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WIENER'S LEMMA FOR INFINITE MATRICES

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ABSTRACT. The classical Wiener lemma and its various generalizations are important and have numerous applications in numerical analysis, wavelet theory, frame theory, and sampling theory. There are many different equivalent formulations for the classical Wiener lemma, with an equivalent formulation suitable for our generalization involving commutative algebra of infinite matrices $\mathcal{W} := \{(a(j-j'))_{j,j'\in\mathbf{Z}^d} : \sum_{j\in\mathbf{Z}^d} |a(j)| < \infty\}$. In the study of spline approximation, (diffusion) wavelets and affine frames, Gabor frames on non-uniform grid, and non-uniform sampling and reconstruction, the associated algebras of infinite matrices are extremely non-commutative, but we expect those non-commutative algebras to have a similar property to Wiener's lemma for the commutative algebra $\mathcal{W}$. In this paper, we consider two non-commutative algebras of infinite matrices, the Schur class and the Sjöstrand class, and establish Wiener's lemmas for those matrix algebras.

1. INTRODUCTION

The classical Wiener lemma states that *if a periodic function f has an absolutely convergent Fourier series and never vanishes, then $1/f$ has an absolutely convergent Fourier series* ([39]).

There are many different equivalent formulations for the classical Wiener lemma. An equivalent formulation of the classical Wiener lemma suitable for our generalization involves matrix algebra: *$A \in \mathcal{W}$ and $A^{-1} \in \mathcal{B}^2$ imply $A^{-1} \in \mathcal{W}$.* Here

$$\mathcal{W} := \Big\{A := (a(j-j'))_{j,j'\in\mathbf{Z}^d},\ \|A\|_{\mathcal{W}} := \sum_{j\in\mathbf{Z}^d} |a(j)| < \infty\Big\}, \tag{1.1}$$

and $\mathcal{B}^r, 1 \le r \le \infty$, is the set of all bounded operators on the space ℓ^r of all r-summable sequences and is equipped with the usual operator norm $\|\cdot\|_{\mathcal{B}^r}$.

The matrix algebra $\mathcal{W}$ in the matrix formulation of the classical Wiener lemma is a *commutative Banach algebra*. In the study of spline approximation and projection ([15, 16]), wavelets and affine frames ([9, 26]), Gabor frame ([3, 23, 24]), and non-uniform sampling, the associated matrix algebras are *extremely non-commutative* but they are still expected to have the same property as the commutative matrix algebra $\mathcal{W}$. We are motivated by the above expectation, and by the importance of Wiener's lemma in the study of (Gabor) frames on non-uniform grids and of non-uniform sampling and the reconstruction problem. For instance, we apply Wiener's lemma established in this paper to show the well-localization of dual (tight) frame generators of a locally finitely-generated space ([37]), and robustness

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and finite implementation of an average (ideal) sampling and the reconstruction process ([38]).

In this paper, we introduce two non-commutative matrix algebras of infinite matrices of the form

$$A := (a(x,y))_{x,y\in X}, \tag{1.2}$$

the Schur class and the Sjöstrand class, and establish Wiener's lemmas for those matrix algebras. The Schur class and the Sjöstrand class have

$$\begin{aligned} \mathcal{A}_{p,w} \quad := \quad & \Big\{(a(j,j'))_{j,j'\in \mathbf{Z}^d}: \ \sup_{j\in\mathbf{Z}^d}\Big(\sum_{j'\in\mathbf{Z}^d}|(aw)(j,j')|^p\Big)^{1/p} \\ & \qquad + \sup_{j'\in\mathbf{Z}^d}\Big(\sum_{j\in\mathbf{Z}^d}|(aw)(j,j')|^p\Big)^{1/p}<\infty\Big\} \end{aligned} \tag{1.3}$$

and

$$\mathcal{C}_{p,w} := \Big\{(a(j,j'))_{j,j'\in\mathbf{Z}^d}: \ \Big(\sum_{k\in\mathbf{Z}^d}\sup_{j-j'=k}|(aw)(j,j')|^p\Big)^{1/p}<\infty\Big\} \tag{1.4}$$

as their models respectively, where $1\le p\le\infty$ and w is a weight. We use a space of homogenous type as the index set of the infinite matrix A instead of $\mathbf{Z}^d$ in the above model (see Sections 2 and 6 for details), because the index set of the infinite matrix A in frame theory and sampling theory carries some important information, such as the center of generators in frame theory and the sampling location in the average (ideal) sampling process, and hence it is unsuitable to be re-indexed as $\mathbf{Z}^d$ or its subsets, [37, 38].

The classical Wiener lemma and its various generalizations (see, for instance, [3, 4, 5, 16, 23, 24, 25, 26, 33]) are important and have numerous applications in numerical analysis, wavelet theory, frame theory, and sampling theory. For example, the classical Wiener lemma and its weighted variation ([25]) were used to establish the decay property at infinity for dual generators of a shift-invariant space ([1, 27]); the Wiener lemma for matrices associated with twisted convolution was used in the study the decay properties of the dual Gabor frame for L^2 ([3, 23, 24]); the Jaffard's result ([26]) for infinite matrices with polynomial decay was used in numerical analysis ([8, 34, 35]), wavelet analysis ([26]), time-frequency analysis ([19, 20, 21]) and sampling ([2, 14, 21]); and Sjöstrand's result ([33]) for infinite matrices was used in the study of pseudo-differential operators and Gabor frames ([3, 22, 33]). We will apply Wiener's lemma for infinite matrices of Schur type and of Sjöstrand type in the study of (Gabor) frame property for locally finitely-generated spaces, and non-uniform sampling and stable reconstruction for signals with finite rate of innovations; see the subsequent papers [37, 38].

The paper is organized as follows. In the first part of the paper (Sections 2 – 5), we introduce the Schur class $\mathcal{A}_{p,w}$ of infinite matrices (Section 2), study the asymptotic behavior of $A^n, n\ge 1$, for a matrix $A\in\mathcal{A}_{p,w}$ (Section 3), establish Wiener's lemma for infinite matrices A in the Schur class: *$A\in A_{p,w}$ and $A^{-1}\in\mathcal{B}^2$ implies* $A^{-1}\in\mathcal{A}_{p,w}$ (Section 4), and discuss some useful variations and generalizations of the above Wiener lemma (Section 5). In the second part of the paper (Section 6), we introduce the Sjöstrand class of infinite matrices and establish Wiener's lemma for infinite matrices in the Sjöstrand class.

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In this paper, the capital letter C, if not specified, denotes an absolute constant which may be different at a different occurrence.

2. Infinite matrices of Schur type

In this section, we discuss the index set X that has a quasi-metric ρ and a Borel measure μ, introduce the Schur class $\mathcal{A}_{p,w}(X,\rho,\mu)$ of infinite matrices, and establish some basic results for the Schur class $\mathcal{A}_{p,w}(X,\rho,\mu)$ with different exponents p and weights w (Theorem 2.4).

2.1. **Index set of infinite matrices.** For infinite matrices of the form (1.2), we introduce a quasi-metric ρ on the index set X to measure how far a location (x,y) in the matrix A is from the diagonal. Here we recall that a *quasi-metric* on X is a function $\rho: X\times X \longmapsto [0,\infty)$ such that (i) $\rho(x,x)=0$ for all $x\in X$; (ii) $\rho(x,y)=\rho(y,x)$ for all $x,y\in X$; and (iii) there exists a positive constant L such that $\rho(x,y)\le L(\rho(x,z)+\rho(z,y))$ for all $x,y,z\in X$. A qausi-metric ρ is said to be a *metric* if the triangle inequality (iii) holds for $L=1$. The space (X,ρ) is known as a *quasi-metric space*.

We introduce a non-negative Borel measure μ on the quasi-metric space (X,ρ) that has *polynomial growth property*: there exist positive constants D and d such that

$$\mu(B(x,\tau))\le D\tau^d \quad \text{for all } x\in X \text{ and } \tau\ge 1, \tag{2.1}$$

where $B(x,\tau):=\{y\in X:\ \rho(x,y)<\tau\}$ is the open ball of radius τ around x. The polynomial growth assumption is an important assumption for our establishment of Wiener's lemma. For the model cases (1.3) and (1.4), we use $\mathbf{Z}^d$ as the index set and the counting measures μ_c as the Borel measure on the index set.

Remark 2.1. A non-negative Borel measure μ on a quasi-metric space (X,ρ) is said to be a *doubling measure* if

$$0<\mu(B(x,\tau))\le D_1\mu(B(x,\tau/2))<\infty \text{ for all } \tau>0 \text{ and } x\in X, \tag{2.2}$$

where D_1 is a positive constant. A quasi-metric space (X,ρ) with a non-negative doubling Borel measure μ, to be denoted by (X,ρ,μ), is known as *a space of homogenous type* ([13, 28, 29]). If the Borel measure μ on the space of homogenous type (X,ρ,μ) further satisfies *uniform boundedness conditions*, that is, there exist two positive constants D_2, D_3 such that $D_2\le \mu(B(x,1))\le D_3$ for all $x\in X$, then it has polynomial growth property. We remark that for any space of homogenous type (X,ρ,μ), an equivalent quasi-metric $\tilde\rho$ to the quasi-metric ρ, in the sense that quasi-metric spaces (X,ρ) and $(X,\tilde\rho)$ have the same topology, can be found so that the new quasi-metric $\tilde\rho$ satisfies the uniform boundedness condition ([28, 29]).

2.2. **Admissible weights.** For infinite matrices of the form (1.2), we use a *weight* w on $X\times X$ to measure the importance of a location $(x,y)\in X\times X$, while in most situations it is used to measure the off-diagonal decay. Here a weight w is a positive symmetric measurable function on $X\times X$ that satisfies

$$1\le w(x,y)=w(y,x)<\infty \text{ for all } x,y\in X, \tag{2.3}$$

$$D(w):=\sup_{x\in X} w(x,x)<\infty, \tag{2.4}$$

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and

$$\sup_{\rho(x,\tilde{x})+\rho(y,\tilde{y})\le C_0} \frac{w(x,y)}{w(\tilde{x},\tilde{y})} \le D(C_0,w) < \infty \tag{2.5}$$

for all $C_0 \in (0,\infty)$.

Example 2.2. Typical examples of weights include:

(i) The functions $w_\alpha, \alpha \ge 0$,

$$w_\alpha(x,y) = (1+\rho(x,y))^\alpha, \tag{2.6}$$

are weights on $X \times X$, which are known as *polynomial weights*; see also Example A.2.

(ii) The functions $e_{D,\delta}$ with $D \in (0,\infty)$ and $\delta \in (0,1)$,

$$e_{D,\delta}(x,y) = \exp(D\rho(x,y)^\delta), \tag{2.7}$$

are weights on $X \times X$ provided that ρ is a quasi-metric, which are known as *subexponential weights*; see also Example A.3.

To establish Wiener's lemma for infinite matrices in the Schur class and the Sjöstrand class, we need a technical assumption on the weight w. Let $1 \le p, r \le \infty$. We say that a weight w is (p,r)-*admissible* if there exist another weight v and two positive constants $D \in (0,\infty)$ and $\theta \in (0,1)$ such that

$$w(x,y) \le D(w(x,z)v(z,y) + v(x,z)w(z,y)) \quad \text{for all } x,y,z \in X, \tag{2.8}$$

$$\sup_{x\in X} \|(vw^{-1})(x,\cdot)\|_{p'} + \sup_{y\in X} \|(vw^{-1})(\cdot,y)\|_{p'} \le D, \tag{2.9}$$

and

$$\inf_{\tau>0} a_{r'}(\tau) + b_{p'}(\tau)t \le Dt^\theta \text{ for all } t \ge 1, \tag{2.10}$$

where $p' = p/(p-1), r' = r/(r-1)$,

$$a_{r'}(\tau) = \sup_{x\in X} \|v(x,\cdot)\chi_{B(x,\tau)}(\cdot)\|_{r'} + \sup_{y\in X} \|v(\cdot,y)\chi_{B(y,\tau)}(\cdot)\|_{r'}, \tag{2.11}$$

$$b_{p'}(\tau) = \sup_{x\in X} \|(vw^{-1})(x,\cdot)\chi_{X\setminus B(x,\tau)}(\cdot)\|_{p'} + \sup_{y\in X} \|(vw^{-1})(\cdot,y)\chi_{X\setminus B(y,\tau)}(\cdot)\|_{p'}, \tag{2.12}$$

χ_E is the characteristic function on the set E, and $\|\cdot\|_p$ is the usual norm on $L^p := L^p(X,\mu)$, the space of all p-integrable functions on X. Clearly a (p,r)-admissible weight is $(p,\tilde{r})$-admissible for any $r \le \tilde{r} \le \infty$. More discussion on the (p,r)-admissibility of a weight w will be given in the Appendix. For instance, we show in Examples A.2 and A.3 that the polynomial and subexponential weights in Example 2.2 are (p,r)-admissible if the exponent α in the polynomial weight w_α and the exponents D and δ in the subexponential weight $e_{D,\delta}$ satisfy certain conditions.

2.3. Schur class of infinite matrices.

Take $p \in [1,\infty]$; we define

$$\mathcal{A}_{p,w}(X,\rho,\mu) := \big\{A := (a(x,y))_{x,y\in X} : \ \|A\|_{\mathcal{A}_{p,w}} < \infty\big\}, \tag{2.13}$$

where

$$\|A\|_{\mathcal{A}_{p,w}} := \sup_{x\in X} \|(aw)(x,\cdot)\|_p + \sup_{y\in X} \|(aw)(\cdot,y)\|_p. \tag{2.14}$$

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We use $\mathcal{A}_{p,w}$ for the brevity of $\mathcal{A}_{p,w}(X,\rho,\mu)$. We call $\mathcal{A}_{p,w}(X,\rho,\mu)$ the *Schur class* and an infinite matrix in $\mathcal{A}_{p,w}(X,\rho,\mu)$ as being of Schur type because, for $p=1$, $\mathcal{A}_{p,w}(X,\rho,\mu)$ becomes the usual Schur class; see Remark 2.3 below.

Remark 2.3. A large class of non-commutative matrix algebras has been introduced recently, and Wiener's lemma is established on those matrix algebras. Here are some examples of those non-commutative matrix algebras:

(1) The Schur class

$$\mathcal{A}_v^1=\Big\{(a_{k,l})_{k,l\in X}:\sup_{k\in X}\sum_{l\in X}|a_{k,l}|v(k-l)+\sup_{l\in X}\sum_{k\in X}|a_{k,l}|v(k-l)<\infty\Big\}$$

where v is a positive (radial) function $\mathbf{R}^d$ and $X\subset\mathbf{R}^d$ ([24]).

(2) The Sjöstrand class

$$\mathcal{C}=\Big\{(a_{k,l})_{k,l\in\mathbf{Z}^d}:\sum_{l\in\mathbf{Z}^d}\sup_{k\in\mathbf{Z}^d}|a_{k,k-l}|<\infty\Big\}$$

([5, 6, 33]).

(3) The Jaffard class

$$\mathcal{A}_s=\Big\{(a_{k,l})_{k,l\in X}:\sup_{k,l\in X}|a_{k,l}|(1+|k-l|)^s<\infty\Big\}$$

where $s\in\mathbf{R}$ and $X\subset\mathbf{R}^d$ ([24, 26]).

2.4. **Schur classes with different exponents and weights.** Let $\mathcal{B}^r$ be the space of all bounded operators on $L^r(X,\mu), 1\le r\le\infty$, and $\|\cdot\|_{\mathcal{B}^r}$ denote the usual operator norm. For any $A=(a(x,y))_{x,y\in X}\in\mathcal{A}_{p,w}(X,\rho,\mu)$, we define its transpose A^* by $A^*:=\big(\overline{a(y,x)}\big)_{x,y\in X}$.

For Schur classes $\mathcal{A}_{p,w}(X,\rho,\mu)$ with different exponents p and weights w, we have the following basic properties, which will be used later in the proofs.

Theorem 2.4. *Let $1\le p\le\infty$, let X be an index set that has a pseudo-metric ρ and a Borel measure μ with the polynomial growth property, let the weights u,v and w on $X\times X$ satisfy* (2.3)*,* (2.4) *and* (2.5)*, let the weight w_0 be as in* (2.6)*, and let the Schur class $\mathcal{A}_{p,w}(X,\rho,\mu)$ be as in* (2.13)*. Then*

(i) *$A\in\mathcal{A}_{p,w}(X,\rho,\mu)$ if and only if its transpose $A^*\in\mathcal{A}_{p,w}(X,\rho,\mu)$.*
(ii) *$\mathcal{A}_{p,u}(X,\rho,\mu)\subset\mathcal{A}_{q,v}(X,\rho,\mu)$ if $vu^{-1}\in\mathcal{A}_{(p/q)',w_0}(X,\rho,\mu)$ and $p\ge q$, where $(p/q)'=pq/(p-q)$.*
(iii) *$\mathcal{A}_{1,w_0}(X,\rho,\mu)\subset\mathcal{B}^r$ for all $1\le r\le\infty$.*
(iv) *$\mathcal{A}_{p,w}(X,\rho,\mu)\subset\mathcal{B}^r$ for all $1\le r\le\infty$ and $w^{-1}\in\mathcal{A}_{p/(p-1),w_0}$.*

Proof. The first statement follows from the definition of the transpose A^* of a matrix $A\in\mathcal{A}_{p,w}(X,\rho,\mu)$. Moreover, the transpose A^* has the same norm in $\mathcal{A}_{p,w}(X,\rho,\mu)$ as the one of A,

$$\|A^*\|_{\mathcal{A}_{p,w}}=\|A\|_{\mathcal{A}_{p,w}}. \qquad (2.15)$$

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Take any $A = (a(x,y))_{x,y\in X} \in \mathcal{A}_{q,v}(X,\rho,\mu)$; we have

$$\|(av)(x,\cdot)\|_q \le \|(au)(x,\cdot)\|_p \|(vu^{-1})(x,\cdot)\|_r, \quad x \in X, \tag{2.16}$$

where $r^{-1} = q^{-1} - p^{-1}$. Thus

$$\|A\|_{\mathcal{A}_{p,u}} \le \|vu^{-1}\|_{\mathcal{A}_{r,w_0}} \|A\|_{\mathcal{A}_{q,v}} \text{ for all } A \in \mathcal{A}_{q,v}. \tag{2.17}$$

The second statement then follows.

A matrix in $\mathcal{A}_{1,w_0}(X,\rho,\mu)$ clearly defines a bounded operator on $L^r, 1 \le r \le \infty$. Moreover,

$$\max\left(\|A\|_{\mathcal{B}^r}, \|A^*\|_{\mathcal{B}^r}\right) \le \|A\|_{\mathcal{A}_{1,w_0}} \tag{2.18}$$

for all $A \in \mathcal{A}_{1,w_0}(X,\rho,\mu)$ and $1 \le r \le \infty$. Hence the third statement is proved.

The fourth statement holds because the second and third statements are true. □

3. Power of infinite matrices in the Schur class

In this section, we establish an asymptotic estimate for the power of infinite matrices in the Schur class (Theorem 3.1), and an equality of various spectral radii for infinite matrices in the Schur class (Theorem 3.3). The asymptotic estimate (3.2) of $A^n, n \ge 1$, for an infinite matrix A in the Schur class is *crucial* for the establishment of our Wiener lemma in the next section.

Theorem 3.1. *Let $1 \le p, r \le \infty$, let X be an index set that has a pseudo-metric ρ and a non-negative Borel measure μ with the polynomial growth property, let w be a (p,r)-admissible weight on $X \times X$, and let the Schur class $\mathcal{A}_{p,w}(X,\rho,\mu)$ be as in (*2.13*). Assume that $A \in \mathcal{A}_{p,w}(X,\rho,\mu)$, and that*

$$\|A^n\|_{\mathcal{A}_{r,w_0}} \le P(n) \max\left(\|A^n\|_{\mathcal{B}^r}, \|(A^*)^n\|_{\mathcal{B}^r}\right) \text{ for all } n \ge 1, \tag{3.1}$$

where $P(n) \ge 1$ is a polynomial. Then the asymptotic estimate,

$$\begin{aligned} \|A^n\|_{\mathcal{A}_{p,w}} \quad &\le \quad C_0 \left(\frac{C_0 \|A\|_{\mathcal{A}_{p,w}}}{\max(\|A\|_{\mathcal{B}^r}, \|A^*\|_{\mathcal{B}^r})}\right)^{\frac{1+\theta}{\theta} n^{\log_2(1+\theta)}} \\ &\qquad \times \max\left(\|A\|^n_{\mathcal{B}^r}, \|A^*\|^n_{\mathcal{B}^r}\right), \quad n \ge 1, \end{aligned} \tag{3.2}$$

*holds for some positive constant C_0, which depends only on the parameters $p, r \in [1,\infty]$, the constants D and θ in (*2.8*)–(*2.10*), and the polynomial $P(n)$ in (*3.1*).*

Proof. Take any $A := (a(x,y))_{x,y\in X}, B := (b(x,y))_{x,y\in X} \in \mathcal{A}_{p,w}(X,\rho,\mu)$, and set $AB := (c(x,y))_{x,y\in X}$. Noting that

$$\begin{aligned} |(cw)(x,y)| \quad &\le \quad D \int_X |(aw)(x,z)||(bv)(z,y)| d\mu(z) \\ &\qquad + D \int_X |(av)(x,z)||(bw)(z,y)| d\mu(z) \end{aligned}$$

by (2.8), we then have the following compensated compactness estimate:

$$\|AB\|_{\mathcal{A}_{p,w}} \le D\|A\|_{\mathcal{A}_{p,w}} \|B\|_{\mathcal{A}_{1,v}} + D\|A\|_{\mathcal{A}_{1,v}} \|B\|_{\mathcal{A}_{p,w}} \quad \text{for all } A, B \in \mathcal{A}_{p,w}(X,\rho,\mu). \tag{3.3}$$

(A similar compensated compactness estimate has been used in [7] to establish an equality for spectral radii of an operator in two Banach algebras.)

By (2.9) and Theorem 2.4, we obtain

$$\|A\|_{\mathcal{A}_{1,v}} \le \|vw^{-1}\|_{\mathcal{A}_{p',w_0}} \|A\|_{\mathcal{A}_{p,w}} \quad \text{for all} \quad A \in \mathcal{A}_{p,w}(X,\rho,\mu). \tag{3.4}$$

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For any $n \geq 1$, it follows from (2.9), (2.17), (2.18), and (3.1) that

$$\max(\|A^n\|_{\mathcal{B}^r}, \|(A^*)^n\|_{\mathcal{B}^r}) \leq \|A^n\|_{\mathcal{A}_{1,w_0}} \leq \|w^{-1}\|_{\mathcal{A}_{p',w_0}} \|A^n\|_{\mathcal{A}_{p,w}}, \tag{3.5}$$

and

$$\begin{aligned}
&\int_X |a^n(x,y)|v(x,y)d\mu(y) \\
= \ & \Big(\int_{\rho(x,y)\leq\tau} + \int_{\rho(x,y)>\tau}\Big)|a^n(x,y)|v(x,y)d\mu(y) \\
\leq \ & \|a^n(x,\cdot)\|_r \|v(x,\cdot)\chi_{B(x,\tau)}(\cdot)\|_{r'} \\
& \quad + \|a^n(x,\cdot)w(x,\cdot)\|_p \|(vw^{-1})(x,\cdot)\chi_{X\setminus B(x,\tau)}(\cdot)\|_{p'} \\
\leq \ & a_{r'}(\tau)\|A^n\|_{\mathcal{A}_{r,w_0}} + b_{p'}(\tau)\|A^n\|_{\mathcal{A}_{p,w}} \\
\leq \ & a_{r'}(\tau)P(n)\max(\|A^n\|_{\mathcal{B}^r}, \|(A^*)^n\|_{\mathcal{B}^r}) + b_{p'}(\tau)\|A^n\|_{\mathcal{A}_{p,w}} \\
\leq \ & P(n)\big(a_{r'}(\tau)\max(\|A^n\|_{\mathcal{B}^r}, \|(A^*)^n\|_{\mathcal{B}^r}) + b_{p'}(\tau)\|A^n\|_{\mathcal{A}_{p,w}}\big), \quad (3.6)
\end{aligned}$$

where $\tau > 0$ and $A^n = (a^n(x,y))_{x,y\in X}$. Thus combining (2.10), (3.5) and (3.6), we reach the following interpolating estimate:

$$\begin{aligned}
\|A^n\|_{\mathcal{A}_{1,v}} \leq \ & P(n)\inf_{\tau>0}\big(a_{r'}(\tau)\max(\|A^n\|_{\mathcal{B}^r}, \|(A^*)^n\|_{\mathcal{B}^r}) + b_{p'}(\tau)\|A^n\|_{\mathcal{A}_{p,w}}\big) \\
\leq \ & CP(n)\big(\max(\|A^n\|_{\mathcal{B}^r}, \|(A^*)^n\|_{\mathcal{B}^r})\big)^{1-\theta}(\|A^n\|_{\mathcal{A}_{p,w}})^\theta \\
\leq \ & CP(n)\max\big((\|A\|_{\mathcal{B}^r})^{n(1-\theta)}, (\|A^*\|_{\mathcal{B}^r})^{n(1-\theta)}\big)(\|A^n\|_{\mathcal{A}_{p,w}})^\theta. \quad (3.7)
\end{aligned}$$

(A similar interpolating estimate has been established by Jaffard in [26].)

By (3.3), (3.4), and (3.7), we obtain that

$$\begin{aligned}
\|A^{2n}\|_{\mathcal{A}_{p,w}} \leq \ & C\|A^n\|_{\mathcal{A}_{p,w}}\|A^n\|_{\mathcal{A}_{1,v}} \\
\leq \ & D_1 n^{D_2}\|A^n\|^{1+\theta}_{\mathcal{A}_{p,w}}\big(\max\big(\|A\|_{\mathcal{B}^r}, \|A^*\|_{\mathcal{B}^r}\big)\big)^{n(1-\theta)} \quad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
\|A^{2n+1}\|_{\mathcal{A}_{p,w}} \leq \ & C\|A^{2n}\|_{\mathcal{A}_{p,w}}\|A\|_{\mathcal{A}_{p,w}} \\
\leq \ & D_1 n^{D_2}\|A\|_{\mathcal{A}_{p,w}}\|A^n\|^{1+\theta}_{\mathcal{A}_{p,w}}\big(\max\big(\|A\|_{\mathcal{B}^r}, \|A^*\|_{\mathcal{B}^r}\big)\big)^{n(1-\theta)} \quad (3.9)
\end{aligned}$$

for all $n \geq 1$, where D_1, D_2 are positive constants. Therefore the sequence $\{b_n\}$, to be defined by

$$b_n = D_1^{1/\theta} n^{D_2/\theta}\|A^n\|_{\mathcal{A}_{p,w}}\big(\max(\|A\|_{\mathcal{B}^r}, \|A^*\|_{\mathcal{B}^r})\big)^{-n}, \quad n \geq 1, \tag{3.10}$$

satisfies

$$b_{2n} \leq b_n^{1+\theta} \quad \text{and} \quad b_{2n+1} \leq c_0 b_n^{1+\theta} \text{ for all } n \geq 1, \tag{3.11}$$

where $c_0 = \max(3^{D_2/\theta}, D_1^{1/\theta})\|A\|_{\mathcal{A}_{p,w}}\big(\max\big(\|A\|_{\mathcal{B}^r}, \|A^*\|_{\mathcal{B}^r}\big)\big)^{-1}$. This implies that

$$b_n \leq c_0^{\sum_{i=0}^k \epsilon_i(1+\theta)^i}$$

for $n = \sum_{i=0}^k \epsilon_i 2^i$, where $\epsilon_i \in \{0,1\}, 0 \leq i \leq k$. Therefore the desired estimate (3.2) follows. □

From the proof of Theorem 3.1, particularly (3.3) and (3.4), we conclude that $\mathcal{A}_{p,w}(X,\rho,\mu)$ is a Banach algebra.

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Proposition 3.2. *Let $1 \le p \le \infty$, let X be an index set that has a pseudo-metric ρ and a non-negative Borel measure μ with the polynomial growth property, and let w be a weight on $X \times X$ such that there exists another weight v satisfying (2.8) and (2.9). Then $\mathcal{A}_{p,w}(X, \rho, \mu)$ is a Banach algebra. Moreover, the inequality,*

$$\|AB\|_{\mathcal{A}_{p,w}} \le C_1 \|A\|_{\mathcal{A}_{p,w}} \|B\|_{\mathcal{A}_{p,w}} \text{ for all } A, B \in \mathcal{A}_{p,w}(X, \rho, \mu), \tag{3.12}$$

holds for some positive constant C_1 that depends only on the exponent p and the constant D in (2.8) and (2.9).

Define

$$\rho_{p,w}(A) := \limsup_{n\to\infty} (\|A^n\|_{\mathcal{A}_{p,w}})^{1/n} \tag{3.13}$$

for any $A \in \mathcal{A}_{p,w}(X, \rho, \mu)$, and

$$\rho_{\mathcal{B}^r}(A) := \limsup_{n\to\infty} (\|A^n\|_{\mathcal{B}^r})^{1/n} \tag{3.14}$$

for $A \in \mathcal{B}^r$. Using (3.3), (3.4), and (3.6), we have the following equality for various spectral radii associated with infinite matrices in the Schur class $\mathcal{A}_{p,w}(X, \rho, \mu)$.

Theorem 3.3. *Let $X, \rho, \mu, p, r, w, v, w_0$ be as in Theorem 3.1. Assume that $A \in \mathcal{A}_{p,w}(X, \rho, \mu)$. Then*

$$\rho_{1,v}(A) = \rho_{p,w}(A). \tag{3.15}$$

If A is further assumed to satisfy (3.1), then

$$\rho_{r,w_0}(A) = \max(\rho_{\mathcal{B}^r}(A), \rho_{\mathcal{B}^r}(A^*)) = \rho_{p,w}(A). \tag{3.16}$$

Proof. By (3.3) and (3.4), there is a positive constant C such that

$$\|A^n\|_{\mathcal{A}_{1,v}} \le C \|A^n\|_{\mathcal{A}_{p,w}} \tag{3.17}$$

and

$$\|A^n\|_{\mathcal{A}_{p,w}} \le C \|A^{[n/2]}\|_{\mathcal{A}_{p,w}} \|A^{[n/2]}\|_{\mathcal{A}_{1,v}},\ n \ge 1, \tag{3.18}$$

where $[x]$ is the integral part of the real number x. This proves (3.15).

If A is further assumed to satisfy (3.1), then

$$\|A\|_{\mathcal{A}_{r,w_0}} \le P(n) \max(\|A^n\|_{\mathcal{B}^r}, \|(A^*)^n\|_{\mathcal{B}^r}) \le P(n) \|A^n\|_{\mathcal{A}_{1,v}}, \tag{3.19}$$

and

$$\|A^n\|_{\mathcal{A}_{1,v}} \le C(\|A^n\|_{\mathcal{A}_{r,w_0}})^{1-\theta} (\|A^n\|_{\mathcal{A}_{p,w}})^{\theta} \quad \text{for all } 1 \le n \in \mathbf{Z} \tag{3.20}$$

by (2.18), (3.1), and (3.6). Thus (3.16) follows from (3.15), (3.19), and (3.20). □

We conclude this section with two remarks.

Remark 3.4. For the special case that $p = \infty$, X is a relatively-separated subset of $\mathbf{R}^d$ (see (A.2) for the definition), ρ is the usual Euclidean metric $|\cdot|$, μ is the counting measure μ_c and w is the polynomial weight $(1 + |\cdot|)^\alpha$ with $\alpha > d$, Jaffard used a rather delicate bootstrap argument, which depends highly on the geometrical structure of the Euclidean space, to prove that

$$\rho_{\mathcal{B}^2}(A) = \rho_{\infty, \alpha_0+\epsilon}(A) \tag{3.21}$$

for some sufficiently small number ϵ, where α_0 is the largest integer strictly smaller than α ([26, Lemma 4]). Clearly, the above result (3.21) follows from (3.16) and

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the observation: $\rho_{\mathcal{B}^2}(A) \le \rho_{\infty,\alpha_0+\epsilon}(A) \le \rho_{\infty,\alpha}(A)$. For the special case that $p = 1, r = 2$ and w is the polynomial weight $(1+\rho(\cdot,\cdot))^\alpha$ with $0 < \alpha \le 1$, the equality

$$\rho_{\mathcal{B}^2}(A) = \rho_{p,w}(A), \tag{3.22}$$

the second part of equality (3.16), follows from Lemma 4.6 in [4]. For the case that $p = 1, r = 2$, the index set X is a relatively-separated subset of $\mathbf{R}^d$, the weight w is of the form $\exp(\kappa(|x-y|))$ for some continuous concave function κ that satisfies $\kappa(0) = 0, \lim_{t\to\infty} \kappa(t)/t = 0$ and $\kappa(t) \ge \delta \ln(1+t) - D$ for some positive constants δ and D, and the equality in (3.22) is established in [24, Theorem 3.1]. For the above case, the weight w is of the form $\exp(\kappa(|x-y|))$. We see from Theorem A.1 that the above assumption on the concave function κ is a bit weaker than our assumption on the existence of another weight v satisfying (2.8), (2.9), and (2.10) in Theorem 3.1, but both conditions are satisfied for polynomial weights and subexponential weights; see Examples A.2 and A.3.

Remark 3.5. For the special case that $p = 1$, $w = w_0$ and the matrix $A = (a(j-j'))_{j-j'\in\mathbf{Z}} \in \mathcal{W}$ in (1.1) with $\sum_{j\in\mathbf{Z}} a(j)e^{-ij\xi}$ being the reciprocal of a trigonometric polynomial Q, Newman proved the following better estimate than the one in (3.2) for the $\mathcal{A}_{1,w_0}$ norm of A^n: $\|A^n\|_{\mathcal{A}_{1,w_0}} \le Cn^2\|A\|^n_{\mathcal{B}^2}$ for all $n \ge 1$, where C is a positive constant depending on the degree of the polynomial Q ([30]). We also remark that from the equality of various spectral radii in Theorem 3.1, a weaker estimate than the one in (3.2) follows: for any $A \in \mathcal{A}_{p,w}(X,\rho,\mu)$, there exists a nonnegative number $q_n(A)$ for any $n \ge 1$ such that $\lim_{n\to\infty} q_n(A)^{1/n} = 1$ and $\|A^n\|_{\mathcal{A}_{p,w}} \le q_n(A)(\max(\rho_{\mathcal{B}^r}(A), \rho_{\mathcal{B}^r}(A^*)))^n$ for all $n \ge 1$ (cf. [4, 7, 24, 26]).

4. Wiener's lemma for infinite matrices in the Schur class

In this section, we establish the principal result of the paper, Wiener's lemma for infinite matrices in the Schur class.

Theorem 4.1. *Let $1 \le p \le \infty$, let X be a discrete set that has a pseudo-metric ρ and a usual counting measure μ_c with polynomial growth property, let w be a $(p,2)$-admissible weight, and let $\mathcal{A}_{p,w}(X,\rho,\mu_c)$ be the Schur class defined by (2.13). If $A \in \mathcal{A}_{p,w}(X,\rho,\mu_c)$ and $A^{-1} \in \mathcal{B}^2$, then $A^{-1} \in \mathcal{A}_{p,w}(X,\rho,\mu_c)$. Moreover,*

$$\|A^{-1}\|_{\mathcal{A}_{p,w}} \le C_2 \tag{4.1}$$

holds for some constant C_2 which depends only on the exponent $p \in [1,\infty]$, the constant $D(w)$ in (2.4), the constant $D(C_0,w)$ in (2.5), the constants D,θ in (2.8)–(2.10), and the norms $\|A\|_{\mathcal{A}_{p,w}}$ and $\|A^{-1}\|_{\mathcal{B}^2}$.

The above Wiener lemma for infinite matrices in the Schur class follows in a standard way from the Banach algebraic properties (Proposition 3.2), and the asymptotic estimate of $A^n, n \ge 1$, for $A \in \mathcal{A}_{p,w}(X,\rho,\mu)$ (Theorems 3.1 and 3.3). We include a proof for the completeness.

Proof. By (2.3), (2.4), and (2.5), the unit matrix I belongs to $\mathcal{A}_{p,w}(X,\rho,\mu_c)$,

$$I \in \mathcal{A}_{p,w}(X,\rho,\mu_c). \tag{4.2}$$

By Theorem 2.4 and Proposition 3.2, we have that $A^*A \in \mathcal{A}_{p,w}(X,\rho,\mu_c)$. Therefore the matrix B, to be defined by

$$B := I - \frac{A^*A}{\|A^*A\|_{\mathcal{B}^2}}, \tag{4.3}$$

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satisfies

$$\|B\|_{\mathcal{B}^2} \le 1 - \frac{1}{\|A\|_{\mathcal{B}^2}^2 \|A^{-1}\|_{\mathcal{B}^2}^2} \tag{4.4}$$

and

$$\|B\|_{\mathcal{A}_{p,w}} \le C_3 \frac{\|A\|_{\mathcal{A}_{p,w}}^2}{\|A\|_{\mathcal{B}^2}^2}, \tag{4.5}$$

where C_3 is a positive constant that depends on the constants D in (2.9), C_1 in (3.12), and $D(w)$ in (4.1).

Since X is a discrete set and μ_c is the counting measure, the delta sequence δ_{x_0} belongs to $\ell^2(X)$ and has norm one for any $x_0 \in X$. This together with the fact that $B^n \delta_{x_0} = (b^n(x, x_0))_{x \in X}$ implies that

$$\|B^n\|_{\mathcal{A}_{2,w_0}} \le \max(\|B^n\|_{\mathcal{B}^2}, \|(B^*)^n\|_{\mathcal{B}^2}) = \|B^n\|_{\mathcal{B}^2}, \ n \ge 1, \tag{4.6}$$

where, as usual, $B^n =: (b^n(x,y))_{x,y \in X}$. Therefore the estimate (3.1) in Theorem 3.1 holds for the matrix B and $r = 2$.

By (4.2), (4.4), (4.5), (4.6), and Theorem 3.1, we then obtain the following crucial estimate for $B^n, n \ge 1$:

$$\begin{aligned} \|B^n\|_{\mathcal{A}_{p,w}} \quad &\le \quad C_4 \left(\frac{C_4 \|A\|_{\mathcal{A}_{p,w}}^2}{\|A\|_{\mathcal{B}^2}^2} \right)^{\frac{1+\theta}{\theta} n^{\log_2(1+\theta)}} \\ &\quad \times \left(1 - \frac{1}{\|A\|_{\mathcal{B}^2}^2 \|A^{-1}\|_{\mathcal{B}^2}^2}\right)^{n - \frac{1+\theta}{\theta} n^{\log_2(1+\theta)}}, \end{aligned} \tag{4.7}$$

where C_4 is a positive constant that depends only on the constants C_0 in (3.2) and C_3 in (4.5). Hence

$$\|(I-B)^{-1}\|_{\mathcal{A}_{p,w}} \quad = \quad \Big\| \sum_{n=0}^{\infty} B^n \Big\|_{\mathcal{A}_{p,w}} \le C_5 < \infty,$$

where the constant C_5 depends only on $\|A\|_{\mathcal{A}_{p,w}}$, $\|A^{-1}\|_{\mathcal{B}^2}$, the condition number $\|A\|_{\mathcal{B}^2} \|A^{-1}\|_{\mathcal{B}^{-1}}$, and the constant C_4 in (4.7). This together with (4.3) yields that $(A^*A)^{-1} \in \mathcal{A}_{p,\alpha}(X, \rho, \mu_c)$. Therefore the desired conclusion $A^{-1} \in \mathcal{A}_{p,w}(X, \rho, \mu_c)$ follows from (3.3), (3.4), and the facts that $A^{-1} = (A^*A)^{-1}A^*$ and $(A^*A)^{-1} \in \mathcal{A}_{p,w}(X, \rho, \mu_c)$. □

We conclude this section with some remarks.

Remark 4.2. Let $\mathcal{A}_{1,w_0}^0(X, \rho, \mu)$ contain all infinite matrices

$$(a(x,y))_{x,y \in X} \in \mathcal{A}_{1,w_0}(X, \rho, \mu)$$

such that $\lim_{m \to \infty} \|A_m\|_{\mathcal{A}_{1,w_0}} = 0$, where $A_m = (a_m(x,y))_{x,y \in X}$ and

$$a_m(x,y) = \begin{cases} a(x,y), & \text{if } \rho(x,y) \le m, \\ 0, & \text{if } \rho(x,y) > m. \end{cases}$$

Define

$$\mathcal{A}_{2,w_\alpha}^0(X, \rho, \mu) = \mathcal{A}_{2,w_\alpha}(X, \rho, \mu) \cap \mathcal{A}_{1,w_0}^0(X, \rho, \mu).$$

Clearly

$$\mathcal{A}_{2,w_\alpha}(X, \rho, \mu) = \mathcal{A}_{2,w_\alpha}^0(X, \rho, \mu) \text{ if } \alpha > d(X, \rho, \mu)/2,$$

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where $d(X,\rho,\mu)$ is defined as in (A.16). For $0<\alpha\le 1$, Barnes ([4, Theorem 4.8]) proved that $A\in \mathcal{A}^0_{2,w_\alpha}(X,\rho,\mu)$ and $A^{-1}\in \mathcal{B}^2$ imply $A^{-1}\in \mathcal{A}^0_{2,w_\alpha}(X,\rho,\mu)$. Therefore for $p=2$, and $w=w_\alpha$ with $d(X,\rho,\mu)/2<\alpha\le 1$, the conclusion in Theorem 4.1 follows from Theorem 4.8 in [4].

Remark 4.3. Let $p=1$ and X is a relatively-separated subset of $\mathbf{R}^d$. For the case that the weight w is of the form $w(x,y)=\exp(\kappa(|x-y|))$ for a continuous, concave function κ on $[0,\infty)$ satisfying $\kappa(0)=1, \kappa(t)\ge \delta\ln(1+t)-D$ for some $D,\delta\in(0,\infty)$ and $\lim_{t\to\infty}\kappa(t)/t=0$, Gröchenig and Leinert proved in [24] that $A\in\mathcal{A}_{1,w}(X,|\cdot|,\mu_c)$ and $A^{-1}\in\mathcal{B}^2$ imply $A^{-1}\in\mathcal{A}_{1,w}(X,|\cdot|,\mu_c)$ (see [4] for the case that $\kappa(t)=\delta\ln(1+t)$ for some $\delta\in(0,1]$). The difference between the conclusion in Theorem 4.1 with $p=1$ and the above conclusion by Gröchenig and Leinert is on the weight w: (i) the weight w in Theorem 4.1 need not be radical but satisfy some complicated conditions not easily verified in general, while the weight w in [24] is essentially radical (hence it could be difficult to establish similar results for infinite matrices whose index set is not a discrete subset of $\mathbf{R}^d$); (ii) the polynomial weights with $\kappa(t)=\alpha\ln(1+t), \alpha>0$, and the subexponential weights w with $\kappa(t)=Dt^\delta, D\in(0,\infty), \delta\in(0,1)$, satisfy the requirements in Gröchenig and Leinert's result and also the ones in Theorem 4.1 by Examples A.2 and A.3; (iii) the weight function w with $\kappa(t)=t/\ln(e+t)$ satisfies all requirements in Gröchenig and Leinert's result, but not the ones in Theorem 4.1 by Theorem A.1.

Remark 4.4. For the case that $p=\infty$, X is a relatively-separated subset of $\mathbf{R}^d$, and the weight w is of the form $w(x,y)=\exp(\kappa(|x-y|)+s\ln(1+|x-y|))$ with $s>d$ and a continuous, concave function κ on $[0,\infty)$ satisfying $\kappa(0)=1, \kappa(t)\ge\delta\ln(1+t)-D$ for some $D,\delta\in(0,\infty)$ and $\lim_{t\to\infty}\kappa(t)/t=0$, Gröchenig and Leinert proved in [24] that $A\in\mathcal{A}_{\infty,w}(X,|\cdot|,\mu_c)$ and $A^{-1}\in\mathcal{B}^2$ imply $A^{-1}\in\mathcal{A}_{\infty,w}(X,|\cdot|,\mu_c)$ (see [26, Proposition 3] for the case that $\kappa(t)=\delta\ln(1+t)$ for some $\delta>0$, and [5, 6] for the generalization of Jaffard's result in Hilbert spaces).

Remark 4.5. A baby version of Theorem 4.1 is established in [36], where $X=\mathbf{Z}^d$, ρ is the usual Euclidean metric $|\cdot|$, μ is the standard accounting measure μ_c, and w is a polynomial weight $(1+|\cdot|)^\alpha$ with $\alpha>d(1-1/p), 1\le p\le\infty$.

Remark 4.6. The upper bound estimate (4.1) for A^{-1} is important to obtain a scale-independent Riesz and frame bounds in the study of non-stationary multiresolution analysis, wavelets on intervals (domains), and diffusion wavelets on manifolds, where the index sets X and the quasi-metrics ρ vary according to the scales ([9, 11, 10, 12, 26, 31, 37]). For the case that $p=\infty$, X is a relatively-separated subset of $\mathbf{R}^d$, and the weight w is of the form $w(x,y)=(1+|x-y|)^s$ with $s>d$, the upper bound estimate (4.1) is not mentioned in [26], but can be obtained by keeping track of the constants in the argument.

Remark 4.7. There are many different approaches to establish Wiener's lemma for infinite matrices in the Schur class $\mathcal{A}_{p,w}(X,\rho,\mu)$. Here are three of them: (i) the indirect approach, such as using Gelfand's technique to estimate spectral radius $\rho_{p,w}(A)$ ([4, 17, 18, 23, 24]); (ii) the semi-direct approach, such as the bootstrap argument ([26]); (iii) the direct approach, such as the direct estimate of $\|A^n\|_{\mathcal{A}_{p,w}}$ in Theorem 3.1 (see [36] for the baby version of that approach). Each approach has its advantages and disadvantages. For instance, the indirect approach works for the extreme case $\mathcal{A}_{1,w_0}(X,\rho,\mu)$, but provides an upper bound estimate for A^{-1}

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depending on the structure of the space X *implicitly*. The semi-direct and direct approaches do not work for the extreme case, but they work for infinite matrices with index set X having complicated geometrical structures, and they give the upper bound estimate for A^{-1} depending on the space X *explicitly* (hence Wiener's lemma can be used to obtain certain uniformity; see Remark 4.6).

5. Variations and generalizations of Wiener's lemma

In this section, we discuss some straightforward (but useful) variations and generalizations of Wiener's lemma for infinite matrices in the Schur class (Theorem 4.1), including Moore-Penrose pseudo-inverse (Theorem 5.1), Wiener-Levy lemma (Theorem 5.3), Wiener's lemma on product spaces (Theorem 5.5), and Wiener's lemma for twisted convolutions (Theorem 5.7).

5.1. **Moore-Penrose pseudo-inverse.**

Theorem 5.1. *Let X, ρ, μ_c, p, w and $\mathcal{A}_{p,w}(X,\rho,\mu_c)$ be as in Theorem* 4.1*. Assume that $A \in \mathcal{A}_{p,w}(X,\rho,\mu_c)$ satisfies*

$$A^* = A, \tag{5.1}$$

and that there exist a Hilbert subspace H of $\ell^2(X)$ and a positive constant D with the property that

$$\|Ac\| \ge D\|c\| \text{ for all } c \in H, \tag{5.2}$$

and

$$PA = AP = A, \tag{5.3}$$

where P is the projection operator from ℓ^2 to H. Then the Moore-Penrose pseudo-inverse $A^\dagger$ of the matrix A, that is, $PA^\dagger = A^\dagger P = A^\dagger$ and $AA^\dagger = A^\dagger A = P$, belongs to $\mathcal{A}_{p,w}(X,\rho,\mu_c)$.

Theorem 5.1 follows from Wiener's lemma for $\mathcal{A}_{p,w}(X,\rho,\mu_c)$ (Theorem 4.1), the Banach algebra property of $\mathcal{A}_{p,w}(X,\rho,\mu_c)$ (Proposition 3.2), and the standard holomorphic calculus ([32]). We include a proof for the completeness of the paper.

Proof of Theorem 5.1. By (5.2) and (5.3), the spectrum of the positive operator A on $\ell^2(X)$ is contained in $\{0\} \cap [m, M]$, where $0 < D \le m \le M = \|A\|_{\mathcal{B}^2} < \infty$. Let $\mathcal{C}$ be the circle of radius $M/2$ centered at $(m+M)/2$ in the complex plane. Define the operator B, as an operator on $\ell^2(X)$, by

$$B = \frac{1}{2\pi i}\int_{\mathcal{C}} (zI - A)^{-1} dz. \tag{5.4}$$

By standard holomorphic calculus ([32]), B is the Moore-Penrose pseudo-inverse of the matrix A. Therefore it suffices to prove that $B \in \mathcal{A}_{p,w}(X,\rho,\mu_c)$. Since $zI - A$ is invertible as an operator on $\ell^2(X)$ for any $z \in \mathcal{C}$, we then have that

$$(zI - A)^{-1} \in \mathcal{A}_{p,w}(X,\rho,\mu_c) \tag{5.5}$$

by Theorem 4.1. Given any $z_0 \in \mathcal{C}$, we obtain from (3.3) and (3.4) that

$$\begin{aligned} & \|(zI - A)^{-1} - (z_0I - A)^{-1}\|_{\mathcal{A}_{p,w}} \\ \le\ & \|(z_0 I - A)^{-1}\|_{\mathcal{A}_{p,w}} \sum_{n=1}^{\infty} |z_0 - z|^n \big(C\|(z_0I - A)^{-1}\|_{\mathcal{A}_{p,w}}\big)^n \\ \le\ & C|z - z_0| \|(z_0I - A)^{-1}\|^2_{\mathcal{A}_{p,w}} \big(1 - C|z - z_0|\|(z_0 I - A)^{-1}\|_{\mathcal{A}_{p,w}}\big)^{-1} \end{aligned} \tag{5.6}$$

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for any complex number z with $|z - z_0|$ sufficiently small, where C is a positive constant. Therefore the claim $B \in \mathcal{A}_{p,w}(X, \rho, \mu_c)$ follows from (5.4), (5.5), and (5.6). $\square$

Remark 5.2. The generalization of the Wiener lemma (Theorem 5.1), the replacement of the invertibility of the matrix A by the pseudo-invertibility, is important for the applications to (Gabor) frame ([37]), since the Gram matrix associated with frame generators does not have bounded inverse in general, but does have bounded pseudo-inverse. We remark that the above generalization from invertibility to pseudo-invertibility in Wiener's lemma holds for any Banach algebra; see [18] for the holomorphic calculus approach used in the proof, and also for another approach from Banach algebra.

5.2. **Wiener-Levy lemma and non-integer power.**

Theorem 5.3. *Let X, ρ, μ_c, p, w and $\mathcal{A}_{p,w}(X, \rho, \mu_c)$ be as in Theorem* 4.1. *If $A \in \mathcal{A}_{p,w}(X, \rho, \mu_c)$ and f is an analytic function in an open domain $\mathcal{O}$ containing the spectrum of the matrix A as an operator on $\ell^2(X)$, then $f(A) \in \mathcal{A}_{p,w}(X, \rho, \mu_c)$.*

Proof. We may use the same argument as the one used in the proof of Theorem 5.1 except replacing the circle in the proof of Theorem 5.1 by a contour $\tilde{\mathcal{C}}$ in the open domain $\mathcal{O}$ that contains the spectrum of the matrix A inside the contour, and applying the following formula $f(A) = \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}} f(z)(zI - A)^{-1}dz$ instead of (5.4). It is safe to omit the details of the proof here. $\square$

Combining Theorems 5.1 and 5.3, we have the following result about non-integer power of a matrix in the Schur class, which will be useful in the study of localized tight frames ([37]).

Corollary 5.4. *Let X, ρ, μ_c, p, w and $\mathcal{A}_{p,w}(X, \rho, \mu_c)$ be as in Theorem* 4.1. *Assume that $A \in \mathcal{A}_{p,w}(X, \rho, \mu_c)$ satisfies* (5.1)*,* (5.2)*, and* (5.3)*. Then for any $\gamma > 0$, A^γ and its Moore-Penrose pseudo-inverse $(A^\gamma)^\dagger$ belong to $\mathcal{A}_{p,w}(X, \rho, \mu_c)$.*

5.3. **Wiener's lemma on product spaces.** Let $N \geq 1$, $(X_n, \rho_n), 1 \leq n \leq N$, be quasi-metric spaces with non-negative Borel measures μ_n that have polynomial growth property. On the product space $X := X_1 \times X_2 \times \cdots \times X_N$, we define a quasi-metric ρ,

$$\rho((x_1, \ldots, x_N), (y_1, \ldots, y_N)) = \max_{1 \leq n \leq N} \rho_n(x_n, y_n), \tag{5.7}$$

and a non-negative Borel measure $\mu := \mu_1 \times \cdots \times \mu_N$. For $\mathbf{p} = (p_1, \ldots, p_N) \in [1, \infty]^N$, we let $L^{\mathbf{p}}(X)$ denote the space of all $\mathbf{p}$-integrable functions,

$$L^{\mathbf{p}}(X) := \big\{f(x_1, \ldots, x_N) : \ \|f\|_{\mathbf{p}} < \infty\big\},$$

with standard norm

$$\|f\|_{\mathbf{p}} = \|\cdots\|\|f(x_1, \ldots, x_N)\|_{L^{p_N}(X_N)}\|_{L^{p_{N-1}}(X_{N-1})}\cdots\|_{L^{p_1}(X_1)}.$$

For $\mathbf{p} = (p_1, \ldots, p_N) \in [1, \infty]^N$, a product space $X = X_1 \times X_2 \times \cdots \times X_N$ with a quasi-metric ρ and a non-negative Borel measure μ, and a weight w on the quasi-metric space (X, ρ), we define the Schur class on the product space X by

$$\mathcal{A}_{\mathbf{p},w}(X, \rho, \mu) := \Big\{A := (a(\mathbf{x}, \mathbf{y}))_{\mathbf{x},\mathbf{y}\in X} : \\ \|A\|_{\mathcal{A}_{\mathbf{p},w}} := \sup_{\mathbf{x}\in X} \|(aw)(\mathbf{x}, \cdot)\|_{\mathbf{p}} + \sup_{\mathbf{y}\in X} \|(aw)(\cdot, \mathbf{y})\|_{\mathbf{p}} < \infty\Big\}.$$

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Using an argument similar to the one used in the proofs of Theorems 3.1 and 4.1, we have the following Wiener lemma for infinite matrices in the Schur class on a product space, which is convenient in the study of Gabor frames in the time-frequency domain.

Theorem 5.5. *Let $N \geq 1$, $\mathbf{p} = (p_1, \ldots, p_N) \in [1,\infty]^N$, and $X_n, 1 \leq n \leq N$, be discrete sets embedded with quasi-metrics ρ_n and counting measures $\mu_{n,c}$ that have polynomial growth property, let $X := X_1 \times X_2 \times \cdots \times X_N$ be the product space of $X_n, 1 \leq n \leq N$, with a quasi-metric ρ in (5.7) and a usual counting measure $\mu_c := \mu_{1,c} \times \cdots \times \mu_{N,c}$, and let w be a weight on $X \times X$. Set $\mathbf{p}' = (p_1/(p_1-1), \ldots, p_N/(p_N-1))$ and $\mathbf{2}^* = (2, \ldots, 2)$. Assume that there exist another weight u on $X \times X$ and two positive constants $D \in (0,\infty)$ and $\theta \in (0,1)$ so that (i) $w(\mathbf{x},\mathbf{y}) \leq Dw(\mathbf{x},\mathbf{z})u(\mathbf{z},\mathbf{y}) + Du(\mathbf{x},\mathbf{z})w(\mathbf{z},\mathbf{y})$ for all $\mathbf{x},\mathbf{y},\mathbf{z} \in X$; (ii) $\|uw^{-1}\|_{\mathcal{A}_{\mathbf{p}',w_0}} < \infty$; and (iii) $\inf_{\tau \geq 1} \|uc_\tau\|_{\mathcal{A}_{\mathbf{2}^*,w_0}} + \|uw^{-1}(1-c_\tau)\|_{\mathcal{A}_{\mathbf{p}',w_0}} t \leq Dt^\theta$ for all $t \geq 1$, where $c_\tau(x,y) = \chi_{B(x,\tau)}(y)$. Then $A^{-1} \in \mathcal{A}_{\mathbf{p},w}(X,\rho,\mu_c)$ if $A \in \mathcal{A}_{\mathbf{p},w}(X,\rho,\mu_c)$ and $A^{-1} \in \mathcal{B}^2$.*

Remark 5.6. The following weights $w(\mathbf{x},\mathbf{y})$ satisfy the assumptions (i) – (iii) in Theorem 5.5:

- (i) $w(\mathbf{x},\mathbf{y}) = (1+\rho(\mathbf{x},\mathbf{y}))^\alpha$ for some $\alpha > \sum_{n=1}^N d_n(X_n,\rho_n,\mu_{n,c})(1-1/p_n)$, where $d_n(X_n,\rho_n,\mu_{n,c}), 1 \leq n \leq N$, are defined in (A.16).
- (ii) $w(\mathbf{x},\mathbf{y}) = \prod_{n=1}^N (1+\rho(x_n,y_n))^{\alpha_n}$, where $\mathbf{x} = (x_1,\ldots,x_N), \mathbf{y} = (y_1,\ldots,y_N) \in X$, $\alpha_n > d_n(X_n,\rho_n,\mu_{n,c})(1-1/p_n)$ for all $1 \leq n \leq N$, and $d_n(X_n,\rho_n,\mu_{n,c})$, $1 \leq n \leq N$, are defined in (A.16).
- (iii) $w(\mathbf{x},\mathbf{y}) = \exp(D\rho(\mathbf{x},\mathbf{y})^\theta)$, where $D \in (0,\infty), \theta \in (0,1)$ and ρ is a metric on the product space X.
- (iv) $w(\mathbf{x},\mathbf{y}) = \prod_{n=1}^N \exp(D_n\rho_n(x_n,y_n)^{\theta_n})$, where $\mathbf{x} = (x_1,\ldots,x_N) \in X, \mathbf{y} = (y_1,\ldots,y_N) \in X$, and $D_n \in (0,\infty), \theta_n \in (0,1)$, and ρ_n are metrics on $X_n, 1 \leq n \leq N$.

5.4. **Wiener's lemma for twisted convolutions.** Given a $d \times d$ matrix A, the *twisted convolution* of two sequences $a = (a(j))_{j\in\mathbf{Z}^d}$ and $c = (c(j))_{j\in\mathbf{Z}^d}$ is given by

$$(a \natural_A c)(j) = \sum_{j' \in \mathbf{Z}^d} a(j-j')c(j')e^{2\pi i j'^T A(j-j')}, \tag{5.8}$$

and the *twisted convolution operator* L_a associated with a summable sequence $a \in \ell^1$ is defined by

$$L_a c := a \natural_A c, \ c \in \ell^r, \tag{5.9}$$

where, as usual, B^T denotes the transpose of a matrix or a vector B. Clearly the twisted convolution $\natural_A$ becomes the usual discrete convolution $*$ when $A = 0$, and the twisted convolution $\natural_\theta$ in [23] when $A = \theta \begin{pmatrix} 0 & I_{d/2} \\ 0 & 0 \end{pmatrix}$ and d is an even integer, where $\theta \in \mathbf{R}$ and I_l is the $l \times l$ unit matrix.

Applying Theorem 5.1, we have the following Wiener lemma for twisted convolutions, suitable for the study of a Gabor frame on a uniform grid.

Theorem 5.7. *Let $1 \leq p \leq \infty$, A be a $d \times d$ matrix, and $u = (u(j))_{j\in\mathbf{Z}^d}$ be a sequence of positive numbers such that the weight w, to be defined by*

$$w(j,k) = u(j-k), \ j,k \in \mathbf{Z}^d,$$

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satisfies (2.3) – (2.5) and (2.8) – (2.10) with $X = \mathbf{Z}^d, \rho = |\cdot|$ *and* $\mu = \mu_c$*. If* $a \in \ell^p_u(\mathbf{Z}^d)$*, the space of all sequences* $a := (a(j))_{j\in\mathbf{Z}^d}$ *with finite norm* $\|a\|_{\ell^p_u} := \big\|\big(a(j)u(j)\big)_{j\in\mathbf{Z}^d}\big\|_{\ell^p}$*, and if the associated twisted convolution operator* L_a *is invertible on* $\ell^2(\mathbf{Z}^d)$*, then there exists* $b \in \ell^p_u(\mathbf{Z}^d)$ *such that the associated twisted convolution* L_b *is the inverse of* L_a*, that is,* $L_b = (L_a)^{-1}$*.*

Proof. Given a sequence $a = (a(j))_{j\in\mathbf{Z}^d}$, we correspond it to a matrix $\tau_A(a)$,

$$\tau_A(a) = (a(j-j')e^{2\pi i j'^T A(j-j')})_{j,j'\in\mathbf{Z}^d}. \tag{5.10}$$

Clearly we have that

$$\tau_A(a \natural_A b) = \tau_A(a)\tau_A(b) \quad \text{for all } a, b \in \ell^1, \tag{5.11}$$

and

$$L_a c = \tau_A(a)c \quad \text{for all } a \in \ell^1 \text{ and } c \in \ell^r, \tag{5.12}$$

where $1 \le r \le \infty$. One may easily verify that

$$\tau_A(a) \in \mathcal{A}_{p,w}(\mathbf{Z}^d, |\cdot|, \mu_c) \text{ if and only if } a \in \ell^p_u(\mathbf{Z}^d). \tag{5.13}$$

Then we obtain from (5.12), (5.13), Theorem 4.1, and the assumptions on the sequence a and its corresponding twisted convolution L_a that

$$B\tau_A(a) = \tau_A(a)B = I \tag{5.14}$$

for some $B \in \mathcal{A}_{p,w}(\mathbf{Z}^d, |\cdot|, \mu_c)$. Write $B = (b(j,j'))_{j,j'\in\mathbf{Z}^d}$ and define

$$B' = (b(j-j',0)e^{2\pi i j'^T A(j-j')})_{j,j'\in\mathbf{Z}^d}.$$

By direct computation, we have that the matrix B' also satisfies (5.14), which implies that

$$b(k,k') = b(k-k',0)e^{2\pi i k'^T A(k-k')} \text{ for all } k, k' \in \mathbf{Z}^d. \tag{5.15}$$

Thus $B = \tau_A(b)$ for $b = (b(k,0))_{k\in\mathbf{Z}^d}$. Therefore $(L_a)^{-1} = L_b$ for some sequence $b \in \ell^p_u(\mathbf{Z}^d)$ by (5.13), (5.14), (5.15) and $B \in \mathcal{A}_{p,w}(\mathbf{Z}^d, |\cdot|, \mu)$. □

By taking $A = 0$ and letting u be the polynomial weight u_α or the subexponential weight $e_{D,\delta}$ in Theorem 5.7, we have the following result similar to the classical Wiener lemma ([39]), which is applicable in the study of dual generators in a shift-invariant space.

Corollary 5.8. *Let* $1 \le p \le \infty$*, and let* $u = ((1+|j|)^\alpha)_{j\in\mathbf{Z}^d}$ *for some* $\alpha > d(1-1/p)$ *or* $u = (\exp(D|j|^\delta)_{j\in\mathbf{Z}^d}$ *for some* $D \in (0,\infty)$ *and* $\delta \in (0,1)$*. If* $f(\xi)$ *is a periodic function that never vanishes and has its Fourier series in* $\ell^p_u(\mathbf{Z}^d)$*, then* $1/f(\xi)$ *has its Fourier series in* $\ell^p_u(\mathbf{Z}^d)$*.*

Remark 5.9. For $p = 1$, $A = B^T \begin{pmatrix} 0 & I_d \\ 0 & 0 \end{pmatrix} B$ for some non-singular $d\times d$ matrix B, and the sequence $u := (u(j))_{j\in\mathbf{Z}^d}$ is of the form $(\exp(\kappa(j))_{j\in\mathbf{Z}^d}$ for some continuous, concave function κ on $[0,\infty)$ that satisfies $\kappa(0) = 1$, the conclusion in Theorem 5.7 follows from [23, Theorem 2.14 and Remark 4.2] and [24, Theorem 5.1], where the approach from Banach algebra is used.

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6. Wiener's lemma for infinite matrices in the Sjöstrand class

In this section, we introduce the Sjöstrand class of infinite matrices, establish the Wiener's lemma for infinite matrices in the Sjöstrand class (Theorem 6.1), and discuss some variations and generalizations of the above Wiener's lemma (Theorems 6.4–6.7).

In order to define the Sjöstrand class of infinite matrices $(a(x,y))_{x,y\in X}$, we further assume that *the index set X is a discrete group with a left-invariant quasi-metric ρ and a counting measure μ_c that has the polynomial growth property.* Here a function g on $X\times X$ is said to be *left-invariant* if

$$g(zx,zy)=g(x,y)\quad \text{for all } x,y,z\in X.$$

Clearly a left-invariant function g on $X\times X$ is determined by a function $\tilde g$ on X by $g(x,y)=\tilde g(y^{-1}x), x,y\in X$.

Given $1\le p\le\infty$, a discrete group X with a left-invariant quasi-metric ρ and a counting measure μ_c, and a left-invariant weight w on $X\times X$, we define

$$\|A\|_{\mathcal{C}_{p,w}}:=\Big\|\Big(\sup_{z\in X}|(aw)(zx,z)|\Big)_{x\in X}\Big\|_p \tag{6.1}$$

for $A:=(a(x,y))_{x,y\in X}$, and let

$$\mathcal{C}_{p,w}(X,\rho,\mu_c):=\big\{A:=(a(x,y))_{x,y\in X},\ \|A\|_{\mathcal{C}_{p,w}}<\infty\big\}. \tag{6.2}$$

We use $\mathcal{C}_{p,w}$ for the brevity of $\mathcal{C}_{p,w}(X,\rho,\mu)$. We call $\mathcal{C}_{p,w}$ the Sjöstrand class and a matrix in $\mathcal{C}_{p,w}$ as being of Sjöstrand type, since for $p=1$ and $w=w_0$, $\mathcal{C}_{p,w}$ becomes the matrix algebra studied in [33]; see also Remark 2.3. We remark that for $p=\infty$, the Sjöstrand class $\mathcal{C}_{p,w}$ is the same as the Schur class $\mathcal{A}_{p,w}$.

Now we state the Wiener's lemma for infinite matrices in the Sjöstrand class, whose proof is postponed to the end of this section.

Theorem 6.1. *Let $1\le p\le\infty$, let X be a discrete group with a left-invariant quasi-metric ρ and with the counting measure μ_c having polynomial growth property, and let w be a left-invariant weight on $X\times X$. Assume that there is another left-invariant weight v such that (2.8) – (2.10) hold for $r=\max(2,p/(p-1))$. Then $A^{-1}\in\mathcal{C}_{p,w}(X,\rho,\mu_c)$ provided that $A\in\mathcal{C}_{p,w}(X,\rho,\mu_c)$ and $A^{-1}\in\mathcal{B}^2$.*

Remark 6.2. Let $1\le p\le\infty$, and let X be a discrete group with a left-invariant quasi-metric ρ and with the counting measure μ_c having the polynomial growth property. From Example A.2 and Example A.3, there exist left-invariant weights v associated with the following weights w such that (2.8) – (2.10) hold for $r=\max(2,p/(p-1))$:

(i) The polynomial weights $(1+\rho(x,y))^\alpha$ with $\alpha>d(X,\rho,\mu_c)(1-1/p)$.
(ii) The subexponential weights $\exp(D\rho(x,y)^\delta)$ with $D\in(0,\infty)$ and $\delta\in(0,1)$ if we further assume that ρ is a left-invariant metric on X.

Remark 6.3. For the special case that $(X,\rho,\mu_c)=(\mathbf{Z},|\cdot|,\mu_c)$ and $p=1$, Wiener's lemma for infinite matrices in the Sjöstrand class $\mathcal{C}_{p,w}(X,\rho,\mu_c)$ was proved by Sjöstrand [33] for unweighted case ($w=w_0$), and by Baskakov [5, 6] for those left-invariant weights w satisfying $\lim_{|k|\to\infty}\frac{\ln\tilde w(k)}{|k|}=0$, $\tilde w(k)\ge 1$ for all $k\in\mathbf{Z}$, and $\tilde w(k+l)\le\tilde w(k)\tilde w(l)$ for all $k,l\in\mathbf{Z}$, where $\tilde w(k)=w(k,0)$. By Theorem A.1, the assumption on the weight w in [5, 6] is weaker than the assumption on the weight w in Theorem 6.1, and hence the result in Theorem 6.1 follows from the result in [5, 6]

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for the special case that $p = 1$ and $(X, \rho, \mu_c) = (\mathbf{Z}, |\cdot|, \mu_c)$. It is remarked in [3] that Wiener's lemma for infinite matrices in the Sjöstrand class in [5, 6, 33] can be generalized to the setting that the index set X is a discrete abelian group of the form $\prod_{i=1}^d a_i\mathbf{Z} \times \prod_{j=1}^e (\mathbf{Z}/n_j\mathbf{Z})$ in a straightforward way, where $0 \neq a_i \in \mathbf{R}, 1 \leq i \leq d$, and $n_j \in \mathbf{Z}, 1 \leq j \leq e$ ([3]). So for $p = 1$, comparing our results in Theorem 6.1 with the above known results, we see that there is more restriction on the weight w in Theorem 6.1 than the ones in [5, 6], while the index set X in Theorem 6.1 could be a non-abelian group. For $1 < p < \infty$, our results in Theorem 6.1 are totally new. For $p = \infty$, we have that $\mathcal{C}_{\infty,w}(X, \rho, \mu_c) = \mathcal{A}_{\infty,w}(X, \rho, \mu_c)$. The reader may refer to Remark 4.4 for the comparison between the results in Theorem 6.1 and the known results for $p = \infty$.

Before we start the proof of Theorem 6.1, let us state some variations and generalizations of Wiener's lemma for infinite matrices in the Sjöstrand class, whose proofs are left to the readers.

Theorem 6.4. *Let X, ρ, μ_c, p, w, v and $\mathcal{C}_{p,w}(X, \rho, \mu_c)$ be as in Theorem 6.1. Assume that $A \in \mathcal{C}_{p,w}(X, \rho, \mu_c)$ satisfies (5.1), (5.2), and (5.3) for some Hilbert subspace H of $\ell^2(X)$ and a projection operator P from $\ell^2(X)$ to H. Then A^γ and its Moore-Penrose pseudo-inverse $(A^\gamma)^\dagger$ belong to $\mathcal{C}_{p,w}(X, \rho, \mu_c)$ for any $\gamma > 0$.*

Theorem 6.5. *Let X, ρ, μ_c, p, w, v and $\mathcal{C}_{p,w}(X, \rho, \mu_c)$ be as in Theorem 6.1. If $A \in \mathcal{C}_{p,w}(X, \rho, \mu_c)$ and f is an analytic function in an open domain $\mathcal{O}$ containing the spectrum of the matrix A as an operator on $\ell^2(X)$, then $f(A) \in \mathcal{C}_{p,w}(X, \rho, \mu_c)$.*

Let $N \geq 1$, $X_n, 1 \leq n \leq N$, be discrete groups embedded with left-invariant quasi-metric ρ_n and counting measures $\mu_{n,c}$ that have polynomial growth property. Define the left-invariant quasi-metric ρ on the product space $X = X_1 \times X_2 \times \cdots \times X_N$ by $\rho(\mathbf{x}, \mathbf{y}) = \max_{1 \leq n \leq N} \rho_n(x_n, y_n)$, where $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N) \in X$. Given $\mathbf{p} = (p_1, \ldots, p_N) \in [1, \infty]^N$ and a left-invariant weight w on the product space X, we define the Sjöstrand class on the product space X by

$$\begin{aligned} \mathcal{C}_{\mathbf{p},w}(X, \rho, \mu_c) \quad := \quad & \Big\{ A := (a(\mathbf{x}, \mathbf{y}))_{\mathbf{x},\mathbf{y} \in X}, \\ & \qquad \|A\|_{\mathcal{C}_{\mathbf{p},w}} := \|(\sup_{\mathbf{z} \in X} |(aw)(\mathbf{zx}, \mathbf{z})|)_{\mathbf{x} \in X}\|_{\mathbf{p}} < \infty \Big\}. \end{aligned} \tag{6.3}$$

Theorem 6.6. *Let $N \geq 1$, $\mathbf{p} = (p_1, \ldots, p_N) \in [1, \infty]^N$, $X_n, 1 \leq n \leq N$, be discrete groups embedded with quasi-metrics ρ_n and counting measures $\mu_{n,c}$ that have polynomial growth property. Set $\mathbf{p}' = (p_1/(p_1 - 1), \ldots, p_N/(p_N - 1))$ and $\min(\mathbf{2}, \mathbf{p}) = (\min(2, p_1), \ldots, \min(2, p_N))$. Assume that the left-invariant weights w and u on the product space $X = X_1 \times X_2 \times \cdots \times X_N$ satisfy (i) $\tilde{w}(\mathbf{xy}) \leq D\tilde{w}(\mathbf{x})\tilde{u}(\mathbf{y}) + D\tilde{u}(\mathbf{x})\tilde{w}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$, (ii) $\|\tilde{u}(\tilde{w})^{-1}\|_{\mathbf{p}'} < \infty$, and (iii) $\inf_{\tau \geq 1} \|\tilde{u}\chi_{B(\mathbf{I}_0, \tau)}\|_{\min(\mathbf{2},\mathbf{p})} + \|\tilde{u}(\tilde{w})^{-1}\chi_{X \setminus B(\mathbf{I}_0, \tau)}\|_{\mathbf{p}'} t \leq Dt^\theta$, $t \geq 1$, where $\tilde{w}(\mathbf{x}) = w(\mathbf{x}, \mathbf{I}_0), \tilde{u}(\mathbf{x}) = u(\mathbf{x}, \mathbf{I}_0)$, $\mathbf{I}_0$ is the unit of the group X, and $D \in (0, \infty)$ and $\theta \in (0, 1)$ are positive constants. Then $A^{-1} \in \mathcal{C}_{\mathbf{p},w}(X, \rho, \mu_c)$ if $A \in \mathcal{C}_{\mathbf{p},w}(X, \rho, \mu_c)$ and $A^{-1} \in \mathcal{B}^2$.*

Let X be a discrete group with a left-invariant quasi-metric ρ, and let $\tilde{X}$ be a set with a map M from $\tilde{X}$ to X satisfying $1 \leq \#\{y \in \tilde{X} : M(y) = x\} \leq C_0$ for some positive constant C_0. Given $1 \leq p \leq \infty$ and a left-invariant weight w on the group X, we define the Sjöstrand class on the set $\tilde{X}$ by

$$\mathcal{C}_{p,w}(\tilde{X}, M, X, \rho, \mu_c) \quad := \quad \Big\{ A := (a(x, y))_{x,y \in \tilde{X}}, \quad \|A\|_{\mathcal{C}_{\mathbf{p},w}} < \infty \Big\},$$

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where

$$\|A\|_{\mathcal{C}_{p,w}} = \Big\| \big(\sup_{x',y'\in\tilde X \text{ with } M(x')=M(y')x} w(M(x'),M(y'))|a(x',y')| \big)_{x\in X} \Big\|_p.$$

Using an argument similar to the one used in the proof of Theorem 6.1, we have the following result, convenient for the study of a (Gabor) frame on a non-uniform grid and a (non-uniform) sampling for signals with a finite rate of innovation ([3, 37, 38]).

Theorem 6.7. *Let X, ρ, μ_c, p, w, v be as in Theorem 6.1. Assume that $\tilde X$ is a set with a map $M : \tilde X \longmapsto X$ satisfying $1 \le \#\{y \in \tilde X : M(y) = x\} \le C_0$ for some positive constant C_0. Then $A^{-1} \in \mathcal{C}_{p,w}(\tilde X, M, X, \rho, \mu_c)$ provided that $A \in \mathcal{C}_{p,w}(\tilde X, M, X, \rho, \mu_c)$ and $A^{-1} \in \mathcal{B}^2$.*

6.1. **Proof of Theorem 6.1.** To prove Theorem 6.1, we need some properties of the matrix algebra $\mathcal{C}_{p,w}(X, \rho, \mu)$ (see Proposition 3.2 for a similar result for the Schur class).

Lemma 6.8. *Let X, ρ, μ_c, p, w, v be as in Theorem 6.1. Then $\mathcal{C}_{p,w}(X, \rho, \mu_c)$ is a Banach algebra, that is, there exists a positive constant C such that*

$$\|AB\|_{\mathcal{C}_{p,w}} \le C\|A\|_{\mathcal{C}_{p,w}}\|B\|_{\mathcal{C}_{p,w}} \text{ for all } A, B \in \mathcal{C}_{p,w}(X, \rho, \mu_c). \tag{6.4}$$

Proof. Take $A = (a(x, y))_{x,y\in X}, B = (b(x, y))_{x,y\in X} \in \mathcal{C}_{p,w}(X, \rho, \mu_c)$, and write $AB = (c(x, y))_{x,y\in X}$. Then

$$\begin{aligned} \widetilde{c}(x) &:= \sup_{y\in X} |c(yx, y)| \\ &\le \sup_{y\in X} \sum_{z\in X} |a(yx, yz)||b(yz, y)| \le \sum_{z\in X} \tilde a(z^{-1}x)\tilde b(z), \end{aligned}$$

where $\tilde a(x) = \sup_{y\in X} |a(yx, y)|$ and $\tilde b(x) = \sup_{y\in X} |b(yx, y)|$. This together with the assumptions (i)–(iii) on the weight w proves (6.4). □

To prove Theorem 6.1, we need an asymptotic estimate of $A^n, n \ge 1$, for a matrix A in the Sjöstrand class (see Theorem 3.1 for a similar result for the Schur class).

Lemma 6.9. *Let X, ρ, μ_c, p, w, v be as in Theorem 6.1. Then there exists a positive constant C such that*

$$\|A^n\|_{\mathcal{C}_{p,w}} \le C \left(\frac{C\|A\|_{\mathcal{C}_{p,w}}}{\|A\|_{\mathcal{B}^2}} \right)^{\frac{1+\theta}{\theta} n^{\log_2(1+\theta)}} (\|A\|_{\mathcal{B}^2})^n \tag{6.5}$$

holds for all $A \in \mathcal{C}_{p,w}(X, \rho, \mu_c)$ and $n \ge 1$.

Proof. Using the argument used in the proof of Theorem 3.1, it suffices to prove the following compensated compactness estimate:

$$\|A^2\|_{\mathcal{C}_{p,w}} \le C\|A\|^{1+\theta}_{\mathcal{C}_{p,w}}\|A\|^{1-\theta}_{\mathcal{B}^2} \quad \text{for all } A \in \mathcal{C}_{p,w}(X, \rho, \mu_c), \tag{6.6}$$

where C is a positive constant.

Take $A := (a(x, y))_{x,y\in X} \in \mathcal{C}_{p,w}(X, \rho, \mu_c)$. Let $\tilde w(x) = w(x, I_0), \tilde v(x) = v(x, I_0)$, and $\tilde a(x) = \sup_{z\in X} |a(zx, z)|, x \in X$, where I_0 is the unit of the discrete group X. By the assumptions on the weights w and v, we have

$$\begin{aligned} \|A\|_{\mathcal{B}^2} &\le \max\Big(\sup_{x\in X}\sum_{y\in X} |a(x, y)|, \sup_{y\in X}\sum_{x\in X} |a(x, y)|\Big) \\ &\le C\|A\|_{\mathcal{C}_{p,w}}\|\tilde w^{-1}\|_{p/(p-1)} \le C\|A\|_{\mathcal{C}_{p,w}}\|\tilde v\tilde w^{-1}\|_{p/(p-1)}. \end{aligned} \tag{6.7}$$

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Since

$$
\begin{aligned}
& \sup_{y\in X}\sum_{z\in X}|a(yx,z)|v(yx,z)|a(z,y)|w(z,y) \\
\le\ & \sup_{y\in X}\Big(\sum_{z\in X}\big(v(yx,z)\chi_{\rho(yx,z)<\tau}|a(z,y)|w(z,y)\big)^2\Big)^{1/2}\times\Big(\sum_{z\in X}|a(yx,z)|^2\Big)^{1/2} \\
& +\sup_{y\in X}\sum_{z\in X}\tilde a(z^{-1}yx)\tilde v(z^{-1}yx)\chi_{\rho(yx,z)\ge\tau}|\tilde a(y^{-1}z)|\tilde w(y^{-1}z) \\
\le\ & \Big(\sum_{z\in X}\big(\tilde v(z^{-1}x)\chi_{\rho(x,z)<\tau}|\tilde a(z)|\tilde w(z)\big)^{\min(2,p)}\Big)^{1/\min(2,p)}\|A\|_{\mathcal{B}^2} \\
& +\sum_{z\in X}\tilde a(z^{-1}x)\tilde v(z^{-1}x)\chi_{\rho(x,z)\ge\tau}|\tilde a(z)|\tilde w(z)
\end{aligned}
$$

for all $\tau\ge 1$, we obtain

$$
\begin{aligned}
& \Big\|\Big(\sup_{y\in X}\sum_{z\in X}|(av)(yx,z)||(aw)(z,y)|\Big)_{x\in X}\Big\|_p \\
\le\ & \inf_{\tau\ge1}\Big(\|\tilde a\tilde w\|_p\|\tilde v\chi_{B(I_0,\tau)}\|_{\min(2,p)}\|A\|_{\mathcal{B}^2}+\|\tilde a\tilde w\|_p\|(\tilde a\tilde v)\chi_{X\backslash B(I_0,\tau)}\|_1\Big) \\
\le\ & C\|A\|_{\mathcal{C}_{p,w}}\|A\|_{\mathcal{B}^2}\inf_{\tau\ge1}\Big(\|\tilde v\chi_{B(I_0,\tau)}\|_{\min(2,p)} \\
& \qquad +\frac{\|A\|_{\mathcal{C}_{p,w}}}{\|A\|_{\mathcal{B}^2}}\|\tilde v\tilde w^{-1}\chi_{X\backslash B(I_0,\tau)}\|_{p/(p-1)}\Big) \\
\le\ & C\|A\|^{1+\theta}_{\mathcal{C}_{p,w}}\|A\|^{1-\theta}_{\mathcal{B}^2}, \qquad (6.8)
\end{aligned}
$$

where we have used (6.7) and the assumptions on the weights w and v to obtain the last inequality. Similarly, we have

$$
\Big\|\Big(\sup_{y\in X}\sum_{z\in X}|(aw)(yx,z)||(av)(z,y)|\Big)_{x\in X}\Big\|_p\le C\|A\|^{1+\theta}_{\mathcal{C}_{p,w}}\|A\|^{1-\theta}_{\mathcal{B}^2}. \qquad (6.9)
$$

By the assumption on left-invariant weights w and v,

$$
\begin{aligned}
\sup_{y\in X}|(cw)(yx,y)|\ \le\ & C\sup_{y\in X}\sum_{z\in X}|(av)(yx,z)||(aw)(z,y)| \\
& +C\sup_{y\in X}\sum_{z\in X}|(aw)(yx,z)||(av)(z,y)|, \qquad (6.10)
\end{aligned}
$$

where $A^2:=(c(x,y))_{x,y\in X}$. Therefore the compensated compactness estimate (6.6) follows from (6.8), (6.9), and (6.10). This establishes the asymptotic estimate (6.5) and completes the proof. □

Proof of Theorem 6.1. We use the same argument as the one used in the proof of Theorem 4.1 except using Lemmas 6.8 and 6.9 instead of Proposition 3.2 and Theorem 3.1. We omit the details of the proof here. □

Appendix A. (p,r)-admissible weights

In the Appendix, we discuss the technical assumption, the (p,r)-admissibility, on the weight w. Particularly, we consider the (p,r)-admissibility of the radial weights

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w of the form

$$w(\lambda, \lambda') = \exp(\kappa(\rho(\lambda, \lambda'))), \quad \lambda, \lambda' \in \Lambda, \tag{A.1}$$

where κ is a continuous concave function κ on $[0, \infty)$ with $\kappa(0) = 0$, and Λ is a relatively-separated subset of a quasi-metric space (X, ρ). Here we recall that a *relatively-separated* subset Λ of a quasi-metric space (X, ρ) means that there exists a positive constant $D(\Lambda)$ such that

$$\sum_{\lambda \in \Lambda} \chi_{B(\lambda,1)}(x) \le D(\Lambda) \quad \text{for all } x \in X. \tag{A.2}$$

Theorem A.1. *Let $1 \le p, r \le \infty$, $(X, \rho, \mu) = (\Lambda, |\cdot|, \mu_c)$ for some relatively-separated subset Λ of $\mathbf{R}^d$, and let the weight w be of the form (A.1) for some continuous concave function κ on $[0, \infty)$ with $\kappa(0) = 0$, and set $\tilde\kappa(t) = \kappa(2t) - \kappa(t)$. Then we have*

(i) *If the functions $\tilde a_s(\tau)$ and $\tilde b_s(\tau), \tau > 0$, to be defined by*

$$\tilde a_s(\tau) = \begin{cases} \exp(\kappa(\tau)) & \text{if } s = \infty, \\ \left(\int_0^\tau \exp(s\kappa(t) + (d-1)\ln t)dt\right)^{1/s} & \text{if } 1 \le s < \infty, \end{cases} \tag{A.3}$$

and

$$\tilde b_s(\tau) = \begin{cases} \exp(\kappa(2\tau) - 2\kappa(\tau)) & \text{if } s = \infty, \\ \left(\int_\tau^\infty \exp(s\kappa(2t) - 2s\kappa(t) + (d-1)\ln t)dt\right)^{1/s} & \text{if } 1 \le s < \infty, \end{cases} \tag{A.4}$$

satisfy

$$\inf_{\tau \ge 1} \tilde a_{r'}(\tau) + \tilde b_{p'}(\tau) t \le D t^\theta, \quad t \ge 1, \tag{A.5}$$

for some positive constants $D \in (0, \infty)$ and $\theta \in (0, 1)$, then the weight w is (p, r)-admissible, and furthermore the weight v,

$$v(x, y) = \exp(\tilde\kappa(|x - y|)), \ x, y \in X, \tag{A.6}$$

satisfies (2.8), (2.9), and (2.10).

(ii) *If $1 \le p \le r \le \infty, d > 1$, $X = \mathbf{Z}^d$, κ and $\tilde\kappa$ are strictly increasing and $\lim_{\tau\to+\infty} \kappa(\tau) = +\infty$, and the weight w is (p, r)-admissible, then $\lim_{t\to\infty} \kappa(t)/t = 0$ and $\kappa(t) \ge \delta \ln(1 + t) - D$ for some positive constants δ and D.*

(iii) *If $1 \le p \le r \le \infty, d > 1$, $X = \mathbf{Z}^d$, and $\kappa(t) = t/\ln(2 + t)$, then the weight w is not (p, r)-admissible.*

Proof. (i) Clearly it suffices to prove that the weight v in (A.6) satisfies (2.8), (2.9), and (2.10). By the concavity of the function κ,

$$\kappa(t + s) - \kappa(t) \le \kappa(2s) - \kappa(s) = \tilde\kappa(s) \quad \text{for all } t \ge s \ge 0.$$

Thus for any $x, y, z \in X$ with $|x - y| \ge |y - z|$,

$$\begin{aligned} w(x, y)v(y, z) + w(y, z)v(x, y) &\ge w(x, y)v(y, z) \\ &\ge \exp(\kappa(|x - y| + |y - z|)) \ge w(x, y), \end{aligned}$$

which proves that the weight v satisfies (2.8). Noting that $\kappa(2t)-2\kappa(t)$ is decreasing on $[0, \infty)$ and $\kappa(t) \le \kappa(t + s) \le \kappa(t) + \kappa(s)$ for all $t, s \ge 0$ by the concavity of the function κ and $\kappa(0) = 0$, we then obtain the following for any $y \in X$ and $\tau \ge 1$:

$$\sup_{x \in X, |x-y| \ge \tau} (vw^{-1})(x, y) \le \exp(\kappa(2\tau) - 2\kappa(\tau)),$$

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and

$$\sum_{x\in X,|x-y|\geq\tau}((vw^{-1})(x,y))^s \leq \sum_{j=[\tau]}^{\infty}\sum_{j\leq|x-y|\leq j+1}((vw^{-1})(x,y))^s$$
$$\leq C\sum_{j=[\tau]}^{\infty}\exp(s\kappa(2j)-2s\kappa(j))(j+1)^{d-1}$$
$$\leq C\int_{\tau}^{\infty}\exp(s\kappa(2t)-2s\kappa(t)+(d-1)\ln t)dt,$$

where $1\leq s<\infty$ and $[\tau]$ denotes the integral part of the real number τ. Similarly, for any $y\in X$ and $\tau\geq 1$,

$$\sup_{x\in X,|x-y|\leq\tau} v(x,y)\leq \exp(\kappa(\tau)),$$

and

$$\sum_{x\in X,|x-y|\leq\tau}(v(x,y))^s \leq \sum_{j=0}^{[\tau]}\sum_{j\leq|x-y|\leq j+1}(v(x,y))^s \leq C\int_0^{\tau}\exp(s\kappa(t)+(d-1)\ln t)dt,$$

where $1\leq s<\infty$. Therefore we have the following upper bound estimates for $a_{r'}(\tau)$ and $b_{p'}(\tau),\tau\geq 1$:

$$a_{r'}(\tau)\leq C\tilde a_{r'}(\tau), \tag{A.7}$$

and

$$b_{p'}(\tau)\leq C\tilde b_{p'}(\tau). \tag{A.8}$$

This together with (A.5) implies that the weight v in (A.6) satisfies (2.9) and (2.10).

(ii) By the assumption on the weight w, there exists a weight v that satisfies (2.8), (2.9), and (2.10). By (2.8) and the symmetry of the weight v, there exists a positive constant D such that

$$v(x,x+y)+v(x,x-y)\geq D\exp(\tilde\kappa(|y|))\quad\text{for all } x,y\in X:=\mathbf{Z}^d. \tag{A.9}$$

Thus,

$$\sum_{j\leq|x-y|\leq j+1}((vw^{-1})(x,y))^s$$
$$\geq C\sum_{j\leq|z|\leq j+1,z\in\mathbf{Z}^d}(v(x,x+z)+v(x,x-z))^s\exp(-s\kappa(j+1))$$
$$\geq Cj^{d-1}\exp(s\tilde\kappa(j+1)-s\kappa(j+1))$$

and

$$\sum_{j\leq|x-y|\leq j+1}(v(x,y))^s\geq Cj^{d-1}\exp(s\tilde\kappa(j+1))$$

for all $1\leq j\in\mathbf{Z}$. Applying the above two estimates and using the same argument as the one used in the proof of the inequalities (A.7) and (A.8), we obtain a lower bound estimate for $a_{r'}(\tau)$ and $b_{p'}(\tau)$:

$$a_{r'}(\tau)\geq C_0\tilde a_{r'}(\tau), \tag{A.10}$$

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and

$$\text{(A.11)} \qquad b_{p'}(\tau) \geq C_0 \tilde{b}_{p'}(\tau),\ \tau \geq 1,$$

where C_0 is a positive constant. Therefore

$$\text{(A.12)} \qquad a_{r'}(\tau) \geq C_1 \exp(\tilde{\kappa}(\tau))\tau^{(d-1)/r'}$$

and

$$\text{(A.13)} \qquad b_{p'}(\tau) \geq C_1 \exp(\tilde{\kappa}(\tau) - \kappa(\tau))\tau^{(d-1)/p'}$$

for some positive constant C_1. Combining (2.10), (A.12), and (A.13), we have

$$\text{(A.14)} \qquad \inf_{\tau\geq 1} e^{\tilde{\kappa}(\tau)}\tau^{(d-1)/p'}(\tau^{(d-1)(1/r'-1/p')} + e^{-\kappa(\tau)}t) \leq Ct^{\theta},\ t \geq 1.$$

Noting that

$$\begin{aligned} &\inf_{\tau\geq 1} e^{\tilde{\kappa}(\tau)}\tau^{(d-1)/p'}(\tau^{(d-1)(1/r'-1/p')} + e^{-\kappa(\tau)}t) \\ \geq\ &\inf_{\tau\geq 1} e^{\tilde{\kappa}(\tau)}(1 + e^{-\kappa(\tau)}t) = 2e^{\tilde{\kappa}(\tau_t)} \end{aligned}$$

for the unique τ_t such that $e^{\kappa(\tau_t)} = t$ (the existence from the assumptions that κ and $\tilde{k}$ are strictly monotone, and κ satisfies $\kappa(0) = 0$ and $\lim_{\tau\to\infty} \kappa(\tau) = +\infty$), we then obtain from (A.14) that

$$\kappa(2\tau) \leq (1+\theta)\kappa(\tau) + D \text{ for all } \tau \geq 1$$

where D is a positive constant. Thus

$$\text{(A.15)} \qquad \kappa(\tau) \leq C\tau^{\ln_2(1+\theta)},\ \tau \geq 1,$$

which implies that $\lim_{\tau\to\infty} \kappa(\tau)/\tau = 0$. Let $\tilde{\tau}_t, t \geq 1$, be the unique solution of the equation $\kappa(\tau) + (d-1)(1/r' - 1/p')\ln\tau = \ln t$. The existence of such a $\tilde{\tau}_t$ follows from the assumption $r \geq p$ and the monotonic property of the functions $\kappa(\tau)$ and $\ln\tau$. Then it follows from (A.14) and the estimate

$$\begin{aligned} &\inf_{\tau\geq 1} e^{\tilde{\kappa}(\tau)}\tau^{(d-1)/p'}(\tau^{(d-1)(1/r'-1/p')} + e^{-\kappa(\tau)}t) \\ \geq\ &\tau^{(d-1)(1/r'-1/p')} + e^{-\kappa(\tau)}t \geq e^{-\kappa(\tilde{\tau}_t)}t \end{aligned}$$

that

$$\kappa(\tilde{\tau}_t) \geq (1-\theta)\ln t - C \geq (1-\theta)\kappa(\tilde{\tau}_t) + (1-\theta)(d-1)(1/r' - 1/p')\ln\tilde{\tau}_t - C.$$

Therefore the conclusion $\kappa(\tau) \geq \delta\ln(1+\tau) - D$ follows.

(iii) Suppose, on the contrary, that there exists a weight v that satisfies (2.8), (2.9), and (2.10). Then (A.15) holds from the proof of the second conclusion, which is a contradiction. □

In the following two examples, we show that given a polynomial weight or a subexponential weight w, a weight v can be found to satisfy (2.8), (2.9), and (2.10).

Example A.2. For the index set X with a quasi-metric ρ and a Borel measure μ having polynomial growth property, we define $d(X,\rho,\mu)$ as follows:

$$\text{(A.16)} \qquad d(X,\rho,\mu) := \inf\{d : (2.1) \text{ holds for some positive } C\}.$$

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Clearly $d(X,\rho,\mu)$ becomes the dimension d of the Euclidean space when $X=\mathbf{R}^d$ and μ is the Lebesgue measure m. Take $\beta>d(X,\rho,\mu)$. We then obtain (2.1) and (A.16) such that

$$\begin{aligned}
&\int_{\rho(x,y)\ge\tau} w_{-\beta}(x,y)d\mu(y)\\
&= \sum_{j=1}^\infty \int_{2^{j-1}\tau\le\rho(x,y)<2^j\tau} w_{-\beta}(x,y)d\mu(y)\\
&\le \sum_{j=1}^\infty (1+2^{j-1}\tau)^{-\beta}\mu(B(x,2^j\tau))\le C_\epsilon\tau^{-(\beta-d(X,\rho,\mu))/2} \qquad \text{(A.17)}
\end{aligned}$$

for all $x\in X$ and $\tau\ge1$, where C is a positive constant independent of $x\in X$ and $\tau\ge1$. Therefore for the polynomial weight $w_\alpha(x,y):=(1+\rho(x,y))^\alpha$ with $\alpha>d(X,\rho,\mu)(1-1/p)$, the weight $v:=w_0$ satisfies (2.8), (2.9), and (2.10) by the quasi-metric property of ρ and (A.17), where

$$\theta=\frac{(d(X,\rho,\mu)+\delta)(1-1/r)}{\alpha-(d(X,\rho,\mu)+\delta)(1-1/p)+(d(X,\rho,\mu)+\delta)(1-1/r)},$$

and $\delta=(\alpha p'-d)/2$ if $p'<\infty$ and $\delta=1$ if $p'=\infty$.

Example A.3. Let (X,ρ) be a metric space with a Borel measure μ having polynomial growth property. For weights w of the form

$$w(x,y)=\exp(D\rho(x,y)^\delta),\quad x,y\in X, \qquad \text{(A.18)}$$

with $D\in(0,\infty)$ and $\delta\in(0,1)$, we let

$$v(x,y)=\exp(D(2^\delta-1)\rho(x,y)^\delta). \qquad \text{(A.19)}$$

Recalling that ρ is a metric and applying the trivial inequality $1\le s^\delta+(2^\delta-1)(1-s)^\delta$ for all $1/2\le s\le1$, we have that the weights w and v satisfy (2.8). By the polynomial growth property of the Borel measure μ,

$$\begin{aligned}
\|vw^{-1}\|_{\mathcal{A}_{p/(p-1),w_0}} &\le C+C\sup_{x\in X}\Big(\sum_{j=1}^\infty e^{Dp(2^\delta-2)2^j/(p-1)}\mu(B(x,2^{j+1})\Big)^{(p-1)/p}\\
&\le C+C\sup_{x\in X}\Big(\sum_{j=1}^\infty e^{Dp(2^\delta-2)2^j/(p-1)}2^{j(d(X,\rho,\mu)+\epsilon)}\Big)^{(p-1)/p}<\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\inf_{\tau\ge1}\tilde a_{r'}(\tau)+\tilde b_{p'}(\tau)t\\
&\le C\inf_{\tau\ge1} e^{D\tau^\delta}\tau^{(d(X,\rho,\mu)+\epsilon)/r'}+e^{-D(2-2^\delta)\tau^\delta}\tau^{(d(X,\rho,\mu)+\epsilon)/p'}t\\
&\le Ct^{1/(3-2^\delta)}(1+\ln t)^{(d(X,\rho,\mu)+\epsilon)/(\min(r',p'))}\le Ct^\theta
\end{aligned}$$

for all $t\ge1$ and $\theta\in(0,1/(3-2^\delta))$, where $\epsilon>0$ is sufficiently small. Therefore for the subexponential weight w in (A.18) for some $\delta\in(0,1)$, the weight v in (A.19) satisfies (2.8), (2.9), and (2.10). We remark that the metric assumption on ρ in the above cannot be replaced by the quasi-metric assumption on ρ in general. For instance, one may easily show that for the setting $(X,\rho,\mu)=(\mathbf{Z},|\cdot|^2,\mu_c)$, there does not exist another weight v associated with the weight $w(x,y)=\exp(\rho(x,y)^{1/2})=\exp(|x-y|)$ such that (2.8) and (2.9) hold.

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Example A.4. Let (X, ρ) be a quasi-metric space with a Borel measure μ having polynomial growth property. Assume that the quasi-metric ρ is Hölder continuous in the sense that there exists $C \in (0, \infty)$ and $\beta \in (0, 1]$ such that

$$|\rho(x, y) - \rho(x, z)| \le C\rho(y, z)^\beta (\rho(x, y) + \rho(x, z))^{1-\beta} \text{ for all } x, y, z \in X$$

([13, 28, 29]). One may show that there exists a positive constant δ_0 such that for a subexponential weight w of the form $\exp(D\rho(x, y)^\delta)$ with $\delta \in (0, \delta_0)$ and $D \in (0, \infty)$ there exists another subexponential weight v of the form $\exp(D\theta_0\rho(x, y)^\delta)$ with $\theta_0 \in (0, 1)$ satisfying (2.8) – (2.10).

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