, S_3, C_2^3, $C_2 \times C_4$, D_8, $C_2 \times C_6$, D_{12}, A_4, S_4, $D_8 \times C_3$, $D_6 \times C_4$ or $A_4 \times C_2$.*

Theorem 1.2 *Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1\rangle$, $n = 2m+1$, $d \mid n$ $(1 < d < n)$ and $S = \{a^k \mid k \in B(1,n)\} \cup \{a^{dk} \mid k \in B(1, \frac{n}{d})\} \cup \{ba^k \mid k \in B(1,n)\} \cup \{ba^{dk} \mid k \in B(1, \frac{n}{d})\}$. Then $\Gamma(S : D_{2n})$ is integral.*

Theorem 1.3 *Let $T_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1}\rangle$, $n = 2m+1$ $(n \neq 1)$ and $S = \{a^k \mid 1 \leqslant k \leqslant 2n-1, k \neq n\} \cup \{ab, a^{n+1}b\}$. Then $\Gamma(S : T_{4n})$ is integral.*

Theorem 1.4 *Let $U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$, $n = 2m+1$ $(n \neq 1)$ and $S = \{a^{2k}b \mid 1 \leqslant k \leqslant n-1\} \cup \{a^{2k}b^2 \mid 1 \leqslant k \leqslant n-1\} \cup \{a^{2k+1}b \mid 0 \leqslant k \leqslant n-1\}$. Then $\Gamma(S : U_{6n})$ is integral.*

2 Preliminaries

First we give some facts that are needed in the next section. Let n be a positive integer. Then $B(1,n)$ denotes the set $\{j \mid 1 \leqslant j < n, (j,n) = 1\}$. Let $\omega = e^{\frac{2\pi i}{n}}$ and

$$C(r,n) = \sum_{j \in B(1,n)} \omega^{jr}, \qquad 0 \leqslant r \leqslant n-1. \tag{2.1}$$

The function $C(r,n)$ is a Ramanujan sum. For integers r and n, $(n > 0)$, Ramanujan sums have only integral values (See [15] and [18]).

Lemma 2.1 *Let $\omega = e^{\frac{\pi i}{n}}$, where $i^2 = -1$. Then*

i) $\displaystyle\sum_{j=1}^{2n-1} \omega^j = -1.$

ii) If l is even, then $\displaystyle\sum_{j=1}^{n-1} \omega^{lj} = -1.$

iii) If l is odd, then $\displaystyle\sum_{j=1}^{n-1} \omega^{lj} + \omega^{-lj} = 0.$

Proof. The proof is straightforward. □

Lemma 2.2 *Let $G = C_n = \langle a \rangle$, $d \mid n$ $(1 < d < n)$ and $A_d = \{a^{dk} \mid k \in B(1, \frac{n}{d})\}$. Then $A_d^{-1} = A_d$.*

Proof. Let $n = dk'$ and a^{dk} be an arbitrary element of A_d. Since $(k-k', k') = 1$ and $(a^{dk})^{-1} = a^{n-dk} = a^{dk'-dk} = a^{(k'-k)d}$, $(a^{dk})^{-1} \in A_d$. So $A_d^{-1} \subseteq A_d$. It is easy to see that $|A_d^{-1}| = |A_d|$. Hence $A_d^{-1} = A_d$. □

Lemma 2.3 *[2] Let G be a finite group of order n whose irreducible characters (over $\mathbb{C}$) are $\rho_1, \ldots, \rho_h$ with respective degrees $n_1, \ldots, n_h$. Then the spectrum of the Cayley graph $\Gamma(S : G)$ can be arranged as $\Lambda = \{\lambda_{ijk} \mid i = 1, \ldots, h; j, k = 1, \ldots, n_i\}$ such that $\lambda_{ij1} = \ldots = \lambda_{ijn_i}$ and*

$$\lambda_{i1}^t + \ldots + \lambda_{in_i}^t = \sum_{s_1, \ldots, s_t \in S} \rho_i(\Pi_{l=1}^t s_l), \tag{2.2}$$

for any natural number t.

Lemma 2.4 *[13] Let $C_n = \langle a \rangle$. Then irreducible characters of C_n are $\rho_j : a^k \mapsto \omega^{jk}$, where $j, k = 0, 1, \ldots, n-1$.*

Lemma 2.5 *[13] Let $G = C_{n_1} \times \cdots \times C_{n_r}$ and $C_{n_i} = \langle a_i \rangle$, so that for any $i, j \in \{1, \ldots, r\}$, $(n_i, n_j) \neq 1$. If $\omega_t = e^{\frac{2\pi i}{n_t}}$, then $n_1 \cdots n_r$ irreducible characters of G are*

$$\rho_{l_1 \cdots l_r}(a_1^{k_1}, \ldots, a_r^{k_r}) = \omega_1^{l_1 k_1} \omega_2^{l_2 k_2} \cdots \omega_r^{l_r k_r} \tag{2.3}$$

where $l_i = 0, 1, \ldots, n_i - 1$ and $i = 1, 2, \ldots, r$.

Lemma 2.6 *Let G be a group and $G = \langle S \rangle$, where $S = S^{-1}$ and $1 \notin S$. If $a \in S$ and $o(a) = m > 2$, then $\Gamma(S : G)$ has the cycle with m vertices as a subgraph.*

Proof. Observe that $1 - a - a^2 - \cdots - a^{m-1} - a^m = 1$ is a cycle with m vertices. $\square$

Lemma 2.7 *Let $G = \langle S \rangle$ be a group, $|G| = n$, $|S| = 2$, $S = S^{-1} \not\ni 1$. Then $\Gamma(S : G)$ is integral if and only if $n \in \{3, 4, 6\}$.*

Proof. It is clear that $\Gamma(S : G)$ is a connected 2-regular graph. Thus $\Gamma(S : G)$ is the cycle with n vertices. By checking the eigenvalues of the cycles, one can easily see that the only integral cycles are ones with 3, 4 or 6 vertices. This completes the proof. $\square$

Lemma 2.8 *Let G be the cyclic group $\langle a \rangle$, $|G| = n > 3$ and let S be a generating set of G such that $|S| = 3$, $S = S^{-1}$ and $1 \notin S$. Then $a^{n/2} \in S$. Also if $a^r \in S$ and $o(a^r) = m > 2$, then $(n, r) = 1$ or $(n/2, r) = 1$.*

Proof. Let $(n, r) \neq 1$ and $(n/2, r) \neq 1$. Then $\langle a^r \rangle \neq G$. Suppose $(n/2, r) = d$, where $d \neq 1$, then $\langle a^r, a^{n/2} \rangle = \langle a^d \rangle$. Since $d \mid n$, $G \neq \langle a^d \rangle$. Hence $\langle a^r, a^{n/2} \rangle \neq G$. This contradicts the fact that S generates G. $\square$

Lemma 2.9 *Let G be the cyclic group $\langle a \rangle$, $|G| = n > 3$ and let S be a generating set of G such that $|S| = 3$, $S = S^{-1}$ and $1 \notin S$. Then $\Gamma(S : G)$ is integral if and only if $n \in \{4, 6\}$.*

Proof. Let $\Gamma(S : G)$ be integral. Then $S = \{a^{n/2}, a^r, a^{-r}\}$, where $(n, r) = 1$ or $(n/2, r) = 1$. If λ is the eigenvalue of $\Gamma(S : G)$ corresponding to irreducible character of ρ_1. Then by Lemmas 2.3 and 2.4, $\lambda = \rho_1(a^r) + \rho_1(a^{-r}) + \rho_1(a^{n/2}) = 2\cos(2\pi r/n) - 1$. Since λ is integer, $\cos(2\pi r/n) \in \{\pm 1/2, \pm 1, 0\}$. We consider the following cases:
Case1: Let $(n, r) = 1$. Then if $\cos(2\pi r/n) \in \{-1/2, -1, 1\}$, then $n \in \{1, 2, 3\}$, which is false. If $\cos(2\pi r/n) = 0$, then $n = 4$ and $r = 1$ or 3. So $S = \{a, a^2, a^3\}$. If $\cos(2\pi r/n) = 1/2$, then $n = 6$ and $r = 1$ or 5. So $S = \{a, a^3, a^5\}$.
Case2: Let $(n, r) \neq 1$ and $(n/2, r) = 1$. Without loss of generality we can assume $r < n/2$. Similarly if $\cos(2\pi r/n) \in \{-1, 0, 1/2, 1\}$, then $r = 1$, which is false. If $\cos(2\pi r/n) = -1/2$, then $n = 6$ and $r = 2$ or 4. So $S = \{a^2, a^3, a^4\}$.
Conversely, if $n = 4$, then $\Gamma(S : G)$ is complete graph K_4 and so is integral.
If $n = 6$, $S_1 = \{a, a^3, a^5\}$ and $S_2 = \{a^2, a^3, a^4\}$, then by Lemmas 2.3 and 2.4, $\Gamma(S_1 : G)$ and $\Gamma(S_2 : G)$ are integral with spectra of $[-3, 0^4, 3]$ and $[-2^2, 0^2, 1, 3]$ respectively. $\square$

Lemma 2.10 *Let G_1 and G_2 be two groups and $G = G_1 \times G_2$ such that $\Gamma(S : G)$ is integral, where $S = S^{-1} \not\ni 1$ with three elements. Let $S_1 = \{s_1 \mid (s_1, g_2) \in S, g_2 \in G_2\} \setminus \{1\}$. Then $\Gamma(S_1 : G_1)$ is integral.*

Proof. Let χ_0 and ρ_0 be the trivial irreducible characters of G_1 and G_2, respectively. Let λ_{i0} and λ_i be the eigenvalues of $\Gamma(S : G)$ and $\Gamma(S_1 : G_1)$ corresponding to irreducible characters of $\chi_i \times \rho_0$ and χ_i, respectively. Since S generates G and $S = S^{-1} \not\ni 1$ with three elements, $|S_1| = 2$ or 3. If $|S_1| = 2$, then by Lemma 2.3,

$$\lambda_{i0} = \sum_{(g_1,g_2)\in S} (\chi_i \times \rho_0)(g_1, g_2) = \sum_{s_1\in S_1} \chi_i(s_1) + 1$$

and so $\lambda_{i0} = \lambda_i + 1$. If $|S_1| = 3$, then by Lemma 2.3,

$$\lambda_{i0} = \sum_{(g_1,g_2)\in S} (\chi_i \times \rho_0)(g_1, g_2) = \sum_{s_1\in S_1} \chi_i(s_1) = \lambda_i$$

and so $Spec(\Gamma(S_1 : G_1)) \subseteq Spec(\Gamma(S : G))$. However $\Gamma(S_1 : G_1)$ is integral. Furthermore if $|S_1| = 2$, then $-1 \leqslant \lambda_{i0}$. □

Lemma 2.11 *Let G be a finite abelian group such that is not cyclic and let $G = \langle S\rangle$, where $|S| = 3$, $S = S^{-1}$ and $1 \notin S$. Then $\Gamma(S : G)$ is integral if and only if $|G| \in \{4, 8, 12\}$.*

Proof. Let $\Gamma(S : G)$ be integral. If all of elements of S are of order two, then $G = C_2^2$ or $G = C_2^3$. So $|G| = 4$ or 8. Otherwise $G = C_m \times C_2$ where m is even. By Lemmas 2.7, 2.9 and 2.10, we conclude that $m \in \{3, 4, 6\}$. Since m is even, $m \in \{4, 6\}$. Hence $|G| \in \{4, 8, 12\}$.
Conversely, if $|G| = 4$, then $\Gamma(S : G) = K_4$ and so is integral.
Let $|G| = 8$. Then $G = C_2^3$ or $C_4 \times C_2$. If $G = C_2^3$ and $S = \{(b, 1, 1), (1, b, 1), (1, 1, b)\}$, then by Lemma 2.3, $\Gamma(S : C_2^3)$ is integral with spectrum of $[-3, -1^3, 1^3, 3]$. If $G = C_4 \times C_2$ and $S = \{(a, 1), (a^3, 1), (1, b)\}$, then by Lemma 2.3, $\Gamma(S : C_4 \times C_2)$ is integral with spectrum of $[-3, -1^3, 1^3, 3]$.
Let $|G| = 12$. Then $G = C_6 \times C_2$. If $S = \{(a, 1), (a^5, 1), (1, b)\}$, then by Lemma 2.3, $\Gamma(S : C_6 \times C_2)$ is integral with spectrum of $[-3, -2^2, -1, 0^4, 1, 2^2, 3]$. □

Lemma 2.12 *Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1\rangle$, $n = 2m + 1$, and $\Gamma(S : D_{2n})$ be integral, where $D_{2n} = \langle S\rangle$, $|S| = 3$, $S = S^{-1}$ and $1 \notin S$. Then*

- *i) -3 is the simple eigenvalue of $\Gamma(S : D_{2n})$ if and only if all of elements of S are of order two.*
- *ii) If $[-3, -2^{l_1}, -1^{l_2}, 0^{l_3}, 1^{l_4}, 2^{l_5}, 3]$ is the spectrum of $\Gamma(S : D_{2n})$, then $l_1 = l_4$, $l_2 = l_5$ and $4 \mid l_3$. Furthermore l_1, l_2 are even.*
- *iii) If $n \neq 3$, then $\Gamma(S : D_{2n})$ is bipartite.*

Proof. i) Let -3 be the simple eigenvalue of $\Gamma(S : D_{2n})$. By Lemma 2.3 and using characters table D_{2n}, -3 is the eigenvalue of $\Gamma(S : D_{2n})$ corresponding to irreducible character χ_{m+1}. So all of elements of S are in conjugacy class of b.
Conversely, if all of elements of S are of order two, then $S \subseteq \overline{b}$ *(the bar indicates conjugacy class)*. By Lemma 2.3 and using characters table of D_{2n}, the eigenvalue of $\Gamma(S : D_{2n})$ corresponding to irreducible character χ_{m+1} is -3.

ii) Since -3 is the simple eigenvalue of $\Gamma(S : D_{2n})$, $S \subseteq \overline{b}$. By Lemma 2.3 and using characters table of D_{2n}, the eigenvalues of $\Gamma(S : D_{2n})$ corresponding to irreducible characters χ_j $(1 \leqslant j \leqslant m)$, are negative. Thus $l_1 = l_4$ and $l_2 = l_5$. Furthermore since the multiplicity of the eigenvalues of corresponding to irreducible characters of degree two is 2, l_1 and l_2 are even and $4 \mid l_3$.

iii) Let $a^r \in S$, where $1 \leqslant r \leqslant m$. It is clear that $(n, r) = 1$. Since $n \neq 3$ and $(n, r) = 1$, $2\cos(2\pi r/n)$ is not integer. Let λ_{11} and λ_{12} be eigenvalues of $\Gamma(S : D_{2n})$ corresponding to irreducible character χ_1. By Lemma 2.3 and using characters table of D_{2n}, $\lambda_{11} + \lambda_{12} = 2\cos(2\pi r/n)$. This contradicts the fact that $\Gamma(S : D_{2n})$ is integral. Thus $S \subseteq \overline{b}$ and so -3 is an eigenvalue of $\Gamma(S : D_{2n})$. Therefore, $\Gamma(S : D_{2n})$ is bipartite. □

Lemma 2.13 *Let $S = \{(12), (13), \ldots, (1n)\}$ and $n \in \{3, 4, 5, 6\}$. Then $\Gamma(S : S_n)$ is integral.*

Proof. It is clear that $\Gamma(S : S_3)$ is a cycle with six vertices and so is integral with spectrum of $[-2, -1^2, 1^2, 2]$. By using the following program written in GAP [19] and thanks to the GRAPE package of L.H. Soicher, one can easily see that $\Gamma(S : S_4)$, $\Gamma(S : S_5)$ and $\Gamma(S : S_6)$ are integral graphs with spectra as follows:

$$[-3, -2^6, -1^3, 0^4, 1^3, 2^6, 3],$$

$$[-4, -3^{12}, -2^{28}, -1^4, 0^{30}, 1^4, 2^{28}, 3^{12}, 4],$$

$$[-5, -4^{20}, -3^{105}, -2^{120}, -1^{30}, 0^{168}, 1^{30}, 2^{120}, 3^{105}, 4^{20}, 5],$$

respectively.

```
LoadPackage("grape");
### The following function admat constructs the adjacency matrix
### of a given graph G with n vertices
admat:=function(G,n)
local B,A,i,j;
A:=[];
for i in [1..n] do
B:=[];
for j in [1..n] do
if (j in Adjacency(G,i))=true then Add(B,1); else
Add(B,0); fi;
od;
Add(A,B);
od;
return A;
end;
#### The following function listcompress converts a multiset to a set
#### of ordered pairs whose  first components  are exactly the
#### elements of the corresponding set to the multiset
```

```
#### and the second one is the multiplicity of the first
#### component in the multiset
listcompress:=function(L)
local l;
l:=Set(L);
return List(l,i->[i,Size(Filtered(L,j->j=i))]);
end;
## Example: Computing the spectrum of the Cayley graph of
## the symmetric group of degree 6 on the set
## [(1,2),(1,3),(1,4),(1,5),(1,6)]
G:=CayleyGraph(SymmetricGroup(6),[(1,2),(1,3),(1,4),(1,5),(1,6)]);
### Construct the required Cayley graph
A:=admat(G,720);
p:=CharacteristicPolynomial(A);
r:=RootsOfUPol(p); #roots of the characteristic polynomial of A
SpectrumOfS6:=listcompress(r); #Spectrum of G
```

□

We end this section by the following conjecture.

Conjecture 2.14 *Let $n \geqslant 4$ be an arbitrary integer and $S = \{(12), (13), \ldots, (1n)\}$ be the subset of the symmetric group S_n of degree n. Then $\Gamma(S : S_n)$ is integral. Moreover, $\{0, \pm 1, \ldots, \pm(n-1)\}$ is the set of all distinct eigenvalues of $\Gamma(S : S_n)$.*

3 Proof of Our main results

In this section we prove our main results.
Proof of Theorem 1.1. Let $\Gamma(S : G)$ be integral. Since $\Gamma(S : G)$ is a cubic integral graph, $\Gamma(S : G)$ is of type G_i, for $1 \leqslant i \leqslant 13$ (see [16]). Since the number of vertices of G_i, for $1 \leqslant i \leqslant 13$, are 4, 6, 8, 10, 12, 20, 24 or 30, $|G| \in \{4, 6, 8, 10, 12, 20, 24, 30\}$. Hence we have the following cases:
Case1: Let $|G| = 4$. Then $\Gamma(S : G) = K_4 = G_1$.
Case2: Let $|G| = 6$. Then $G = C_6$ or D_6.
If $C_6 = \langle a \rangle$, $S_1 = \{a, a^3, a^5\}$ and $S_2 = \{a^2, a^3, a^4\}$, then by using the program written in Lemma 2.13, $\Gamma(S_1 : C_6)$ and $\Gamma(S_2 : C_6)$ are integral with spectra of $[-3, 0^4, 3]$ and $[-2^2, 0^2, 1, 3]$ respectively. So $\Gamma(S_1 : C_6) = G_2$ and $\Gamma(S_2 : C_6) = G_5$.
If $G = D_6 = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1 \rangle$, $S_1 = \{b, ab, a^2b\}$ and $S_2 = \{a, a^2, b\}$, then by using the program written in Lemma 2.13, $\Gamma(S_1 : D_6)$ and $\Gamma(S_2 : D_6)$ are integral with spectra of $[-3, 0^4, 3]$ and $[-2^2, 0^2, 1, 3]$ respectively. So $\Gamma(S_1 : D_6) = G_2$ and $\Gamma(S_2 : D_6) = G_5$.
Case3: Let $|G| = 8$. Then $G = C_8$, C_2^3, $C_4 \times C_2$, D_8 or $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$. We show that the graph G_4 is only and only cayley graph of C_2^3, $C_4 \times C_2$ and D_8.

Let $G = C_2^3$ or $C_4 \times C_2$, by the proof of Lemma 2.11, $\Gamma(S : G) = G_4$.
Let $G = D_8 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1\rangle$ and $S = \{a, a^{-1}, b\}$ or $\{b, a^2b, ab\}$. Then by using the program written in Lemma 2.13, $\Gamma(S : D_8)$ is integral with spectrum of $[-3, -1^3, 1^3, 3]$ and so $\Gamma(S : D_8) = G_4$.
Let $G = C_8$. By Lemma 2.9, $\Gamma(S : C_8)$ is not isomorphic to G_4.
Let $G = Q_8$. Since a^2 is the unique element of degree two, $a^2 \in S$. Since S is generator and $S = S^{-1}$, $S = \{a^2, b, a^2b\}$ or $\{a^2, ab, a^3b\}$. If $S = \{a^2, b, a^2b\}$, then by Lemma 2.3 and using characters table of Q_8, the eigenvalue of $\Gamma(S : Q_8)$ corresponding to the irreducible character χ_3 is 3. If $S = \{a^2, ab, a^3b\}$, then the eigenvalue of $\Gamma(S : Q_8)$ corresponding to the irreducible character χ_4 is 3. However the multiplicity 3 as an eigenvalue of $\Gamma(S : Q_8)$ is greater than one. So $\Gamma(S : Q_8)$ is not isomorphic to G_4.
Case4: Let $|G| = 10$. Then by Lemmas 2.9 and 2.11, G is a non-abelian group and so $G = D_{10}$. Since $\Gamma(S : D_{10})$ is integral, $\Gamma(S : D_{10}) = G_3$, G_7 or G_{11}. If $\Gamma(S : D_{10}) = G_3$ or G_7, then $\Gamma(S : D_{10})$ is not bipartite graph, which by Lemma 2.12 (iii), is a contradiction. If $\Gamma(S : D_{10}) = G_{11}$, then by Lemma 2.12 (ii), it is a contradiction. Therefore, the graphs of G_3, G_7 and G_{11} are not Cayley graphs.
Case5: Let $|G| = 12$. By Lemmas 2.9 and 2.11, $G = C_6 \times C_2$, T_{12}, A_4 or D_{12}. First we show G_{12} is only and only cayley graph of $C_6 \times C_2$ and D_{12}.
Let $G = C_6 \times C_2$ and $S = \{(a, c), (a^{-1}, c), (a^3, c)\}$ where $C_6 = \langle a\rangle$ and $C_2 = \langle c\rangle$. Then by using the program written in Lemma 2.13, $\Gamma(S : C_6 \times C_2)$ is integral with spectrum of $[-3, -2^2, -1, 0^4, 1, 2^2, 3]$. So $\Gamma(S : C_6 \times C_2) = G_{12}$.
Let $G = D_{12} = \langle a, b \mid a^6 = b^2 = (ab)^2 = 1\rangle$ and $S = \{a, a^5, b\}$. Then by using the program written in Lemma 2.13, $\Gamma(S : D_{12})$ is integral with spectrum of $[-3, -2^2, -1, 0^4, 1, 2^2, 3]$. So $\Gamma(S : D_{12}) = G_{12}$.
Let $\Gamma(S : T_{12}) = G_{12}$. It is easy to see that a^3 is the unique element of order two, so $a^3 \in S$. Since S generates G, $a^r \notin S$. By Lemma 2.3 and using characters table of T_{4n}, we conclude that the eigenvalues of $\Gamma(S : T_{12})$ corresponding to linear irreducible characters of T_{12} are distinct from -3. Therefore, G_{12} have not -3 as an eigenvalue, which is not true. So G_{12} is not Cayley graph of T_{12}.
Let $\Gamma(S : A_4) = G_{12}$. By Lemma 2.3 and using characters table of A_4, $\Gamma(S : A_4)$ has an eigenvalue with multiplicity greater than 6 or three eigenvalues with multiplicities greater than 3. Which is impossible.
Therefor the graph G_{12} is only and only Cayley graph of $C_6 \times C_2$ and D_{12}.
We continue by showing that G_8 is only and only Cayley graph A_4.
Let $G = A_4$ and $S = \{(1 \ \ 2)(3 \ \ 4), (1 \ \ 2 \ \ 3), (1 \ \ 3 \ \ 2)\}$. By using the program written in Lemma 2.13, $\Gamma(S : A_4)$ is integral with spectrum of $[-2^3, -1^3, 0^2, 2^3, 3]$, and so $\Gamma(S : A_4) = G_8$.
Let $\Gamma(S : T_{12}) = G_8$. Since G_8 does not have C_4 as a subgraph, by Lemma 2.6, $S = \{a^3, a^r, a^{-r}\}$ for $r = 1, 2$. This contradicts the fact that S generates G.
Let $\Gamma(S : C_6 \times C_2) = G_8$ and $S_1 = \{s_1 \mid (s_1, c) \in S, c \in C_2\} \setminus \{1\}$. Then by Lemma 2.10 and case 2, $|S_1| = 2$ and so $-1 \leqslant \lambda_{i0}$, where λ_{i0} is the eigenvalue of $\Gamma(S : C_6 \times C_2)$ corresponding to a linear irreducible character of $C_6 \times C_2$. This contradicts the fact that -2 is an eigenvalue of G_8.

Let $\Gamma(S : D_6 \times C_2) = G_8$ and $S_1 = \{s_1 \mid (s_1, c) \in S, c \in C_2\} \setminus \{1\}$. Then by Lemma 2.10 and case 2, $|S_1| = 2$ and so $-1 \leqslant \lambda_{i0}$, where λ_{i0} is the eigenvalue of $\Gamma(S : D_6 \times C_2)$ corresponding to a linear irreducible character of $D_6 \times C_2$. This contradicts the fact that -2 is an eigenvalue of G_8.

Therefor the graph G_8 is only and only Cayley graph of A_4.

Case6: Let $|G| = 20$. By Lemmas 2.9 and 2.11, G is a non-abelian group and so it is $D_{20} = D_{10} \times C_2$, T_{20} or $F_{5,4} = \langle a, b \mid a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle$. Since $\Gamma(S : G)$ is integral, $\Gamma(S : G) = G_9$ or G_{10}.

Let $G = F_{5,4}$. Since the graphs G_9 and G_{10}, does not have C_4 and C_5 as a subgraph, by Lemma 2.6, all of the elements of S are of order 2 or 10. It is clear that $F_{5,4}$ does not have any element of order 10, so $S \subseteq \overline{b}$ *(the bar indicates conjugacy class)*. By Lemma 2.3 and using characters table of $F_{5,4}$, we see that the eigenvalues of $\Gamma(S : F_{5,4})$ corresponding to irreducible characters χ_1 and χ_3 are 3. which is impossible.

Let $G = D_{10} \times C_2$ and $S_1 = \{s_1 \mid (s_1, c) \in S, c \in C_2\} \setminus \{1\}$, then by Lemma 2.10 and Case 4, $|S_1| = 2$ and so $-1 \leqslant \lambda_{i0}$, where λ_{i0} is the eigenvalue of $\Gamma(S : D_{10} \times C_2)$ corresponding to a linear irreducible character of $D_{10} \times C_2$. This contradicts the fact that -3 is an eigenvalue of G_9 and G_{10}.

Let $G = T_{20}$. Since $a^5 \in T_{20}$ is the unique element of order two and G_9, G_{10}, does not have C_4 and C_5 as a subgraph, $S = \{a^5, a^r, a^{-r}\}$. This contradicts the fact that S generates G. Hence the graphs G_9 and G_{10} are not Cayley graphs.

Case7: Let $|G| = 24$. By Lemmas 2.9 and 2.11, G is a non-abelian group and so $G = D_{12} \times C_2$, $T_{12} \times C_2$, $Q_8 \times C_3$, $SL(2, 3)$, D_{24}, T_{24}, U_{24}, V_{24}, S_4, $D_8 \times C_3$, $D_6 \times C_4$ or $A_4 \times C_2$. We show that G_{13} is only and only Cayley graph of groups S_4, $A_4 \times C_2$, $D_8 \times C_3$, $D_6 \times C_4$.

Let $G = S_4$. By Lemma 2.13, $\Gamma(S : S_4) = G_{13}$.

Let $G = A_4 \times C_2$ and $S = \{((1\ \ 2)(3\ \ 4), c), ((1\ \ 2\ \ 3), c), ((1\ \ 3\ \ 2), c)\}$, where $C_2 = \langle c \rangle$. Then by using the program written in Lemma 2.13, $\Gamma(S : A_4 \times C_2)$ is integral with spectrum of $[-3, -2^6, -1^3, 0^4, 1^3, 2^6, 3]$. So $\Gamma(S : A_4 \times C_2) = G_{13}$.

Let $G = D_8 \times C_3$ and $S = \{(a, c), (a^3, c), (b, 1)\}$, where $D_8 = \langle a, b \rangle$ and $C_2 = \langle c \rangle$. Then by using the program written in Lemma 2.13, $\Gamma(S : D_8 \times C_3)$ is integral with spectrum of $[-3, -2^6, -1^3, 0^4, 1^3, 2^6, 3]$ and so $\Gamma(S : D_8 \times C_3) = G_{13}$.

Let $G = D_6 \times C_4$. In the same manner we can see that $\Gamma(S : D_6 \times C_4) = G_{13}$, where $D_6 = \langle a, b \rangle$, $C_4 = \langle c \rangle$ and $S = \{(a, c), (a^3, c), (b, 1)\}$.

It remains to prove that $\Gamma(S : G)$ is not integral, for others. On the contrary, let $\Gamma(S : G) = G_{13}$, for $G = Q_8 \times C_3$, $T_{12} \times C_2$, $D_{12} \times C_2$, T_{24}, D_{24}, $SL(2, 3)$ or V_{24}.

Let $G = Q_8 \times C_3$ or $T_{12} \times C_2$ and $S_1 = \{s_1 \mid (s_1, c) \in S, c \in C_3\} \setminus \{1\}$ or $\{s_1 \mid (s_1, c) \in S, c \in C_2\} \setminus \{1\}$, then by Lemma 2.10 and Cases 3, 5, we have $|S_1| = 2$ and so $-1 \leqslant \lambda_{i0}$, where λ_{i0} is the eigenvalue of $\Gamma(S : G)$ corresponding to a linear irreducible character of G. This contradicts the fact that -3 is an eigenvalue of G_{13}.

Let $G = D_{12} \times C_2$ and $S_1 = \{s_1 \mid (s_1, c) \in S, c \in C_2\} \setminus \{1\}$. One can check that $(1, c) \in S$ where $C_2 = \langle c \rangle$. So $|S_1| = 2$ and $-1 \leqslant \lambda_{i0}$, where λ_{i0} is the eigenvalue of $\Gamma(S : D_{12} \times C_2)$ corresponding to a linear irreducible character of $D_{12} \times C_2$. This contradicts the fact that -3 is an eigenvalue of G_{13}.

Let $G = T_{24}$. Since $a^6 \in T_{24}$ is the unique element of order two and G_{13} does not have C_4 as a subgraph, $S = \{a^6, a^r, a^{-r}\}$ for $1 \leqslant r \leqslant 5$. This contradicts the fact that S generates G.

Let $G = U_{24} = \langle a, b \mid a^8 = b^3 = 1, a^{-1}ba = b^{-1}\rangle$. Since a^4 is the unique element of order two, $a^4 \in S$ and so $a^r \notin S$ for $r \neq 4$ because of S generates G. It is easy to see that $(a^{2r}b)^{-1} = a^{8-2r}b^2$ and $(a^{2r+1}b)^{-1} = a^{8-2r-1}b$. So $S = \{a^4, a^{2r}b, a^{8-2r}b^2\}$, $\{a^4, a^{2r+1}b, a^{8-2r-1}b\}$ or $\{a^4, a^{2r+1}b^2, a^{8-2r-1}b^2\}$ $(0 \leqslant r \leqslant 3)$. If $S = \{a^4, a^{2r}b, a^{8-2r}b^2\}$, then by Lemma 2.3 and using characters table of U_{6n}, the eigenvalue of $\Gamma(S : U_{24})$ corresponding to χ_4 is equal to 3, which is not true. If $S = \{a^4, a^{2r+1}b, a^{8-2r-1}b\}$ or $\{a^4, a^{2r+1}b^2, a^{8-2r-1}b^2\}$, then by Lemma 2.3 and using characters table of U_{6n}, the eigenvalue of $\Gamma(S : U_{24})$ corresponding to χ_1 is $-1+2\cos((2r+1)\pi/4)$ for $0 \leqslant r \leqslant 3$, obviously is not integer. Which is a contradiction.

Let $G = D_{24}$. First consider $a^6 \in S$. Since S generates G, $a^r \notin S$. By Lemma 2.3, it is immediate that $\Gamma(S : D_{24}) = G_{13}$ does not have -3 as an eigenvalue, which is impossible. Thus $a^6 \notin S$. Now suppose $a^r \in S$ where $1 \leqslant r \leqslant 5$. Since S generates D_{24}, $(r, 12) = 1$. So $S = \{a^r, a^{-r}, a^{2l}b\}$ or $\{a^r, a^{-r}, a^{2l+1}b\}$ where $r = 1$ or 5. By Lemma 2.3 and using characters table of D_{2n}, the sum of the eigenvalues of $\Gamma(S : D_{24})$ corresponding to χ_1 is $\sqrt{3}$ or $-\sqrt{3}$, which is impossible. Therefore, all of the elements of S are in conjugacy class of b or ab. Let $S = \{a^{2s}b, a^{2r+1}b, a^{2l+1}b\}$ $(1 \leqslant l, r, s \leqslant 5)$ and ρ be an irreducible character of degree two of D_{24}. If λ and μ are the eigenvalues $\Gamma(S : D_{24})$ corresponding to ρ, then by Lemma 2.3 and using characters table of D_{2n}, we have:

$\lambda + \mu = 0$

$\lambda^2 + \mu^2 = 6 + 2[\rho(a^{2s-2r-1}) + \rho(a^{2s-2l-1}) + \rho(a^{2r-2l})]$.

A trivial verification shows that if $\omega = e^{\frac{2\pi i}{12}}$, then $\omega+\omega^{-1} = \sqrt{3}$, $\omega^2+\omega^{-2} = 1$, $\omega^3+\omega^{-3} = 0$, $\omega^4 + \omega^{-4} = -1$ and $\omega^5 + \omega^{-5} = -\sqrt{3}$. From this and using characters table of D_{2n} we conclude that $\lambda^2 + \mu^2 \neq 0$. It follows that $\Gamma(S : D_{24})$ does not have 0 as an eigenvalue . Therefore, G_{13} does not have 0 as an eigenvalue, which is impossible.

Let $G = SL(2, 3)$. It is easy to see that g_2 is the unique element of order two, so $g_2 \in S$. On the other hand, since $g_6g_7 = 1$ and the graph G_{13} does not have C_3 and C_4 as a subgraph, $S = \{g_2, x, x^{-1}\}$, such that x is in conjugacy class of g_6 and x^{-1} in conjugacy class of g_7. By Lemma 2.3 and using characters table of $SL(2, 3)$, it is easily seen that the eigenvalues of corresponding to irreducible linear characters of $SL(2, 3)$ are equal to zero. This contradicts the fact that -3 is an eigenvalue of G_{13}

Let $G = V_{24} = \langle a, b \mid a^6 = b^4 = (ba)^2 = (a^{-1}b)^2 = 1\rangle$. Since the graph G_{13} does not have C_3 and C_4 as a subgraph, $S \cap \overline{b} = \phi$ and $S \cap \overline{a^2} = \phi$ *(the bar indicates conjugacy class)*. If $S \cap \overline{ab} = \phi$, then by Lemma 2.3 and using characters table of V_{24}, we see that the eigenvalues of corresponding to linear irreducible characters of χ_1 and χ_2 are equal to 3. Which is impossible. So $S \cap \overline{ab} \neq \phi$. Also if $b^2 \in S$ or $a^2b^2 \in S$, then by Lemma 2.3, we check at once that $\Gamma(S : V_{24})$ does not have -3 as an eigenvalue, which is not true. Hence $S = \{a, a^{-1}, a^rb^s\}$, $\{ab^2, a^{-1}b^2, a^rb^s\}$ or $\{a^3, a^3b^2, a^rb^s\}$, where $r \in \{1, 3, 5\}$ and $s \in \{1, 3\}$. Let λ and μ be the eigenvalues of $\Gamma(S : V_{24})$ corresponding to irreducible character χ_5.

If $S = \{a^3, a^3b^2, a^rb^s\}$, then by Lemma 2.3 and using characters table of V_{24}, $\lambda+\mu = 0$ and $\lambda^2 + \mu^2 = \chi_5(a^6) + \chi_5(a^3b^2)^2 + \chi_5(a^rb^s)^2 + 2[\chi_5(a^3a^3b^2) + \chi_5(a^{r+3}b^s) + \chi_5(a^{r+3}b^{s+2})] = 10$.

If $S = \{a, a^{-1}, a^r b^s\}$ or $\{ab^2, a^{-1}b^2, a^r b^s\}$, then by Lemma 2.3, $\lambda + \mu = 0$ and $\lambda^2 + \mu^2 = \chi_5(a^2) + \chi_5(a^{-2}) + \chi_5(a^r b^s)^2 + 2[\chi_5(aa^{-1}) + \chi_5(a^{r+1}b^s) + \chi_5(a^{r-1}b^s)]$ or $\lambda + \mu = 0$ and $\lambda^2 + \mu^2 = \chi_5(ab^2)^2 + \chi_5(a^{-1}b^2)^2 + \chi_5(a^r b^s)^2 + 2[\chi_5(1) + \chi_5(a^{r+1}b^{s+2}) + \chi_5(a^{r-1}b^{s+2})]$, respectively.
By using character table of V_{24}, we have $\chi_5(a^{r+1}b^s) = \chi_5(a^{r-1}b^s) = \chi_5(a^{r+1}b^{s+2}) = \chi_5(a^{r-1}b^{s+2}) = \chi_5(a^{r+3}b^s) = \chi_5(a^{r+3}b^{s+2}) = 0$. So $\lambda^2 + \mu^2 = 10$.
This gives λ and μ are not integers, which is false.
Case8: Let $|G| = 30$ and $\Gamma(S : G) = G_6$. By Lemmas 2.9 and 2.11, G is a non-abelian group and so $G = D_{10} \times C_3$, D_{30} or $U_{30} = \langle a, b \mid a^{10} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$.
Let $G = D_{10} \times C_3$ and $S_1 = \{s_1 \mid (s_1, c) \in S, c \in C_3\} \setminus \{1\}$. By Lemma 2.10 and Case 4, $|S_1| = 2$ and so $-1 \leqslant \lambda_{i0}$, where λ_{i0} is the eigenvalue of $\Gamma(S : D_{10} \times C_3)$ corresponding to a linear irreducible character of $D_{10} \times C_3$. This contradicts the fact that -3 is an eigenvalue of G_6.
Let $\Gamma(S : D_{30}) = G_6$. By Lemma 2.12 (ii), we have $4 \mid 10$, which is impossible.
Let $\Gamma(S : U_{30}) = G_6$. It is obvious that U_{30} has exactly three elements of order two and they are a^5, a^5b and a^5b^2. So $S \cap \{a^5, a^5b, a^5b^2\} \neq \phi$. If $S = \{a^5, a^5b, a^5b^2\}$, then by Lemma 2.3 and using characters table of U_{6n}, the eigenvalues of $\Gamma(S : U_{30})$ corresponding to irreducible characters χ_1 and χ_5 are -3. This contradicts the fact that the multiplicity -3 as an eigenvalue of G_6 is one. If $a^{2r} \in S$ or $a^{2r}b \in S$ $(0 \leqslant r \leqslant 4)$, then by Lemma 2.3 and using characters table of U_{6n}, the eigenvalue of $\Gamma(S : U_{30})$ corresponding to irreducible character χ_5 is 1, this show that 1 is an eigenvalue of G_6, which is not true. Thus $S = \{a^5b^k, a^{2r+1}b^s, (a^{2r+1}b^s)^{-1}\}$, where $k, s \in \{0, 1, 2\}$ and $r \in \{0, 1, 3, 4\}$. By Lemma 2.3 and using characters table of U_{6n}, the eigenvalue of $\Gamma(S : U_{30})$ corresponding to irreducible character χ_1 is $-1 + 2\cos((2r + 1)\pi/5)$ for $r \in \{0, 1, 3, 4\}$. This is not integer. which is a contradiction. Therefor G_6 is not Cayley graph.
Hence there are exactly seven connected, cubic integral Cayley graphs. This proves the theorem. □

Theorem 3.1 (See [14]) *Let $C_n = \langle a\rangle$. If $S = \{a^j \mid j \in B(1, n)\}$, then $\Gamma(S : C_n)$ is integral.*

Proof. By Lemma 2.2, $\Gamma(S : C_n)$ is connected graph. By Lemmas 2.3 and 2.4, n eigenvalues of $\Gamma(S : C_n)$ are $\lambda_r = \sum_{j \in B(1,n)} \omega^{jr}$, $(1 \leqslant r < n)$. By equation (2.1), $\lambda_r = C(r, n)$, $(1 \leqslant r < n)$. Hence $\Gamma(S : C_n)$ is integral. □

Corollary 3.2 *For any natural number n, there is at least an connected, $\varphi(n)$-regular integral graph with n vertices.*

Theorem 3.3 *Let $C_n = \langle a\rangle$, $d \mid n$ $(1 < d < n)$ and $A_d = \{a^{dj} \mid j \in B(1, \frac{n}{d})\}$. If $S = A_1 \cup A_d$, then $\Gamma(S : C_n)$ is integral.*

Proof. By Lemma 2.2, $\Gamma(S : C_n)$ is connected graph. Let λ_r $(0 \leqslant r \leqslant n - 1)$ be the eigenvalues of $\Gamma(S : C_n)$. By Lemmas 2.3 and 2.4, we have:

$$\lambda_r = \sum_{g\in A_1}\rho_r(g) + \sum_{g\in A_d}\rho_r(g) = \sum_{j\in B(1,n)}\rho_r(a^j) + \sum_{j\in B(1,\frac{n}{d})}\rho_r(a^{dj}) = \sum_{j\in B(1,n)}\omega^{jr} + \sum_{j\in B(1,\frac{n}{d})}\omega^{djr}.$$

By equation (2.1), $\sum_{j\in B(1,n)}\omega^{jr}$ and $\sum_{j\in B(1,\frac{n}{d})}\omega^{djr}$ are integer. Hence $\Gamma(S : C_n)$ is integral. □

Corollary 3.4 *For any natural number n, there is at least a connected, $(\varphi(n) + \varphi(\frac{n}{d}))$-regular integral graph with n vertices, where $d \mid n$ $(1 < d < n)$.*

Lemma 3.5 *Let $G = C_m \times C_n$, $C_m = \langle a\rangle$ and $C_n = \langle b\rangle$ so that $(m, n) \neq 1$. If $S = \{(a^j, b^{j'}) \mid j \in B(1, m), j' \in B(1, n)\} \cup \{(a^j, 1) \mid j \in B(1, m)\} \cup \{(1, b^{j'}) \mid j' \in B(1, n)\}$, then $\Gamma(S : G)$ is integral.*

Proof. It is clear that $\Gamma(S : G)$ is connected graph. By Lemma 2.3, mn eigenvalues of $\Gamma(S : G)$ are $\lambda_{kr} = \sum_{g\in S}\rho_{kr}(g)$, $(0 \leqslant k \leqslant m-1)$ and $(0 \leqslant r \leqslant n-1)$. By Lemma 2.5,

$$\lambda_{kr} = \sum_{j\in B(1,m)}\Big(\sum_{j'\in B(1,n)}\omega_1^{kj}\omega_2^{rj'}\Big) + \sum_{j\in B(1,m)}\omega_1^{kj} + \sum_{j'\in B(1,n)}\omega_2^{rj'}.$$

An easy computation shows:
$\lambda_{kr} = \sum_{j\in B(1,m)}\omega_1^{kj}\sum_{j'\in B(1,n)}\omega_2^{rj'} + \sum_{j\in B(1,m)}\omega_1^{kj} + \sum_{j'\in B(1,n)}\omega_2^{rj'}$. By equation (2.1), $\sum_{j\in B(1,m)}\omega_1^{kj}$ and $\sum_{j'\in B(1,n)}\omega_2^{rj'}$ are integer. Hence $\Gamma(S : G)$ is integral. □

Theorem 3.6 *Let $G = C_{n_1} \times \ldots \times C_{n_l}$ and $C_{n_i} = \langle a_i\rangle$, so that for any $i, j \in \{1, \ldots, l\}$, $(n_i, n_j) \neq 1$. If $S = \{(a_1^{j_1}, a_2^{j_2}, \ldots, a_l^{j_l}) \mid j_i \in B(1, n_i), i = 1, \ldots, l\} \cup \{(a_1^{j_1}, 1, \ldots, 1) \mid j_1 \in B(1, n_1)\} \cup \ldots \cup \{(1, 1, \ldots, a_l^{j_l}) \mid j_l \in B(1, n_l)\}$, then $\Gamma(S : G)$ is integral.*

Proof. Suppose $\alpha = \sum_{j_1\in B(1,n_1)}\cdots\sum_{j_l\in B(1,n_l)}\omega_1^{r_1j_1}\cdots\omega_l^{r_lj_l}$, where $\omega_t = e^{\frac{2\pi i}{n_t}}$, for $t = 1, \ldots, l$. One can check that $\alpha = \big(\sum_{j_1\in B(1,n_1)}\omega_1^{r_1j_1}\big)\cdots\big(\sum_{j_l\in B(1,n_l)}\omega_l^{r_lj_l}\big)$. By Lemma 2.3, $n_1n_2\ldots n_l$ eigenvalues of $\Gamma(S : G)$ are $\lambda_{r_1\ldots r_l} = \sum_{g\in S}\rho_{r_1r_2\ldots r_l}(g)$, where $0 \leqslant r_i \leqslant n_i - 1$ and $1 \leqslant i \leqslant l$. By Lemma 2.5, $\lambda_{r_1\ldots r_l} = \alpha + \sum_{j_1\in B(1,n_1)}\omega_1^{r_1j_1} + \ldots + \sum_{j_l\in B(1,n_l)}\omega_l^{r_lj_l}$.

By equation (2.1), $\sum_{j_i\in B(1,n_i)}\omega_i^{r_ij_i}$, $(1 \leqslant i \leqslant l)$ is integer. Hence $\Gamma(S : G)$ is integral. □

Corollary 3.7 *Let $n = n_1\cdots n_l$ such that $(n_i, n_j) \neq 1$, where $1 \leqslant i, j \leqslant l$. Then there is at least a connected $(\sum_{i=1}^{l}\varphi(n_i))(\prod_{i=1}^{l}\varphi(n_i))$-regular integral graph with n vertices.*

Theorem 3.8 *Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1\rangle$, $n = 2m + 1$ $(n \neq 1)$ and $S = \{a^k \mid k \in B(1, n)\} \cup \{ba^k \mid k \in B(1, n)\}$. Then $\Gamma(S : D_{2n})$ is integral.*

Proof. Since S generates D_{2n}, $\Gamma(S : D_{2n})$ is connected graph. We know that $\{1\}$, $\{a^r, a^{-r}\}$, $1 \leqslant r \leqslant (n-1)/2$ and $\{a^s b \mid 0 \leqslant s \leqslant n-1\}$ are the conjugacy classes of D_{2n}. Let $A_j = \sum_{k\in B(1,n)} \omega^{jk}$, $S_1 = \{a^k \mid k \in B(1,n)\}$ and $S_2 = \{ba^k \mid k \in B(1,n)\}$. If λ_{j1} , λ_{j2} (Each one 2 times)for $1 \leqslant j \leqslant m$, λ_{m+1} and λ_{m+2} are $2n$ eigenvalues of $\Gamma(S : D_{2n})$, then by Proposition 4.1 from [2], $\lambda_{j1} + \lambda_{j2} = 2A_j$ and $\lambda_{j1}^2 + \lambda_{j2}^2 = 4A_j^2$. So $\lambda_{j1} = 0$, $\lambda_{j2} = 2A_j$ or $\lambda_{j1} = 2A_j$, $\lambda_{j2} = 0$. Also $\lambda_{m+1} = 0$ and $\lambda_{m+2} = |S_1| + |S_2| = 2\varphi(n)$. By equation (2.1), A_j is integer. So all of the eigenvalues of $\Gamma(S : D_{2n})$ are integers. Hence $\Gamma(S : D_{2n})$ is integral. □

Corollary 3.9 *For any odd natural number n $(n \neq 1)$, there is at least a connected, $(2\varphi(n))$-regular integral graph with $2n$ vertices.*

Proof of Theorem 1.2. It is clear that $\Gamma(S : D_{2n})$ is connected graph. Let $C_1 = \{k \mid k \in B(1,n)\} = \{k_1, \ldots, k_{\varphi(n)}\}$ and $C_2 = \{dk \mid k \in B(1, \frac{n}{d})\} = \{k'_1, \ldots, k'_{\varphi(\frac{n}{d})}\}$. Then $C_1 \cap C_2 = \phi$. Suppose $C_1 \cup C_2 = \{k_i \mid 1 \leqslant k_1 < \cdots < k_t \leqslant n-1\}$ and $A_j = \sum_{u=1}^{t} \omega^{jk_u}$ for $j = 1, \ldots, m$. Then $A_j = \sum_{k\in C_1} \omega^{jk} + \sum_{k'\in C_2} \omega^{jk'}$ and by equation (2.1), A_j is integer. If λ_{j1} , λ_{j2} for $1 \leqslant j \leqslant m$ (Each one 2 times), λ_{m+1} and λ_{m+2} are $2n$ eigenvalues of $\Gamma(S : D_{2n})$, then by Proposition 4.1 from [2], we have $\lambda_{j1} + \lambda_{j2} = 2A_j$ and $\lambda_{j1}^2 + \lambda_{j2}^2 = 4A_j^2$. So $\lambda_{j1} = 0$, $\lambda_{j2} = 2A_j$ or $\lambda_{j1} = 2A_j$, $\lambda_{j2} = 0$. Also $\lambda_{m+1} = 0$ and $\lambda_{m+2} = 2|C_1| + 2|C_2| = 2\varphi(n) + 2\varphi(\frac{n}{d})$. Since A_j is integer, $2n$ eigenvalues of $\Gamma(S : D_{2n})$ are integers. Hence $\Gamma(S : D_{2n})$ is integral. □

Corollary 3.10 *For any odd natural number n, there is at least a connected, $(2\varphi(n) + 2\varphi(\frac{n}{d}))$-regular integral graph with $2n$ vertices, where $d \mid n$ $(1 < d < n)$.*

Proof of Theorem 1.3. We know that $\{1\}$, $\{a^r, a^{-r}\}$, $(1 \leqslant r \leqslant n-1)$, $\{a^{2k}b \mid 0 \leqslant k \leqslant n-1\}$ and $\{a^{2k+1}b \mid 0 \leqslant k \leqslant n-1\}$ are all of the conjugacy classes of T_{4n}. It is clear that $T_{4n} = \langle S \rangle$, $S = S^{-1}$ and $1 \notin S$. Let λ_{j1} , λ_{j2} for $1 \leqslant j \leqslant n-1$ (Each one 2 times) and μ_l for $1 \leqslant l \leqslant 4$ be $4n$ eigenvalues of $\Gamma(S : T_{4n})$. Then by Lemma 2.3 and using characters table of T_{4n}, we have:

$\mu_1 = \sum_{g\in S} \chi_1(g) = 2n$, $\mu_2 = \sum_{g\in S} \chi_2(g) = 0$, $\mu_3 = \sum_{g\in S} \chi_3(g) = 2n-4$ and $\mu_4 = \sum_{g\in S} \chi_4(g) = 0$.

$$\lambda_{j1} + \lambda_{j2} = \sum_{g\in S} \rho_j(g) = 2\sum_{k=1}^{n-1} \rho_j(a^k) = 2\sum_{k=1}^{n-1} \omega^{jk} + \omega^{-jk}$$

$$\lambda_{j1}^2 + \lambda_{j2}^2 = \sum_{s_1,s_2\in S} \rho_j(s_1 s_2) = (4n-8)\sum_{k=1}^{n-1} \rho_j(a^k) + 2n[\rho_j(a^n) + \rho_j(1)]=$$

$$(4n-8)\sum_{k=1}^{n-1}(\omega^{jk} + \omega^{-jk}) + 2n[2(-1)^j + 2].$$

By Lemma 2.1, if j is odd, then $\lambda_{j1} + \lambda_{j2} = 0$ and $\lambda_{j1}^2 + \lambda_{j2}^2 = 0$ and so $\lambda_{j1} = \lambda_{j2} = 0$ (Each one two times).

If j is even, then $\lambda_{j1}+\lambda_{j2}=-4$ and $\lambda_{j1}^2+\lambda_{j2}^2=16$ and so $\lambda_{j1}=0$ and $\lambda_{j2}=-4$ (Each one two times). Therefore, the spectrum of $\Gamma(S:T_{4n})$ is: $[-4^{n-1}, 0^{3n-1}, 2n-4, 2n]$. $\square$

Corollary 3.11 *For any odd natural number n, $(n\neq 1)$, there is at least a connected, $2n$-regular integral graph with $4n$ vertices.*

Proof of Theorem 1.4. Consider $A=\{a^{2k}b \mid 1\leqslant k\leqslant n-1\}\cup\{a^{2k}b^2 \mid 1\leqslant k\leqslant n-1\}$ and $B=\{a^{2k+1}b \mid 0\leqslant k\leqslant n-1\}$. We know that $\{a^{2r}\}$, $\{a^{2r}b, a^{2r}b^2\}$ and $\{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2\}$, $(0\leqslant r\leqslant n-1)$ are the conjugacy classes of U_{6n}. An easy computation shows that $ba^2=a^2b$, $(a^{2r}b)^{-1}=a^{-2r}b^2$, $(a^{2r+1}b^2)^{-1}=a^{-2r-1}b^2$ and $(a^{2r+1}b)^{-1}=a^{-2r-1}b$. So $U_{6n}=\langle S\rangle$, $S=S^{-1}$ and $1\notin S$. Let λ_{j1} , λ_{j2} for $0\leqslant j\leqslant n-1$ (Each one 2 times) and μ_l for $0\leqslant l\leqslant 2n-1$ be $6n$ eigenvalues of $\Gamma(S:U_{6n})$ corresponding to the characters of ρ_j and χ_l of U_{6n}, respectively. Then by Lemma 2.3 and using characters table of U_{6n}, we have:

$\mu_0=\sum\limits_{s\in S}\chi_0(s)=3n-2$, $\mu_n=\sum\limits_{s\in S}\chi_n(s)=n-2$,

$\mu_l=\sum\limits_{s\in S}\chi_l(s)=-2$, for $1\leqslant l\leqslant 2n-1$ and $l\neq n$.

Also for $0\leqslant j\leqslant n-1$, we have:

$$\lambda_{j1}^2+\lambda_{j2}^2=\sum_{s_1,s_2\in A}\rho_j(s_1s_2)+\sum_{s_1,s_2\in B}\rho_j(s_1s_2)+\sum_{s_1\in A,s_2\in B}\big(\rho_j(s_1s_2)+\rho_j(s_2s_1)\big).$$

One can check that :

$$\sum_{s_1,s_2\in A}\rho_j(s_1s_2)=(2n-4)[\sum_{k=1}^{n-1}-\omega^{2kj}+\sum_{k=1}^{n-1}2\omega^{2kj}]+(2n-2)[2+\sum_{k=1}^{n-1}-\omega^{2kj}].$$

$$\sum_{s_1,s_2\in B}\rho_j(s_1s_2)=n\sum_{k=1}^{n-1}2\omega^{2kj}.$$

$$\sum_{s_1\in A,s_2\in B}\big(\rho_j(s_1s_2)+\rho_j(s_2s_1)\big)=0.$$

By Lemma 2.1, $\lambda_{01}^2+\lambda_{02}^2=4n^2-4n+2$ and $\lambda_{j1}^2+\lambda_{j2}^2=2$ for $1\leqslant j\leqslant n-1$. On the other hand, it is clear that $\lambda_{01}+\lambda_{02}=\sum\limits_{s\in S}\rho_0(s)=-2n+2$ and $\lambda_{j1}+\lambda_{j2}=\sum\limits_{s\in S}\rho_j(s)=2$ for $1\leqslant j\leqslant n-1$. So $\lambda_{01}=1$ and $\lambda_{02}=1-2n$ (Each one two times).
Also $\lambda_{j1}=1=\lambda_{j2}$ for $1\leqslant j\leqslant n-1$ (Each one two times). Therefore, the spectrum of $\Gamma(S:U_{6n}))$ is: $[(1-2n)^2, -2^{2n-2}, 1^{4n-2}, n-2, 3n-2]$. $\square$

Corollary 3.12 *For any odd natural number n $(n\neq 1)$, there is at least a connected $(3n-2)$-regular integral graph with $6n$ vertices.*

Character Table of Q_8

	1	a^2	a	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	1	−1	1	−1
χ_4	1	1	−1	−1	1
χ_5	2	−2	0	0	0

Character Table of A_4

	(1)	(1 2)(3 4)	(1 2 3)	(1 3 2)
χ_1	1	1	1	1
χ_2	1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
χ_3	1	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
χ_4	3	-1	0	0

Character Table of $F_{5,4}$

	1	a	b	b^2	b^3
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4	-1	0	0	0

Character Table of $SL(2,3)$

	$g_1 = 1$	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
χ_3	1	1	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	-1	-1	1	1
χ_6	2	-2	0	$-e^{\frac{2\pi i}{3}}$	$-e^{\frac{4\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
χ_7	2	-2	0	$-e^{\frac{4\pi i}{3}}$	$-e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$

$$g_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$g_5^{-1} = g_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, g_7^{-1} = g_6 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Character Table of V_{24}

	1	b^2	a	a^3	a^5	a^2	a^2b^2	b	ab
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	-1	-1
χ_3	1	1	-1	-1	-1	1	1	1	-1
χ_4	1	1	-1	-1	-1	1	1	-1	1
χ_5	2	-2	0	0	0	2	-2	0	0
χ_6	2	-2	$i\sqrt{3}$	0	$-i\sqrt{3}$	-1	1	0	0
χ_7	2	-2	$-i\sqrt{3}$	0	$i\sqrt{3}$	-1	1	0	0
χ_8	2	2	1	-2	1	-1	-1	0	0
χ_9	2	2	-1	2	-1	-1	-1	0	0

Character Table of D_{2n}, $n = 2m + 1$ odd

	1	a^r	b
χ_j	2	$\omega^{jr} + \omega^{-jr}$	0
χ_{m+1}	1	1	-1
χ_{m+2}	1	1	1

$\omega = e^{\frac{2\pi i}{n}}$, $1 \leqslant j \leqslant m$ and $1 \leqslant r \leqslant m$

Character Table of D_{2n}, $n = 2m$ even

	1	a^m	a^r	b	ab
χ_j	2	$2(-1)^j$	$\omega^{jr} + \omega^{-jr}$	0	0
χ_{m+1}	1	1	1	-1	-1
χ_{m+2}	1	$(-1)^m$	$(-1)^r$	1	-1
χ_{m+3}	1	$(-1)^m$	$(-1)^r$	-1	1
χ_{m+4}	1	1	1	1	1

$\omega = e^{\frac{2\pi i}{n}}$, $1 \leqslant j \leqslant m$ and $1 \leqslant r \leqslant m - 1$

Character Table of T_{4n}, $n = 2m + 1$ odd

Characters	a^n	a^r	$a^{2r}b$	$a^{2r+1}b$
χ_1	1	1	1	1
χ_2	-1	$(-1)^r$	i	$-i$
χ_3	1	1	-1	-1
χ_4	-1	$(-1)^r$	$-i$	i
ρ_j	$2(-1)^j$	$\omega^{jr} + \omega^{-jr}$	0	0

$\omega = e^{\frac{2\pi i}{2n}}$, $1 \leqslant j \leqslant n - 1$ and $0 \leqslant r \leqslant n - 1$

Character Table of T_{4n}, $n = 2m$ even

	a^n	a^r	$a^{2r}b$	$a^{2r+1}b$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	$(-1)^r$	1	-1
χ_4	1	$(-1)^r$	-1	1
ρ_j	$2(-1)^j$	$\omega^{jr} + \omega^{-jr}$	0	0

$\omega = e^{\frac{2\pi i}{2n}}$, $1 \leqslant j \leqslant n - 1$ and $0 \leqslant r \leqslant n - 1$

Character Table of U_{6n}

	a^{2r}	$a^{2r}b$	a^{2r+1}
χ_l	ω^{2lr}	ω^{2lr}	ω^{2lr+l}
ρ_j	$2\omega^{2jr}$	$-\omega^{2jr}$	0

$\omega = e^{\frac{2\pi i}{2n}}$, $0 \leqslant l \leqslant 2n - 1$, $0 \leqslant j \leqslant n - 1$ and $0 \leqslant r \leqslant n - 1$

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