Wiener-Hopf factorization for Lévy processes having negative jumps with rational transforms

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Abstract

We give the closed form of the ruin probability for a Lévy processes, possibly killed at a constant rate, with completely arbitrary positive distributed jumps, and finite intensity negative jumps with distribution characterized by having a rational Laplace or Fourier transform.

Abbreviated Title: WH-factors of Lévy processes with rational jumps.

1 Introduction

1.1 Lévy processes and Wiener-Hopf factorization

Let $X = \{X_t\}_{t\geq 0}$ be a real valued stochastic process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, P)$. Assume that X is càdlàg, adapted, $X_0 = 0$, and for $0 \leq s < t$ the random variable $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$ with a distribution that only depends on the difference $t - s$. Assume that $\mathcal{B}$ satisfies the usual conditions. The stochastic process X is a process with stationary independent increments (PIIS), or a Lévy process.

If $u \in \mathbb{R}$, Lévy-Khinchine formula states $E(e^{iuX_t}) = e^{t\psi(u)}$, where the characteristic exponent of the process is

$$\psi(u) = iau - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \big(e^{iux} - 1 - iuh(x)\big)\Pi(dx). \tag{1}$$

Here the truncation function $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$ is fixed, and the parameters characterizing the law of the process are: the drift a, an arbitrary real number; the standard deviation of the Gaussian part of the process $\sigma \geq 0$; and the Lévy jump measure Π, a non negative measure, defined on $\mathbb{R} \setminus \{0\}$ such that

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$\int(1\wedge x^2)\Pi(dx) < +\infty$. We always assume that the process does not degenerate, i.e. $\sigma \neq 0$ or $\Pi \neq 0$.

Denote by $\tau(q)$ an exponential random variable with parameter $q > 0$, independent of the process X, and for $q = 0$, denote $\tau(0) = \infty$. Our main interest in this paper is the determination of the law of the following random variables:

$$M_q = \sup_{0\leq t<\tau(q)} X_t \qquad \text{and} \qquad I_q = \inf_{0\leq t<\tau(q)} X_t \tag{2}$$

called the *supremum* and the *infimum* of the process, respectively, killed at rate q if $q > 0$. When $q = 0$, the random variables (possibly degenerated) M_0 and I_0 are the overall supremum and infimum of the process respectively. Their characteristic functions are given by

$$\phi_q^+(u) = E(e^{iuM_q}), \quad \phi_q^-(u) = E(e^{iuI_q}). \tag{3}$$

A relevant instrument to study these distributions is the Wiener-Hopf factorization, obtained by Rogozin (1966), that, for $q > 0$, states

$$\frac{q}{q-\psi(u)} = \phi_q^+(u)\phi_q^-(u), \tag{4}$$

where $\phi_q^+(u)$ is also called the *positive Wiener-Hopf factor* and $\phi_q^-(u)$ the *negative Wiener-Hopf factor*. Although this factorization is valid only in case $q > 0$, we use the definitions in (3) for all $q \geq 0$. A feature of using this the Wiener-Hopf factorization in order to find the distributions of M_q and I_q is that both factors must be determined simultaneously, and, in general, this requires more restrictions on the class of processes considered. Another relevant instrument in order to compute the distribution of the supremum (or infimum) of a Lévy process is the formula obtained by Baxter and Donsker (1957), that we analyze in 4.2.

The main result of our paper is Theorem 2.1 where we give a closed formula for $\phi_q^-(u)$, that can be easily inverted to yield the density of I_q for a wide class of Lévy processes. This class is characterized by having finite intensity negative jumps with a distribution with rational Laplace transform, completely arbitrary positive jumps, and possibly gaussian part. The results presented generalize the ones obtained in Mordecki (2003), Mordecki (2002a), and can be applied to compute prices of Perpetual American Options in Lévy markets applying the results in Mordecki (2002b).

For general reference on the subject we refer to Jacod and Shiryaev (1987), Skorokhod (1991), Bertoin (1996) or Sato (1999).

1.2 Rational transform type random variables

In order to introduce the class of Lévy processes to be considered we need the following definition.

Definition 1.1. *We say that a random variable U is of the* rational transform type *when its characteristic function is a rational function, i.e. the quotient of two polynomials.*

This type of distribution is widely known and used in applications. It must be said that in this paper we do not address the question of which rational functions are the characteristic function of some random variable. Let us recall some facts about rational type distributions, also in order to fix some notations.

Lemma 1.1. *Consider a negative random variable U with density $p(x)$ $(x < 0)$ and characteristic function $\hat{p}(u) = \int_{-\infty}^{0} e^{iux} p(x) dx$. The following two statements are equivalent:*

(a) *The characteristic function has the form*

$$\hat{p}(u) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} c_{kj} \Big(\frac{-i\alpha_k}{u - i\alpha_k} \Big)^j = \frac{Q(u)}{R(u)}, \tag{5}$$

where $0 < \alpha_1 \le \Re(\alpha_2) \le \cdots \le \Re(\alpha_n)$,

$$R(u) = (u - i\alpha_1)^{m_1} \cdots (u - i\alpha_n)^{m_n},$$
$$R_k(u) = \frac{R(u)(-i\alpha_k)^{m_k}}{(u - i\alpha_k)^{m_k}} \ (k = 1, \dots, n).$$

and $Q(u)$ is a complex polynomial of degree strictly less than the pole count $P = m_1 + \cdots + m_n$*. The relation between coefficients and polynomials is given by*

$$c_{k,m_k-j} = \frac{(-i\alpha_k)^j}{j!} \Big[\frac{\partial^j}{\partial u^j} \frac{Q}{R_k} \Big]_{u=i\alpha_k} \qquad (k = 1, \dots, n; j = 0, \dots, m_k - 1).$$

(b) *The density function has the form*

$$p(x) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} c_{kj} \big(\alpha_k\big)^j \frac{(-x)^{j-1}}{(j-1)!} e^{\alpha_k x} \qquad (x < 0). \tag{6}$$

Remark 1.1. *The Lemma follows from routine (omitted) computations. However, the Lemma does not cover the whole class of non-negative distributions of the rational transform type. The remaining situation is when the random variable has an atom in $x = 0$ with a certain probability c_{00}. In this case the density in (6) has an additional term of the form $c_{00}\delta(dx)$, where $\delta(dx)$ is Dirac delta at point $x = 0$, and, accordingly, a constant term c_{00} should be added in its Fourier Transform in (5).*

1.3 Lévy processes with negative jumps of the rational transform type

We now consider the class of processes of interest in the paper. Assume that X is a Lévy process with jump measure given by

$$\Pi(dx) = \begin{cases} \pi^+(dx) & \text{if} \quad x > 0, \\ \\ \pi^-(dx) = \lambda p(x) dx & \text{if} \quad x < 0. \end{cases} \tag{7}$$

where π^+ is an arbitrary Lévy measure concentrated on the set $(0,+\infty)$ describing the behaviour of positive jumps. Negative jumps have finite intensity $\lambda > 0$,

$$p(x) = \sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}\big(\alpha_k\big)^j \frac{(-x)^{j-1}}{(j-1)!}e^{\alpha_k x}, \quad x<0, \tag{8}$$

is the density of a negative random variable of the rational transform type, and the paramenters α_k and c_{jk} are as in Lemma 1.1

The characteristic exponent of the process is

$$\psi(u) = iau - \frac{1}{2}\sigma^2u^2 + \int_0^\infty (e^{iux} - 1 - iuh(x))\pi^+(dx) + \lambda(\hat{p}(u) - 1). \tag{9}$$

where, according to Lemma 1.1

$$\hat{p}(u) = \int_{-\infty}^0 e^{izu}p(x)dx = \sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}\Big(\frac{-i\alpha_k}{u - i\alpha_k}\Big)^j. \tag{10}$$

Observe that, for a Lévy process with negative jumps of the rational transform type the expectation can always be defined as

$$E(X_1) = -i\lim_{u\to 0+}\frac{\psi(iu)}{u} = a + \int_1^\infty x\pi^+(dx) + \lambda\int_{-\infty}^0 xp(x)dx, \tag{11}$$

that can take the value $+\infty$.

In order to formulate the following result consider a Lévy process X^+ with no negative jumps and characteristic exponent given by

$$\psi^+(u) = iau - \frac{1}{2}\sigma^2u^2 + \int_0^\infty (e^{iux} - 1 - iuh(x))\pi^+(dx).$$

Observe that the characteristic exponent in (1) can be analytically continued to the strip $0 \le \Im(z) < \alpha_1$, and more generally, it can be continued to a meromorphic function

$$\psi(z) = \psi^+(z) + \lambda\Big[\sum_{k=1}^{n}\sum_{j=1}^{m_k} c_{kj}\Big(\frac{-i\alpha_k}{z - i\alpha_k}\Big)^j - 1\Big].$$

defined for $\Im(z) \ge 0$ with the exception of the poles located at $i\alpha_1,\dots,i\alpha_n$. The following result is important in what follows.

Lemma 1.2 (Roots). *Given $q \ge 0$ the equation $q - \psi(z) = 0$ has, in the half-plane $\Im(z) > 0$, N distinct roots*

$$i\beta_1(q),\dots,i\beta_N(q), \text{ with multiplicities } n_1,\dots,n_N, \tag{12}$$

ordered such that $0 < \Re(\beta_1) \le \Re(\beta_2) \le \cdots \le \Re(\beta_N)$. The root $i\beta_1$ is purely imaginary. Furthermore, the total root count $Z = \sum_{j=1}^N n_j$ satisfies:

(S) $Z = P$ *when* X^+ *is a subordinator.*

(NS) $Z = P + 1$ *when* X^+ *is not a subordinator.*

All proofs may be found in Sec. 4. Note that we frequently omit the explicit display of q-dependencies in roots.

2 Main Results

Theorem 2.1. *Let X be a Lévy process with characteristic exponent given by* (9)*, and $q \geq 0$. Assume that $E(X_1) > 0$ when $q = 0$. Then, the characteristic function ϕ_q^- of the infimum I_q is of rational transform type, and satisfies*

$$\phi_q^-(u) = \prod_{k=1}^{n} \left(\frac{u - i\alpha_k}{-i\alpha_k} \right)^{m_k} \prod_{j=1}^{N} \left(\frac{-i\beta_j}{u - i\beta_j} \right)^{n_j}, \tag{13}$$

where the $\alpha_1, \dots, \alpha_n$ are the parameters of the density in (8)*, and $\beta_1, \dots, \beta_N$ are the roots given in* (12)*.*

Remark 2.1. *We want to examine when the distribution of I_q has an atom at $x = 0$. As $I_q \leq 0$, the function $\phi_q^-(u) = E(e^{iuI_q})$ can be analytically extended to the half-space $\Im u < 0$. This, in particular, gives*

$$\lim_{u\to\infty} \phi_q^-(-iu) = \lim_{u\to\infty} E(e^{uI_q}) = P(I_q = 0).$$

We then conclude that in case (NS)*, when $Z = P + 1$ the density has no atom at $x = 0$.*

In case (S)*, as $Z = P$, we have the limit*

$$P(I_q = 0) = \lim_{u\to\infty} \phi_q^-(-iu) = \frac{\prod_{j=1}^{N} (\beta_j)^{n_j}}{\prod_{k=1}^{n} (\alpha_k)^{m_k}}.$$

In other terms, in case (S) *the process is not recurrent, and in case* (NS) *it is recurrent (at $x = 0$).*

Remark 2.2. *Let us examine in particular the case $q = 0$, included in the previous result. In this case I_0 is the overall infimum of the process. Furthermore, condition $0 < E(X_1) \leq +\infty$ ensures that I_0 is a proper random variable, i.e. the process drifts to $+\infty$, or equivalently $P(I_0 > -\infty) = 1$ (see Rogozin (1966)).*

The density of I_0 can be obtained applying Lemma 1.1, and can be integrated to yield the ruin probability,

$$R(x) = P(\exists t \geq 0 \colon x + X_t \leq 0) = P(I_0 \leq -x) = \int_{\infty}^{-x} f_0(y)dy \quad x \geq 0. \tag{14}$$

For instance, in case (NS)*, when $Z = P + 1$, we have*

$$f_0(x) = \sum_{k=1}^{N} \sum_{j=1}^{n_k} d_{kj}(0) \big(\beta_k\big)^j \frac{(-x)^{j-1}}{(j-1)!} e^{\beta_k x} \qquad (x < 0),$$

with the various $d_{kj}(0)$ given by applying Lemma 1.1 to (13) at $q = 0$. Notice finally that $f_0(x)$ essentially replicates the functional form of $p(x)$ from (7); that is: exponentials times polynomials. Now the exponents have switched from the α_ks to β_js, and the polynomials have generally different orders and coefficients. Both expressions have rational transforms.

In particular, the leading coefficient determines the $x \to -\infty$ asymptotics, which we discuss in Sec. 3. For now, we simply note that, by 'leading', we mean coefficient(s) associated to β_1, whose exponentials have has the slowest decay. Moreover, as it turns out (see the remark in page 8), there is really only one such coeffciient (the multiplicity $n_1 = 1$). Given that, we find, using Lemma 1.1, that the leading behavior comes with a coefficient:

$$d_{11}(0) = \frac{\prod_{k=1}^{n}(1 - \frac{\beta_1}{\alpha_k})^{m_k}}{\prod_{j=2}^{N}(1 - \frac{\beta_1}{\beta_j})^{n_j}}. \tag{15}$$

2.1 Phase type distributed jumps

In this section we assume that the negative jumps of the process X are phase type distributed. Consider then a random variable of phase-type, and representation $(\mathbf{a}, \mathbf{T}, d)$, where d is a positive integer, $\mathbf{a} = (a_1, \dots, a_d)$ is the initial probability distribution, and the intensity matrix $\mathbf{T}$, and asociated exit rates vector $\mathbf{t}$, are given by

$$\mathbf{T} = \begin{bmatrix} t_{11} & \cdots & t_{1d} \\ \vdots & \vdots & \vdots \\ t_{d1} & \cdots & t_{dd} \end{bmatrix}, \qquad \mathbf{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_d \end{bmatrix},$$

where $\mathbf{t}$ satisfies $t_j + \sum_{k=1}^{d} t_{jk} = 0$ $(j = 1, \dots, d)$. For details and general results on phase-type distributions we refer to Asmussen (1997).

In this case, the distribution of the negative jumps in (9) is determined by

$$\hat{p}(u) = \int_{-\infty}^{0} e^{iux} p(x)dx = \mathbf{a}(-iu\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}.$$

Phase type distributions are a wide class of probability distributions, including practically all exponential-like distributions (exponential, mixtures of exponentials, Erlang and Cox distributions, see Asmussen (1997)). However, the class of rational type distributions is wider, as shown by the example density defined for $t \geq 0$ by

$$Ke^{-t}(1 - \cos t),$$

for a convenient $K > 0$, that is not of phase type, due to the fact that it vanishes (see Shaked and Shanthikumar (1985)).

Computing the inverse of the matrix $-iu\mathbf{I} - \mathbf{T}$ by Cramer's rule we obtain that

$$\hat{p}(u) = \frac{Q(z)}{\det(-iu\mathbf{I} - \mathbf{T})},$$

where Q is a polynomial of degree less or equal than $d-1$. Furthermore, as the matrix $\mathbf{T}$ has dimension $d \times d$, it has d complex eiguenvalues, say $i\alpha_1, \dots, i\alpha_d$, possibly coincident. In consequence

$$\det(-iz\mathbf{I} - \mathbf{T}) = \prod_{k=1}^{d}(-iz + \alpha_k).$$

Applying Lemma 1.2, we know that the equation $q - \psi(z) = 0$ has N different roots, say $i\beta_1, \dots, i\beta_N$ with multiplicities $n_1, \dots, n_N$, and $Z = P$ or $P+1$ according to the behavior of X^+. Then, the characteristic function of I_q is given by

$$\phi_q^-(z) = \prod_{k=1}^{d}\left(\frac{z - i\alpha_k}{-i\alpha_k}\right)\prod_{k=1}^{N}\left(\frac{-i\beta_k}{z - i\beta_k}\right)^{n_k} = \frac{\det(-iz - \mathbf{T})}{\det(-\mathbf{T})}\prod_{k=1}^{N}\left(\frac{-i\beta_k}{z - i\beta_k}\right)^{n_k},$$

obtaining the the infimum of the Lévy process with phase type negative jumps has a rational transform type distribution. This result was obtained in the paper by Asmussen, Avram, and Pistorius (2004).

The distribution of the maximum of a Lévy processes with positive phase-type jumps was obtained in Mordecki (2002a). There, using fluctuation Theory for Lévy processes, a phase type representation for the distribution of the maximum is found, such that if the jump distribution has d phases, the distribution of the maximum has d or $d+1$ phases.

3 Asymptotics for the ruin probability

There is a large literature and interest, especially in insurance, regarding the asymptotic behavior of (ultimate) ruin probabilities $R(x) = R_0(x)$ as $x \to \infty$ in various models. In the classic insurance model for a firm's capital, X is compound Poisson; i.e., the Lévy process with $\sigma^2 = 0$, drift to $+\infty$, no positive jumps, and $\pi^-(dx)$ admitting an intensity λ. Asymptotics for the insurance model were developed in the 1930s under *Cramér's condition*: there exists $0 < \omega < \infty$ with $\psi(i\omega) = 0$. Under that condition, Cramér and Lundberg established that

$$\lim_{x\to\infty} e^{\omega x}R(x) = \lim_{x\to\infty} e^{\omega x}P(I < -x) = C,$$

where I is the ultimate minimum and C is an explicitly given constant.

Many extensions have been developed. A very attractive one is Doney (1991) generalization: the Cramér-Lundberg result holds for any Lévy process with (i) no positive jumps, (ii) drift to $+\infty$, and (iii) with the very simple $C = \mu/|\mu^*|$. Here $\mu = E(X_1) > 0$ is precisely the condition that X_t drifts to $+\infty$. And $\mu^* = E(X_1^*) < 0$ is the drift of an *associated* Lévy process X^*, namely the one with Lévy exponent $\psi^*(z) = \psi(z + i\omega)$. Of course, since $E(X_1) = -i\psi'(0)$, Doney's result may be written $C = -\psi'(0)/\psi'(i\omega)$.

It is simple to verify that our main result is in agreement with Doney's formula in the special case where there are no positive jumps. In the rational transform model with drift to $+\infty$, that Cramér's condition holds and $R(x) \sim Ce^{-\omega x}$, with $\omega = \beta_1 = \beta_1(0)$ is clear. It then remains to show that Doney's C equals our $d_{11}(0)$ from (15). To see that, note that with no positive jumps, the rational transform model reduces to

$$\psi(z) = \frac{1}{2}\sigma^2 z(z - i\omega)\ \frac{\prod_{k=2}^{N}(z - i\beta_k)^{n_k}}{\prod_{k=1}^{n}(z - i\alpha_k)^{m_k}}, \tag{16}$$

where $\beta_k = \beta_k(0)$. Simple differentiation yields

$$C = -\frac{\psi'(0)}{\psi'(i\omega)} = \frac{\prod_{k=1}^{n}(1 - \frac{\omega}{\alpha_k})^{m_k}}{\prod_{k=2}^{N}(1 - \frac{\omega}{\beta_k})^{n_k}},$$

which is indeed (15).

The restriction to one-sided jumps was afterwards removed in the paper by Bertoin and Doney (1994), who showed $R(x) \sim Ce^{-\omega x}$ in any Lévy process admitting Cramér's condition. However, in that general case, their formula for C is much less explicit. Here we give an alternative general formula for C, not just for the rational transform model, but for any Lévy process admitting Cramér's condition.

Remark 3.1. *Bertoin and Doney's result establishes the earlier fact, used at (15) that the leading term comes with unit multiplicity.*

Consider (13) again , but in the limit $q \to 0$. This limit will exist under the positive drift condition, and writing $\beta_1(0) = \omega$,

$$\phi_0^-(z) = \int_{-\infty}^{0} e^{izx} f_I(x)dx, \quad \Im z < \omega. \tag{17}$$

Now by Bertoin and Doney's result, we know that there exists a $C < \infty$ such that $f_I(x) \sim C\omega\, e^{\omega x}$ as $x \to -\infty$. Suppose that, in fact, $f_I(x) = C\omega\, e^{\omega x}$. Then, the integral is easy, and it shows that $\phi_0^-(z)$ is indeed analytic for $\Im z < \omega$, and develops a simple pole as $z \to i\omega$ from below, with residue $-iC\omega$. This suggests the following statement

Lemma 3.1. *Suppose $F(dx)$ is the density of a distribution with support on $(-\infty, 0]$, such that $F(-\infty, x) \sim Ae^{\omega x}$ as $x \to -\infty$ for some $\omega > 0$. Then, its characteristic function $\Phi(z) = \int_{-\infty}^{0} e^{izx}F(dx)$ is analytic in $\mathcal{H} = \{z : \Im z < \omega\}$. Moreover, $\lim_{z\to i\omega}(z - i\omega)\Phi(z) = -iA\omega$ as z approaches $i\omega$ from within $\mathcal{H}$.*

The analyticity statement is a well-known consequence of the fact that $F(dx)$ is of exponential type. The statement about the pole is established in Korevar (2002, Proposition 3.1). As a simple consequence, the asymptotic constant C in general Lévy models with positive drift (admitting Cramér's condition), is given by

$$C = \frac{i}{\omega} \lim_{z \uparrow i\omega} (z - i\omega)\phi_0^-(z), \tag{18}$$

where the up-arrow means that the approach is from below. Note that one could also define an inverse factor $\chi(z) = 1/\phi_0^-(z)$. Then, an alternative, but equivalent formula is $C = i/(\omega\,\chi'(i\omega))$, where the prime denotes the derivative obtained with an approach from below.

Finally, specializing again to our rational transform model, the last formula again yields $C = d_{11}(0)$.

4 Proofs

4.1 Proof of Lemma 1.2

We begin recalling some standard results.

Theorem 4.1 (Extended Rouche's Theorem). *Let two functions $f(z)$ and $g(z)$ be meromorphic inside and analytic on a simple closed contour C, and suppose that $|g(z)| < |f(z)|$ at each point on C. Then, $f(z)$ and $f(z) + g(z)$ have the same winding number $W = Z - P$, counting multiplicities, inside C.*

We take following classical result from Petrov (1987, Lemma I.3.2),

Lemma 4.1. *A distribution with characteristic function $\hat{p}(u)$ is lattice if and only if there exists $u_0 \neq 0$ such that $|\hat{p}(u_0)| = 1$.*

Proof of Lemma 1.2. The proof of Lemma 1.2 is based on Extended Rouche's Theorem, choosing

$$f(z) = \lambda + q - \psi^+(z) = q + \lambda - iaz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (1 + izh(x) - e^{izx})\pi^+(dx),$$
$$g(z) = -\lambda\hat{p}(z),$$

with $f(z)+g(z) = q-\psi(z)$, the full model. The idea is to establish that $|g| < |f|$ on a contour of the form

$$\{z = re^{i\theta}, 0 \le \theta \le \pi\} \cup \{\Im(z) = 0, r \le |z| \le R\} \cup \{z = Re^{i\theta}, 0 \le \theta \le \pi\} \quad (19)$$

with $0 \le r < R$ (r small, R big), that contains all the poles of $q - \psi$, in order to obtain that the winding numbers of f and $f + g = q - \psi$ coincide, in our notation $W_f = W_{q-\psi}$. This would ensure the proof of the Lemma. To prove $|g| < |f|$ on (19) we proceed by steps.

Step 1. We verifiy $W_f = 0$ in case (S). In fact, as X^+ is a subordinator

$$\psi^+(z) = iaz + \int_0^\infty (e^{izx} - 1)\pi^+(dx),$$

for some $a \ge 0$. Denote $z = u + iv$ and observe that when $v = \Im z > 0$ we have

$$|f(z)| \ge \Re f(z) = \lambda + q + av + \int_0^\infty \big(1 - e^{-vx}\cos ux\big)\pi^+(dx) \ge \lambda + q. \quad (20)$$

This means that f has no zeros in $\Im z > 0$. It obviously has no poles, so the winding number for any simple closed curve in the half plane $\Im z > 0$ is zero, in other terms $W_f = 0$.

Step 2. We verify $W_f = 1$ in case (NS). When X^+ is not a subordinator, the equation $f(z) = q - \psi^+(z) = 0$ has exactly one purely imaginary root in the half-plane $\Im(z) > 0$ if $q > 0$, or, if $q = 0$ and $E(X_1) > 0$ (Bertoin (1996, see VII.1)).

Step 3. Assume $q > 0$. We prove that $|g| < |f|$ when $\Im(z) = 0$, in both cases (S) and (NS). Denote $z = u + iv$ and observe that, due to the fact that the distribution with rational transform is non lattice (as it has density), due to Lemma 4.1

$$|g(u)| < \lambda \text{ if } u \neq 0.$$

On the other side,

$$|f(u)| \geq \Re f(u) \geq f(0) = \lambda + q > \lambda,$$

completing the proof of the step.

Step 3. Assume $q = 0$. The preceding argument does not work only in case $u = 0$, when $|g(0)| = \lambda$. But in this case, condition

$$EX_1 = -i\psi'(i0+) = a + \int_1^\infty x\pi^+(dx) + \lambda \int_{-\infty}^0 xp(x)dx > 0$$

ensures that

$$\begin{aligned} f'(i0+) = -i\big(\psi^+\big)'(i0+) &= a + \int_1^\infty x\pi^+(dx) \\ &> -\lambda \int_{-\infty}^0 xp(x)dx = g'(i0+). \end{aligned}$$

As the limit in the derivative is taken with $z \to 0$ and $\Im(z) > 0$, this tell us that there exists a small enough $r > 0$ such that the inequality $|f(z)| > |g(z)|$ holds for $z = re^{i\theta}$ $(0 \leq \theta \leq \pi)$. This means that in this case the countor should have a modification to exclude $z = 0$ of the form $z = re^{i\theta}$ $(0 \leq \theta \leq \pi)$.

Step 4. Let us now verify, for R big enough, first in case $\sigma > 0$, that

$$|g(z)| < |f(z)| \text{ when } z = Re^{i\theta} \quad (0 \leq \theta \leq \pi). \tag{21}$$

To begin notice that, as the distribution of U is of rational transform type, with no atom at zero

$$\lim_{|z|\to\infty} g(z) = -\lambda \lim_{|z|\to\infty} \hat{p}(z) = 0. \tag{22}$$

As, by dominated convergence we obtain that

$$\lim_{|z|\to\infty} \int_0^\infty \frac{e^{izx} - 1 - izh(x)}{|z|^2} \pi^+(dx) = 0,$$

we conclude that $f(z) = -(\sigma^2/2)z^2 + o(|z|^2)$ when $\Im z > 0$. Taking (22) into account we obtain (21).

Step 5. Let us consider now the case $\sigma = 0$. In case (S) equation (20) gives that $|f| \geq \lambda$, and this ensures condition (21) to hold. Assume now that X^+ is not a subordinator, i.e. we are in case (NS). In this situation we know that

$$\int_0^1 x\pi^+(dx) = \infty.$$

This ensures the existence of $c \in (0,1)$ such that

$$a_c = a - \int_c^1 x\pi^+(dx) < 0.$$

Consider now

$$\frac{\psi^+(z)}{iz} = a_c + \int_0^c \frac{e^{izx} - 1 - izx}{iz}\pi^+(dx) + \int_c^{+\infty} \frac{e^{izx} - 1}{iz}\pi^+(dx).$$

We will consider $|z| \to \infty$ with $\Im z > 0$. It is clear that

$$\lim_{|z|\to\infty} \left| \int_c^\infty \frac{e^{izx} - 1}{iz}\pi^+(dx) \right| \leq \lim_{|z|\to\infty} \frac{2\int_c^\infty \pi^+(dx)}{|z|} = 0.$$

Furthermore, we check that

$$\Re\Big(\frac{1 + izx - e^{izx}}{iz}\Big) = \int_0^x \Re(1 - e^{izt})dt \geq 0$$

to obtain:

$$\begin{aligned} \frac{|f(z)|}{|z|} &\geq \Re\frac{f(z)}{iz} \\ &= -a_c + \int_0^c \Re\Big(\frac{1 + izx - e^{izx}}{iz}\Big)\pi^+(dx) + o(1) \geq -a_c + o(1). \end{aligned}$$

As $-a_c > 0$, in view of (22) we obtain that $|g| < |f|$ over $Re^{i\theta}$ $(0 \leq \theta \leq \pi)$, for R big enough. This concludes the proof of the Lemma. □

4.2 Baxter and Donsker formula revisted

Our main tool to prove (13) is a formula obtained by Baxter and Donsker (1957) that gives the double Laplace transform of the distribution of the supremum of a Lévy process in terms of a double integral. In the following result we adapt this formula to our context, and in the proof of the Theorem we simply compute both integrals.

More precisely, the first step of the proof is a result of independent interest. We show that in formula (1.3) of Baxter and Donsker, giving the Laplace transform of M_q, it is always possible, under their assumptions, to reverse the order of integration, to obtain the “slightly more elegant result” they suggest. Observe that in this result we use Laplace instead of Fourier transforms. The precise result follows.

Lemma 4.2. *Consider a Lévy process $X = \{X_t\}_{t\geq 0}$ with characteristic exponent ψ given in* (1) *and such that,*

$$\text{for some } \delta > 0, \text{ the integral} \int_{-\delta}^{\delta} \left| \frac{\psi(\xi)}{\xi} \right| d\xi < \infty. \tag{23}$$

Then, for $u > 0$

$$\phi^{+}(iu) = E(e^{-uM_q}) = \exp\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u}{\xi(\xi - iu)} \log\left(\frac{q}{q - \psi(\xi)} \right) d\xi \right\}, \tag{24}$$

and for $u < 0$

$$\phi^{-}(iu) = E(e^{-uI_q}) = \exp\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u}{\xi(\xi - iu)} \log\left(\frac{q}{q - \psi(\xi)} \right) d\xi \right\}. \tag{25}$$

Proof. We begin by (24). Formula (1.3) in Baxter and Donsker (1957) states

$$E(e^{-uM_q}) = \exp\left\{ \frac{1}{2\pi} \int_{q}^{\infty} ds \int_{-\infty}^{\infty} \frac{u}{\xi(\xi - iu)} \frac{\psi(\xi)}{s\big(s - \psi(\xi)\big)} d\xi \right\}. \tag{26}$$

As

$$\int_{q}^{\infty} \frac{\psi(\xi)}{s\big(s - \psi(\xi)\big)} ds = \log\left(\frac{q}{q - \psi(\xi)} \right),$$

we only need to prove that, for the interated integral in (26), the hypothesis of Fubini's Theorem for iterated integrals hold (see, for instance, Rudin (1987)). This amounts to prove

$$\int_{-\infty}^{\infty} d\xi \int_{q}^{\infty} |I(s,\xi)| ds < \infty$$

where we denote the integrand by

$$I(s,\xi) = \frac{u}{\xi(\xi - iu)} \frac{\psi(\xi)}{s\big(s - \psi(\xi)\big)}.$$

We first integrate on the set $|\xi| \leq \delta$. As

$$|I(s,\xi)| \leq \frac{1}{s^2} \left| \frac{\psi(\xi)}{\xi} \right|,$$

we obtain

$$\int_{-\delta}^{\delta} d\xi \int_{q}^{\infty} |I(s,\xi)| ds \leq \frac{1}{q} \int_{-\delta}^{\delta} \left| \frac{\psi(\xi)}{\xi} \right| d\xi < \infty,$$

according to our assumption (23) on the behaviour of the characteristic exponent in a neighborhood of the origin.

To estimate the integral over the set $|\xi| \geq \delta$, denote

$$a(\xi) = \Re\big(\psi(\xi)\big), \qquad b(\xi) = \Im\big(\psi(\xi)\big),$$

and observe the following bounds, that hold for all $s \geq q$, and all real ξ:

$$
\begin{aligned}
a(\xi) &= -\frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \big(\cos(\xi x) - 1\big)\Pi(dx) \leq 0, \\
\big|s - \psi(\xi)\big| &= \sqrt{(s - a(\xi))^2 + b(\xi)^2} \geq s - a(\xi) > 0, \\
\big|s - \psi(\xi)\big| &\geq \sqrt{s^2 + b(\xi)^2}.
\end{aligned}
$$

Using this bounds, we obtain

$$
\left|\frac{\psi(\xi)}{s\big(s - \psi(\xi)\big)}\right| \leq \frac{-a(\xi)}{s\big(s - a(\xi)\big)} + \frac{\big|b(\xi)\big|}{s\sqrt{s^2 + b(\xi)^2}},
$$

and, as $\max\big(-a(\xi), |b(\xi)|\big) \leq k\xi^2$ for a certain constant k, we have

$$
\begin{aligned}
\int_q^\infty \left|\frac{\psi(\xi)}{s\big(s - \psi(\xi)\big)}\right| ds &\leq \int_q^\infty \frac{-a(\xi)}{s\big(s - a(\xi)\big)} ds + \int_q^\infty \frac{\big|b(\xi)\big|}{s\sqrt{s^2 + b(\xi)^2}} ds \\
&= \log\left(\frac{q - a(\xi)}{q}\right) + \log\left(\frac{2}{q}\left(|b(\xi)| + \sqrt{q^2 + b(\xi)^2}\right)\right) \\
&\leq 2\log(k_1 + k_2\xi^2),
\end{aligned}
$$

for k_1 and k_2 convenient constants. From this follows that

$$
\int_{|\xi|\geq\delta} d\xi \int_q^\infty |I(s,\xi)| ds \leq \int_{|\xi|\geq\delta} \frac{2|u|}{|\xi|\sqrt{u^2 + \xi^2}} \log(k_1 + k_2\xi^2) d\xi < \infty,
$$

concluding the proof of (24).

The formula (25) for the negative factor is obtained from the previous one with the help of the dual process $\hat{X} = \{-X_t\}_{t\geq 0}$. Observe that $\hat{M}_q = \sup_{0\leq t<\tau(q)}(-X_t) = -I_q$, and $\hat{\psi}(u) = \psi(-u)$, where $\hat{\psi}$ is the characteristic exponent of the dual process. Take then $u < 0$ and apply (24) to the dual process, to obtain

$$
\begin{aligned}
E(e^{-uI_q}) = E(e^{-(-u)\hat{M}_q}) &= \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-u)}{\xi(\xi - i(-u))} \log\left(\frac{q}{q - \hat{\psi}(\xi)}\right) d\xi\right\} \\
&= \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{u}{v(v - iu)} \log\left(\frac{q}{q - \psi(v)}\right) dv\right\}
\end{aligned}
$$

where we changed variables according to $\xi = -v$ in the last integral, concluding the proof. □

4.3 Proof of the Theorem

Proof of the Theorem. Consider the case $q > 0$, and first assume that our characteristic exponent satisfies condition (23). By Lemma 1.2 we know that the equation $q-\psi(z)=0$ has Z roots $i\beta_1,\dots,i\beta_N$ with multiplicities $n_1,\dots,n_N$, with root count $Z=n_1+\dots+n_N=P$ in case (S) or $P+1$ in case (NS).

Denote

$$G_q^-(z)=\prod_{k=1}^{n}\left(\frac{z-i\alpha_k}{-i\alpha_k}\right)^{m_k}\prod_{j=1}^{N}\left(\frac{-i\beta_j}{z-i\beta_j}\right)^{n_j},$$

and define $G_q^+(z)$ by the relation

$$\frac{q}{q-\psi(z)}=G_q^+(z)G_q^-(z). \tag{27}$$

As G_q^- is an infinitely divisibly characteristic function with support on $(-\infty,0]$, Rogozin's factorization (4) suggests that this is the correct factorization. If we knew that $G_q^+(z)$ is also an infinitely divisibly characteristic function (with support on $[0,\infty)$), the uniquness of the factorization in (4) would give the answer. As we do not have this information, we apply Baxter and Donsker formula (25). From the definition (27) we observe that

- $G_q^+(0)=1$,
- $G_q^+(z)$ is a nonvanishing analytic function on the half-plane $\Im(z)>0$, and continuous on $\Im(z)\geq 0$.
- There exists $\delta>0$ such that $\int_{-\delta}^{\delta}\left|\frac{\log G_q^+(u)}{u}\right|du<\infty$,
- $G_q^+(z)$ is a bounded function on the half-plane $\Im(z)\geq 0$.

This properties ensures that both integrals

$$I^{\pm}(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u}{\xi(\xi-iu)}\log\left(G_q^{\pm}(\xi)\right)d\xi$$

are convergent. As (25) states that $E(e^{-uI_q})=\exp(I^+(u)+I^-(u))$ for $u<0$, our result (13) will follow from the evaluation of the integrals

$$I^+(u)=0,\qquad I^-(u)=\log G^-(iu). \tag{28}$$

For $z\in\mathbb{C}$, define the contours

$$C_R^+=\{-R\leq\Re(z)\leq R,\Im(z)=0\}\cup\{z=Re^{i\theta}\colon 0\leq\theta\leq\pi\},$$
$$C_R^-=\{-R\leq\Re(z)\leq R,\Im(z)=0\}\cup\{z=Re^{i\theta}\colon \pi\leq\theta\leq 2\pi\},$$

In order to obtain the first integral in (28) observe that

$$\int_{C_R^+}\frac{u}{z(z-iu)}\log\left(G_q^+(z)\right)dz=0,$$

because the integrand is analytic in the interior of the contour.

In order to obtain the second integral in (28) observe that

$$\frac{1}{2\pi i}\int_{C_R^-}\frac{iu}{z(z-iu)}\log\left(G_q^-(z)\right)dz$$
$$=\mathbf{Res}\left[\frac{iu}{z(z-iu)}\log\left(G_q^-(z)\right);z=iu\right]=\log G_q^-(iu).$$

This concludes the proof under condition (23). As our process does not necessary satisfies (23), to conclude the proof we proceed by approximation, considering a Lévy process with positive jumps restricted to an interval $[\epsilon,1/\epsilon]\quad(0<\epsilon<1)$, and obtain the general result taking $\epsilon\to 0$.

Denote then, for $0<\epsilon<1$,

$$\pi^{+,\epsilon}(dy)=\mathbf{1}_{[\epsilon,1/\epsilon]}(y)\pi^+(dy).$$

All notations with the superscript ϵ refer to a Lévy process with jump measure given in (7) with $\pi^{+,\epsilon}$ instead of π^+. In particular

$$\psi^\epsilon(u)=iau-\frac{1}{2}\sigma^2u^2+\int_{[\epsilon,1/\epsilon]}(e^{iux}-1-iuh(x))\pi^+(dx)+\lambda(\hat{p}(u)-1). \quad (29)$$

Lemma 4.3. *The characteristic exponent defined in* (29) *satisfies the following properties.*

(a) *There exists $\delta>0$ such that $\int_{-\delta}^{\delta}\left|\frac{\psi^\epsilon(u)}{u}\right|du<\infty$.*

(b) *The equation $q-\psi^\epsilon(z)=0$ has $P+1$ roots, say $\beta_1^\epsilon,\dots\beta_{P+1}^\epsilon$, possibly coincident, and such that*

$$\prod_{j=1}^{P+1}(z-i\beta_j^\epsilon)\to\prod_{j=1}^{N}(z-i\beta_j)^{n_j}\qquad(\epsilon\to 0).$$

Proof. We use the following bounds, valid for all real u:

$$\left|e^{iu}-1\right|\le\min(|u|,2),\qquad\left|e^{iu}-1-iu\right|\le\frac{1}{2}u^2. \quad (30)$$

In order to verify (a) we have, for $|u|<2\epsilon$,

$$\Big|\int_{[\epsilon,1/\epsilon]}(e^{iux}-1-iuh(x))\pi^+(dx)\Big|$$
$$\le\int_{[\epsilon,1)}\left|e^{iux}-1-iux\right|\pi^+(dx)+\int_{[1,1/\epsilon]}\left|e^{iux}-1\right|\pi^+(dx)$$
$$\le|u|^2\int_{[\epsilon,1)}y^2\pi^+(dx)+|u|\int_{[1,1/\epsilon]}|x|\pi^+(dx).$$

Furthermore $\frac{\hat{p}(u)-1}{u} \to \hat{p}'(0)$ $(u \to 0)$, so for δ small enough, and for all $|u| \leq \delta$

$$\Big|\frac{\psi^\epsilon(u)}{u}\Big| \leq |b| + \frac{1}{2}\sigma^2|u| + \lambda\big(|\hat{p}'(0)|+1\big) + |u|\int_{[\epsilon,1)} y^2\pi^+(dy) + \int_{[1,1/\epsilon]} |y|\pi^+(dy),$$

giving (a). In order to conclude (b) it is enough to verify that for each $j = 1,\dots,N$ equation $q - \psi^\epsilon(z) = 0$ has n_j roots converging to β_j as $\epsilon \to 0$.

In order to do this, take δ arbitrarily small and consider the disk $D = D(\beta_j,\delta)$ with center β_j and radius δ. In particular δ is small enough such that the equation $q - \psi(z) = 0$ has the only root β_j in D, and D is contained in $\Im(z) > 0$.

Based on (30) and (30), and for ϵ small enough,

$$\begin{aligned}\big|\psi^\epsilon(z) - \psi(z)\big| &\leq \int_{[0,\epsilon)} \big|e^{izy} - 1 - izy\big|\pi^+(dy) + \int_{(1/\epsilon,\infty)} \big|e^{izy} - 1\big|\pi^+(dy)\\ &\leq |z|^2\int_{[0,\epsilon)} y^2\pi^+(dy) + 2\int_{(1/\epsilon,\infty)} \pi^+(dy) \to 0 \ (\epsilon \to 0),\end{aligned}$$

where the convergence is uniform over compact sets. So, taking $g = \psi^\epsilon - \psi$ and $f = q - \psi$ we obtain $|g| < |f|$ on ∂D, and in consequence $q - \psi^\epsilon$ has n_j roots inside D. These roots converge to β_j becuse δ was arbitrarily small, concluding (b) and the proof of the Lemma. □

In order to conclude the proof of the Theorem, we must take limits as $\epsilon \to 0$ in both sides of

$$E\big(e^{izI^\epsilon}\big) = \prod_{k=1}^{n}\Big(\frac{z - i\alpha_k}{-i\alpha_k}\Big)^{m_k}\prod_{j=1}^{P+1}\Big(\frac{-i\beta_j^\epsilon}{z - i\beta_j^\epsilon}\Big),$$

to obtain our result, i.e.

$$\phi_q^-(z) = E\big(e^{izI}\big) = \prod_{k=1}^{n}\left(\frac{z - i\alpha_k}{-i\alpha_k}\right)^{m_k}\prod_{j=1}^{N}\left(\frac{-i\beta_j}{z - i\beta_j}\right)^{n_j}.$$

The limit of the r.h.s. is given by (b) in Lemma 4.3.

In order to obtain the limit in the l.h.s., as $\psi^\epsilon(u) \to \psi(u)$ for all real u we have $X_1^\epsilon \Rightarrow X_1$ (where $\Rightarrow$ denotes weak convergence). From Corollary VII.3.6 in Jacod and Shiryaev (1987) we obtain $X^\epsilon \Rightarrow X$, the weak convergence of the processes. As the infimum over an interval is a continuous functional in the Skorokhod space, we also obtain the weak convergence

$$\inf_{0\leq s\leq t} X_s^\epsilon \Rightarrow \inf_{0\leq s\leq t} X_s \qquad (\epsilon \to 0).$$

Finally, to obtain $I^\epsilon \Rightarrow I$, observe that

$$\begin{aligned}E\big(e^{iuI^\epsilon}\big) &= \int_0^\infty E\big(e^{iu\inf_{0\leq s\leq t} X_s^\epsilon}\big)qe^{-qt}dt\\ &\to \int_0^\infty E\big(e^{iu\inf_{0\leq s\leq t} X_s}\big)qe^{-qt}dt = E\big(e^{iuI}\big),\end{aligned}$$

concluding the proof of the Theorem in case $q > 0$.

Case $q = 0$ follows by approximation as follows. Observe that $(1/q)\tau(1)$ has exponential distribution with parameter q if $\tau(1)$ has exponential distribution with parameter 1. As $\tau(q) \to \infty$ $(a.s.)$, this shows that we can find exponential times such that $I_q \to I_0$ $(a.s.)$. In consequence,

$$\phi_q^-(u) \to \phi_0^-(u) \qquad (q \to 0).$$

Furthermore, as when $q \to 0$ we know that $q - \psi(z)$ converges uniformly over compacts to $-\psi(z)$, and we know the number of roots of equation $-\psi(z) = 0$ on the set $\Im z > 0$, a similar reasoning as the one done in the proof of Lemma 4.3, denoting in this case by $\beta(q)_j$ the roots of $q - \psi(z) = 0$, gives

$$\prod_{j=1}^{P+1} \Big(\frac{-i\beta(q)_j}{z - i\beta(q)_j} \Big) \to \prod_{j=1}^{N} \Big(\frac{-i\beta(0)_j}{z - i\beta(0)_j} \Big)^{n_j} \quad (q \to 0).$$

This concludes the proof of the Theorem. □

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