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Why the Universe is not a fractal

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SUMMARY

There is overwhelming evidence from the CfA redshift survey that the distribution of galaxies in the Universe obeys a scaling law on length scales less than $\sim 5h^{-1}$ Mpc. Despite this scale invariance, the Universe is not well represented by a homogeneous fractal on these scales. The dependence of the correlation length r_0 with sample depth and luminosity is studied.

We present a method based on the minimal spanning tree for determining the Hausdorff dimension, D_{H}, of a point distribution. The technique is applied in order to find the Hausdorff dimension of the CfA redshift survey. The obtained value is $D_{\mathrm{H}} \simeq 2.1 \pm 0.1$. The correlation dimension differs from this value, $D_2 \simeq 1.3 \pm 0.1$, therefore the Universe is not well characterized by only one exponent: it is not a simple fractal. It is a more complex structure, a multifractal.

1 INTRODUCTION

It is thought that the initial conditions for galaxy formation can be described by a scale-free power spectrum extending over many decades in scalelength. The scaling invariance is only broken by the galaxy formation process itself, when baryonic material is converted into luminous material. It is quite reasonable to ask whether there is still any evidence for this primordial scaling, and over what range of scales this applies.

There are two aspects to the problem. The distribution of the gravitating material must tend to a finite average density when averaged over large volumes. This is a fundamental tenet of the standard cosmology, and to abandon this would involve coping with other problems like the homogeneity and isotropy of the microwave background radiation. The distribution of the luminous material will reflect that of the gravitating material, but it need not be the same (indeed, we know that the mass-to-light ratio is scale dependent). It is true, however, that any scaling invariance in the distribution of luminous material must eventually break down as we get to the largest scale on which the Universe will look homogeneous. There cannot be any 'lacunarity' (Mandelbrot 1982), if we are to remain consistent with the standard model. We shall show the evidence for this breakdown of scaling.

Even if we find the range of scales on which there is some kind of scaling invariance, we should ask what is the nature of that invariance. The most simple scaling invariance is of the kind exemplified by homogeneous fractals. Scaling invariance in general does not imply that the Universe is such a fractal. Evidence for more complex scaling has been introduced by Jones *et al.* (1988) who show that the notion of the simple fractal is not supported by the available data. The appropriate scaling concept is the multifractal (Mandelbrot 1974; Frisch & Parisi 1985; Halsey *et al.* 1986).

Recently, different authors have been arguing about the probable fractal structure of the large-scale distribution of galaxies. Whereas some people claim that 'the spatial distribution of galaxies can be described as a simple fractal' (Coleman, Pietronero & Sanders 1988), others 'can see no hope for a pure renormalizable fractal universe' (Peebles 1988, 1989). In this paper we shall pursue the agument on the basis of the available data on the three-dimensional distribution of galaxies.

2 DEFINITIONS

The definition of a fractal given by Mandelbrot (1977) is simple: a fractal is a set which has Hausdorff dimension strictly larger than the topological dimension. The Hausdorff dimension is well defined (Falconer 1985) for every set $\mathscr{A}$ by considering for each given value $\varepsilon > 0$, all possible coverings of $\mathscr{A}$ formed by sets with diameters $\varepsilon_i \le \varepsilon$. Let us call this family of coverings, $\Gamma^{\varepsilon}_{\mathscr{A}}$. The Hausdorff β-dimensional outer measure of $\mathscr{A}$ is defined for each $\beta \ge 0$ by

$$H^{\beta}(\mathscr{A}) = \lim_{\varepsilon \to 0} \inf_{\Gamma^{\varepsilon}_{\mathscr{A}}} \sum_i \varepsilon_i^{\beta}. \tag{1}$$

This measure is only non-trivial for uncountable sets. (Without this condition, the infinum requirement makes the β-dimensional outer measure zero!) Now the Hausdorff dimension of $\mathscr{A}$, $D_{\rm H}(\mathscr{A})$ is defined as follows

$$H^\beta(\mathscr{A})=\infty \text{ if } \beta<D_{\rm H}(\mathscr{A}) \text{ and } H^\beta(\mathscr{A})=0 \text{ if } \beta>D_{\rm H}(\mathscr{A}). \tag{2}$$

This definition is a mathematical one due to Hausdorff (1919), but it is clearly not applicable to finite sample realizations of fractals. The goal is to estimate with confidence the dimensionality of the support of the point distribution, using only the data from the point set.

A second definition of a fractal, more qualitative than the previous one, but probably more useful, is the following: 'a fractal is a set which looks similar to itself at every scale'. This property leads to a scale invariance, and this has something to do with the standard statistical analysis of the spatial distribution of galaxies, the two-point spatial correlation function, $\xi(r)$. This function represents the excess probability over a random distribution to find a galaxy at distance r from a given one. The power-law expression found for $\xi(r)$ is (Davis & Peebles 1983)

$$\xi(r)=\left(\frac{r}{r_0}\right)^{-\gamma}, \gamma=1.77\pm0.04, \tag{3}$$

which implies that the second-order intensity function is scale invariant. $\xi(r)$ is related to the radial distribution function $g(r)$ by

$$\xi(r)+1=g(r), \tag{4}$$

where $\rho g(r)4\pi r^2\, dr$ is the mean number of galaxies lying in a shell of thickness dr at distance r from any given one (ρ being the number density). In a Poisson process $g(r)=1$.

If $g(r)$ is a power law, $g(r)=Ar^{D_2-3}$. The exponent D_2 is the so-called correlation dimension, and $D_2\simeq 3-\gamma$. This number is not always equivalent to the fractal or capacity dimension defined by

$$D_0=\lim_{r\to 0}\frac{\log N(r)}{\log(1/r)}\simeq\frac{d\log N(r)}{d\log(1/r)}, \tag{5}$$

where $N(r)$ is the number of occupied cells when a disjoint partition of cells of size r has been performed. (D_0 has been the most popular estimator of the Hausdorff dimension $D_{\rm H}$.) For homogeneous fractals $D_0=D_2$, but in general and in the case of our interest, the spatial distribution of galaxies, the equality does not hold. What actually happens (Jones *et al.* 1988) is the inequality $D_2<D_0$.

Now we are able to give an operational definition of a fractal, by saying that a fractal is a set for which D_2 or D_0 or both have constant values over a broad range of scales; in other words, this quantity is scale-independent within a certain scaling region.

The problem is that the small amount of data available in galaxy redshift catalogues makes it difficult to estimate D_0 with reasonable accuracy. It is therefore better to work with $\xi(r)$ or D_2. We will return to the subject of D_0 later.

3 THE DATA SAMPLE

The data samples we study are all drawn from the Center for Astrophysics (CfA) galaxy redshift survey (Huchra *et al.* 1983). The survey we use is complete down to apparent magnitude $m_B=14.5$ over a limited area of sky covering $\delta>0°$ and $b\geq 40°$ in the northern hemisphere. For absolute magnitude-limited subsamples, we investigate the range of scales where the observational data accord well with fractal behaviour. Different tests have been applied for this purpose.

We use the ten complete subsamples listed in Table 1. The limiting volume for each subsample is given in the table, together with the faintest absolute magnitude to which the subsample is complete. The absolute magnitude limit $M_B^{\rm max}=-0.5-5\log V_{\rm max}$ (km s^{-1}) for $m_B=14.5$. $M_B^{\rm max}$ decreases with increasing sample volume. Velocities have been corrected as in Einasto *et al.* (1984). This includes solar motion, Virgocentric flow, peculiar velocities due to galaxy clusters and relativistic effects (Harrison 1974). [We use a distance scale based on the Hubble constant ($H_0=100$ km s^{-1} Mpc^{-1}).]

4 SPATIAL CORRELATION ANALYSIS

4.1 Radial density

We have studied the density $n(r)$ in concentric volumes centred in our Galaxy, as a function of the radius, r. In a fractal object embedded in a three-dimensional euclidean space, the density decreases from any arbitrary point as $n(r)\propto r^{D-3}$, where $D<3$ is the fractal dimension (de Vaucouleurs 1970; Mandelbrot 1982). In Fig. 1, the solid line shows how $n(r)$ changes with the distance r, for a volume completed subsample with depth $V_{\rm max}=10\,000$ km s^{-1} and absolute magnitude limit $M_B^{\rm max}=-20.5$. The dotted line is the best fit to fractal behaviour in the range where the power law is significant. Beyond a certain distance, one can appreciate a clear breakdown in the fractal behaviour and the density tends to be roughly constant on much larger scales. The average density for galaxies with absolute magnitude $M_B-20.5$ is $3.6\times10^{-4}\,h^3$ Mpc^{-3}.

4.2 Power law shapes

The second test we have performed in order to investigate the linear scaling range has been the study of the radial correlation function $g(r)$. In Fig. 2, we have plotted $\log g(r)$ versus r for the sample S65 (open circles). The power-law behaviour (dotted line) can be appreciated for small values of r, but clearly it breaks away for values larger than $r_f\sim 5h^{-1}$ Mpc. Clearly the scaling region is substantially less than the whole range plotted on the x-axis of Fig. 2, $20h^{-1}$ Mpc. For

Table 1. Sample specifications and results.

Sample	$V_{\rm min}$	$V_{\rm max}$	$M_B^{\rm max}$	*n.gal.*	r_0	γ	D_2
S35	1,700	3,500	−18.22	353	3.97 ± 0.06	1.69 ± 0.03	1.39 ± 0.07
S40	1,700	4,000	−18.51	389	4.11 ± 0.06	1.74 ± 0.03	1.37 ± 0.04
S45	1,700	4,500	−18.77	379	4.78 ± 0.08	1.68 ± 0.03	1.37 ± 0.05
S50	1,700	5,000	−18.99	372	4.88 ± 0.09	1.69 ± 0.03	1.39 ± 0.07
S55	1,700	5,500	−19.20	353	4.98 ± 0.12	1.67 ± 0.03	1.34 ± 0.12
S60	1,700	6,000	−19.39	342	5.20 ± 0.13	1.72 ± 0.04	1.31 ± 0.16
S65	1,700	6,500	−19.56	359	5.83 ± 0.11	1.88 ± 0.03	1.16 ± 0.11
S70	1,700	7,000	−19.73	377	6.38 ± 0.15	1.77 ± 0.03	1.30 ± 0.12
S75	1,700	7,500	−19.86	413	8.12 ± 0.15	1.76 ± 0.02	1.15 ± 0.19
S80	1,700	8,000	−20.02	384	7.39 ± 0.16	1.81 ± 0.03	1.19 ± 0.16

All the subsamples have been drawn from the CfA redshift survey. This catalogue is complete down to apparent magnitude $m_b=14.5$. The errors in r_0, γ and D_2 are statistical.

$r > r_f$, $g(r) \to 1$ very quickly, ruling out the simple fractal behaviour on such scales.

We have calculated D_2 by fitting the relation $g(r) \propto r^{D_2-3}$ in the range $r_1 \leq r \leq r_2$, where r_1 is the mean distance between nearest neighbour points and r_2 is the greatest range over which a power-law fit to the data seems significant. In Table 1 we have listed the values of D_2 for all the subsamples. D_2 takes values around 1.15–1.4 in agreement with the ones reported in Coleman *et al.* (1988). However, we disagree with the scaling range where the invariance is valid. Our figure shows clearly that the scaling range is much less broad than is claimed by Coleman *et al.* (1988). These authors used a particular method of surmounting the problem of the edge correction: when counting pairs of galaxies at separation r, they remove from the average those galaxies which have distances to the boundary of the sample less than $r + dr$; therefore, in their computation of $g(r)$ the sample is not uniformly selected. In fact, for large values of r, only a small number of galaxies are taken into account as centres of the counting shells. In any case, we have also tested their method of calculating $g(r)$. The results are plotted with asterisks in Fig. 2. Although the uncertainties are larger using this procedure for the border correction, the behaviour of $g(r)$ does not differ too much from the results of the calculation done using the standard method (Davis & Peebles 1983; open circles). In any case the departure of the simple fractal model is clearly appreciated.

In Fig. 3 we show the two-point correlation functions $\xi(r)$ for some of these samples. We show a typical slope γ that can be assigned to such two-point spatial correlation functions on the assumption that they are indeed power laws over some range of scales. γ values are listed in Table 1 for these subsamples. One can see that $D_2 \simeq 3 - \gamma$ for each sample.

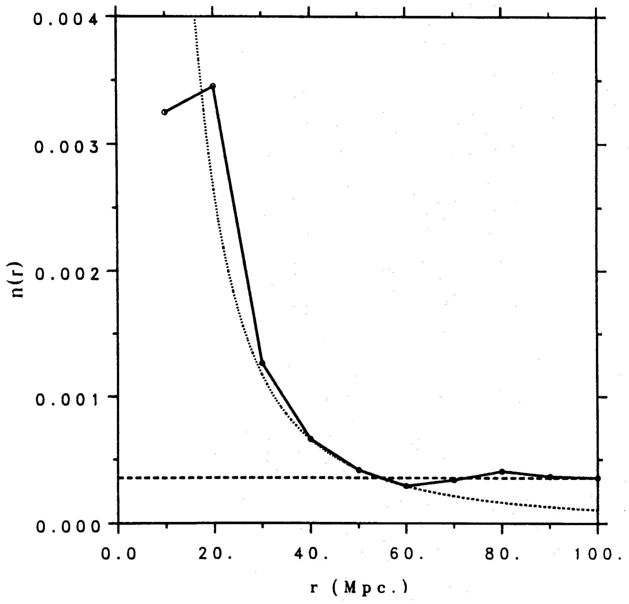

Figure 1. The radial number density in concentric volumes (solid line). The dotted line shows the homogeneous fractal behaviour, while the dashed line is the expected constant density for a Poisson distribution with the same number of galaxies. At small distances, the distribution seems to be well represented by a fractal model, but at much larger scales, the behaviour of $n(r)$ is consistent with homogeneity.

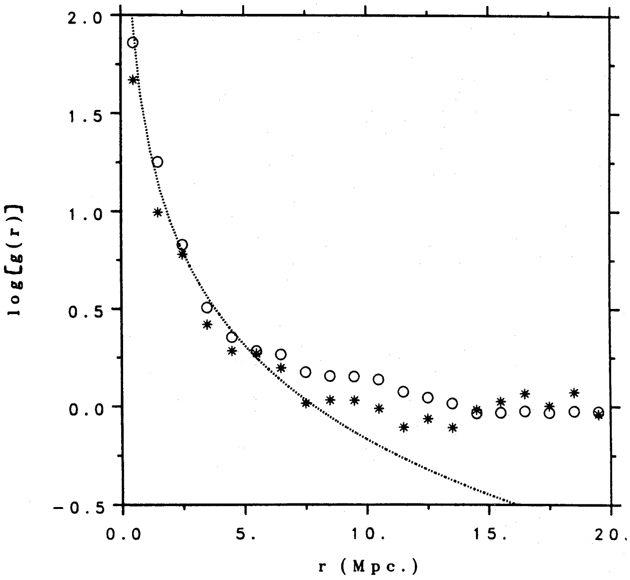

Figure 2. The structure function $g(r) = 1 + \xi(r)$, plotted as a function of *linear* scale r. The dotted line is the best-fit power law to the data (S65) on scales $r < 5h^{-1}$ Mpc, where a power law seems to be a reasonable approximation to the function. There is no question of this, or any other power law being a reasonable representation of the $g(r)$ beyond this range of scales. The open circles and asterisks represent different procedures for the edge corrections (see text).

4.3 Amplitudes

The so-called correlation length, r_0, is the scale at which the correlation function falls to unity $\xi(r_0) = 1$. (r_0 is really a normalization parameter, but conventional abuse dictates that it be referred to as a correlation length.) The estimated (Davis & Peebles 1983) value of r_0 for the observational data (CfA) is $r_0 = 5.4 \pm 0.3h^{-1}$ Mpc. It is interesting that the upper

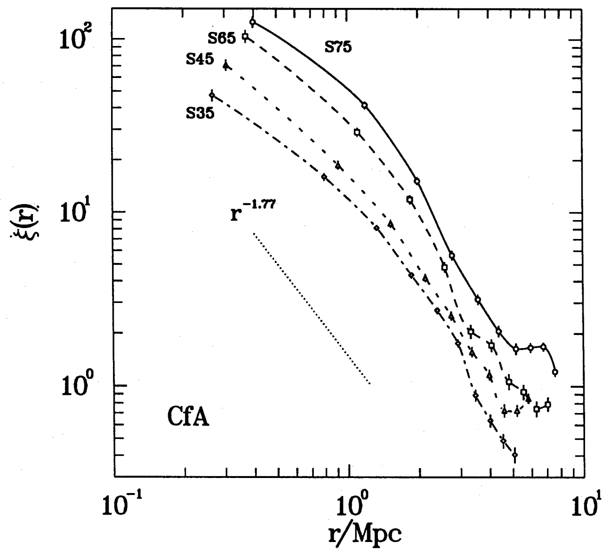

Figure 3. The correlation function $\xi(r)$ plotted as a function of scale r on a log–log plot. The samples are reasonable fits to the simple power law shown on scales $r < 5h^{-1}$ Mpc. The different normalizations for the curves is the basis of the argument that the Universe is a fractal. However, the phenomenon is most probably due to the luminosity dependence of the galaxy clustering process.

cut-off for fractal behaviour is approximately this correlation length, $r_f \sim r_0$.

It has been argued that the correlation length increases with the depth of the sample for complete volume-limited samples drawn from an apparent magnitude-limited survey (Einasto, Klypin & Saar 1986; Davis *et al.* 1988). We have repeated this calculation with our ten subsamples obtaining basically the same qualitative result as the previous authors. [Note that we are using the same catalogue as Einasto *et al.* (1986), so the agreement is not surprising.] The increase of r_0 with sample depth is shown in Table 1 (see also Fig. 3).

Notice that these results are not in disagreement with the value obtained by Davis & Peebles (1983) of $r_0 \sim 5.4$ for a sample, drawn from the CfA catalogue, with $100\,h^{-1}$ Mpc depth. In fact, their sample was not a volume-limited one and they used a selection function in the normalization of the correlation function. This permits them to surmount the problem of missing fainter galaxies ($M_B > -18.5$) at large distances ($r > 40\,h^{-1}$ Mpc). Instead, our S-subsamples are volume-limited, which means that they are intrinsically brighter if they are deeper. In this type of sample there is no question about the increase of r_0 with the sample depth. There are two possible interpretations of this fact.

(i) If all the samples had the same clustering properties, and the distribution of galaxies were well represented by a simple fractal, a *linear* increase of r_0 with the sample depth would be completely natural (Pietronero 1987). The data are not really good enough to say whether or not the increase is linear as expected in this case.

(ii) The second interpretation lies in the fact that clustering properties of galaxies may not be independent of luminosity. Such luminosity segregation is a natural consequence of the galaxy formation process. See for example the N-body simulations of cold dark matter universes (White *et al.* 1987).

Recently, it has been argued (Davis *et al.* 1988; Hamilton 1988; Domínguez-Tenreiro & Martínez 1989) using different techniques, that brighter galaxies are more strongly correlated than fainter galaxies. If the clustering properties change with luminosity, the increase of r_0 with depth may be a fingerprint of such a segregation, because in volume-limited samples the deeper the sample, the brighter are the galaxies in it.

In order to choose between these two interpretations, we can consider subsamples of the volume S80 ($M_B < -20.02$), taken at different depths. From the luminosity point of view, all these subsamples are equivalent and now we can study the behaviour of r_0 without the luminosity bias. In this case it has been shown (Martínez, Portilla & Jones, in preparation) that r_0 is no longer a linearly increasing function of the sample depth. In fact the value of r_0 is rather constant over a large range of velocity limits. The effect of the local density fluctuations is also very important in the behaviour of the correlation length. This result adds support to the view that the second hypothesis above is the correct one.

5 HAUSDORFF DIMENSION

We have seen that there is a scaling range, where D_2 is roughly constant, that implies fractal behaviour, but we still do not know the real value of the Hausdorff dimension D_H, which can be different from D_2. In this section we have tried to measure the Hausdorff dimension D_H of the samples. As has been already reported (Jones *et al.* 1988), box-counting algorithms are not the best ones for measuring D_H directly. The estimator used here for D_H depends on the minimal spanning tree (MST) (Barrow, Bhavsar & Sonoda 1983; Martínez *et al.* 1989). The minimal spanning tree of a point process is the graph connecting all the points in the process (without closed loops) with minimal length. The algorithm goes as follows: we take N_R points (galaxies) randomly selected from the sample, and we calculate in this set of points the lengths of the MST branches, $\{l_i\}_{i=1}^{i=m}$, $m = N_R - 1$. The moments of those distances are related to the Hausdorff dimension, in a way equivalent to that given by Badii & Politi (1984), by using the nearest neighbour distances.

$$H^{\beta}(\mathscr{A}) \equiv \sum_{i=1}^{m} l_i^{\beta}(m) = K(\beta)\, m^{1-\beta/h(\beta)}. \tag{6}$$

The fixed point of the function $h(\beta)$, $h(\tilde{D}_H) = \tilde{D}_H$ is a rather good estimator of the Hausdorff dimension. This statement can be easily proved by simple inspection of the definition of the Hausdorff dimension given in equation (2), if

$$\beta < \tilde{D}_H \Rightarrow h(\beta) > \beta \Rightarrow 1 - \frac{\beta}{h(\beta)} > 0 \Rightarrow H^{\beta} \to \infty \text{ if } m \to \infty, \tag{7}$$

and equivalently

$$\beta > \tilde{D}_H \Rightarrow h(\beta) < \beta \Rightarrow 1 - \frac{\beta}{h(\beta)} < 0 \Rightarrow H^{\beta} \to 0 \text{ if } m \to \infty. \tag{8}$$

In Fig. 4 we show the behaviour of $h(\beta)$ for the subsample S80 and for a Poisson distribution with the same number of objects in the same volume. In the first case (the CfA subsample) the fixed point is $\tilde{D}_H \sim 2$ while in the case of a Poisson distribution, the fixed point is $\tilde{D}_H \sim 3$, as would be expected.

The mean value obtained for the observed samples is $\tilde{D}_H \simeq 2.1 \pm 0.1$. The estimation of the Hausdorff dimension is independent of the absolute magnitude limit. The value obtained by means of the minimal spanning tree method is in agreement with other estimates obtained using different algorithms (Saar 1989).

The difference between D_H and D_2 is a key piece of information on the degree of inhomogeneity of the fractal measure on the set. This difference tells us that the set is not completely characterized by its main dimension; instead it needs more careful attention because it is a multifractal (Martínez *et al.* 1989).

6 CONCLUSIONS

The power-law scaling behaviour of the distribution of galaxies on scales $r < 5\,h^{-1}$ Mpc is a remarkable fact of cosmology, established by numerous analyses of both three-dimensional and projected galaxy distribution data. The fact that the Universe tends towards global homogeneity and isotropy on very large scales is also evident in these data samples, and that fact alone speaks most strongly against any suggestion that the Universe resembles a fractal on arbitrarily large scales. The dependence of clustering on galaxy morphology and luminosity makes this analysis a little delicate, but we believe that we have handled it correctly.

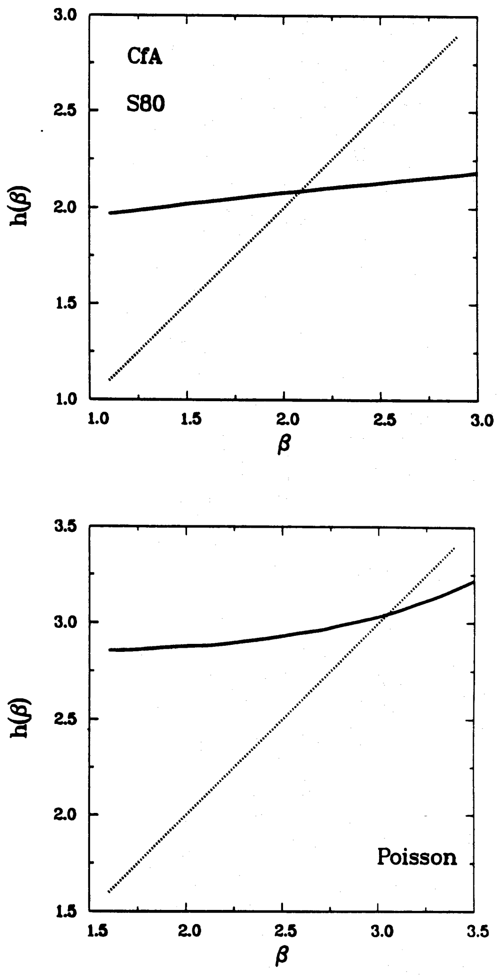

Figure 4. The Hausdorff dimension is obtained by means of the minimal spanning tree as the fixed point of the function $h(\beta)$. This value is ~ 3 for a Poisson distribution and ~ 2 for the real catalogue of galaxies. This result shows that the main contribution to the dimensionality of the large-scale distribution of galaxies are the sheet-like structures.

We have shown here, with an analysis of the best three-dimensional data available, that the structure on scales less than $5h^{-1}$ Mpc is more complex than the homogeneous fractal structure. We show how, despite the relatively small size of the data sample, the minimal spanning tree construction provides a robust estimate of the dimensionality distribution. In the scaling range, the correlation dimension $D_2 \simeq 1.3 \pm 0.1$ differs from the Hausdorff dimension $D_H \simeq 2.1 \pm 0.1$, ruling out homogeneous fractal models. The value of around 2 obtained for the Hausdorff dimension shows that the dimensionality of the Universe is sheet-like, in agreement with a 'bubble' universe (de Lapparent, Geller & Huchra 1986). Even in the power-law scaling domain ($r < 5h^{-1}$ Mpc) it is possible to show that the data can be represented as a *multifractal*, rather than a simple fractal (Jones *et al.* 1988; Martínez *et al.* 1989).

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