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Why Mixture of Probability Distributions?

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Abstract

If we have two random variables ξ_1 and ξ_2, then we can form their *mixture* if we take ξ_1 with some probability w and ξ_2 with the remaining probability $1-w$. The probability density function (pdf) $\rho(x)$ of the mixture is a convex combination of the pdfs of the original variables: $\rho(x) = w \cdot \rho_1(x) + (1-w) \cdot \rho_2(x)$. A natural question is: can we use other functions $f(\rho_1, \rho_2)$ to combine the pdfs, i.e., to produce a new pdf $\rho(x) = f(\rho_1(x), \rho_2(x))$? In this paper, we prove that the only combination operations that always lead to a pdf are the operations

$$f(\rho_1, \rho_2) = w \cdot \rho_1 + (1-w) \cdot \rho_2$$

corresponding to mixture.

1 Formulation of the Problem

What is mixture. If we have two random variables ξ_1 and ξ_2, then, for each probability $w \in [0,1]$, we can form a *mixture* ξ of these variables by selecting ξ_1 with probability w and ξ_2 with the remaining probability $1-w$; see, e.g., [1].

In particular, if we know the probability density function (pdf) $\rho_1(x)$ corresponding to the first random variable and the probability density function $\rho_2(x)$ corresponding to the second random variable, then the probability density function $\rho(x)$ corresponding to their mixture has the usual form

$$\rho(x) = w \cdot \rho_1(x) + (1-w) \cdot \rho_2(x). \tag{1}$$

A natural question. A natural question is: are there other combination operations $f(\rho_1, \rho_2)$ that always transform two probability distributions $\rho_1(x)$ and $\rho_2(x)$ into a new probability distribution

$$\rho(x) = f(\rho_1(x), \rho_2(x)). \tag{2}$$

Our result. Our result is that the only possible transformation (2) that always generates a probability distribution is the mixture (1), for which

$$f(\rho_1, \rho_2) = w \cdot \rho_1 + (1 - w) \cdot \rho_2 \tag{3}$$

for some $w \in [0, 1]$.

2 Main Result

Definition 1. *We say that a function $f(\rho_1, \rho_2)$ that maps pairs of non-negative real numbers into a non-negative real number is a* probability combination operation *if for every two probability density functions $\rho_1(x)$ and $\rho_2(x)$ defined on the same set X, the function $\rho(x) = f(\rho_1(x), \rho_2(x))$ is also a probability density function, i.e., $\int \rho(x)\,dx = 1$.*

Proposition. *A function $f(\rho_1, \rho_2)$ is a probability combination operation if and only if it has the form $f(\rho_1, \rho_2) = w \cdot \rho_1 + (1 - w) \cdot \rho_2$ for some $w \in [0, 1]$.*

Proof.

1°. Let us first prove that $f(0, 0) = 1$.

Indeed, let us take $X = \mathbb{R}$, and the following pdfs:

- $\rho_1(x) = \rho_2(x) = 1$ for $x \in [0, 1]$ and
- $\rho_1(x) = \rho_2(x) = 0$ for all other values x.

Then, the combined function $\rho(x) = f(\rho_1(x), \rho_2(x))$ has the following form:

- $\rho(x) = f(1, 1)$ when $x \in [0, 1]$ and
- $\rho(x) = f(0, 0)$ for $x \notin [0, 1]$.

Let us use the condition $\int \rho(x)\,dx = 1$ to prove that $f(0, 0) = 0$.

We can prove it by contradiction. If we had $f(0, 0) \neq 0$, i.e., if we had $f(0, 0) > 0$, then we would have

$$\int \rho(x)\,dx = f(1, 1) \cdot 1 + f(0, 0) \cdot \infty = \infty \neq 1.$$

Thus, we should have $f(0, 0) = 0$.

2°. Let us now prove that $f(0, \rho_2) = k_2 \cdot \rho_2$ for some $k_2 \geq 0$.

Let us take the following function $\rho_1(x)$:

- $\rho_1(x) = 1$ for $x \in [-1, 0]$ and
- $\rho_1(x) = 0$ for all other x.

Let us now pick any number $\rho_2 > 0$ and define the following pdf $\rho_2(x)$:

- $\rho_2(x) = \rho_2$ for $x \in [0, 1/\rho_2]$ and
- $\rho_2(x) = 0$ for all other x.

In this case, the combined function $\rho(x)$ has the following form:

- $\rho(x) = f(1, 0)$ for $x \in [-1, 0]$;
- $\rho(x) = f(0, \rho_2)$ for $x \in [0, 1/\rho_2]$, and
- $\rho(x) = f(0, 0) = 0$ for all other x.

Thus, the condition $\int \rho(x)\,dx = 1$ takes the form

$$f(1, 0) + f(0, \rho_2) \cdot (1/\rho_2) = 1,$$

hence $f(0, \rho_2) \cdot (1/\rho_2) = 1 - f(1, 0)$ and therefore, $f(0, \rho_2) = k_2 \cdot \rho_2$, where we denoted $k_2 \stackrel{\text{def}}{=} 1 - f(1, 0)$.

3°. Similarly, we can prove that $f(\rho_1, 0) = k_1 \cdot \rho_1$ for some $k_1 \geq 0$.

4°. Let us now prove that for all ρ_1 and ρ_2, we have $f(\rho_1, \rho_2) = k_1 \cdot \rho_1 + k_2 \cdot \rho_2$.

We already know, from Parts 1, 2 and 3 of this proof, that the desired equality holds when one of the values ρ_i is equal to 0.

Let us now take any values $\rho_1 > 0$ and $\rho_2 > 0$. Let us then pick a positive value $\Delta \leq 1/\max(\rho_1, \rho_2)$ and define the following pdfs. The first pdf $\rho_1(x)$ is defined by the following formulas:

- $\rho_1(x) = \rho_1$ for $x \in [0, \Delta]$,
- $\rho_1(x) = 1$ for $x \in [-(1 - \Delta \cdot \rho_1), 0]$, and
- $\rho_1(x) = 0$ for all other x.

The second pdf $\rho_2(x)$ is defined by the following formula:

- $\rho_2(x) = \rho_2$ for $x \in [0, \Delta]$,
- $\rho_2(x) = 1$ for $x \in [\Delta, \Delta + (1 - \Delta \cdot \rho_2)]$, and
- $\rho_2(x) = 0$ for all other x.

Then, the combined function $\rho(x) = f(\rho_1(x), \rho_2(x))$ has the following form

- $\rho(x) = f(1, 0) = k_1$ for $x \in [-(1 - \Delta \cdot \rho_1), 0]$,
- $\rho(x) = f(\rho_1, \rho_2)$ for $x \in [0, \Delta]$,
- $\rho(x) = f(0, 1) = k_2$ for $x \in [\Delta, \Delta + (1 - \Delta \cdot \rho_2)]$, and

- $\rho(x) = f(0,0) = 0$ for all other x.

For this combined function $\rho(x)$, the condition that $\int \rho(x)\,dx = 1$ takes the form

$$k_1 \cdot (1 - \Delta \cdot \rho_1) + f(\rho_1, \rho_2) \cdot \Delta + k_2 \cdot (1 - \Delta \cdot \rho_2) = 1. \tag{4}$$

Let us now consider a different pair of pdfs, $\rho'_1(x)$ and $\rho'_2(x)$. The first pdf $\rho'_1(x)$ is defined by the following formulas:

- $\rho'_1(x) = 2\rho_1$ for $x \in [0, \Delta/2]$,
- $\rho'_1(x) = 1$ for $x \in [-(1 - \Delta \cdot \rho_1), 0]$, and
- $\rho'_1(x) = 0$ for all other x.

The second pdf $\rho'_2(x)$ is defined by the following formula:

- $\rho'_2(x) = 2\rho_2$ for $x \in [\Delta/2, \Delta]$,
- $\rho'_2(x) = 1$ for $x \in [\Delta, \Delta + (1 - \Delta \cdot \rho_2)]$, and
- $\rho'_2(x) = 0$ for all other x.

Then, the combined function $\rho'(x) = f(\rho'_1(x), \rho'_2(x))$ has the following form

- $\rho'(x) = f(1,0) = k_1$ for $x \in [-(1 - \Delta \cdot \rho_1), 0]$,
- $\rho'(x) = f(2\rho_1, 0) = k_1 \cdot (2\rho_1)$ for $x \in [0, \Delta/2]$,
- $\rho'(x) = f(0, 2\rho_2) = k_2 \cdot (2\rho_2)$ for $x \in [\Delta/2, \Delta]$,
- $\rho'(x) = f(0,1) = k_2$ for $x \in [\Delta, \Delta + (1 - \Delta \cdot \rho_2)]$, and
- $\rho'(x) = f(0,0) = 0$ for all other x.

For this combined function $\rho'(x)$, the condition that $\int \rho'(x)\,dx = 1$ takes the form

$$k_1 \cdot (1 - \Delta \cdot \rho_1) + k_1 \cdot (2\rho_1) \cdot (\Delta/2) + k_2 \cdot (2\rho_2) \cdot (\Delta/2) + k_2 \cdot (1 - \Delta \cdot \rho_2) = 1. \tag{5}$$

If we subtract (5) from (4) and divide the difference by $\Delta > 0$, then we conclude that $f(\rho_1, \rho_2) - k_1 \cdot \rho_1 - k_2 \cdot \rho_2 = 0$, i.e., exactly what we want to prove in this section.

5°. To complete the proof, we need to show that $k_2 = 1 - k_1$, i.e., that $k_1 + k_2 = 1$.

Indeed, let us take:

- $\rho_1(x) = \rho_2(x) = 1$ when $x \in [0, 1]$ and
- $\rho_1(x) = \rho_2(x)$ for all other x.

Then, for the combined pdf, we have:

- $\rho(x) = f(\rho_1(x), \rho_2(x)) = k_1 + k_2$ for $x \in [0,1]$ and
- $\rho(x) = 0$ for all other x.

For this combined function $\rho(x)$, the condition $\int \rho(x)\,dx = 1$ implies that

$$k_1 + k_2 = 1.$$

The proposition is proven.

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