Why Numbers are Sets*

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ABSTRACT: I follow standard mathematical practice and theory to argue that the natural numbers are the finite von Neumann ordinals. I present the reasons standardly given for identifying the natural numbers with the finite von Neumann's (e.g. recursiveness; well-ordering principles; continuity at transfinite limits; minimality, and identification of n with the set of all numbers less than n). I give a detailed mathematical demonstration that 0 is {} and for every natural number n, n is the set of all natural numbers less than n. Natural numbers are sets. They are the finite von Neumann ordinals.

1. Introduction

I challenge Benacerraf's (1965) argument that natural numbers cannot be sets. It is surprising that Benacerraf's argument has seen so much philosophical discussion but so little mathematical criticism.[1] The argument is not mathematically sound. Benacerraf (1965: 277; 1996: 21) says that any reduction of the natural numbers to sets has two and only two parts: (1) the part that enables us to formulate the laws of arithmetic and (2) the part that enables us to analyze cardinality in terms of counting. Lest there be any doubt: "Everything else is extraneous" (1996: 21). An informal analysis of cardinality in terms of counting says that we determine the cardinality of a set by taking "its elements one by one as we say the numbers one by one" (1965: 274 - 275). More formally: the cardinal number of a set S is n if and only if there is a 1-1 correspondence between S and the set of numbers less than n (1996: 21, 46). Although it is usually thought that infinitely many progressions satisfy the conditions that Benacerraf articulates for *being* the natural numbers, I will argue that in fact one and only one progression satisfies those conditions. I will show that the conditions that Benacerraf articulates determine an association of natural numbers with sets of natural numbers in such a way that 0 is {} and n is {0, . . . n-1} for every n > 0. I will produce a *mathematical demonstration* that if α is any progression that satisfies Benacerraf's conditions, then α is the finite von Neumann ordinals. If Benacerraf is right about the natural numbers, then as a matter of mathematical fact every number does indeed have a certain specific internal set-theoretical structure. Benacerraf should not have concluded that numbers cannot be sets. He should have concluded that they *must* be the finite von Neumann ordinals. If Benacerraf's analysis of what it means to *be* the natural numbers is correct, then the natural numbers *are* the finite von Neumann ordinals.

2. Mathematics versus the Non-Uniqueness Argument

The natural numbers are intuitively familiar as 0, 1, 2, 3, and so on. More precisely: the natural number system N consists of the set of entities {0, 1, 2, 3, . . .}; a successor function +1; an initial number 0; and a less than relation <. Technically: N is the 4-tuple ({0, 1, 2, 3, . . .}, +1, 0, <). An ontological reduction of the natural number system N to some set-theoretic structure α = (ω, *f*, e, p) identifies each part of N with the corresponding part of α. Specifically: if N = α, then the numbers{0, 1, 2, 3, . . .} = ω, the successor function +1 = *f*, the initial number 0 = e, and the less than relation < = p.

According to Benacerraf (1965: 277; 1996: 21), a set theoretic structure α = (ω, *f*, e, p) is the natural numbers N if and only if it satisfies both (1) an arithmetical condition and (2) a cardinality condition. A set-theoretic structure (ω, *f*, e, p) satisfies the *arithmetical* condition if and only if (ω, *f*, e) is a model of the Dedekind-Peano Axioms (Hamilton, 1982: 9). A structure (ω, *f*, e, p) satisfies the *cardinality* condition if and only if it identifies the numerical less than relation < with some set theoretic relation p such that the cardinality of set S is n if and only if there is a 1-1 correspondence between S and { m | n p m}. While Benacerraf (1965: 275 - 277) initially argues that (3) it must be effectively decidable for any n and m whether n p m (i.e. the less than relation is "recursive"), he later (1996: Appendix) retracts the recursiveness condition. Letting "NN" stand for "Natural Number", I'll refer to the conjoined arithmetical and cardinality conditions as *the NN-conditions*. Benacerraf (1965: 279 - 281) insists that the NN-conditions are correct: satisfaction of the NN-conditions is both necessary and *sufficient* for being the natural numbers. If α satisfies the NN-conditions, then N = α. For convenience I refer to any set-theoretic structure of the form (ω, *f*, e, p) as a *ω-series*.

Benacerraf alleges there are many ω-series that satisfy the NN-conditions. He claims that the Zermelo ω-series and the von Neumann ω-series (and infinitely many others) all satisfy the NN-conditions. The Zermelo ω-series allegedly satisfies the NN-conditions like this: (1) it identifies 0 with {}; for any number n, it defines the successor n+1 as the set {n}; (2) it identifies the less than relation < with the ancestral [∈] of ∈ and says that the cardinality of set S is n if and only if there is a 1-1 correspondence between S and { m ∈ ω | n [∈] m}. The Zermelo reduction says: 0 = {}, 1 = {{}}, 2 = {{{}}}, 3 = {{{{}}}}, and so on. The von Neumann ω-series allegedly satisfies the NN-conditions like this: (1) it identifies 0 with {}; for any n, it defines n+1 as the set (n ∪ {n}); (2) it identifies the less than relation < with ∈ and says that the cardinality of set S is n if and only if there is a 1-1 correspondence between S and { m ∈ ω | n ∈ m}. The von Neumann reduction says: 0 = {}, 1 = {{}}, 2 = {{} {{}}}, 3 = {{} {{}} {{}{{}}}}, and so on.

If both the Zermelo and von Neumann ω-series satisfy the NN-conditions, then we get a contradiction. The argument goes like this:[2] (1) the Zermelo ω-series satisfies the NN-conditions; (2) since the Zermelo ω-series satisfies the NN-conditions, the set {0, 1, 2, . . .} = {{}, {{}}, {{{}}}, . . .}, the successor function n+1 = {n}, the initial number 0 = {}, and the less than relation < = [∈]; (3) so 2 = {{{}}}; (4) the von Neumann ω-series satisfies the NN-conditions; (5) since the von Neumann ω-series satisfies the NN-conditions, the set {0, 1, 2, . . .} = {{}, {{}}, {{}{{}}}, . . .}, the successor function n+1 = (n ∪ {n}), the initial number 0 = {}, and the less than relation < = ∈; (6) so 2 = {{}{{}}}; (7) but then 2 = {{{}}} and 2 = {{}{{}}}; (8) hence {{{}}} = {{}{{}}}; (9)

but alas {{{}}} = {{}{{}}} is a set-theoretic contradiction. Since infinitely many ω-series seem to satisfy the NN-conditions *equally well,* and since they all cancel each other out, Benacerraf concludes that numbers can't be sets. Most philosophers appear to agree with this *non-uniqueness argument* and to stand together on the thesis that numbers are not sets.[3] Balaguer (1998: 65) summarizes mainstream philosophical thinking when he says: "I think it is more or less beyond doubt that no sequence of *sets* stands out as *the* sequence of natural numbers".[4] Mathematicians as a rule do not share this opinion.

One good reason to doubt the soundness of the Benacerrafian argument is that there is one set of sets that stands out very clearly for the mathematicians as *the* natural numbers. The mathematicians standardly identify the natural numbers with the finite von Neumann ordinals.[5] They make the identification because not all apparent reductions satisfy the NN-conditions equally well. The von Neumann reduction is best. Benacerraf says: "If the numbers constitute one particular set of sets, and not another, then there must be arguments to indicate which" (1965: 281). The mathematicians do give arguments for the superiority of the von Neumann ω-series. If their arguments are right, then the choice of the von Neumann ω-series is neither arbitrary nor conventional. The von Neumann ω-series obviously satisfies the NN-conditions. I show in section 3 that it is the only ω-series that also satisfies all these extra conditions: (1) it is recursively defined; (2) its sets uniquely satisfy certain ordering conditions; (3) it is uniformly extendible to the transfinite; (4) it is a minimal ω-series; (5) its n-th member is the set of all m less than n. I have taken these conditions from the mathematicians themselves. Since the von Neumann ω-series is the only one that satisfies (1) through (5), it is superior to the Zermelo ω-series and to every other ω-series. Either these extra mathematical conditions are not relevant to the analysis of number or (exclusively) the natural numbers are the finite von Neumann ordinals.

I suspect that the hardcore Benacerrafian will reply that these extra conditions are not relevant: they are merely superfluous "stylistic preferences" (1965: 284 - 285). Since these extra conditions are highly relevant to mathematical practice and theory, they are certainly mathematically relevant. So if the Benacerrafian says they are irrelevant to the analysis of number, then they must be irrelevant in some non-mathematical way. Benacerraf is often said to be arguing against an *ontological reduction* of numbers to sets. He is said to be arguing against the *metaphysical identification* of numbers with sets. Balaguer (1998: 63 - 64) aptly comments that the Benacerrafian non-uniqueness argument entails that there is nothing "metaphysically special" about any of the ω-series that makes it stand out as "*the* sequence of natural numbers". I'll suppose then that the Benacerrafian is arguing for the *metaphysical* relevance of the NN-conditions and against the *metaphysical* relevance of any extra conditions. I know of one and only one mathematically precise way to handle metaphysical relevance: if some condition is metaphysically relevant, then it is formalized as an *existence condition*. If all and only the NN-conditions are relevant to the existence of the natural numbers, then the NN-conditions exactly specify the conditions in which we can make existence assertions in our arguments about natural numbers. They specify the available *existence axioms* for the "natural number universe" (the *NN-universe*). I do not suppose that the NN-universe is the whole mathematical universe. The NN-conditions specify that part of the mathematical universe to which we may ontologically reduce the natural numbers. I will give a *mathematical demonstration* in section 4 that if the NN-conditions exactly specify the NN-universe, then the natural numbers *are* the finite von Neumann ordinals.[6] If this is right, then the natural numbers *are* sets.

3. Mathematical Advantages of the Finite Von Neumann Ordinals

I agree with Benacerraf that if any series of sets is the natural numbers, then it satisfies the NN-conditions. I will argue in addition that (1) the series ought to be defined recursively; (2a) the series ought to identify $<$ with $\in$; (2b) the series ought to identify $\leq$ with subset-inclusion; (2c) the members of the series ought to be internally well-ordered; (2d) the relation that internally well-orders each member of the series ought to be the relation that well-orders the whole series; (3) the series ought to be uniformly extendible to the transfinite; (4) the series ought to be set-theoretically minimal; and finally (5) the n-th set in the series ought to be the set of all m less than n. Since the only ω-series that has all these properties is the von Neumann ω-series, these are all good reasons to identify the natural numbers with the finite von Neumann ordinals. These are all good reasons why mathematicians ignore the non-uniqueness problem.

(1) Benacerraf (1965: 275 - 277) says that the relation $<$ ought to be recursive. It might be better to say that the relation $<$ ought to be effectively decidable (i.e. that there ought to be an effective procedure for deciding, for any x and y in the ω-series, whether or not $x < y$).[7] Benacerraf (1996: Appendix) has since hardened his position. He recants his recursiveness requirement and declares that "Any old ω-sequence would do after all". I think he should have stuck with recursiveness (or at least some kind of recursive definition). Counting at its purest involves pure repetition: *again*. So any natural number series ought to be defined by repetition alone. The formalists (Korner, 1968: IV) expressed this pure repetition by means of recursively defined stroke sequences: | is (the first) stroke sequence; if X is a stroke sequence, then X| is (the next) stroke sequence. If you prefer to start with 0, then you might prefer to recursively define stroke sequences like this: # is (the zeroth) stroke sequence; if X is a stroke sequence, then X| is (the next) stroke sequence.[8] One might use a recursive definition of $<$ to rule out non-standard models of arithmetic.[9] Every natural number series must satisfy the NN-conditions and ought to be defined recursively. So if any series of sets is the natural number series, then it will be recursively defined; if it is not, then it does not adequately capture the formal concept of counting. If the series of sets is defined recursively, then $<$ will be recursive. So, for all n, n+1 ought to be $f(n)$ where f is defined by some set-theoretic formula or rule. For instance: for the Zermelo's, $f(n) = \{n\}$; for the von Neumann's, $f(n) = n \cup \{n\}$. We could also define $f(n) = n \cup \{\{n\}\}$ or $f(n) = POW(n)$ where POW produces the power set.

(2) Mathematicians use the ordering properties of the finite von Neumann ordinals to argue that they are *the* natural numbers. Suppes (1972: 127 - 130) identifies the natural numbers with the finite von Neumann ordinals and gives a mathematical argument (taken

from Dana Scott) for the superiority of the von Neumann ordinals based on well-ordering principles. Hamilton (1982: 192-3) and Devlin (1991: 22 - 24) also use well-ordering to motivate the identification of the natural numbers with the von Neumann's. Some mathematicians use ordering principles to *define* the natural numbers. For these mathematicians, the ordering principles are logically prior to the NN-conditions. Say set x is *transitive* if and only if for all $y \in x$, y is a subset of x; let $\in_x$ denote $\in$ restricted to x. Hrbacek & Jech (1978: 51) now define natural numbers like this: "A set x is a *natural number* if (a) x is transitive, (b) $\in_x$ is a strict linear ordering of x, (c) every non-empty subset of x has a least and a greatest element in the ordering $\in_x$." Hrbacek & Jech (52 - 56) proceed to deduce that the set of natural numbers is the finite von Neumann ordinals and that it satisfies the NN-conditions. Order conditions are not mathematically irrelevant. Consider: (2a) we ought to identify < with $\in$. If the numerical relation < is identified with $\in$, then < is the simplest set-theoretic relation $\in$. So if < is $\in$ for some ω-series, then that ω-series is simpler than any for which < is not $\in$. So any ω-series that identifies < with $\in$ is superior to one that does not (see Hamilton, 1982: 141). (2b) we ought to identify ≤ with subset-inclusion $\subseteq$. If we are going to identify numbers with sets, then we ought to identify the ordering relation on numbers with some ordering relation on sets. Since $\subseteq$ (partially) orders the universe of sets (i.e. the universe V of the iterative hierarchy), any ω-series whose numbers are ordered by $\subseteq$ is a superior to one that is not. (2c) The members of the ω-series ought to be internally well-ordered. If the numbers in the ω-series are internally well-ordered, then they can be used to well-order other finite sets. If we construct a 1-1 map from n to some set S, we have thereby well-ordered the members of S without doing any extra work. If all the numbers are internally well-ordered, then counting is equivalent to well-ordering. Finally: (2d) The relation that internally well-orders each number ought to be the relation that well-orders the whole ω-series. If (2d) is true of some ω-series, then each number partly *reflects* the whole series. Property (2d) is both deeply beautiful and highly practical. It enables us to think of every number in the ω-series as some *initial segment* of the whole ω-series. It underwrites the theorems that (i) the union of any set of numbers is a number; (ii) an initial segment of any number is a number; (iii) every member of a number is a number; (iv) if n is any number, n is the set of all numbers less than n. So natural numbers are indeed ordinals. These order properties are mathematically significant; they are all found in the von Neumann series but not in the Zermelo's.

(3) All natural numbers are finite. But modern mathematics recognizes a vast Cantorian paradise of transfinite numbers. What if we want to extend the natural numbers into the transfinite? Is there some set theoretic structure $(\omega, f, e, \mathsf{p})$ that enables us to do this in a regular way, so that we can define generalized transfinite counting procedures and naturally extend recursion and arithmetic to the transfinite? Yes: if the natural numbers are the finite von Neumann ordinals, we can *uniformly* extend our counting procedures and proof techniques to the transfinite. Maddy (1992: 84, fn. 13) recognizes this: "there are reasons why Zermelo's version isn't as good as von Neumann's. For example, von Neumann's account works just as well for infinite numbers as for finite". The fact that the von Neumann ordinals are uniformly extendible into the transfinite is one of the main

reasons why the natural numbers are standardly identified with the finite von Neumann ordinals. There is, moreover, a deep difference between the von Neumann and Zermelo ordinals: the endlessly increasing series of finite von Neumann ordinals converges to the von Neumann ω, while the endlessly increasing series of finite Zermelo ordinals does not converge to the Zermelo ω; in general, the series of von Neumann ordinals is *continuous at limits*, while the series of Zermelo ordinals is not. The von Neumann's provide a uniformly increasing series of numbers from 0 through the alephs and up into the large cardinals. Uniformity is extremely valuable in mathematics: it is the very basis for abstraction. Of all the sequences that satisfy the NN-conditions, von Neumann's is the simplest with this property.

(4) Some ω-series are simpler than others. Some ω-series are simpler than *all* the others: they are *minimal* in the iterative hierarchy of sets. Minimal ω-series are surely distinguished from non-minimal ω-series. I discuss minimality informally. Say the *depth* of any set n is the length of the longest ∈-chain that extends from 0 to n. The depth of {{} {{}} {{} {{}}}} is 3, since {} ∈ {{}} ∈ {{}{{}}} ∈ {{}{{}}{{}{{}}}}. I say that ω-series N is *minimal* if and only if, for all n ∈ N, the depth of n is n. The n-th number in a minimal ω-series starts from {} and counts n steps along the ∈ relation. Every minimal ω-series identifies 0 with {} and identifies 1 with {0}. The series {}, {{}} satisfies both n+1 = {n} and n+1 = n ∪ {n}. We now have two minimal choices for 2. These are: 2 = {{{}}} = {1} or 2 = {{}{{}}} = 1 ∪ {1}. Here is where the Zermelo and von Neumann series divide. If we choose 2 = {{}{{}}}, then we have three choices for 3. The von Neumann 3 is {{} {{}} {{}{{}}}}. If we identify the successor operation with the power set operation, then the Powerset 3 is {{} {{}} {{{}}} {{}{{}}}}. Another possibility is to combine the von Neumann and Zermelo successor operations to form what I'll call the "von Zermano" series. The von Zermano 3 is {{} {{}} {{{}}}}. The Zermelo, von Zermano, von Neumann, and Powerset series are all minimal ω-series. Figure 1 shows these 4 series. We ought to prefer minimal ω-series because (1) if we start counting from 0, then we ought to start counting from {}; and (2) if we count from any n to the next number (and there are none in between), then we ought to count from any set n to the next set (where *next* is directly defined in terms of ∈). We ought to prefer to count sets by starting from {} and counting up links of ∈-chains. We can do this only in minimal ω-series. The von Zermano, von Neumann, and Powerset series converge to their limits; only the von Zermano and von Neumann permit counting by 1-1 maps; only the von Neumann and the Powerset series have all the desired order properties. But the von Neumann series alone satisfies all these conditions. It is therefore the best minimal ω-series.

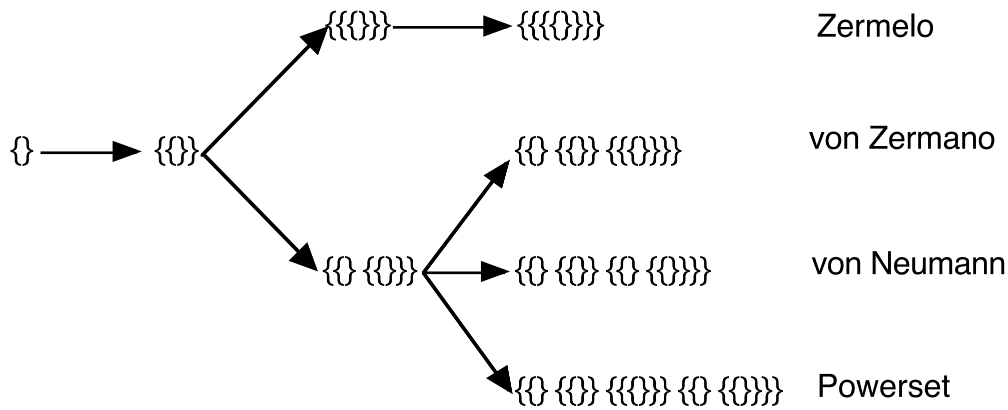

Figure 1. Beginnings of four minimal ω-series.

(5) According to Benacerraf, an account of number has *two and only two* parts: (1) the part that enables us to formulate the laws of arithmetic and (2) the part that enables us to explain cardinality (1965: 277; 1996: 21). Lest there be any doubt: "Everything else is extraneous" (1996: 21; see note 6). Benacerraf says "the account of cardinality must explicitly be included in the account of number" (1965: 293). He contends that "The explanation of cardinality — i.e. of the use of numbers for 'transitive counting', as I have called it — is part and parcel of the explication of number" (1965: 275; note 2). We transitively count the members of some set by taking "its elements one by one as we say the numbers one by one" (1965: 275). We need to attend to the details of this account of cardinality. Benacerraf (1965: 275, 280) says that "a set has n members if and only if it can be put into one-to-one correspondence with *the set of numbers* less than or equal to n". However: there are plenty of collections that have no members.[10] 0 is a perfectly good cardinal number. Benacerraf later decides to count from 0 (1996: 21). He says: "For there are n platypuses iff there is a 1-1 correspondence between the platypuses and the cardinal numbers < n" (1996: 46).[11] So: a set has n members if and only if it can be put into 1-1 correspondence with the set of numbers less than n. If the account of numbers includes an account of cardinality, and if cardinality is analyzed in terms of 1-1 correspondence (if "to count the members of a set is to determine the cardinality of the set" (1965: 275)), then we must work with *sets of numbers* in order to define cardinality.

Suppose we identify the natural number system N with some set-theoretic structure α = (ω, *f*, e, p). We say that N = α. If we choose *any objects of any kind* for the members of ω (whatever those objects might be), then the cardinality part of the NN-conditions compels us to form certain definite *sets* of those objects. Specifically: we have to form, for each n in ω, the set of all numbers less than n. For every number n, we must form the set n* = { m ∈ ω | m p n}. For example: 0* = {}, 1* = {0}, 2* = {0, 1}, 3* = {0, 1, 2}, and so on. We cannot avoid forming all these sets of numbers. Since the account of number includes an account of cardinality, the account of number includes these sets. Let ω* = { n* | n ∈ ω }. The set ω* is neither external nor incidental to the analysis of number. It is internal and it is essential. Since we can easily define successor and less than relations on ω*, we can easily define a ω-series for ω*. Let the zero element of ω* be e* = {}; let the successor function f*(n*) = (n* ∪ {n}); let the less than relation p* be ⊂. So the structure α* = (ω*, f*, e*, p*) satisfies the NN-conditions. Now consider the following informal argument: (1) α satisfies the NN-conditions; (2) since α satisfies the NN-conditions, N = α; (3) if α satisfies the NN-conditions, then α* satisfies the NN-conditions; (4) so α* satisfies the NN-conditions; (5) since α* satisfies the NN-conditions, N = α*; (6) so since α = N = α*, it follows that (ω, f, e, p) = (ω*, f*, e*, p*); (7) so ω = ω* hence n equals n* for every n in ω; (8) if n equals n* for every n in ω, then 0 = {} and for every n, n = {0, . . . n-1}; (9) but if 0 = {} and for every n, n = {0, . . . n-1}, then α is the finite von Neumann ordinals; *consequently:* (10) if α satisfies the NN-conditions, then α is the finite von Neumann ordinals.

This argument is mathematically significant. The equation of n with n* is the main mathematical motivation for the identification of the natural numbers with the finite von Neumann ordinals (see Halmos, 1960: sec. 11; Pinter 1971: 124 - 5; Hrbacek & Jech, 1978: 50; Hamilton, 1982: 134; Devlin, 1991: 66; Just & Weese, 1996: 44 - 45; Ciesielski, 1997: 25 - 26). If this is right, then the von Neumann sequence of sets stands out as the natural numbers because it is the only sequence *of objects of any kind* for which n equals n* for all n.[12] So far the argument that n is n* for every n is merely suggestive. Section 4 makes the argument mathematically precise.

4. The Natural Numbers are the Finite Von Neumann Ordinals

I suspect the hardcore Benacerrafian might well agree with all my points (1) to (5) while still denying that numbers are sets. My points (1) through (5), whatever their mathematical merits, merely indicate my "stylistic preferences" (1965: 285). While the advantages I attribute to the von Neumann series are doubtless mathematically useful, they do not show that numbers *are* the von Neumann ordinals. They do not justify the *metaphysical* conclusion that numbers are sets. I suppose that the hardcore Benacerrafian affirms the metaphysical relevance of the NN-conditions and denies the metaphysical relevance of any other conditions (e.g. denies the metaphysical relevance of my points (1) to (5)).

I suppose that metaphysically relevant conditions determine the boundaries of the natural number universe (the NN-universe). They determine the boundaries of the universe of objects to which we may ontologically reduce the natural numbers. Since all and only the NN-conditions are metaphysically relevant, they and they alone specify the available *existence axioms* for the NN-universe. The arithmetical part of the NN-conditions contains the Dedekind-Peano Axioms. The Dedekind-Peano Axioms contain the following existence axioms: (A1) *there exists* a number 0; (A2) for all x, if x is a number then *there exists* another number y such that y is the successor of x; (A3) there does not exist any number x such that 0 is the successor of x; (A4) *there exists* a set ω of all and only the natural numbers. Other Dedekind-Peano Axioms define the identity conditions for numbers and the set of all numbers.[13] The cardinality part of the NN-conditions involves one existence axiom: (C1) for all x, if x is in ω, then *there exists* a set x* = { z ∈ ω | z < x}. The cardinality part also contains the following definition: (C2) the cardinality of any set S is x if and only if there exists some 1-1 correspondence between x and x*. The definition of cardinality in terms of 1-1 correspondence *does not* assert the existence of any sets; it *does not* say that for every set S, there exists some x and there exists some M such that M is a 1-1 correspondence between S and x*. The definition of cardinality in (C2) does not say that at all. Although the cardinality relation holds among any sets that are in the NN-universe, it does not put any sets into the NN-universe.

Suppose you choose some progression of objects to serve as your natural numbers. I do not presume that these objects are sets. They may be *any* objects at all that satisfy the NN-conditions. Call your choice the *α-progression*. Your α-progression includes a set of α-numbers $\omega = \{ \alpha_0, \alpha_1, \alpha_2, \ldots \}$, a successor function *f*, some zero object e, and some less than relation p. It is the system (ω, *f*, e, p). The NN-universe contains ω, *f*, e, p, and every member of those objects if they have members. So it includes (for example) every a-number α_0, α_1, α_2, and so on. Since the NN-conditions include the cardinality condition, it follows that, for each α-number α_n, the set of α-numbers less than α_n is in the NN-universe. The cardinality condition says that: for all x, if x is in ω, then *there exists* a set y = { z ∈ ω | z p x}. Consequently: for each α-number α_n, there exists the set of all α_m p α_n. If you assume that α is the natural numbers, then the NN-universe contains the cardinality sets {}, $\{\alpha_0\}$, $\{\alpha_0, \alpha_1\}$, $\{\alpha_0, \alpha_1, \alpha_2\}$, and so on. The NN-conditions *do not* allow you to form any other sets of α-numbers. They *do not* assert or imply that the NN-universe contains any other sets built from α-numbers.

The cardinality condition asserts the rule (C1): for all x, if x is in ω, then there exists a set y = { z ∈ ω | z p x}. If the NN-conditions assert some rule R, then the NN-universe contains the domain of R, the range of R, the extension of R, and nothing else. For if we cannot reason to the existence of those objects in the NN-universe, then that rule is meaningless (it plays no role in determining the models of the NN-conditions). The domain of the cardinality rule is just ω. We already know that ω is in the NN-universe. For any x in ω, recall that x* = { z ∈ ω | z p x}. Now let ω* = { x* | x ∈ ω}. The range of the cardinality rule is ω*. So ω* is metaphysically relevant. The NN-conditions entail that ω* is in the NN-universe. Let Q be the extension of the cardinality rule. Q is a function from ω to ω*. For any α-number x in ω, Q(x) is x*. Q maps the α-numbers onto metaphysically relevant sets of α-numbers. Since the cardinality condition defines Q, Q is relevant. Q is in the NN-universe. The analysis of the being of the natural numbers includes ω, ω*, and the function Q: ω → ω*. Since the account of cardinality makes 0* relevant, and since 0* is the zero e* of ω*, the zero e* of ω* is in the NN-universe.

The cardinality condition says that the cardinality of any set S is the number x if and only if there is a 1-1 function from S onto x*. Benacerraf uses "C" to denote this function (1996: 21). So C(S) = x if and only if there is a 1-1 function from S onto x*. The cardinality relation holds among any sets that are in the NN-universe. In particular: since the identity function is a 1-1 correspondence between x* and x*, C(x*) = x. I let K denote the restriction of C to the NN-universe. Since the composition of any two arithmetical functions is an arithmetical function, functional composition preserves metaphysical relevance. If functions g and h stay in the NN-universe, then their composition stays in the NN-universe. If functions g and h are included in the analysis of the being of the natural numbers (if they stay in the NN-universe), then the NN-universe also includes (g∘h) and (h∘g). Since K and *f* are in the NN-universe, (*f*∘K) is relevant; since Q and (*f*∘K) are in the NN-universe, (Q∘(*f*∘K)) is in the NN-universe.[14] Let *f** = (Q∘(*f*∘K)). The function *f** is in the NN-universe. Since K maps x* onto x, and *f* maps x onto (x+1), and Q maps (x+1) onto (x+1)*, *f** maps x* onto (x+1)*. It is clear that *f** is a successor function for the cardinality sets. *f** is the successor function for ω*. Say x* p* y* if and only if K(x*) p K(y*). Since the less than relation p on α-numbers is relevant, and since K and Q are relevant, all their compositions are relevant. Since p* is formed by the composition of Q, K, and p, the relation p* is in the NN-universe.

The analysis of the being of the natural numbers includes ω*, *f**, e*, and p*. Say that "(ω*, *f**, e*, p*) is in the NN-universe" is true if and only if ω* and *f** and e* and p* are in the NN-universe. Since ω*, *f**, e*, and p* are all in the NN-universe, the system (ω*, *f**, e*, p*) is in the NN-universe. Let (ω*, *f**, e*, p*) be α*. I have taken some care to show that the NN-conditions permit us to reason from the existence of α = (ω, *f*, e, p) to the existence of α* = (ω*, *f**, e*, p*). The NN-conditions entail that the analysis of the *being* of the natural numbers includes α*. I have taken some care to show that if α is in the NN-universe, then α* is in the NN-universe. So: for any α, if α satisfies the NN-conditions, then there is some α* in the NN-universe such that α* also satisfies the NN-conditions. The α* system is a perfectly fine natural number system.

While the NN-conditions enable you to reason from the fact that α is in the NN-universe to the fact that α* is in the NN-universe, they *do not* enable you to reason from the fact that α is in the NN-universe to the fact that any other natural number system is in the NN-universe. Suppose you choose some system of objects (any objects) as α = (ω, *f*, e, p). You then define 0^ as {}. For each non-zero n in ω, you define n^ as {n-1}. You define the set ω^ as { n^ | n ∈ ω}. You define the function *f*^ like this: for all n^ in ω^, *f*^(n^) = {n^}. You let e^ be 0^. You define p^ like this: n p^ m if and only if n p m. Finally you define α^ as (ω^, *f*^, e^, p^). Although you are free to *define* α^, the NN-conditions do not permit you to infer that α^ is in the NN-universe. Since only 0^ and 1^ are in the NN-universe, n^ is not generally in the NN-universe; so ω^ is not in the NN-universe; since ω^ is not in the NN-universe, *f*^ and p^ are not in the NN-universe. So α^ is not in NN-universe. Analogous reasoning shows that no other system that satisfies the NN-conditions is in the NN-universe. So: α and α* are the only systems in the NN-universe that satisfy the NN-conditions. Choose any system of objects you like for (ω, *f*, e, p); the NN-conditions entail that (ω*, *f**, e*, p*) is the only other natural number system in the NN-universe. Recall that N is the natural number system ({0, 1, 2, 3, . . .}, +1, 0, <). You can see where this is going: (1) suppose α = N; (2) if α = N, then α satisfies the NN-conditions; (3) if α satisfies the NN-conditions, then α^* satisfies the NN-conditions; (3) if α^* satisfies the NN-conditions, then $\alpha^* = N$; so (4) $\alpha = \alpha^*$.

If my reasoning so far is right, then there is a precise mathematical demonstration that the natural numbers are the finite von Neumann ordinals. Let FVNO abbreviate "the finite von Neumann ordinals". Let α be $(\omega, f, e, \mathsf{p})$. I demonstrate that α is the natural numbers *if and only if* α is the FVNO. I show sufficiency and necessity. For *sufficiency* I show that if α is the natural numbers, then α is the FVNO. Suppose that α is the natural numbers. Since α is the natural numbers, $N = \alpha$. If α is the natural numbers, then α satisfies the NN-conditions. So α satisfies the NN-conditions. Let α^* be $(\omega^*, f^*, e^*, \mathsf{p}^*)$ as defined above. If α satisfies the NN-conditions, then α^* satisfies the NN-conditions. So: α^* is the natural numbers. Since α^* is the natural numbers, $N = \alpha^*$. Therefore: $\alpha = N = \alpha^*$; hence $\alpha = \alpha^*$. If $\alpha = (\omega, f, e, \mathsf{p})$ and $\alpha^* = (\omega^*, f^*, e^*, \mathsf{p}^*)$, then $\alpha = \alpha^*$ means exactly that $\omega = \omega^*$, $f = f^*$, $e = e^*$, and $\mathsf{p} = \mathsf{p}^*$. If $\omega = \omega^*$, then n equals n* for every n in ω. If n equals n* for every n in ω, then 0 = {} and n = {0, . . . n-1} for every n. If 0 = {} and n = {0, . . . n-1} for every n, then 0 = {} and n+1 = (n ∪ {n}) for all n. If 0 = {} and n+1 = (n ∪ {n}) for all n, then α is the FVNO. Therefore: if α is the natural numbers, then α is the finite von Neumann ordinals. For *necessity* I show that if α is the FVNO, then α is the natural numbers. Suppose α is the FVNO. If α is the FVNO, then α satisfies the NN-conditions. If α satisfies the NN-conditions, then α is the natural numbers. Therefore: if α is the finite von Neumann ordinals, then α is the natural numbers. Sufficiency and necessity are thus demonstrated. Consequently: α is the natural numbers if and only if α is the finite von Neumann ordinals.

Consider the Zermelo ordinals. They are eager little sets; they certainly *seem* like they are fit to serve as the natural numbers. The Zermelo structure ζ is the 4-tuple (Z, E, S, L). The set Z is defined like this: {} is in Z; for all x, if x is in Z, then {x} is in Z; no other sets are in Z. The sets in Z are: {}, {{}}, {{{}}}, {{{{}}}}, and so on. The initial object E is {}. The successor function S maps every x in Z onto {x}. The less than function L is the ancestral [∈] of ∈. The natural number structure N is the 4-tuple ({0, 1, 2, 3, . . .}, +1, 0, <). If N is ζ, then 0 = {}, 1 = {{}}, 2 = {{{}}}, 3 = {{{{}}}}, and so on. For every x in Z, the account of cardinality forces us to form the set x* = { m ∈ Z | L(m, x) }. For example: 0* = {}, 1* = {0} = {{}}, 2* = {0, 1} = {{} {{}}}, 3* = {0, 1, 2} = {{} {{}} {{{}}}}. The sets in Z* "must explicitly be included in the account of number" (1965: 293); they are "part and parcel of the explication of number" (1965: 275; note 2). Let Z* = { x* | x ∈ Z}. It is easy to see that {} is the initial object of Z*. Let E* = {}. The successor function S* on Z* is: S*(n*) = (n* ∪ {n}). The less than relation L* is ⊂. If ζ is fit to serve as the natural numbers, then ζ^* is equally fit to serve. The sets and relations in ζ^* are neither external to nor incidental to the analysis: if N is ζ, then N is ζ^*. So if N is ζ, then ζ is ζ^*. If ζ is ζ^*, then for every n in Z, n = n*. Therefore: 2 is 2*. But 2 = {{{}}} while 2* = {{}{{}}}. So if N is ζ, then {{{}}} = {{}{{}}}. But it is not the case that {{{}}} = {{}{{}}}. Consequently: N is not ζ. Since the n-th Zermelo set is not equal to the set of all Zermelo sets less than n, the natural numbers are not the Zermelo ordinals. The Zermelo's are not fit to serve. The natural numbers have an internal set-theoretical structure; the Zermelo sets do not share that structure. Analogous reasoning shows that no sets besides the von Neumann's are fit to serve. Only the finite von Neumann ordinals have the internal set-theoretic structure that the arithmetic and cardinality conditions require. The natural numbers are the finite von Neumann ordinals.

5. Conclusion

If Benacerraf's natural number conditions are correct, then the natural numbers are the finite von Neumann ordinals. Anyone who wants to deny that the natural numbers are the finite von Neumann ordinals must either deny (1) the arithmetical part of the NN-conditions or (2) the cardinality part of the NN-conditions. The natural numbers assuredly satisfy the laws of arithmetic. So anyone who wants to deny that the natural numbers are the finite von Neumann ordinals will have to argue that (1) the analysis of cardinality in terms of counting is not adequate; and (2) some distinct alternative account is superior. I am not aware of any inadequacies in the analysis of cardinality in terms of counting. I am not aware of any superior alternative analysis of cardinality.[15] So, if satisfaction of the NN-conditions is necessary and sufficient for *being* the natural numbers, then the natural numbers *are* the finite von Neumann ordinals. The other systems of objects (e.g. the Zermelo's) that appear to satisfy the NN-conditions are at most *arithmetical counterparts* or simulations of the finite von Neumann ordinals.[16] For all the reasons I have given, the mathematicians identify the natural numbers with the finite von Neumann ordinals. So, contrary to received wisdom, I suggest that philosophers follow mathematical practice and identify the natural numbers with the finite von Neumann ordinals. The Benacerrafian non-uniqueness argument is not sound. Numbers are sets.

Notes.

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[1]Wetzel (1989), Katz (1996) and Balaguer (1998) philosophically criticize Benacerraf's argument; but it has seen very little mathematically motivated criticism.

[2]Benacerraf later (1996: 25) summarizes his argument like this: "(1) that indefinitely many [reductions of numbers to sets] satisfied all the conditions [laid out in 1965]; (2) that, because [the number n = the object o] is an identity, at most one such [reduction of numbers to sets] could be correct; (3) that there was no principled way to choose among them — to decide which sets (or whatever surrogate you might put for sets) the numbers really were, . . . (4) that if one of the accounts were the correct one, there would be a way to discern which one it was . . .; and therefore (5) any claim identifying of a number with a set is a superfluous feature of the account . . . ; consequently (6) 'numbers . . . could not be sets at all'." Benacerraf (1996: 28) suggest that one could either try to weaken premise (3) by defending the old Frege-Russell analysis (against his criticisms in 1965) or deny premise (4) or adopt a kind of set-theoretic holism. I argue, against (3), that there are principled ways to choose among the different reductions of numbers to sets.

[3]Benacerraf's arguments have given rise to structuralism (Resnik, 1997; Shapiro, 1997). I am not arguing against structuralism. There may be good arguments for structuralism; I only claim that the non-uniqueness argument for numbers is not among them.

[4]Balaguer (1998) presents a powerful and general solution to non-uniqueness problems in mathematics. My work here is consistent with his broad approach even though we differ on some of the details. Balaguer argues (sections 1 & 2) that it is sometimes not the case that our mathematical theories truly describe unique systems of objects; he argues (section 3) that structuralism does not solve the problems associated with non-uniqueness; he argues (section 4) that our mathematical theories need not be descriptions

of unique systems of objects and that our mathematical singular terms need not have unique referents. Since Balaguer (1998: 68) says that his solution to the non-uniqueness problem is consistent with the assertion that some ω-sequence *does* stand out as the natural numbers, my special argument that the natural numbers *are* the finite von Neumann ordinals is perfectly consistent with Balaguer's more general view of mathematics. For even if one series of sets does stand out as the natural numbers, non-uniqueness threatens elsewhere. So some larger strategy for handling non-uniqueness is needed. Balaguer's strategy is general and is worthy of further development. Balaguer advocates a view of mathematics as the science of *logical plentitude:* "all the mathematical objects that (logically) possibly *could* exist actually *do* exist" (1998: 75, 78, 83). This is an important alternative to structuralism. I agree that mathematics is the science of logical plentitude.

[5]Since the identification of the natural numbers with the finite von Neumann ordinals is mathematically standard, the list of authors who make it is enormous. Here is a partial list of authors who give *arguments* for the identification of the natural numbers with the finite von Neumann ordinals: Halmos (1960: sec. 11); Eisenberg (1971: 80 - 81); Krivine, J.-L. (1971: 27 - 28); Pinter (1971: 124 - 5); Suppes (1972: 127 - 130); Drake (1974: 25 - 26); Enderton (1977: 67); Hrbacek & Jech (1978: 50); Hamilton (1982: 134, 141, 192 - 193); Devlin (1991: 23 - 24); Just & Weese (1996: 44 - 45); Ciesielski (1997: 25 - 26).

[6]Benacerraf (1965: 281) says the determination of which sets *are* the natural numbers is not amenable to proof. Since the NN-conditions are mathematical premises that logically entail mathematical consequences, they can be used in proofs.

[7]Benacerraf (1996: 57; note 34) clarifies his usage of the term "recursive" as follows: "Strictly speaking, when dealing with sequences of non-numbers, it may be best to speak, instead of recursive sequences, of sequences whose ordering relations have characteristic functions that are decidable, since 'recursive,' on most renderings, applies to numerical

functions, sets, and relations. Also, for the *cognoscenti*, we are assuming Church's Thesis." Benacerraf (1965: 276 - 277) invokes Turing machines.

[8]If the Hilbert stroke-sequence numbers are extended to include an initial zero, then they gain a structure similar to that of the finite von Neumann ordinals. Define extended stroke sequences like this: # is (the zeroth) stroke sequence; if X is a stroke sequence, then X| is (the next) stroke sequence. So: 0 = #; 1 = #|; 2 = #||; 3 = #|||, and so on. If stroke sequences are formed by adding strokes on the right, the parts of any stroke sequence are its initial left to right segments. (Analogous remarks hold if strokes are added on the left.) The parts of #||| are #, #|, #||, and #|||. If the numerical parts of any stroke sequence are all and only its proper parts, then the numerical parts of 3 are #, #|, #||; but then the numerical parts of 3 are 0, 1, and 2. The set theoretic statement 3 = {0, 1, 2} parallels the mereological statement that the proper parts of 3 are 0, 1, and 2. One might say that the commas in {0, 1, 2} match the accumulation of proper parts while wrapping 0, 1, and 2 up in the brackets { and } matches the addition of the rightmost |. A detailed comparison of von Neumann's and Hilbert's accounts of the natural numbers is far beyond the scope of this article; however, if stroke sequences are *anschaulich*, then the similarities between them and the finite von Neumann ordinals suggest that stroke sequences may provide some form of *epistemic access* to the finite von Neumann ordinals.

[9]Non-standard models of arithmetic are well-known (Enderton, 1972, ch. 3; Boolos & Jeffrey, 1974, ch. 17). One might argue that the natural numbers must be anchored to 0 by defining < recursively and then asserting that for every n, 0 < n. Here's one way to define < recursively: for any natural numbers n, m, n < m if and only if either m is the successor of n or there is some z such that n < z and m is the successor of z.

[10]It is clear that Benacerraf's (1965: 280) definition of cardinality does not work for sets with no members. Since the set of numbers less than or equal to 0 is {0}, it would follow

that a set has 0 members if and only if it can be put into a 1-1 correspondence with {0}; but then {A} has 0 members; but that's absurd. If we want to avoid that absurdity, then we need to use the standard definition of (finite) cardinality: for each n, a set has n members if and only if it can be put into 1-1 correspondence with the set of numbers < n.

[11]His modification of this principle to account for non-recursive ω-sequences (1996:48) still starts counting from 0. It just uses a different account of <.

[12]Since ω and ω* are isomorphic, it is hard (at least for me) to see how a structuralist would argue against the identification of n with n* for all n. I think even the most radical structuralist has to agree that n equals n* for all n. Say N is the abstract natural number structure if and only if the n-th *role* in N equals the set of all *roles* less than n in N.

[13]The remaining Dedekind-Peano Axioms can be stated like this: (A4) If m and n are numbers and the successor of m is the successor of n, then m is n; (A5) If A is any set of numbers such that A contains 0 and for every n in A, A contains the successor of n, then A is ω. These axioms are not existence axioms; they are identity axioms.

[14]Since the function f is included in the NN-universe, its composition with itself any number of times is included. So the ranges of these self-compositions are all included. The range of f is {1, 2, 3, . . .}; the range of f^2 is {2, 3, 4, . . .}; the range of f^n is {n, n+1, n+2, . . .}. We cannot reason from the NN-conditions to the existence of the set of all these ranges; we cannot reason to any successor or less than function; so we cannot assemble the ranges of the iterated compositions of f into any relevant model of the Peano-Dedekind Axioms. So these ranges do not form any ω-series. Analogously, the various other iterated compositions of relevant functions do not determine ω-series.

[15]I am not aware of any superior alternative account of cardinality. I am aware of only three available alternative accounts: the theory that numbers are properties of classes (Maddy, 1981); the theory that numbers are quantifiers (see Parsons, 1994: 147); the theory that numbers are relations between aggregates and properties (Kessler, 1980).

Maddy (1981: 503) says "A set S has the number property 3 if and only if $(\exists x)(\exists y)(\exists z)((x \neq y \;\&\; y \neq z \;\&\; x \neq z) \;\&\; (x \in S \;\&\; y \in S \;\&\; z \in S) \;\&\; (\forall w)((w \in S) \Rightarrow (w = x \text{ or } w = y \text{ or } w = z)))$." I say that recursively binding particular members of S to the variables x, y, and z is equivalent to counting off those members of S. Parson gives number-quantifiers like this: $(\exists_0 x)(F(x)) \Leftrightarrow \sim(\exists x)(F(x))$ and $(\exists_{n+1} x)(F(x)) \Leftrightarrow (\exists y)(F(y) \;\&\; (\exists_n z)(F(z) \;\&\; z \neq y))$. Once again the recursive binding of particular members of S to the numerically-quantified variables is equivalent to counting off those members of S. Kessler defines a number n as a relation n(x, p) between an aggregate x and a property p. Example: if x is a full deck of cards, then 52(x, is-a-card); 4(x, is-a-suite-of-cards), 1(x, is-a-deck-of-cards). The evaluation of Kessler's n(x, p) also requires the sequential recursive binding of values to variables (1980: 72); once again this is equivalent to counting. I say Maddy and Kessler respectively define property-theoretic and relation-theoretic *counterparts* of numbers. However: these counterparts of numbers are not numbers. Benacerraf (1965: 282 -284) is skeptical of property-theoretic accounts of number; I share his concerns.

[16]Since the Zermelo ordinals satisfy the Dedekind-Peano Axioms, you can use them to do arithmetic. They satisfy the *arithmetical* part (and *only* that part) of the NN-conditions. So they are arithmetical counterparts of the natural numbers. The arithmetical counterparts of natural numbers are *not* natural numbers — they merely arithmetically *simulate* the natural numbers. Arithmetical simulation is arithmetical isomorphism: the arithmetical part of the Zermelo structure is isomorphic to the arithmetical part (and *only* that part) of the von Neumann structure. Arithmetical simulations of simpler number systems within more complex number systems are well-known in mathematics. The standard definition of integers as equivalence classes of pairs of natural numbers provides a good example. Roughly: any pair of natural numbers (a, b) has some difference; that difference is an integer. If N is the natural numbers (the finite von Neumann ordinals), then the set$\{ (a, b) \in N^2 \mid a + d = b \}$ is the integer -d while $\{ (a, b) \in N^2 \mid a = b + d \}$ is

the integer +d. It is easy to construct a model of the Dedekind-Peano Axioms within the integers (Hamilton, 1982: 16). Within such a model, the integer +d simulates the natural number d; it is *the integer counterpart of* d. But integers are not natural numbers. They have properties and relations that natural numbers do not have. There are simulations of the natural numbers and integers within the rationals; all these simulations are easily extended into the real and complex numbers. Maddy's (1981) properties are property-theoretic simulations of the natural numbers; Kessler's (1980) relations are relation-theoretic simulations of the natural numbers; Church's (1941) λ-calculus numbers are function-theoretic simulations of the natural numbers. None of these simulations are the natural numbers.

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