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Article

Wiener index of uniform hypergraphs induced by trees

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Abstract: The Wiener index $W(G)$ of a graph G is defined as the sum of distances between its vertices. A tree T generates r-uniform hypergraph $H_{r,k}(T)$ by the following way: hyperedges of cardinality r correspond to edges of the tree and adjacent hyperedges have k vertices in common. A relation between quantities $W(T)$ and $W(H_{r,k}(T))$ is established.

Keywords: Tree, hypergraph, Wiener index.

MSC: 05C12

1. Introduction

In this paper we are concerned with undirected connected graphs G with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex is the number of edges that are incident to the vertex. Degree of a vertex v is denoted by $\deg(v)$. If u and v are vertices of G, then the number of edges in a shortest path connecting them is said to be their distance and is denoted by $d_G(u,v)$. Distance of a vertex v is the sum of distances from v to all vertices of a graph, $d_G(v) = \sum_{u \in V(G)} d(v,u)$. The Wiener index is a graph invariant defined as the sum of distances between all vertices of G:

$$W(G) = \sum_{u,v \in V(G)} d(u,v) = \frac{1}{2} \sum_{v \in V(G)} d_G(v).$$

It was introduced as a structural descriptor for tree-like molecular graphs [1]. Details on the mathematical properties and chemical applications of the Wiener index can be found in books [2–7] and reviews [8–13]. A number of articles are devoted to comparing of the index of a graph and its derived graphs such as the line graph, the total graph, thorny and subdivision graphs of various kind (see, for example, [14–17]). Hypergraphs generalize graphs by extending the definition of an edge from a binary to an r-ary relation. Wiener index of some classes of hypergraphs was studied in [18–20]. Chemical applications of hypergraphs were discussed in [21,22].

Define a class of r-uniform hypergraphs $H_{r,k}(T)$ induced by n-vertex trees T. Edges of a tree correspond to hyperedges of cardinality r and adjacent hyperedges have k vertices in common, $1 \le k \le \lfloor r/2 \rfloor$. Examples of a tree and the corresponding hypergraph are shown in Figure 1. The number of vertices of $H_{r,k}(T)$ is equal to $(n-2)(r-k)+r$. We are interesting in finding a relation between quantities $W(T)$ and $W(H_{r,k}(T))$.

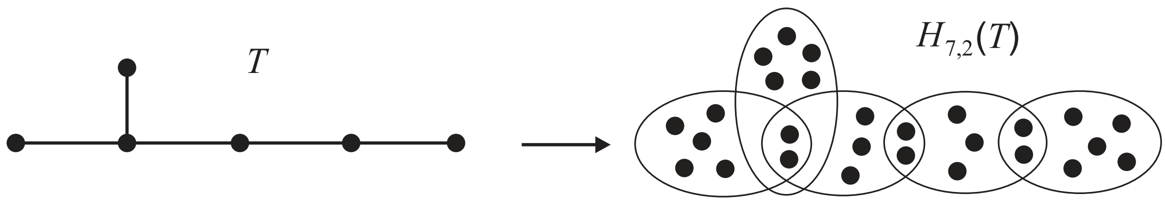

Figure 1. Tree T and the induced hypergraph $H_{7,2}(T)$.

2. Main result

Wiener indices of a tree and its induced hypergraph satisfy the following relation.

Theorem 1. *For the induced hypergraph $H_{r,k}(T)$ of a tree T with n vertices,*

$$W(H_{r,k}(T)) = (r-k)^2\, W(T) + n\binom{k}{2} - (n-1)\binom{r-2k+1}{2}.$$

This result may be useful for ordering of Wiener indices of hypergraphs. If r and k are fixed, then the ordering of the Wiener index of induced hypergraphs $H_{r,k}$ for n-vertex trees is completely defined by the ordering of the index of trees. In particular,

$$W(H_{r,k}(S_n)) \le W(H_{r,k}(T)) \le W(H_{r,k}(P_n))$$

for any n-vertex tree T, where S_n and P_n are the star and the path with n vertices,

$$W(H_{r,k}(S_n)) = \big(r(n-1)[2n(r-2k)+8k-3r-1] + k(n-2)[k(2n-3)+1]\big)/2,$$

$$W(H_{r,k}(P_n)) = n(r-k)[(r-k)n^2 + 10k - 4r - 3]/6 + 2k^2 - k(2r+1) + r(r+1)/2.$$

3. Proof of Theorem 1

The edge subdivision operation for an edge $(x,y) \in E(G)$ is the deletion of (x,y) from graph G and the addition of two edges (x,v) and (v,y) along with the new vertex v. Vertex v is called the subdivision vertex. Denote by T_e the tree obtained from the subdivision of edge e in a tree T. The distance $d_G(v,U)$ from a vertex $v \in V(G)$ to a vertex subset $U \subseteq V(G)$ is defined as $d_G(v,U) = \sum_{u\in U} d_G(v,u)$.

Lemma 2. *Let $T_{e_1}, T_{e_2}, \ldots, T_{e_{n-1}}$ be trees obtained by subdivision of edges $e_1, e_2, \ldots, e_{n-1}$ of n-vertex tree T with subdivision vertices $v_1, v_2, \ldots, v_{n-1}$, respectively. Then*

$$d_{T_{e_1}}(v_1) + d_{T_{e_2}}(v_2) + \cdots + d_{T_{e_{n-1}}}(v_{n-1}) = 2W(T).$$

Proof. Let v be the subdivision vertex of edge $e = (x,y)$ of a tree T. Denote by V_x and V_y the sets of vertices of two connected components after deleting edge e from T where $x \in V_x$ and $y \in V_y$. Since $d_T(x) = d_T(x,V_x) + |V_y| + d_T(y,V_y)$ and $d_T(y) = d_T(y,V_y) + |V_x| + d_T(x,V_x)$, $d_T(x,V_x) + d_T(y,V_y) = (d_T(x) + d_T(y) - n)/2$. Then

$$\begin{aligned} d_{T_e}(v) &= \sum_{u\in V_x} [\, d_{T_e}(v,x) + d_T(x,u)\,] + \sum_{u\in V_y} [\, d_{T_e}(v,y) + d_T(y,u)\,] \\ &= d_T(x,V_x) + d_T(y,V_y) + n = (d_T(x) + d_T(y) + n)/2. \end{aligned}$$

Klein *et al.* [23] proved that $\sum_{v\in V(T)} \deg(v) d_T(v) = 4W(T) - n(n-1)$ for an arbitrary n-vertex tree T. Then

$$\begin{aligned} 2\sum_{i=1}^{n-1} d_{T_{e_i}}(v_i) &= \sum_{(x,y)\in E(T)} (d_T(x) + d_T(y) + n) \\ &= \sum_{v\in V(T)} \deg(v) d_T(v) + n(n-1) = 4W(T). \end{aligned}$$

□

For convenience, we assume that pendent hyperedges are also adjacent with fictitious hyperedges shown by dashed lines in Figure 2. Denote by B_i, $i = 1, 2, \ldots n$, the vertices of a hypergraph $H = H_{r,k}(T)$ belonging to hyperedge intersections and let $A = V(H) \setminus B_1 \cup B_2 \cup \cdots \cup B_n$. We assume that edge E_i of the induced

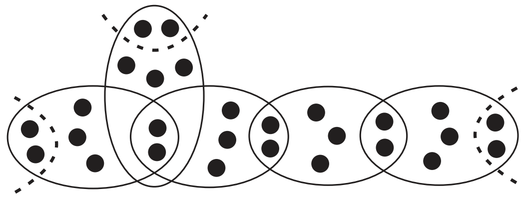

Figure 2. The hyperedges shown by dashed lines

hypergraph corresponds to edge e_i of the source tree T, $i = 1, 2, \ldots, n-1$. Let $d_G(U) = \sum_{u \in U} d_G(u)$ for $U \subseteq V(G)$. Then the Wiener index of H can be represented as follows:

$$W(H) = \frac{1}{2}\left(\sum_{i=1}^{n-1} d_H(E_i \cap A) + \sum_{i=1}^{n} d_H(B_i)\right). \tag{1}$$

Let $u \in E_i \cap A$ and v_i be the subdivision vertex of edge e_i in T, $i = 1, 2, \ldots, n-1$. Then

$$\begin{aligned} d_H(u) &= (r-2k-1) + k + k + 2(r-k) + \cdots + 2(r-k) + 3(r-k) + \cdots + 3(r-k) + \ldots \\ &= (r-2k-1) - 2(r-2k) + (r-k) + (r-k) + 2(r-k) + \cdots + 2(r-k) \\ &\quad + 3(r-k) + \cdots + 3(r-k) + 4(r-k) + \cdots + 4(r-k) + \ldots \\ &= (r-k)d_{T_{e_i}}(v_i) - 2(r-2k) + (r-2k-1). \end{aligned}$$

Summing this equality for all vertices of intersection $E_i \cap A$, we have $d_H(E_i \cap A) = (r-2k)d_H(u) = (r-2k)[(r-k)d_{T_{e_i}}(v_i) - (r-2k+1)]$. Applying Lemma 2, we can write

$$\begin{aligned} \sum_{i=1}^{n-1} d_H(E_i \cap A) &= (r-2k)\left((r-k)\sum_{i=1}^{n-1} d_{T_{e_i}}(v_i) - (n-1)(r-2k+1)\right) \\ &= (r-2k)\left[2(r-k)W(T) - (n-1)(r-2k+1)\right]. \end{aligned} \tag{2}$$

Let $u \in B_i$ and vertex v_i of T corresponds to this hyperedge intersection, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} d_H(u) &= (k-1) + (r-k) + \cdots + (r-k) + 2(r-k) + \cdots + 2(r-k) + 3(r-k) + \cdots + 3(r-k) + \ldots \\ &= (r-k)d_T(v_i) + (k-1). \end{aligned}$$

Summing this equality for all vertices of the hyperedge intersection B_i, we have $d_H(B_i) = kd_H(u) = k[\,(r-k)d_T(v_i) + (k-1)\,]$. For vertices of all intersections,

$$\sum_{i=1}^{n} d_H(B_i) = k\,[\,(r-k)\sum_{i=1}^{n} d_T(v_i) + n(k-1)\,] = 2k(r-k)W(T) + nk(k-1). \tag{3}$$

Substitution expressions (2) and (3) back into Equation (1) completes the proof.

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Conflicts of Interest: "The author declare no conflict of interest."

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