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Wholesale Energy Market in a Smart Grid: Dynamic Modeling and Stability

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Abstract—**The recent paradigm shift in the architecture of a smart grid is driven by the need to integrate renewable energy sources, the availability of information via advanced metering and communication, and an emerging policy of a demand structure that is intertwined with pricing. By using smart grid communication technologies that offer dynamic information, the ability to use electricity more efficiently and provide real-time information to utilities is expected to be significantly improved. The introduction of both renewable energy sources as well as efforts to integrate them through an information processing layer brings in dynamic interactions between the major components of a smart grid. In this paper, a dynamic model of the wholesale energy market due to the network constraints is derived. This dynamic model is fundamentally linked to one of the central features of the energy market, of optimal power flow. Beginning with a framework that includes real-time pricing, an attempt is made in this model to capture the dynamic interactions between generation, demand, locational marginal price, and congestion price near the equilibrium of the optimal dispatch. Conditions under which stability of the market can be guaranteed are derived. Numerical studies are reported to illustrate the dynamic model, and its stability properties.**

NOMENCLATURE

θ_n	Set of indices of generating units at node n
ϑ_n	Set of indices of demands at node n
Ω_n	Set of indices of nodes connected to node n
D_q	Set of indices of Consumers $\{1, 2, \ldots, N_D\}$
G_f	Set of indices of generating units $\{1, 2, \ldots, N_G\}$
N	Number of all buses
N_D	Number of demands
N_t	Number of transmission lines
N_G	Number of generating units
A	Bus incidence matrix$(N_t \times N)$
A_d	Consumers incidence matrix where $A_{d_{ij}} = 1$ if the i^{th} consumer is connected to j^{th} bus and $A_{d_{ij}} = 0$ if the i^{th} consumer is not connected to j^{th} bus
A_g	Generators incidence matrix where $A_{g_{ij}} = 1$ if the i^{th} generator is connected to j^{th} bus and $A_{g_{ij}} = 0$ if the i^{th} generator is not connected to j^{th} bus$(N \times N_g)$
A_r	Reduced bus incidence matrix$(N_t \times N - 1)$
B_{line}	Line admittance matrix with elements B_{nm}
B_{nm}	susceptance of line $n - m$
P^{max}	Vector of maximum capacity limit whose elements are P^{max}_{nm}
P^{max}_{nm}	Transmission capacity limit of line $n - m$
P_{Dj}	Power k that the demand j is consuming
P_{Gi}	Power that the generating unit i is producing
R	Rotating matrix where $Rx1 = [\delta_1 ... \delta_{N-1}]$
C_{Gi}	Cost of Generator Company i
U_{Dj}	Utility of of demand j
c_d	Diagonal matrix of consumers utility coefficient diag$\{c_{Dj}\}$
c_g	Diagonal matrix of generators cost coefficient diag$\{c_{Gi}\}$
b_d	Vector of consumers cost coefficient whose elements are$\{b_{Dj}\}$
b_g	Vector of generators cost coefficient whose elements are$\{b_{Gi}\}$
τ_δ	Diagonal matrix of voltage angle time constant diag$\{\tau_\delta\}$
τ_γ	Diagonal matrix of congestion price time constant diag$\{\tau_\gamma\}$
τ_ρ	Diagonal matrix of Locational Marginal Price time constant diag$\{\tau_\rho\}$
τ_d	Diagonal matrix of demand time constant diag$\{\tau_{Dj}\}$
τ_g	Diagonal matrix of generators time constant diag$\{\tau_{Gi}\}$
γ_{nm}	Dual variable associated with the transmission capacity constraint of line $n - m$
ρ_n	Locational marginal price corresponding to the generating unit i or the demand j that is located at node n
δ_n	voltage angle of bus n

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I. INTRODUCTION

Motivated by the growing energy needs of the sustainability amidst compelling sustainability and environmental concerns, a new architecture for energy management, labeled Smart Grid, is emerging where increasingly energy generation, transmission and distribution are expected to be controlled by cyber-enabled and cyber-secure components. The synthesis and analysis of such a smart grid poses several challenges, many of which are temporal in nature. The need to integrate the emerging, varied sources of renewable energy has been a major driver in the development of a smart grid. The introduction of these renewable sources brings in intermittency and hence dynamics that cannot be neglected.

Electricity market models have become a crucial tool for analyzing and predicting the impact of diverse dynamic drivers (e.g., weather, load, fuel prices, and wind supply), physical constraints (e.g., ramping, transmission congestion), and gaming behaviors (e.g., bidding strategies) on market efficiency and prices [1]. Different market models have been used in the literature to capture various aspects of power market dynamics from bilateral contracts, power exchanges, and Poolco markets [2], [3], to databased time-series models [4], game-theoretical formulations [5], [6], and dynamical modeling of supply, (elastic) demand, and real-time pricing [7]. Among all these models, game-theoretical formulations enable one to understand not only how instability might arise but also how it can be prevented through better market designs. A game-theoretical dynamic model of real-time market was proposed in [8], [9], [10] that assume that the market players (e.g., suppliers and consumers) bid recursively in time in the direction that minimizes their marginal cost. A model based predictive control concept was proposed in [11], [12], where ramping constraints, short foresight horizons, and incomplete gaming solutions are suggested as sources of market instability.

In this paper, we develop a dynamic market model that incorporates the interaction between real-time pricing, physical constraints, and demand response based loads. In contrast to the papers above, this market model is directly linked with the standard market clearing structure so that the relation between the state-variables of the dynamic model and the primal variables of the power dispatch model is transparent. Several constraints that are inevitable in a power systems such as those due to capacity limits on power generation and transmission are explicitly included. And finally, stability of the resulting dynamical model which consists of three main participants, generating company, consumers company, and independent system operator is investigated and the region of attraction around the equilibrium of interest is established. The region of attraction for which the real time market is asymptotically stable places an implicit bound on the congestion price as a congestion price cap. In particular, congestion constraints restrict the bidding procedures and thus effect the overall market performance and stability. In this paper, we proposed a systematic approach to properly adjust the congestion price cap such that Locational Marginal Price (LMP) will remain bounded and rational in the presence of market uncertainties without violating system reliability.

This paper has been organized as follows: In section II, we present the necessary preliminaries. In section III, the underlying dynamic model and its stability properties are derived. Finally in section IV, numerical studies are presented. A summary is presented in Section V.

II. Preliminaries

We consider a convex optimization problem of the form:

$$\begin{aligned} &\text{Minimize} \quad f(x) \\ &\text{s.t. } g_i(x) = 0, \quad \forall i = 1, \dots, N \\ &\sum_{i=1}^{N} R_{ji} h_i(x) \le c_j, \quad \forall j = 1, \dots L \end{aligned} \tag{1}$$

where R is a matrix of constants, f, g_i, h_i are differentiable functions and c_j are constants.

A. Dual Decomposition

In order to derive the dual optimization problem, the Lagrangian function is defined as

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{N} \lambda_i g_i(x) + \sum_{j=1}^{L} \mu_j (R_{ji} h_i(x) - c_j) \tag{2}$$

where λ_i and $\mu_j \geq 0$ are (dual) Lagrangian multipliers for the equality and inequality constraints, and x is the primal variable. Denoting

$$D(\lambda, \mu) = inf_x L(x, \lambda, \mu) \tag{3}$$

the dual optimization problem is formulated as

$$\begin{aligned} &\text{Maximize} \quad D(\lambda, \mu) \\ &\text{s.t. } \mu \geq 0, \ \forall j = 1, \dots, L. \end{aligned} \tag{4}$$

Under the condition that the original problem (1) is strictly feasible, then there is no duality gap (i.e. the original (1) and the dual problems (4) have the same optimum). In this case, the dual problem (4) can be solved instead of the original problem (1). In addition, the constraint set for the optimization problem is convex which allows us to use the method of Lagrange multipliers and the Karush Kuhn Tucker (KKT) theorem in [13], [14].

B. Subgradient Algorithm

Often it is simpler to determine the solution of the above optimization problem in an iterative manner. For this purpose, a gradient approach is often employed, and is briefly described below. Since the ultimate goal of constraint optimization problem in (1) is the minimization of a Lagrangian denoted as $L(x, \lambda, \mu)$ in (2), we progressively change x, λ and μ so that minima-Lagrange multiplier pairs λ and μ satisfy the KKT conditions. We do this by using Primal-Dual interior point method [14], [15] which is given by

$$\begin{aligned} x(t+\varepsilon) &= x(t) - k_x \nabla_x L(x, \lambda, \mu)\varepsilon \\ \lambda(t+\varepsilon) &= \lambda(t) + k_\lambda \nabla_\lambda L(x, \lambda, \mu)\varepsilon \\ \mu(t+\varepsilon) &= \mu(t) + k_\mu \left[\nabla_\mu L(x, \lambda, \mu)\right]_\mu^+ \varepsilon \end{aligned} \tag{5}$$

where k_x, k_λ and k_μ are positive scaling parameters which control the amount of change in the direction of the gradient and

$$\left[h(x,y)\right]_y^+ = \begin{cases} h(x,y) & \text{if } y > 0, \\ max(0, h(x,y)) & \text{if } y = 0. \end{cases} \tag{6}$$

ensure that μ_j's are always nonnegative. Letting $\varepsilon \to 0$, we get

$$\begin{aligned} \tau_x \dot{x}(t) &= -\nabla_x L(x, \lambda, \mu) \\ \tau_\lambda \dot{\lambda}(t) &= \nabla_\lambda L(x, \lambda, \mu) \\ \tau_\mu \dot{\mu}(t) &= \left[\nabla_\mu L(x, \lambda, \mu)\right]_\mu^+ \end{aligned} \tag{7}$$

where $\tau_y = 1/k_y$ for $y = x, \lambda$, and μ.

III. Wholesale Energy Market Structure

The electricity market that we consider in this paper is wholesale and is assumed to function as follows: First, each generating company (GenCo) submits the bidding stacks of each of its units to the pool. Similarly, each consumer (ConCo) submits the bidding stacks of each of its demands to the pool. Then, the ISO clears the market using an appropriate market-clearing procedure resulting in prices and production and consumption schedules.

A. ISO Market-Clearing Model

The market-clearing procedure consists of optimizing a cost function, subject to various network constraints. The most dominant network constraints are due to line capacity limits [16] and network losses [3]. The power flow through any line is often limited due to technical constraints and is said to be congested when it approaches its maximum limit [3]. This constraint is explicitly included in our model below. For ease of exposition, ohmic losses are not modeled in this paper.

The cost function that is typically used is referred to as Social Welfare. Denoted as S_w, Social Welfare is defined as

$$S_W = \sum_{j\in D_q} U_{Dj}(P_{Dj}) - \sum_{i\in G_f} C_{G_i}(P_{Gi}) \tag{8}$$

where the first and second term denote the revenue due to surpluses stemming from bids from GenCo and ConCo, respectively. $U_{Dj}(P_{Dj})$ and $C_{G_i}(P_{Gi})$ correspond to utility of consumers and cost of generators company and is defined as:

$$C_{G_i}(P_{Gi}) = b_{Gi}P_{Gi} + \frac{c_{Gi}}{2}P_{Gi}^2; \quad P_{Gi}^{min} \le P_{Gi} \le P_{Gi}^{max} \tag{9}$$

$$U(P_{Dj}) = b_{Dj}P_{Dj} + \frac{c_{Dj}}{2}P_{Dj}^2; \quad P_{Dj}^{min} \le P_{Dj} \le P_{Dj}^{max} \tag{10}$$

where P_{Gi}^{min} and P_{Gi}^{max} are lower and upper bounds for GenCo and P_{Dj}^{min} and P_{Dj}^{max} are lower and upper bounds for ConCo, respectively. In summary, the market-clearing procedure is given by

$$\text{Maximize } S_W = \text{Minimize } -S_W \tag{11}$$

s.t.

$$-\sum_{i\in\theta_n} P_{Gi} + \sum_{j\in\vartheta_n} P_{Dj} + \sum_{m\in\Omega_n} B_{nm}\left[\delta_n - \delta_m\right] = 0; \ \rho_n \tag{12}$$

$$B_{nm}\left[\delta_n - \delta_m\right] \le P_{nm}^{max}; \quad \gamma_{nm}, \forall n \in N; \forall m \in \Omega. \tag{13}$$

The constraints (12) and (13) are due to power balance and capacity limits, respectively. It can be seen that the associated Lagrange multipliers, ρ_n and γ_{nm}, are indicated in each constraint. The underlying optimization problem of the ISO can therefore be defined as the optimization of (11) subject to constraints (12) and (13).

The resulting solution can be determined, using KKT conditions [13], as P_{Gi}^*, the amounts of power to be generated by each generating unit i, P_{Dj}^*, the amounts of power to be consumed by each consumer j, the locational marginal prices, ρ_n^*, and congestion price γ_{nm}^* that satisfies the following conditions:

$$\frac{d(C_{G_i}(P_{Gi}))}{dP_{Gi}}|_{P_{Gi}^*} - \rho_{n(i)}^* = 0 \quad \forall i \in G_f \tag{14a}$$

$$\rho_{n(j)}^* - \frac{d(U_{Dj}(P_{Dj}))}{dP_{Dj}}|_{P_{Dj}^*} = 0 \quad \forall j \in D_q \tag{14b}$$

$$\sum_{m\in\Omega_n} B_{nm}\left[\rho_n^* - \rho_m^* + \gamma_{nm}^* - \gamma_{mn}^*\right] = 0 \quad \forall n \in N \tag{14c}$$

$$-\sum_{i\in\theta_n} P_{Gi}^* + \sum_{j\in\vartheta_n} P_{Dj}^* + \sum_{m\in\Omega_n} B_{nm}\left[\delta_n^* - \delta_m^*\right] = 0 \, \forall n \in N \tag{14d}$$

$$\gamma_{nm}^*\left(B_{nm}\left[\delta_n^* - \delta_m^*\right] - P_{nm}^{max}\right) = 0 \quad \forall n \in N; \forall m \in \Omega_n. \tag{14e}$$

B. Game-Theoretical Dynamic Model of Wholesale Market

The optimization problem in (11)-(13) can be viewed alternately as a game between the GenCos, ConCos, and the ISO, where each of these three players attempts to maximize their own benefit. Instead of solving Eq. (14a)-(14e) as a static optimization problem, we take a dynamic approach, inspired by Eq. (7). Suppose that the underlying primal and dual variables are perturbed from their corresponding equilibrium to P_{Gi}, P_{Dj}, ρ_n, and γ_{nm}. Using (7) and (9), we can derive a differential equation for the ith GenCo $\forall i \in G_f$ as

$$\tau_{G_i}\dot{P_{Gi}} = \rho_{n(i)} - c_{Gi}P_{Gi} - b_{Gi} \tag{15}$$

with the goal of driving its solution P_{Gi} to the equilibrium P_{Gi}^* which solves (14a). Similarly, using (7) and (10), a differential equation can be derived for the jth ConCo $\forall j \in D_q$ as

$$\tau_{D_j}\dot{P_{Dj}} = c_{Dj}P_{Dj} + b_{Dj} - \rho_{n(j)} \tag{16}$$

where τ_{Gi} and τ_{Dj} are time-constants that can be adjusted so as to result in an optimal convergence of these solutions to the equilibrium in (14a)-(14e). Finally, differential equations for the LMPs, congestion price and phase angles can be determined as

$$\tau_{\delta_n}\dot{\delta_n} = -\sum_{m\in\Omega_n} B_{nm}\left[\rho_n - \rho_m + \gamma_{nm} - \gamma_{mn}\right] \tag{17}$$

$$\tau_{\rho_n}\dot{\rho_n} = -\sum_{i\in\theta_n} P_{Gi} + \sum_{j\in\vartheta_n} P_{Dj} + \sum_{m\in\Omega_n} B_{nm}\left[\delta_n - \delta_m\right] \tag{18}$$

$$\tau_{nm}\dot{\gamma}_{nm} = \left[B_{nm}\left[\delta_n - \delta_m\right] - P_{nm}^{max}\right]_{\gamma_{nm}}^{+}. \tag{19}$$

Equations (15)-(19) represent a dynamic model of the overall wholesale energy market.

Two important points should be made regarding the above model. The solution of this model $P_{Gi}(t)$, $P_{Dj}(t)$, $\delta_n(t)$, $\rho_n(t)$, and $\gamma_{nm}(t)$ converges to the equilibrium in (14a)-(14e), as $t \to \infty$ if the overall system of equations is stable. At all other transient times, the trajectories $P_{Gi}(t)$, $P_{Dj}(t)$, $\delta_n(t)$, $\rho_n(t)$, and $\gamma_{nm}(t)$ represent the specific path that these variables take, when perturbed, as they converge towards the optimal solution. In other words, $(P_{Gi}(t), P_{Dj}(t), \delta_n(t), \rho_n(t), \gamma_{nm}(t))$ is distinct from the optimal solution $(P_{Gi}^*, P_{Dj}^*, \delta_n^*, \rho_n^*, \gamma_{nm}^*)$ and coincides with it at infinity if the market is stable.

The second point that should be noted about the above dynamic model is its decentralized nature. It can be seen that given the LMP at node i, Equation (15) can be assembled and solved completely by GenCo i and Eq. (16) by ConCo j using the LMP at node j. That is, GenCo i decides their generation quantities by estimating their own marginal profit, as exemplified by c_{Gi}, and b_{Gi}. At any given iteration, if the marginal profit of the GenCo is greater than zero, Eq. (15) implies that the GenCo will increase P_{Gi} to obtain a greater economic benefit; if the marginal profit of the GenCo is less than zero, the Genco will decrease P_{Gi}. ConCo j updates its consumption using Eq. (16) in a similar manner. These players then transmit the information to the ISO, over a communication network with low latencies, which then proceeds to solve Eqs. (17) to (19). Eq. (17) implies that the dynamic of voltage angle of bus n depends on the corresponding locational marginal price and also congestion price. Eq. (18) describes the evolution of the locational marginal price, and implies that every ρ_n at a node n is affected by the energy imbalance at that node. Eq. (19) describes the evolution of the congestion price, and implies that for each transmission line from bus n to bus m, the congestion price is affected by the empty capacity that is the difference of the line flow denoted by $B_{nm}\left[\delta_n - \delta_m\right]$ and maximum thermal capacity P_{nm}^{max}. If the overall system is stable, such an iterative procedure between the market participants, GenCos, ConCos, and ISO, evolving according to the strategies given by Eqs. (15)-(19), will guarantee convergence to the optimal solution.

The above dynamic model is a significant departure from the current practice where information is exchanged only once between the GenCos and the ISO following which the ISO clears the market and provides information regarding the price. Our thesis here is that due to the huge volatility and uncertainty of the dynamic drivers such as wind and solar energy sources, and load in the market, such a single iteration will not suffice, and stability cannot be ensured; continued iteration as suggested by the dynamic model above is needed in order to mitigate volatility in real-time price and ensure a stable market design. In the subsequent sections, guidelines for determining stability with such an iterative exchange of information between the different players are discussed.

C. Equilibrium of Wholesale Market

Using Eqs. (15)-(19), the dynamic model of the wholesale energy market can be written compactly as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & 0\end{array}\right]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix} b \\ f_2(x_1,x_2)\end{bmatrix} \tag{20}$$

where

$$x_1(t) = \begin{bmatrix}P_{Gi} & P_{Dj} & \delta_n & \rho_n\end{bmatrix}^T_{(Ng+N_d+2N-1)\times 1} \tag{21}$$

$$x_2(t) = \begin{bmatrix}\gamma_1 & \dots & \gamma_{Nt}\end{bmatrix}^T_{Nt\times 1} \tag{22}$$

$A_1 =$

$$\begin{bmatrix} -\tau_g^{-1}c_g & 0 & 0 & \tau_g^{-1}A_g^T \\ 0 & \tau_d^{-1}c_d & 0 & -\tau_d^{-1}A_d^T \\ 0 & 0 & 0 & -\tau_\delta^{-1}A_r^T B_{line}A \\ -\tau_\rho^{-1}A_g & \tau_\rho^{-1}A_d & \tau_\rho^{-1}A^T B_{line}A_r & 0 \end{bmatrix} \tag{23}$$

$$A_2 = \begin{bmatrix}0 & 0 & -B_{line}^T A_r \tau_\delta^{-1} & 0\end{bmatrix}^T \tag{24}$$

B_{line} denotes the line admittance matrix (N_t by N_t diagonal matrix) with elements B_{nm} and let A denote the $N_t \times N$ bus incidence matrix. Let A_r denote the reduced bus incidence matrix ($N_t \times N-1$) which is A with column corresponding to reference bus removed. A_g is generators incidence matrix where $A_{g_{ij}} = 1$ if the i^{th} generator is connected to j^{th} bus and $A_{g_{ij}} = 0$ if the i^{th} generator is not connected to j^{th} bus, similarly for A_d which is load incident matrix where $A_{d_{ij}} = 1$ if the i^{th} consumer is connected to j^{th} bus and $A_{d_{ij}} = 0$ if the i^{th} consumer is not connected to j^{th} bus. Finally

$$b = \begin{bmatrix}b_g^T\tau_g^{-1} & b_d^T\tau_d^{-1} & 0\end{bmatrix}^T \tag{25}$$

$$f_2(x_1,x_2) = \left[\tau_\gamma^{-1}\left[B_{line}A_rRx_1 - P^{max}\right]^+_{x_2}\right] \tag{26}$$

where $Rx_1 = [\delta_1 \dots \delta_{N-1}]^T$ and R is rotating matrix ($N-1\times N_g+N_d+2N-1$) and P^{max} denotes a vector with maximum capacity limit of transmission lines ($N_t \times 1$) whose elements are P_{nm}^{max}. Before we analyze the stability of the dynamic market model in Eqs. (15)-(19), we evaluate its equilibria.

We denote the equilibrium set of the wholesale market given by the game in Eqs. (15)-(19) as

$$E = \{(x_1,x_2) | A_1x_1 + A_2x_2 + b = 0 \wedge f_2(x_1,x_2) = 0\} \tag{27}$$

and (x_1^*, x_2^*) is an equilibrium point of the wholesale market under the game given by Eqs. (15)-(19) if $(x_1^*, x_2^*) \in E$. The goal of this paper is to study the stability of the game in (15)-(19) around $(x_1^*, x_2^*) \in E$ using Lyapunov stability. For this purpose, we use the following definition.

Definition 1: The equilibrium point of the wholesale energy market denoted as $(x_1^*, x_2^*) \in E$ defined by the dynamical game (15)-(19) is stable if for each $\epsilon > 0$, there is $\sigma = \sigma(\epsilon)$ that given $\left((x_1(0)-x_1^*)^T\ (x_2(0)-x_2^*)^T\right)^T \in \Omega(\sigma) := \{x|\ ||x|| < \sigma\}$, there exist the feasible sequences of P_{G_i}, P_{D_j}, and δ_n such that $\left((x_1(t)-x_1^*)^T\ (x_2(t)-x_2^*)^T\right)^T \in \Omega(\epsilon)\ \forall t \geq 0$.

D. Nominal Stability of Electrical Market

We now establish the stability property of the equilibrium using the Lyapunov approach.

In what follows, it is assumed that strong duality holds and there exists $(x_1^*, x_2^*) \in E$. The stability and region of attraction around this equilibrium is established in Theorem 1.

Theorem 1: Let strong duality hold. Then the equilibrium $(x_1^*, x_2^*) \in E$ of the wholesale market defined by the dynamic game (20) is asymptotically stable for all initial conditions in $\Omega_{c_{max}} = \{(y_1,y_2)\ |\ V(y_1,y_2) \leq c\}$ for a $c_{max} > 0$ where $\Omega_{c_{max}} \subsetneq D = \{(y_1,y_2)\ |\ ||y_2|| \leq d\}$ with

$$d = \frac{2\lambda_{min}(P_2)\psi_{min}\lambda_{min}(Q)}{\tau_{\gamma_{max}}\beta^2},$$

if A_1 is Hurwitz.

Proof: Since strong duality holds, it follows that the dynamic game (20) has an equilibrium point denoted by $(x_1^*, x_2^*) \in E$. We first establish stability of this equilibrium point based on Definition 1 and then proceed to its asymptotic stability.

(i) Stability: Differentiating positive definite Lyapunov function $V(y_1,y_2) = y_1^TP_1y_1 + y_2^TP_2y_2$ with respect to time where $y_1 = x_1 - x_1^*$, and $y_2 = x_2 - x_2^*$ and due to the fact that $\left[B_{line}A_rRy_1 - P^{max}\right]^+_{y_2} \leq B_{line}A_rRy_1 - P^{max}$, we have

$$\begin{aligned}\dot{V}(y_1,y_2) &\leq y_1^T(P_1A_1 + A_1^TP_1)y_1 + y_1^TP_1A_2y_2 + y_2^TA_2^TP_1y_1 \\ &+ y_2^TP_2\left(\tau_\gamma^{-1}B_{line}A_rRy_1 - \tau_\gamma^{-1}P^{max}\right) + \\ &\left(\tau_\gamma^{-1}B_{line}A_rRy_1 - \tau_\gamma^{-1}P^{max}\right)^TP_2y_2\end{aligned} \tag{28}$$

If A_1 is Hurwitz, for any $Q > 0$, there exists a positive definite matrix P_1 such that $P_1 A_1 + A_1^T P_1 = -Q$. Let $\lambda_{min}(Q)$ denote the minimum eigenvalue of Q. Since P_2 is a symmetric positive definite matrices, with a set of N_t orthogonal, real and nonzero eigenvectors $x_1, ...x_n$, can be written $P_2 = \sum_{i=1}^{N_t} \lambda_i x_i x_i^T$ where $\lambda_i > 0$ is the eigenvalue corresponding to x_i. We can expand the vector P^{max} using the orthogonal vector w_i as $P^{max} = \sum_{i=1}^{N_t} \psi_i w_i$, this implies that

$$P^{max^T} \tau_\gamma^{-1} P_2 y_2 \geq \frac{\lambda_{min}(P_2)\psi_{min}}{\tau_{\gamma_{max}}} ||y_2|| \quad (29)$$

where $\psi_{min} = min(\psi_i) \quad \forall i = 1, ..., N_t$.

Let $||P_1 A_2 + R^T A_r^T B_{line}^T \tau_\gamma^{-1} P_2|| \leq \beta$, we obtain that

$$\begin{aligned} & y_1^T (P_1 A_2 + R^T A_r^T B_{line}^T \tau_\gamma^{-1} P_2) y_2 + \\ & y_2^T (A_2^T P_1 + P_2 \tau_\gamma^{-1} B_{line} A_r R) y_1 \leq 2\beta ||y_1|| ||y_2||. \end{aligned} \quad (30)$$

Using Eqs. (29)-(30), we have

$$\begin{aligned} \dot{V}(y_1, y_2) \leq & -\lambda_{min}(Q) \left(||y_1|| - \frac{\beta}{\lambda_{min}(Q)} ||y_2|| \right)^2 \\ & - ||y_2|| \left(2 \frac{\lambda_{min}(P_2)\psi_{min}}{\tau_{\gamma_{max}}} - \frac{\beta^2}{\lambda_{min}(Q)} ||y_2|| \right). \end{aligned} \quad (31)$$

where $\tau_{\gamma_{max}} = \max(\tau_{\gamma_{nm}})$. For all $\Omega_{c_{max}} \subsetneq D$, it follows that for all solutions beginning in $\Omega_{c_{max}}$, $\dot{V} \leq 0$. Hence the equilibrium is stable, and $\Omega_{c_{max}}$ is the region of attraction.

(ii) Asymptotic stability: We now show that all solutions beginning in $\Omega_{c_{max}}$ will converge to the equilibrium point. Eq. (31) can be rewritten as

$$\dot{V}(y_1, y_2) \leq -a(||y_1|| - b||y_2||)^2 - ||y_2||(e - f||y_2||)$$

where $a = \lambda_{min}(Q)$, $b = \frac{\beta}{\lambda_{min}(Q)}$, $e = 2\frac{\lambda_{min}(P_2)\psi_{min}}{\tau_{\gamma_{max}}}$, and $f = \frac{\beta^2}{\lambda_{min}(Q)}$.

That is, $\dot{V}$ can be zero if $(||y_1||, ||y_2||) = \left(\frac{be}{f}, \frac{e}{f}\right)$ or if $(||y_1||, ||y_2||) = (0, 0)$. Note that $||y_2|| = \frac{e}{f}$ implies that the solution lies on D. However, since the initial conditions start in $\Omega_{c_{max}}$ and the latter is a strict subset of D, y_2 cannot be equal to $\frac{e}{f}$ in $\Omega_{c_{max}}$. This in turn implies that $(||y_1||, ||y_2||) = (0, 0)$ is the only invariant set. Hence all solutions (y_1, y_2), starting in $\Omega_{c_{max}}$ converge to the equilibrium point $(x_1, x_2) = (x_1^*, x_2^*)$, which establishes asymptotic stability. ■

Remark 1: The region of attraction Ω_{max} for which stability and asymptotic stability hold places an implicit bound on the congestion price. In particular, it implies that the congestion price needs to be smaller than d, which is proportional to P_{max}. We note that this is similar to the standard interpretation of congestion price, that it is the marginal cost for relieving one MW of congestion in a given constraint line [17]. This condition also provides a warning signal for the ISO, that the system reaches its stability limit as the congestion price approaches the congestion price cap d.

IV. Illustrative Example

We now numerically evaluate the stability of the equilibrium of the energy market using a standard 4-bus network in [2]. The network includes two generating units located at node 1 which corresponds to a base-load generator with the low cost coefficients $c_{G_1} = 0.25^{[\$/MW^2h]}$, $b_{G_1} = 47.2^{[\$/MWh]}$ and has a slow dynamics corresponding to the higher time constant $\tau_{G_1} = 48^{[\$/MW]}$ and a peaking generator located at node 2 has a fast dynamics corresponding to the lower time constant $\tau_{G_2} = 10^{[\$/MW]}$ but high cost coefficients $c_{G_2} = 0.53^{[\$/MW^2h]}$, $b_{G_2} = 48.8^{[\$/MWh]}$. The latter can be assumed to be a spinning reserve to compensate for demand fluctuations that may occur in bus 2. There are power consumption at nodes 3 and 4 with the utility coefficients $c_{D_1} = c_{D_2} = -0.41^{[\$/MW^2h]}$, $b_{D_1} = 70^{[\$/MWh]}$, $b_{D_2} = 73^{[\$/MWh]}$ and their respective time constants are $\tau_{D_1} = \tau_{D_2} = 5^{[\$/MW]}$. Transmission line parameters such as B_{nm} and the line capacity limits P_{nm}^{max} can be found in [2]. We assume that the market time constants $\tau_\rho = 5^{MWh^2/\$}$, and $\tau_\delta = 5^{\$/MW}$. We consider two cases, labeled Case 1 and Case 2, different cases with the same parameters and all initial conditions being the same except for $\delta_2(0)$ (see Table I). The results obtained are shown in Figures 1 to 4. Figures 1 and 2 show responses of the critical state variables, P_{G1}, P_{G2}, P_{D1}, P_{D2}, and $\rho_n \quad \forall n = \{1, ..., 4\}$ for the initial conditions these two cases. It can be seen in Figure 1 that P_{G1} supplies the base-load consumption and P_{G2} is dispatched to follow up load fluctuations. Since transmission lines are not congested, Locational Marginal Prices (LMPs), ρ_n, converge to the same value for all n buses, and the wholesale market is stable with the given parameters. However, when $\delta_2(0)$ is increased from 8^{deg} to 12^{deg}, the wholesale market exhibits instability, as illustrated in Figure 2. The difference in the stable and unstable solutions for Cases 1 and 2 is also illustrated in Figure 3 using a projection of the phase-plane.

A more detailed study of the sensitivity to initial conditions was also carried out. Starting with the initial conditions in Case 1, we perturbed each of the fifteen state variables as $x_i + \Delta_i$ while keeping all $j \neq i$ constant, and determined the maximum Δ_{max_i} that led to instability. It follows that larger the Δ_{max_i}, the higher the robustness to perturbations in that particular state x_i. The values Δ_{max_i} are shown in Figure 4 for each of the fifteen state variables. As Figure 4 shows, the most sensitive states, i.e. the states that possess the smallest set Δ_{max_i}'s correspond to the phase angles $\delta_n \forall n = \{1...4\}$. These sensitivity studies can provide guidance for the design of robust control.

V. Summary

Increasing demand for electrical power generation and the current energy crisis have created an urgent need in incorporating renewable energy sources into the power grid, using available information via communication networks and regulating the overall active agents. In this paper, we begin with the standard market clearing procedure and capture the dynamics of the real-time market using Primal-Dual interior point method. In particular, a gradient-based algorithm is used to derive the dynamic evolution of the primal variables and dual variables to reach the optimum solution of the real-time market. The stability of the resulting dynamical model of the real-time market is investigated and the region of attraction around the equilibrium of interest is established. This region for which the real-time market is asymptotically stable places an implicit bound on the congestion price.

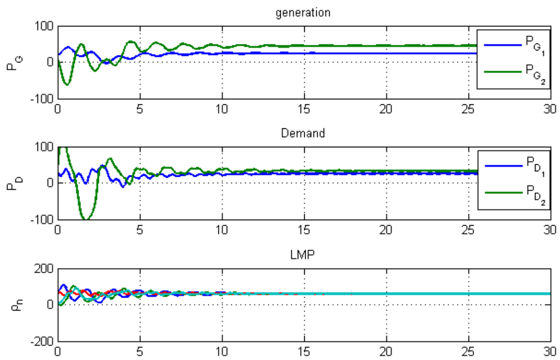

Fig. 1. Market dynamic transients for GenCos P_{G1}, and P_{G2}, ConCo P_{D1}, and P_{D2}, and Locational Marginal Prices $\rho_n \quad \forall n = 1, ..., 4$ with initial conditions in Table III, Case 1.

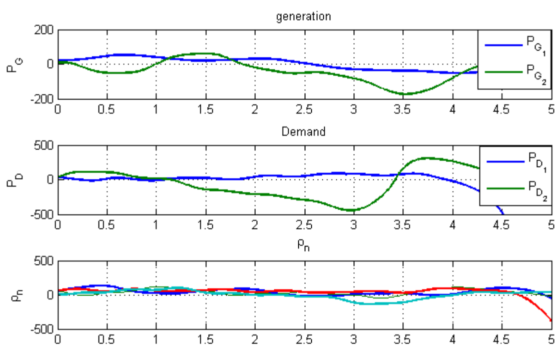

Fig. 2. Market dynamic instability after a sudden increase of load in bus 3 for GenCos P_{G1}, P_{G2}, ConCo P_{D1}, P_{D2}, and Locational Marginal Prices $\rho_n \quad \forall n = \{1, ..., 4\}$ with initial conditions in Table III, Case 2.

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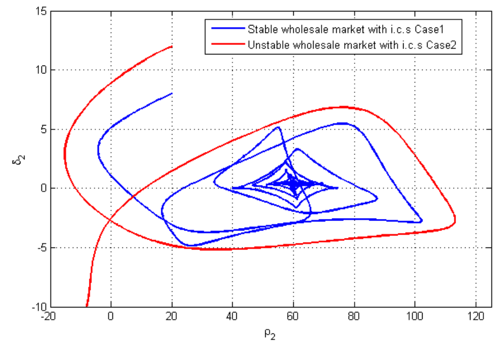

Fig. 3. Blue curve is stable phase plane for initial conditions belong to $\Omega_c = \{(y_1, y_2) \mid V(y_1, y_2) \leq 12.787\}$, Red curve is unstable phase plane due the disturbance.

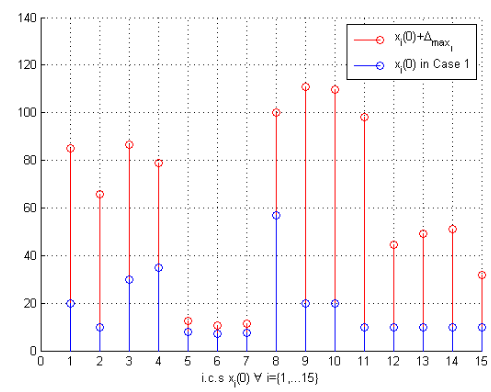

Fig. 4. Relative size of sensitivity Δ_{max_i} to perturbation in $x_i(0)$ with $x_j(0), j \neq i$ fixed, for $i = \{1, \ldots, 15\}$.

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