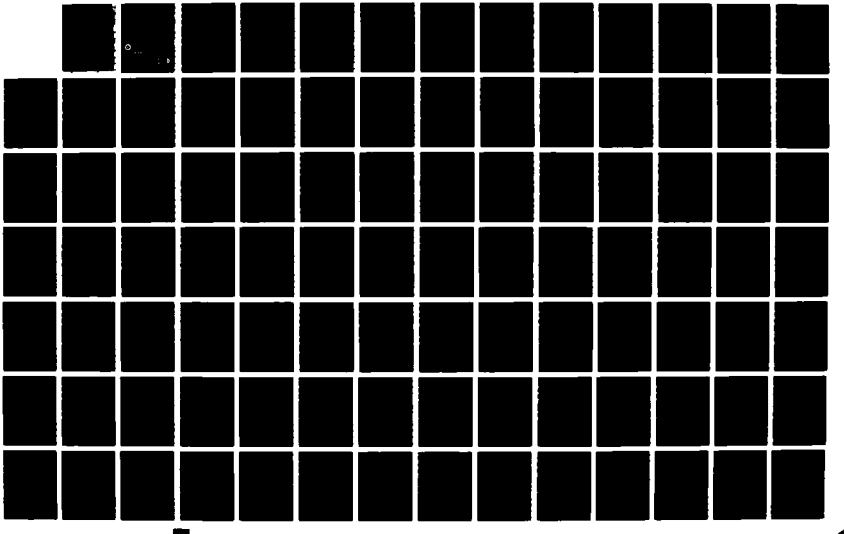
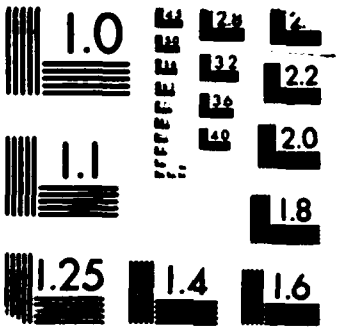
AD-A193 865 WIGNER DISTRIBUTION FUNCTION: RELATION TO SHORT-TERM SPECTRAL ESTIMATION..(U) NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CT NEW LONDON LAB.. A H NUTTALL 1/2

UNCLASSIFIED 16 FEB 88 NUSC-TR-8225 F/G 12/3 NL





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NUSC Technical Report 8225
16 February 1988

Wigner Distribution Function: Relation to Short-Term Spectral Estimation, Smoothing, and Performance in Noise

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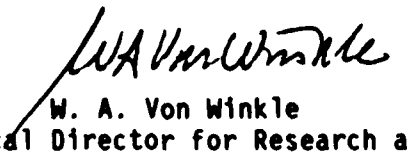
Preface

This research was conducted under NUSC Project Nos. A60080 and G20021, "CUARP3," Principal Investigator Dr. Kent Scarbrough (Code 3331), Program Manager Mr. Ronald Tompkins (Code 33A). The sponsoring activities are the Office Of Naval Technology (Code 231), Program Manager Dr. Tom Warfield, and the Naval Sea Systems Command (Code 63D), Program Manager CDR L. Schneider. Also, this research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 304). This technical report was prepared with funds provided by the NUSC In-House Independent Research and Independent Exploratory Development Program, sponsored by the Chief of Naval Research.

The Technical Reviewer for this report was Dr. John W. Fay (Code 3331).

Reviewed and Approved:



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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 8225	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Underwater Systems Center 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code). New London Laboratory New London, CT 06320	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING / SPONSORING ORGANIZATION 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
WIGNER DISTRIBUTION FUNCTION: RELATION TO SHORT-TERM SPECTRAL ESTIMATION, SMOOTHING, AND PERFORMANCE IN NOISE

12. PERSONAL AUTHOR(S)
Albert H. Nuttall

13a. TYPE OF REPORT	13b. TIME COVERED FROM ____ TO ____	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
		1988 February 16	

16. SUPPLEMENTARY NOTATION

17 COSATI CODES		
FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Wigner Distribution Function; Noise Performance; Short-Term Spectral Estimation; Linear Frequency Modulation; Smoothing; → next page

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The properties and behavior of the Wigner Distribution Function (WDF) are investigated both analytically and by means of a number of simple informative examples. The lack of local temporal averaging when obtaining the instantaneous correlation function, and the lack of weighting the longer delay values when transforming to the instantaneous spectrum, are shown to be the causes of the deleterious interference effects that are inherent to the WDF. The equivalence of short-term spectral estimation to the smoothed WDF offers an attractive alternative with guaranteed positive distribution values and no interference effects.

The performance of a processor which estimates the WDF of a signal waveform in the presence of additive noise is investigated in terms of the output mean, bias, and variance. Dependence on filtering the input and time-weighting is allowed and included in the analysis. Numerical application to a particular example is carried out.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Albert H. Nuttall	22b. TELEPHONE (Include Area Code) (203) 440-4618
	22c. OFFICE SYMBOL Code 304

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

18. SUBJECT TERMS (Cont'd.)

Interference Effects
Gaussian Amplitude Modulation
Marginals
Moments
Spread of Distribution
Positive Distributions
Ambiguity Function

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

WDF	Wigner Distribution Function
t	Time, (1)
$s(t)$	Waveform, function of time, (1)
τ	Time separation or lag variable, (1)
$R(t,\tau)$	Instantaneous correlation, (1)
f	Frequency, (3)
$W(t,f)$	WDF at time t and frequency f, (3), (10)
E	Waveform energy, (5), (14)
$S(f)$	Voltage density spectrum of s(t), (9)
δ	Delta function, (13)
$\circledast$	Convolution, (19), (20)
υ	Frequency shift variable, (22)
$\chi(\upsilon,\tau)$	Ambiguity function, (22)
$A(\upsilon,f)$	Spectral function, (24)
Im	Imaginary part, (28)
δ'	Derivative of delta function, (29)
$\mu_f(t)$	Frequency center at time t, (30)
$M(t)$	Magnitude of s(t), (31)
$\Theta(t)$	Phase of s(t), (31)
$\mu_t(f)$	Time center at frequency f, (33)
$A(f)$	Magnitude of S(f), (34)
$\phi(f)$	Phase of S(f), (34)
$\bar{f}$	Frequency center, (36)
$\bar{t}$	Time center, (37)

LIST OF SYMBOLS (Cont'd)

$\sigma_f^2(t)$	Instantaneous mean-square frequency spread, (42)
$W_{ab}(t,f)$	Cross WDF, (51)
$a(t)$, $A(f)$	Fourier transform pair, (51)
$\chi_{ab}(\nu,\tau)$	Cross ambiguity function, (53)
$c(t)$	Complex envelope, (55)
f_o	Carrier frequency, (55)
Re	Real part, (55)
Δ_t	Sampling increment in time, (57)
$\tilde{W}$	Approximation to W, (57), (59)
m,n,N_f	Integers, (60)
T	Time spread, figure 2
B	Frequency spread, figure 3
f_H	Highest frequency in s(t), (65)
t_o,f_o	Time and frequency shifts, (66)
Θ_o	Time-bandwidth product, (88)-(90)
$u(t)$	Weighting in time, (102)
$S_u(t,f)$	Voltage spectrum of s, relative to u, (102)
STSE	Short-term spectral estimate, (103)
E_s	Energy of waveform s(t), (111)
E_u	Energy of weighting u(t), (111)
σ_c	Effective duration of weighting, (146)
α_c	Linear frequency modulation parameter, (146)
Θ_c	Time-bandwidth product, (146)
$v_1(t)$	Time weighting, (148)
$\hat{R}$	Locally averaged correlation, (148), (150)

LIST OF SYMBOLS (Cont'd)

$\hat{W}$	Locally averaged WDF, (149)
$v_2(t,\tau)$	Time and lag weighting, (150)
$V_2(t,f)$	Transform (on τ) of v_2, smoothing function, (151)
I	Spread measure, (163)
β_c	Specified slope in t,f plane, (163)
α_c	$2\pi\beta_c$, (163)
s(t)	Signal, (165)
n(t)	Additive noise, (165)
x(t)	Observed waveform, (165)
$C_n(\tau)$	Noise correlation, (166)
$G_n(f)$	Noise spectrum, (166)
v(t)	Weighting of x, (167)
y(t)	Weighted waveform, (167)
a,b,c,d	Components of W_{yy}, (168), (169)
overbar	Ensemble average, (172), (173)
$G_n^{(2)}(f)$	Modified noise spectral quantity, (178)
B(t,f)	Auxiliary function, (183)
Var	Variance of WDF estimate, (188)
$s_o(t)$	Input signal, figure 7
$n_o(t)$	Input noise, figure 7
H(f)	Filter, figure 7
N_d	Noise spectral level, (189)
B	Passband of filter H, figure 8
L	Duration of weighting v, figure 8
LFM	Linear Frequency Modulation
sgn(x)	1 for $x > 0$, -1 for $x < 0$

WIGNER DISTRIBUTION FUNCTION: RELATION TO SHORT-TERM SPECTRAL ESTIMATION, SMOOTHING, AND PERFORMANCE IN NOISE

INTRODUCTION

The potential of the Wigner Distribution Function (WDF) for characterizing the short-term local time and frequency content of a transient waveform has been amply demonstrated in a series of papers; for example, see the recent publications [1,2,3] and the extensive references listed therein. In particular, [1] contains numerical examples of the WDF for rectangularly gated linear frequency modulation and a version which has been smoothed with a square window in the time-frequency plane, in order to yield positive distribution values. Here, we will be concerned with smoothing so as to minimally spread the WDF, but will not presume all the information that is required for implementation via [2], nor do we limit consideration to a constant-magnitude function. We will then use the close connection between short-term spectral estimation and smoothed WDFs to suggest a possible analysis procedure and philosophy to extract information about a given waveform without an extensive search in waveform parameters. Finally, the performance of a particular WDF estimator in the presence of additive noise will be analyzed, both in terms of bias and variance.

This report summarizes and compiles many of the results in the publications noted above in a unified framework and notation. Also, numerous examples are presented in the various sections of this report to

illustrate and bring out some of the fundamental concepts and limitations of the WDF; these examples can be evaluated analytically in closed form, allowing for close investigation of the behavior of the WDF, and as control cases on any computer-written program for numerical evaluation of the WDF.

BASIC PROPERTIES OF THE WDF

DEFINITIONS

A natural definition of the time-varying correlation of a nonstationary complex stochastic process s(t) is

$$R(t,\tau) = \overline{s(t + \frac{\tau}{2})\ s^*(t - \frac{\tau}{2})}\ , \qquad (1)$$

where the overbar denotes an ensemble average. The "center" time in (1) is t, while the "separation" time is τ. However, if an ensemble is not available, or if s(t) is a deterministic waveform, the obvious extension of (1) is simply

$$R(t,\tau) = s(t + \frac{\tau}{2})\ s^*(t - \frac{\tau}{2})\ . \qquad (2)$$

This quantity is interpreted as the instantaneous correlation of waveform s(t) at time t, for separation (or lag) τ.

The associated "spectrum" at time t is then available, as usual, by Fourier transforming (2) on separation variable τ, to get at frequency f,

$$W(t,f) = \int d\tau\ \exp(-i2\pi f\tau)\ R(t,\tau) =$$

$$= \int d\tau\ \exp(-i2\pi f\tau)\ s(t + \frac{\tau}{2})\ s^*(t - \frac{\tau}{2})\ . \qquad (3)$$

(Integrals without limits are over the range of nonzero integrand. Also, it is presumed that s(t) and its derivatives decay fast enough to zero at $t = \pm\infty$ for all the integrals to converge.) This time-frequency function W(t,f) is called the Wigner Distribution Function (WDF). It is a real function, even when s(t) is complex, since

$$W^*(t,f) = \int d\tau \exp(i2\pi f\tau)\, s^*(t + \tfrac{\tau}{2})\, s(t - \tfrac{\tau}{2}) =$$

$$= \int du \exp(-i2\pi fu)\, s^*(t - \tfrac{u}{2})\, s(t + \tfrac{u}{2}) = W(t,f) \ . \qquad (4)$$

However, it is not necessarily positive, as the simple example of a rectangularly gated pulse quickly shows: for

$$s(t) = \left\{ \begin{array}{ll} a & \text{for } |t| < T/2 \\ 0 & \text{otherwise} \end{array} \right\} \ ,$$

then

$$W(t,f) = 2E\, \frac{\sin[2\pi f(T - 2|t|)]}{2\pi fT} \qquad \text{for } |t| < \frac{T}{2} \ , \ \text{all } f \ ,$$

and zero otherwise, where E is the waveform energy:

$$E = \int dt\, |s(t)|^2 = |a|^2\, T \ . \qquad (5)$$

An even simpler example is furnished by waveforms with odd symmetry, $s(-t) = -s(t)$. Substitution in (3) immediately yields $W(0,0) = -\int d\tau\, |s(\tau/2)|^2 = -2E$. Thus the origin value of the WDF is always negative for an odd waveform.

More generally, when waveform s(t) is expressed in terms of its even and odd parts according to

$$s(t) = e(t) + o(t) \ , \tag{6}$$

then the origin value of the WDF is

$$W(0,0) = \int d\tau \ \ s(\tau/2)\ s^*(-\tau/2) = 2\int dt\ s(t)\ s^*(-t) =$$

$$= 2\int dt \left[e(t) + o(t)\right] \left[e^*(t) - o^*(t)\right] = 2E_e - 2E_o \ , \tag{7A}$$

where

$$E_e = \int dt \left|e(t)\right|^2 \ , \qquad E_o = \int dt \left|o(t)\right|^2 \tag{7B}$$

are the energies of the even and odd parts respectively. For nonzero t,f, it can readily be shown that the magnitude of the WDF is upper bounded by $2E = 2(E_e + E_o)$.

PROPERTIES OF WDF

For s(t) real, it readily follows from the definition of the WDF in (3) that

$$W(t,-f) = W(t,f) \quad \text{for } s(t) \text{ real} . \tag{8}$$

In this special case, it is only necessary to evaluate W(t,f) for $f \geq 0$.

Define the voltage density spectrum of waveform s(t) as

$$S(f) = \int dt \exp(-i2\pi ft)\, s(t) . \tag{9}$$

Then an alternative form for the WDF in (3) is

$$W(t,f) = \int d\tau \exp(-i2\pi f\tau)\, s(t + \tfrac{\tau}{2})\, s^*(t - \tfrac{\tau}{2}) =$$

$$= \int d\nu \exp(i2\pi\nu t)\, S(f + \tfrac{\nu}{2})\, S^*(f - \tfrac{\nu}{2}) , \tag{10}$$

in terms of S(f).

Integration on (10) immediately yields the marginals

$$\int dt\, W(t,f) = |S(f)|^2 , \tag{11}$$

and

$$\int df\, W(t,f) = |s(t)|^2 , \tag{12}$$

where we used the result

$$\int dx \exp(i2\pi xy) = \delta(y) \ . \tag{13}$$

The quantity in (11) is the energy density function, while that in (12) is the instantaneous power. If we complete the integrations on the remaining variables in (11) and (12), they both yield

$$\iint dt\ df\ W(t,f) = E =$$

$$= \int dt\ |s(t)|^2 = \int df\ |S(f)|^2 \ , \tag{14}$$

where E is the total waveform energy.

If waveform s(t) satisfies a time-limited restriction, namely

$$s(t) \neq 0 \quad \text{only for } t_1 < t < t_2 \ , \tag{15}$$

then (3) reduces to

$$W(t,f) = \int_{-\tau_m}^{\tau_m} d\tau \exp(-i2\pi f\tau)\ s(t + \tfrac{\tau}{2})\ s^*(t - \tfrac{\tau}{2}) \quad \text{for } t_1 < t < t_2 \ , \tag{16}$$

and zero otherwise, where

$$\tau_m = 2 \min(t_2 - t,\ t - t_1) \quad \text{for } t_1 < t < t_2 \ . \tag{17}$$

Thus the WDF is time-limited if waveform s(t) is time-limited; however, if there are gaps in s(t), the behavior of the WDF is more complicated, as will be demonstrated later.

PRODUCT AND CONVOLUTION

If waveform s(t) is the product of two other waveforms,

$$s(t) = a(t)\ b(t)\ , \tag{18}$$

then the WDF of s(t) is (inserting subscripts as needed)

$$W_s(t,f) = \int d\tau\ \exp(-i2\pi f\tau)\ R_s(t,\tau) =$$

$$= \int d\tau\ \exp(-i2\pi f\tau)\ R_a(t,\tau)\ R_b(t,\tau) =$$

$$= \int dv\ W_a(t,v)\ W_b(t,f - v) =$$

$$= W_a(t,f) \overset{f}{\circledast} W_b(t,f)\ , \tag{19}$$

which is a convolution on frequency f, for fixed t.

In a similar fashion, if s(t) is the convolution in time, of two other waveforms,

$$s(t) = a(t) \overset{t}{\circledast} b(t) = \int d\tau\ a(\tau)\ b(t - \tau)\ , \tag{20}$$

then the WDF of s(t) is

$$W_s(t,f) = W_a(t,f) \overset{t}{\circledast} W_b(t,f) = \int d\tau\, W_a(\tau,f)\, W_b(t-\tau,f)\ , \qquad (21)$$

which is a convolution on time t, for fixed f.

AMBIGUITY FUNCTION

The WDF is closely related to the complex ambiguity function of s(t), which is defined here as [4; section 7.2]

$$\chi(\nu,\tau) = \int dt\, \exp(-i2\pi\nu t)\, s(t+\tfrac{\tau}{2})\, s^*(t-\tfrac{\tau}{2}) =$$

$$= \int dt\, \exp(-i2\pi\nu t)\, R(t,\tau) =$$

$$= \int df\, \exp(i2\pi f\tau)\, S(f+\tfrac{\nu}{2})\, S^*(f-\tfrac{\nu}{2})\ . \qquad (22)$$

In fact, the two are double Fourier transforms of each other:

$$\iint dt\, df\, \exp(i2\pi\tau f - i2\pi\nu t)\, W(t,f) =$$

$$= \iint dt\, df\, \exp(i2\pi\tau f - i2\pi\nu t) \int d\tau_1\, \exp(-i2\pi f\tau_1)\, s(t+\tfrac{\tau_1}{2})\, s^*(t-\tfrac{\tau_1}{2}) =$$

$$= \iint dt\, d\tau_1\, \exp(-i2\pi\nu t)\, s(t+\tfrac{\tau_1}{2})\, s^*(t-\tfrac{\tau_1}{2})\, \delta(\tau-\tau_1) =$$

$$= \int dt\, \exp(-i2\pi\nu t)\, s(t+\tfrac{\tau}{2})\, s^*(t-\tfrac{\tau}{2}) = \chi(\nu,\tau)\ . \qquad (23)$$

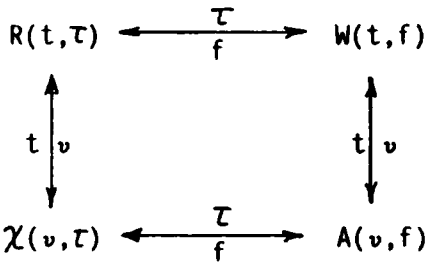
Here we used (3), (13), and (22).

In a similar fashion, the following (single) Fourier transform relationships on the WDF hold:

$$\int df\ \exp(i2\pi f\tau)\ W(t,f) = s(t + \frac{\tau}{2})\ s^*(t - \frac{\tau}{2}) = R(t,\tau)\ ,$$

$$\int dt\ \exp(-i2\pi\upsilon t)\ W(t,f) = S(f + \frac{\upsilon}{2})\ S^*(f - \frac{\upsilon}{2}) = A(\upsilon,f)\ . \qquad (24)$$

These properties are summarized in the following diagram, where an arrow denotes a Fourier transform:



Not every function of t,f is a (legal) WDF; in fact, from (24) there follows

$$\int df\ \exp\left(i2\pi f(t_1 - t_2)\right)\ W\left(\frac{t_1 + t_2}{2}\ ,f\right) = s(t_1)\ s^*(t_2) = R\left(\frac{t_1 + t_2}{2}\ ,\ t_1 - t_2\right)\ . \qquad (25)$$

Thus, in order for a candidate function W(t,f) to be a WDF, the function resulting on the right-hand side of (25) must be separable in the variables t_1 and t_2. When and only when that separability occurs, the waveform s(t) can be recovered from correlation R or W (within a constant unknown phasor) as follows: let

$$s(t_o) = |s(t_o)| \exp\left(i\theta(t_o)\right) , \qquad R(t_o,0) = |s(t_o)|^2 , \tag{26}$$

where t_o is arbitrary, except that $s(t_o) \neq 0$. Then, from the right-hand side of (25),

$$s(t) = \frac{R\left(\frac{t + t_o}{2}, t - t_o\right)}{s^*(t_o)} = \frac{R\left(\frac{t + t_o}{2}, t - t_o\right)}{\sqrt{R(t_o,0)}} \exp\left(i\theta(t_o)\right) \quad \text{for all } t . \tag{27}$$

The special case of $t_o = 0$ was given in [3; (17)]. The fact that the constant phase $\theta(t_o)$ is irretrievably lost in R and W can easily be seen by considering $s(t) = c\, g(t)$, for which $W_s(t,f) = |c|^2 W_g(t,f)$.

The box-like function rect(t/T) rect(f/F) = 1 for $|t| < T/2$ and $|f| < F/2$, zero otherwise, which was employed for smoothing in [1], is not a WDF, since the transform on the left-hand side of (25) yields $F\ \mathrm{sinc}(Ft_1 - Ft_2)$ for $|t_1 + t_2| < T$, which is not separable in t_1 and t_2. Also, the Gaussian function $\exp(-t^2/\sigma^2 - b^2 4\pi^2 f^2)$ is a legal WDF if and only if $b = \sigma$, in which case $s(t) = (4\pi\sigma^2)^{-1/4} \exp(-t^2/(2\sigma^2))$, with σ arbitrary.

FIRST MOMENTS OF W

The marginal integrals of W were given in (11) and (12). The (conditional) first moment of W, with respect to frequency, is

$$\int df\ f\ W(t,f) = \int df\ f \int d\tau\ \exp(-i2\pi f\tau)\ s(t + \tfrac{\tau}{2})\ s^*(t - \tfrac{\tau}{2}) =$$

$$= \int d\tau\ s(t + \tfrac{\tau}{2})\ s^*(t - \tfrac{\tau}{2}) \int df\ f\ \exp(-i2\pi f\tau) =$$

$$= \frac{i}{2\pi} \int d\tau\ s(t + \tfrac{\tau}{2})\ s^*(t - \tfrac{\tau}{2})\ \delta'(\tau) = \frac{1}{2\pi}\ \mathrm{Im}\{s'(t)\ s^*(t)\} \quad . \tag{28}$$

Here we used the result

$$i2\pi \int dx\ x\ \exp(i2\pi xy) = \delta'(y) \quad , \tag{29}$$

obtainable directly from (13) by taking a derivative with respect to y.

Therefore the "frequency center at time t" of waveform s(t) is defined as

$$\mu_f(t) = \frac{\int df\ f\ W(t,f)}{\int df\ W(t,f)} = \frac{1}{2\pi}\ \frac{\mathrm{Im}\{s'(t)s^*(t)\}}{|s(t)|^2} \quad , \tag{30}$$

upon use of (28) and (12). If we let complex waveform s(t) be represented in terms of its amplitude and phase modulations according to

$$s(t) = M(t)\ \exp[i\theta(t)] \quad , \tag{31}$$

then (30) yields simply

$$\mu_f(t) = \frac{1}{2\pi}\,\theta'(t)\ , \tag{32}$$

which is independent of amplitude modulation M(t). (32) can also be interpreted as the instantaneous frequency at time t of waveform s(t).

The "time center at frequency f" follows in an analogous fashion as

$$\mu_t(f) = \frac{\int dt\ t\ W(t,f)}{\int dt\ W(t,f)} = -\frac{1}{2\pi}\ \frac{\mathrm{Im}\{S'(f)\ S^*(f)\}}{|S(f)|^2}\ . \tag{33}$$

If we represent the voltage density spectrum S(f) in terms of its magnitude and phase,

$$S(f) = A(f)\ \exp[-i\phi(f)]\ , \tag{34}$$

then (33) reduces to

$$\mu_t(f) = \frac{1}{2\pi}\,\phi'(f)\ , \tag{35}$$

which is independent of A(f). (35) can also be interpreted as the group delay at frequency f of waveform s(t).

The unconditional first moments of W are frequency center

$$\bar{f} = \frac{\iint dt\ df\ f\ W(t,f)}{\iint dt\ df\ W(t,f)} = \frac{\int df\ f\ |S(f)|^2}{\int df\ |S(f)|^2} = \frac{\int df\ f\ A^2(f)}{\int df\ A^2(f)} \tag{36}$$

and time center

$$\bar{t} \equiv \frac{\iint dt\ df\ t\ W(t,f)}{\iint dt\ df\ W(t,f)} = \frac{\int dt\ t\ |s(t)|^2}{\int dt\ |s(t)|^2} = \frac{\int dt\ t\ M^2(t)}{\int dt\ M^2(t)} \quad . \tag{37}$$

(36) follows directly from (11) and (34), while (37) follows directly from (12) and (31). Thus, $\bar{f}$ is independent of $\phi(f)$, and $\bar{t}$ is independent of $\Theta(t)$.

Alternative forms to (36) and (37), in the complementary domains, are available:

$$\bar{f} = \frac{1}{2\pi} \frac{\int dt\ \mathrm{Im}\{s'(t)\ s^*(t)\}}{\int dt\ |s(t)|^2} = \frac{1}{2\pi} \frac{\int dt\ M^2(t)\ \Theta'(t)}{\int dt\ M^2(t)} \quad , \tag{38}$$

and

$$\bar{t} = -\frac{1}{2\pi} \frac{\int df\ \mathrm{Im}\{S'(f)S^*(f)\}}{\int df\ |S(f)|^2} = \frac{1}{2\pi} \frac{\int df\ A^2(f)\ \phi'(f)}{\int df\ A^2(f)} \quad . \tag{39}$$

The result in (38) follows from the use of (28) and (12) in definition (36); a similar procedure yields (39). The frequency center $\bar{f}$ in (38) is an average of the instantaneous frequency $\mu_f(t)$ in (32), weighted according to the magnitude-squared waveform, $M^2(t)$; similarly, time center $\bar{t}$ in (39) is a weighted average of $\mu_t(f)$ in (35).

SECOND MOMENTS OF W

By taking two partial derivatives with respect to τ in (24), there follows

$$\int df\ f^2\ W(t,f) = \frac{1}{8\pi^2}\left[\left|s'(t)\right|^2 - Re\left\{s''(t)\ s^*(t)\right\}\right] . \tag{40}$$

When we then employ (12) and (31), the (conditional) second moment with respect to f develops into the form

$$\frac{\int df\ f^2\ W(t,f)}{\int df\ W(t,f)} = \frac{1}{8\pi^2}\left[\frac{M'^2(t)}{M^2(t)} - \frac{M''(t)}{M(t)}\right] + \left[\frac{\Theta'(t)}{2\pi}\right]^2 . \tag{41}$$

Therefore the instantaneous "mean-square frequency spread" is

$$\sigma_f^2(t) \equiv \frac{\int df\ [f - \mu_f(t)]^2\ W(t,f)}{\int df\ W(t,f)} =$$

$$= \frac{1}{8\pi^2}\left[\frac{M'^2(t)}{M^2(t)} - \frac{M''(t)}{M(t)}\right] = -\frac{1}{8\pi^2}\frac{d}{dt}\left\{\frac{M'(t)}{M(t)}\right\} , \tag{42}$$

where we employed (32) and (12). This result does not depend on phase modulation $\Theta(t)$. However, it should be observed that this quantity can be negative; consider the example $M(t) = \exp(-t^{\upsilon})$ for $t > 0$, with $0 < \upsilon < 1$. Thus (42) can not be interpreted as a true variance. This unfortunate feature of the WDF is due to the fact that W(t,f) can go negative for some values of t,f.

The unconditional second moment with respect to f follows from (40) and (11), respectively, as

$$\iint dt\, df\, f^2\, W(t,f) = \frac{1}{4\pi^2} \int dt\, \left|s'(t)\right|^2 = \int df\, f^2 \left|S(f)\right|^2 . \tag{43}$$

Analogous relations for the second moments with respect to t can also be derived via a similar approach.

MOMENTS OF W^2

The marginal integral of the square of the WDF with respect to f is, via (3), (13), and (2),

$$\int df\, W^2(t,f) = \int df \iint d\tau\, du\, \exp\left(-i2\pi f(\tau - u)\right)\, R(t,\tau)\, R^*(t,u) =$$

$$= \int d\tau \left|R(t,\tau)\right|^2 = \int d\tau \left|s(t + \tfrac{\tau}{2})\right|^2 \left|s(t - \tfrac{\tau}{2})\right|^2 =$$

$$= 2\left\{\left|s(\tau)\right|^2 \circledast \left|s(\tau)\right|^2\right\}_{\tau = 2t}, \tag{44}$$

which is the convolution of $|s|^2$ with itself, at argument 2t. The complementary result, integrating with respect to t, is

$$\int dt\, W^2(t,f) = \int dv\, \left|S(f + \tfrac{v}{2})\right|^2 \left|S(f - \tfrac{v}{2})\right|^2 =$$

$$= 2\left\{\left|S(v)\right|^2 \circledast \left|S(v)\right|^2\right\}_{v=2f} . \tag{45}$$

If we complete the integrations in (44) and (45) on the remaining variables, both yield the result

$$\iint dt\ df\ W^2(t,f) = E^2 \ ; \tag{46}$$

see (14). Also note, for comparison, that the double integral on W yielded E.

Although the results in (44) and (45) are not overly simple, continued integration does yield a surprisingly simple result; multiplying (44) by t, there follows

$$\iint dt\ df\ t\ W^2(t,f) = \int dt\ t\ 2 \int dx\ |s(x)|^2\ |s(2t - x)|^2 =$$

$$= \int dx\ |s(x)|^2\ 2 \int dt\ t\ |s(2t - x)|^2 = \int dx\ |s(x)|^2 \int dy\ |s(y)|^2 (y + x)/2 =$$

$$= \frac{1}{2} \int dx\ |s(x)|^2\ [\bar{t}E + xE] = \frac{1}{2}[\bar{t}\ E^2 + \bar{t}\ E^2] = E^2\ \bar{t}\ . \tag{47}$$

Here we used (44), (37), and (14). Thus

$$\frac{\iint dt\ df\ t\ W^2(t,f)}{\iint dt\ df\ W^2(t,f)} = \bar{t} = \frac{\int dt\ t\ |s(t)|^2}{\int dt\ |s(t)|^2} \tag{48}$$

from (47), (46), and (37). This result in (48) is the same as (37), but now for W^2 rather than W.

Conditional first moments of W^2 with respect to f and t are also deriveable; for example,

$$\int df\ f\ W^2(t,f) = \frac{1}{i2\pi} \int d\tau\ R^*(t,\tau)\ \frac{\partial}{\partial\tau} R(t,\tau) =$$

$$= \frac{1}{2\pi} \int d\tau\ \left|s(t + \tfrac{\tau}{2})\right|^2\ \mathrm{Im}\left\{s'(t - \tfrac{\tau}{2})\ s^*(t - \tfrac{\tau}{2})\right\} =$$

$$= \frac{1}{\pi} \int dx\ \left|s(2t - x)\right|^2\ \mathrm{Im}\left\{s'(x)\ s^*(x)\right\} =$$

$$= \frac{1}{\pi} \int dx\ M^2(2t - x)\ M^2(x)\ \Theta'(x)\ . \tag{49}$$

Here we used (3), (29), (2), and (31). When normalized by the quantity in (44), the result is considerably more complicated than the corresponding result for W in (30) and (32); nevertheless, continued integrations simplify tremendously. In particular, there follows, from (49), (14), (38), (46), and (36),

$$\frac{\iint dt\ df\ f\ W^2(t,f)}{\iint dt\ df\ W^2(t,f)} = \bar{f} = \frac{\int df\ f\ \left|S(f)\right|^2}{\int df\ \left|S(f)\right|^2}\ . \tag{50}$$

This is the dual relation to (48), but derived by means of a different approach. Comparison of (50) with (36) reveals the same result for W as for W^2.

CROSS WIGNER DISTRIBUTION FUNCTION

The cross WDF of two complex waveforms a(t) and b(t) is a generalization of (3) and (10) according to

$$W_{ab}(t,f) = \int d\tau \exp(-i2\pi f\tau)\, R_{ab}(t,\tau) =$$

$$= \int d\tau \exp(-i2\pi f\tau)\, a(t + \tfrac{\tau}{2})\, b^*(t - \tfrac{\tau}{2}) =$$

$$= \int d\upsilon \exp(i2\pi\upsilon t)\, A(f + \tfrac{\upsilon}{2})\, B^*(f - \tfrac{\upsilon}{2}) , \qquad (51)$$

which is generally complex. If a(t) and b(t) are nonzero only for $a_1 < t < a_2$ and $b_1 < t < b_2$, respectively, then the integral limits on τ in (51) are explicitly τ_1, τ_2, where

$$\tau_1 = 2 \max (a_1 - t,\ t - b_2) , \qquad \tau_2 = 2 \min (a_2 - t,\ t - b_1) .$$

If $\tau_1 \geq \tau_2$, then W_{ab} is zero.

The following properties of the cross WDF result immediately:

$$W_{a^*b^*}(t,f) = W^*_{ab}(t,-f) ,$$

$$W_{ba}(t,f) = W^*_{ab}(t,f) ,$$

$$\int df\, W_{ab}(t,f) = a(t)\, b^*(t) ,$$

$$\int dt\ W_{ab}(t,f) = A(f)\ B^*(f)\ ,$$

$$\iint dt\ df\ W_{ab}(t,f) = \int dt\ a(t)\ b^*(t) = \int df\ A(f)\ B^*(f)\ ,$$

$$\iint dt\ df\ \left|W_{ab}(t,f)\right|^2 = E_a\ E_b\ ,$$

$$\iint dt\ df\ W_{aa}(t,f)\ W_{bb}(t,f) = \left|\int dt\ a(t)\ b^*(t)\right|^2 = \left|\iint dt\ df\ W_{ab}(t,f)\right|^2\ ,$$

$$\iint dt\ df\ W_{ab}(t,f)\ W^*_{cd}(t,f) = \int dt\ a(t)\ c^*(t) * \int dt\ b^*(t)\ d(t)\ . \qquad (52)$$

The last three relations follow upon substitution of (51), interchanging integrals, and the use of (13). Again, the double Fourier transform of the cross WDF yields the cross ambiguity function:

$$\iint dt\ df\ \exp(-i2\pi\nu t + i2\pi f\tau)\ W_{ab}(t,f) =$$

$$= \int dt\ \exp(-i2\pi\nu t)\ a(t + \tfrac{\tau}{2})\ b^*(t - \tfrac{\tau}{2}) =$$

$$= \int df\ \exp(i2\pi f\tau)\ A(f + \tfrac{\nu}{2})\ B^*(f - \tfrac{\nu}{2}) =$$

$$= \chi_{ab}(\nu,\tau) = \int dt\ \exp(-i2\pi\nu t)\ R_{ab}(t,\tau)\ . \qquad (53)$$

The magnitude-squared cross-WDF of waveforms a and b is also related to the auto-WDFs by means of a double convolution:

$$\left|W_{ab}(t,f)\right|^2 = \iint d\tau_1\, d\tau_2\, \exp\left(-i2\pi f(\tau_1 - \tau_2)\right) *$$

$$* R_{aa}\left(t + \frac{\tau_1 + \tau_2}{4}\,,\, \frac{\tau_1 - \tau_2}{2}\right) R_{bb}\left(t - \frac{\tau_1 + \tau_2}{4}\,,\, \frac{\tau_1 - \tau_2}{2}\right) =$$

$$= \iint d\tau'\, d\tau\, \exp(-i2\pi f\tau')\, R_{aa}\left(t + \frac{\tau}{2}\,,\, \frac{\tau'}{2}\right)\, R_{bb}\left(t - \frac{\tau}{2}\,,\, \frac{\tau'}{2}\right) =$$

$$= \iint d\tau'\, d\tau\, \exp(-i2\pi f\tau')\, R_{aa}\left(t + \frac{\tau}{2}\,,\, \frac{\tau'}{2}\right) \int df'\, \exp\left(i2\pi f'\, \frac{\tau'}{2}\right) W_{bb}\left(t - \frac{\tau}{2}\,,\, f'\right) =$$

$$= \iint d\tau\, df'\, W_{bb}\left(t - \frac{\tau}{2}\,, f'\right) \int d\tau'\, \exp\left(-i2\pi(2f - f')\, \frac{\tau'}{2}\right) R_{aa}\left(t + \frac{\tau}{2}\,,\, \frac{\tau'}{2}\right) =$$

$$= 2 \iint d\tau\, df'\, W_{bb}\left(t - \frac{\tau}{2}\,, f'\right)\, W_{aa}\left(t + \frac{\tau}{2}\,, 2f - f'\right) =$$

$$= \iint d\tau\, d\upsilon\, W_{aa}\left(t + \frac{\tau}{2}\,,\, f + \frac{\upsilon}{2}\right) W_{bb}\left(t - \frac{\tau}{2}\,,\, f - \frac{\upsilon}{2}\right), \qquad (54)$$

where we let $\tau = (\tau_1 + \tau_2)/2$, $\tau' = \tau_1 - \tau_2$ in the third line.

NARROWBAND REAL WAVEFORM

When waveform s(t) is narrowband and real, it can be expressed in terms of its low-pass complex envelope c(t) according to

$$s(t) = 2\,\mathrm{Re}\{c(t)\,\exp(i2\pi f_o t)\} =$$

$$= c(t)\,\exp(i2\pi f_o t) + c^*(t)\,\exp(-i2\pi f_o t)\ ,$$

where f_o is the carrier or center frequency of s(t). The WDF of s(t) is then expressible as

$$W_{ss}(t,f) = \int d\tau\,\exp(-i2\pi f\tau)\,s(t+\tfrac{\tau}{2})\,s^*(t-\tfrac{\tau}{2}) =$$

$$= W_{cc}(t,f-f_o) + W_{cc}(t,-f-f_o) +$$

$$+ 2\,\mathrm{Re}\{W_{cc^*}(t,f)\,\exp(i4\pi f_o t)\}\ . \qquad (55)$$

Here, we substituted for s(t), and used (51) and (52). Since complex envelope c(t) is low-pass, a representative contour plot of (55) appears as shown in figure 1. The wiggly lobe centered at f = 0 is subject to rapid oscillations in t, whereas those lobes centered at $\pm f_o$ are slowly varying with f and t. A small amount of averaging in time would wipe out the undesired oscillating lobe, but maintain the desired components at $f = \pm f_o$.

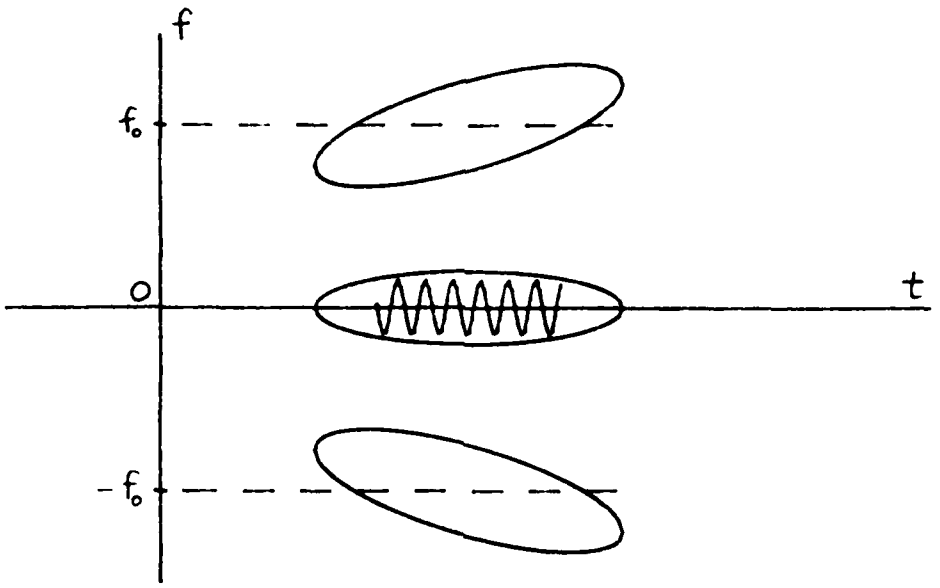


Figure 1. WDF for Narrowband Real Waveform

SAMPLING PROPERTIES

By letting $u = \tau/2$ in (3), the WDF becomes

$$W(t,f) = 2 \int du \exp(-i4\pi fu)\, s(t + u)\, s^*(t - u) , \qquad (56)$$

where we again now allow general complex s(t). If this integral is to be evaluated numerically on a computer, we will need to sample the integrand at some increment Δ_t, and apply some integration rule. In particular, if we use the Trapezoidal rule and carry out the summation over $-\infty$, $+\infty$, we have approximation

$$\tilde{W}(t,f) \cong 2\Delta_t \sum_k \exp(-i4\pi fk\Delta_t)\, s(t + k\Delta_t)\, s^*(t - k\Delta_t) \qquad (57)$$

for all t,f. Since it is immediately seen from (57) that

$$\tilde{W}\left(t,f + \frac{1}{2\Delta_t}\right) = \tilde{W}(t,f) \ , \tag{58}$$

it follows that $\tilde{W}(t,f)$ has period $(2\Delta_t)^{-1}$ in f, when waveform s(t) is sampled at increment Δ_t. In fact, it can be shown that $\tilde{W}$ is the aliased version of W:

$$\tilde{W}(t,f) = \sum_n W(t,f - \frac{n}{2\Delta}) \ . \tag{59}$$

Thus, $\tilde{W}(t,f)$ need be evaluated only over one period, say $(0,.5/\Delta_t)$.

Since (57) cannot be evaluated for all continuous values of t and f, we will limit its evaluation to

$$t = m\Delta_t \ , \ f = \frac{n}{2N_f\Delta_t} \ , \tag{60}$$

where m, n, N_f are integers. Then (57) becomes (exactly)

$$\tilde{W}\left(m\Delta_t, \frac{n}{2N_f\Delta_t}\right) = 2\Delta_t \sum_k \exp(-i2\pi nk/N_f)\ s\left((m + k)\Delta_t\right)\ s^*\left((m - k)\Delta_t\right) , \tag{61}$$

the right side of which is recognized as an N_f-point discrete Fourier transform. If the number of nonzero samples in k is greater than N_f, we simply collapse them mod N_f, without loss of accuracy; see [5; page 7]. Since the period of $\tilde{W}(t,f)$ is $(2\Delta_t)^{-1}$ in f, we only need consider $0 \le n \le N_f - 1$, that is, $0 \le f < (2\Delta_t)^{-1}$. Values of m must be considered wherever the summand of (61) is nonzero.

A plot of two of the infinite number of lobes of $\tilde{W}(t,f)$ in (59) is depicted in figure 2 for a representative bandpass analytic waveform s(t).

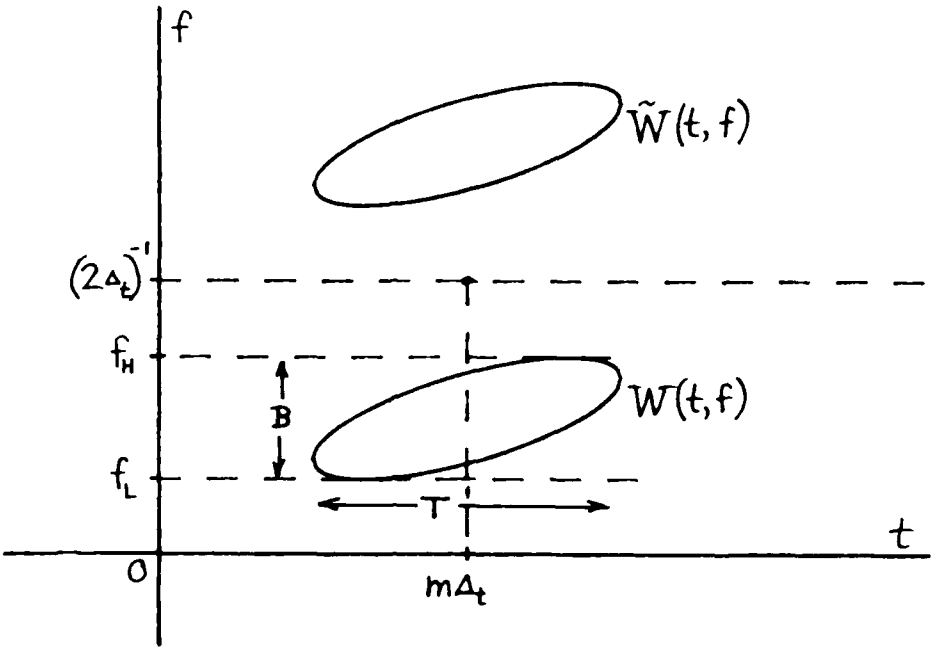


Figure 2. WDF for Sampled Version $\tilde{W}(t,f)$

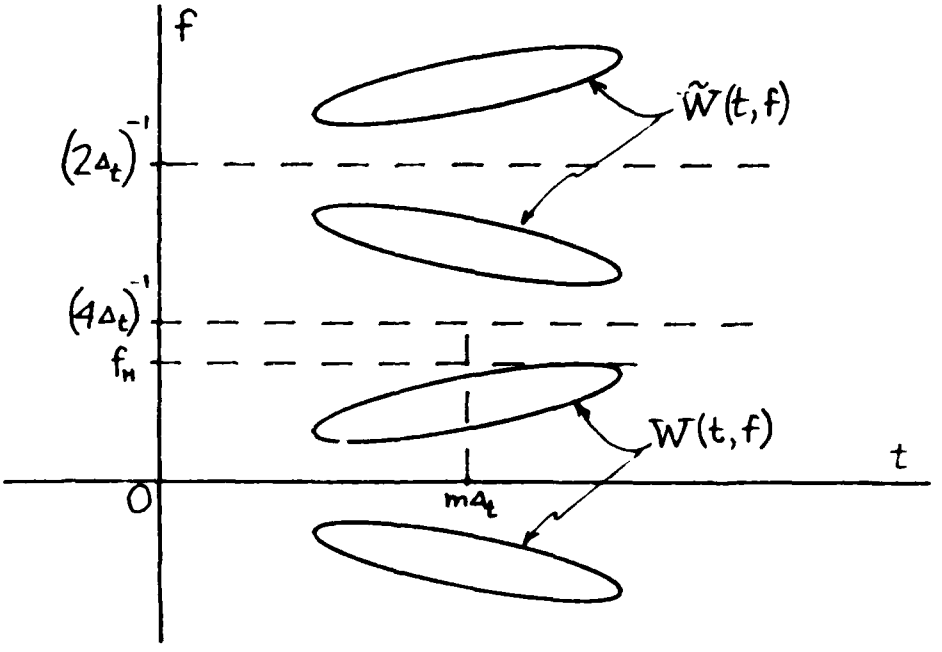


Figure 3. $\tilde{W}(t,f)$ for Real Waveform s(t)

The spreads of the desired WDF term $W(t,f)$ are T and B in time and frequency, respectively. In order to guarantee that aliasing is insignificant in figure 2, we must choose

$$(2\Delta_t)^{-1} > B, \quad \text{that is, } \Delta_t < (2B)^{-1} . \tag{62}$$

For N_f equal to a power of 2 in (61), an FFT can be employed to evaluate $\tilde{W}$ and will give the vertical slice in f indicated in figure 2 between $f = 0$ and $f = (2\Delta_t)^{-1}$, for the particular m value under consideration. Since the spacing of frequency values in (61) is $(2N_f\Delta_t)^{-1}$, then in order to keep track of the wiggles in $\tilde{W}(t,f)$ as a function of f, we must choose

$$(N_f\Delta_t)^{-1} < T^{-1}, \quad \text{that is, } N_f > T/\Delta_t > 2BT. \tag{63}$$

Thus the FFT size may have to be quite large for an extended WDF in t,f space.

If s(t) is real, then (8) applies, meaning that $\tilde{W}$ in (61) need only be computed for

$$0 \le n \le N_f/2, \quad \text{that is, } 0 \le f \le (4\Delta_t)^{-1} . \tag{64}$$

The pertinent approximate WDF $\tilde{W}$ is depicted in figure 3. In order to avoid aliasing now, we must have

$$(4\Delta_t)^{-1} > f_H, \quad \text{that is, } \Delta_t < (4f_H)^{-1} , \tag{65}$$

where f_H is the highest frequency contained in s(t). This sampling rate is twice as fast as the usual Nyquist rate for waveform s(t), and is due to the unavoidable factors of 1/2 in definition (3).

The procedure described above, in (61) et seq., realizes a slice in f, at fixed t, of the WDF; see figures 2 and 3. An alternative procedure for obtaining slices of the WDF in t, at fixed f, is described in appendix A; however, starting with time samples of s(t), it requires an additional large-size FFT to start the calculations.

EXAMPLES OF WDF

In this section, we present several examples of the WDF for waveforms that are likely to be encountered in practice, and that are amenable to simple closed form solution. A significant shortcut in the presentation is possible when it is observed from (3) that if

$$r(t) = s(t - t_o) \exp(i2\pi f_o t + i\theta_o) , \tag{66}$$

which corresponds to a time delay and frequency shift, then

$$W_r(t,f) = W_s(t - t_o, f - f_o) . \tag{67}$$

Thus we can choose any convenient origin for the waveform s in time and frequency, without loss of generality, and then merely shift the WDF according to (67), as appropriate.

We will place heavy emphasis here on combinations of Gaussian pulses, both because of their analytic tractability and due to the fact that any waveform can be expanded into elementary waveforms consisting of Gaussian wavelets; see, for example, Gabor's original paper [6, part 1, section 5].

In the following, frequent use will be made of the following integral:

$$\int dx \exp(-\tfrac{1}{2}\alpha x^2 \pm \beta x) = \left(\frac{2\pi}{\alpha}\right)^{1/2} \exp\left(\frac{\beta^2}{2\alpha}\right) \quad \text{for } \alpha_r > 0 , \tag{68}$$

where α and β can be complex, with components

$$\alpha = \alpha_r + i\alpha_i , \qquad \beta = \beta_r + i\beta_i . \tag{69}$$

Also, as a special case, there follows

$$\left| \int dx \, \exp\left(-\frac{1}{2}\alpha x^2 \pm \beta x \right) \right|^2 = \frac{2\pi}{(\alpha_r^2 + \alpha_i^2)^{1/2}} \exp\left[\frac{\alpha_r(\beta_r^2 - \beta_i^2) + 2\alpha_i\beta_r\beta_i}{\alpha_r^2 + \alpha_i^2} \right] , \tag{70}$$

written out in terms of purely real quantities.

GAUSSIAN WAVEFORM

Let waveform

$$s(t) = a_0 \exp\left(-\frac{t^2}{2\sigma_0^2} \right) ; \quad a_0 \text{ complex} . \tag{71}$$

(Parameters will be real unless indicated otherwise.) Use of (3) and (68) yields WDF

$$W(t,f) = 2E \exp\left[-\frac{t^2}{\sigma_0^2} - (2\pi f\sigma_0)^2 \right] , \tag{72}$$

where E is the waveform energy:

$$E = \sqrt{\pi} |a_0|^2 \sigma_0 . \tag{73}$$

The WDF consists of a single positive lobe in t,f space, centered at the origin.

Observe that W(0,0) is equal to 2E for this example; in fact, from (3),

$$W(0,0) = \int d\tau\ s(\tau/2)\ s^{*}(-\tau/2) = 2E \quad \text{if } s(-t) = s(t)\ . \tag{14}$$

Thus waveforms s(t) with this even symmetry result in peak WDF values of 2E at the origin. However, if s has odd symmetry about zero, s(-t) = -s(t), then W(0,0) = -2E.

The contours of equal height of the WDF in (72) are ellipses. The contour for the case where the levels are down to exp(-1) of their peak value is the ellipse indicated in figure 4. The area of this particular level ellipse is 1/2 in the t,f plane. When this area is multiplied by the peak height of 2E, the product is E, which is just the volume under the WDF; see (14). Thus the "effective extent" of the WDF in (72) is that given in figure 4, for relative level 1/e of the peak.

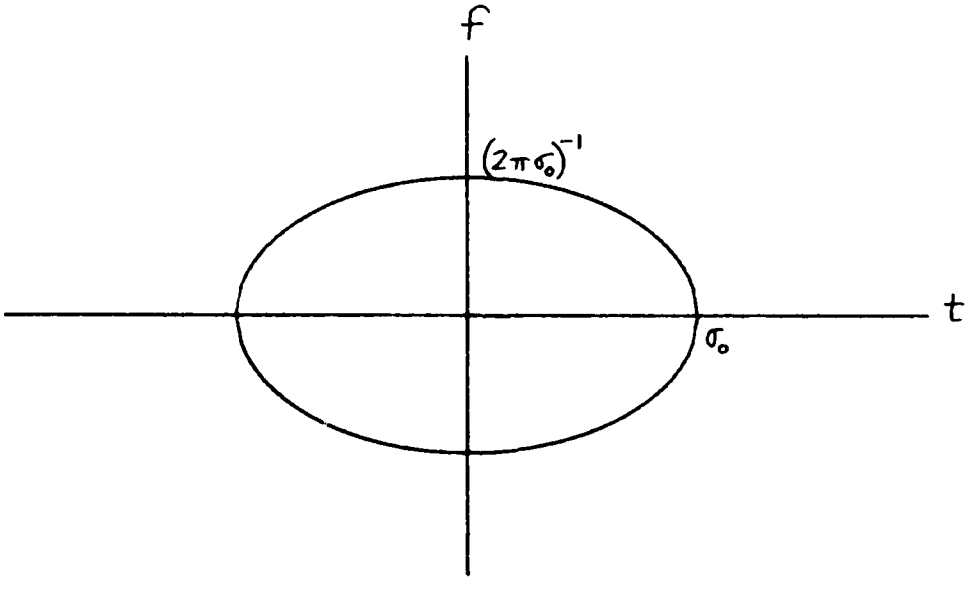


Figure 4. Contour of WDF (72) at 1/e Relative Level

GAUSSIAN-MODULATED TONE

$$s(t) = b_0 \cos(2\pi f_0 t + \phi_0) \exp\left(-\frac{t^2}{2\sigma_0^2}\right). \tag{75}$$

The energy of this waveform is

$$E = \frac{1}{2}\sqrt{\pi}\, b_0^2\, \sigma_0 [1 + \cos(2\phi_0)\exp(-y_0^2)] , \tag{76}$$

and its WDF is

$$\left(\frac{1}{2}\sqrt{\pi}\, b_0^2\, \sigma_0\right)^{-1} W(t,f) = \exp\left[-x^2 - (y - y_0)^2\right] + \exp\left[-x^2 - (y + y_0)^2\right] +$$

$$+ 2\cos(4\pi f_0 t + 2\phi_0)\exp[-x^2 - y^2] , \tag{77}$$

where dimensionless variables

$$x = t/\sigma_0 , \quad y = 2\pi f \sigma_0 , \quad y_0 = 2\pi f_0 \sigma_0 . \tag{78}$$

There are two positive lobes centered at $(t,f) = (0,f_0)$ and $(0,-f_0)$, each of peak height approximately E (if y_0 is large). The contours of each of these lobes are circles in the x,y plane, or ellipses in the t,f plane, as indicated in figure 4. There is also an oscillating lobe centered at the origin; this is an example of the general situation depicted in figure 1.

It should also be observed from (77) that if a slice in frequency is taken of the WDF, at fixed time t, that there is <u>no</u> fast oscillation in any of the three lobes. Whatever value of the cos is encountered, that value is maintained, and the only variation with y (frequency) is the Gaussian dependence. Thus if we locally averaged the WDF with respect to frequency alone, that would <u>not</u> eliminate the undesired oscillating lobe centered at (0,0).

MULTIPLE MODULATED TONES

Consider complex waveform

$$s(t) = \sum_k a_k \exp\left[i2\pi f_k t - \frac{(t - t_k)^2}{2\sigma_k^2}\right]; \quad \{a_k\} \text{ complex} . \tag{79}$$

This is a collection of tone bursts centered at $\{t_k, f_k\}$ in the t,f plane, with energy $E_k = \sqrt{\pi}|a_k|^2 \sigma_k$. The corresponding WDF follows from (3) and (68) as

$$W(t,f) = 2\sum_k E_k \exp\left[-\frac{(t - t_k)^2}{\sigma_k^2} - \sigma_k^2 4\pi^2 (f - f_k)^2\right] +$$

$$+ 4\sqrt{\pi} \sum_{k<\ell} \tilde{\sigma}_{k\ell} \exp\left[-\frac{(t - t_{k\ell})^2}{\bar{\sigma}_{k\ell}^2} - \tilde{\sigma}_{k\ell}^2 4\pi^2 (f - f_{k\ell})^2\right] *$$

$$* \operatorname{Re}\left\{a_k a_\ell^* \exp\left[i2\pi(f_k - f_\ell)t + i\tilde{\sigma}_{k\ell}^2 2\pi(f - f_{k\ell})\left(\frac{t - t_k}{\sigma_k^2} - \frac{t - t_\ell}{\sigma_\ell^2}\right)\right]\right\}, \tag{80}$$

where

$$t_{k\ell} = \frac{1}{2}(t_k + t_\ell), \quad f_{k\ell} = \frac{1}{2}(f_k + f_\ell),$$

$$\bar{\sigma}_{k\ell}^2 = \frac{1}{2}(\sigma_k^2 + \sigma_\ell^2), \quad \frac{1}{\tilde{\sigma}_{k\ell}^2} = \frac{1}{2}\left(\frac{1}{\sigma_k^2} + \frac{1}{\sigma_\ell^2}\right). \tag{81}$$

The first line of (80) represents the desired positive lobes centered at (t_k, f_k), each scaled according to its energy. The remaining undesired lobes are centered at

$$\frac{t_k + t_\ell}{2} \, , \, \frac{f_k + f_\ell}{2} \quad \text{for all } k \neq \ell \, , \tag{82}$$

and oscillate with t and/or f. These locations in (82) are halfway between <u>every</u> possible pair of desired lobes; their amplitudes are proportional to the geometric means of the corresponding interacting lobes, and therefore constitute significant interference effects to interpretation of the computed WDF. Furthermore, the locations in (82) can occur in time where the waveform s(t) is zero, and/or in frequency where the spectrum S(f) is zero. This most undesirable feature of the WDF has been reported previously in [7,8]. The only saving feature, that should allow salvaging the WDF, is that the undesired lobes, $k < \ell$ in (80), oscillate positive-and-negative and can be averaged out by smoothing the WDF. Of course, via this smoothing procedure, the desired lobes will also be smeared somewhat, but this trade-off appears to be required in order to make a meaningful, useful interpretation of the WDF at all points of the t,f plane.

The envelope of the k,ℓ lobe in (80) is proportional to an exponential of an elliptical function. When this exponential has decreased to 1/e of its peak value, the corresponding elliptical contour has area

$$\frac{\pi\bar{\sigma}_{k\ell}}{2\pi\tilde{\sigma}_{k\ell}} = \frac{1}{4}\left(\frac{\sigma_k}{\sigma_\ell} + \frac{\sigma_\ell}{\sigma_k}\right) \geq \frac{1}{2} \tag{83}$$

in the t,f plane, the latter value of 1/2 being the area of every desired lobe. Thus the undesired oscillating lobes are smeared out more than the desired positive lobes.

If we restrict (79) to two equal-duration bursts with the same time center, but different center frequencies, the undesired lobe oscillates only with t, not f. This is similar to example (75)-(77). On the other hand, if (74) is restricted to two equal-duration bursts with different time centers, but the same frequency center, the undesired lobe oscillates only with f, not t.

More generally, for two equal-duration bursts with different time and frequency centers, the undesired lobe has no fast oscillation along lines in the t,f plane which are parallel to the line joining the centers of the two positive lobes in the WDF. For two unequal-duration bursts, the situation is more complicated, and there is generally oscillation along all straight lines in the t,f plane.

What these simple examples demonstrate is that if we want to locally smooth (average) the WDF, in an effort to wipe out the undesired oscillating cross-terms, that smoothing must be applied in both t and f, not either one alone. Of course, such smoothing will also tend to smear the desired positive lobes; thus the minimum amount of smoothing to guarantee a nonnegative WDF is of interest.

Although these conclusions have been drawn from the particular example of Gaussian-modulated tone bursts in (79) (for analytic simplicity), they hold generally. Appendix B demonstrates the oscillating character of the interacting cross-terms of the WDF for a waveform with two separated energy bursts in time of general shape.

The ambiguity function of waveform s(t) in (75) is considered in appendix C. It has some similar properties to the WDF and some significant differences, which make it much less desirable as a descriptor of a signal's concentration in time-frequency space.

LINEAR FREQUENCY MODULATION

Here, we consider waveform

$$s(t) = a_o \exp\left[- \frac{t^2}{2\sigma_o^2} + i \frac{\alpha_o}{2} t^2 \right] ; \quad \alpha_o \geq 0 , \quad a_o \text{ complex} . \tag{84}$$

The instantaneous frequency, according to (31) and (32), is a linear function of time,

$$\nu_f(t) = \frac{\alpha_o}{2\pi} t , \tag{85}$$

while the envelope is Gaussian. When

$$t = t_o = \pm \sigma_o \sqrt{\pi/2} , \tag{86}$$

the magnitude of the waveform s(t) is

$$|s(t_o)| = |a_o| \exp(-\pi/4) = .456 |s(0)| . \tag{87}$$

If we define the duration, Δt, of $s(t)$ as the time between these function values, then

$$\Delta t = \sigma_0\sqrt{2\pi} . \tag{88}$$

During this time interval, the instantaneous frequency in (85) sweeps through a bandwidth

$$\Delta f = \frac{\alpha_0}{2\pi}\,\Delta t = \alpha_0\sigma_0\sqrt{2\pi} . \tag{89}$$

The time-bandwidth product of waveform $s(t)$ is therefore

$$\Delta t\,\Delta f = \alpha_0\sigma_0^2 \equiv \Theta_0 \quad (\geq 0) , \tag{90}$$

when the time duration is defined as the interval between the function values in (87). This quantity, Θ_0, is an important parameter of the linear frequency modulation waveform (84).

The WDF of (84) follows, upon use of (3) and (68), as

$$W(t,f) = 2E\,\exp\left[-\frac{t^2}{\sigma_0^2} - \sigma_0^2(2\pi f - \alpha_0 t)^2\right] =$$

$$= 2E\,\exp\left[-x^2 - (y - x\Theta_0)^2\right] =$$

$$= 2E\,\exp\left[-x^2(1 + \Theta_0^2) + 2xy\,\Theta_0 - y^2\right] , \tag{91}$$

where we employed (78) and (90). This is an everywhere-positive lobe centered at the origin of time-frequency space, with contours that are tilted ellipses. The peak value, $2E$, is independent of the amount of frequency modulation.

For a given value of time t, the frequency f that maximizes the WDF in (91) is

$$f = \frac{\alpha_0}{2\pi} t \ , \quad \text{that is, } y = x\Theta_0 \ , \tag{92}$$

which is just the instantaneous frequency in (85). However, this line, (92), in the t,f plane is <u>not</u> the major axis of the elliptical contours of the WDF. A similar observation regarding the ambiguity function $\chi(\nu,\tau)$, (23), of the linear frequency modulation waveform, namely

$$\chi(\nu,\tau) = E \exp\left[-\frac{1}{4}\left\{x'^2(1+\Theta_0^2) - 2x'y'\Theta_0 + y'^2\right\}\right] , \tag{93}$$

where

$$x' = \frac{\tau}{\sigma_0} \ , \qquad y' = 2\pi\nu\sigma_0 \ , \tag{94}$$

has been made in [4; page 124].

What this means is that, if the WDF of a waveform is evaluated numerically from a given data sequence (via (61) for example), then the tilt of the major axis of the contours of the computed WDF is not <u>directly</u> the amount of linear frequency modulation in the waveform. Rather, the major axis of the ellipse in (91) lies along the line

$$y = x \tan\psi \tag{95}$$

in the x,y plane, where

$$\tan\psi = \frac{\sqrt{\Theta_0^2 + 4} + \Theta_0}{2} \geq \Theta_0 \ . \tag{96}$$

(See appendix D for detailed derivations on the rotation of coordinate axes.) Thus, the major axis (95) of the ellipse is more tilted than the instantaneous frequency line (92). Equation (96) can be inverted and solved for the linear frequency modulation parameter Θ_o according to

$$\Theta_o = \tan\psi - 1/\tan\psi \ , \tag{97}$$

in terms of the measured or calculated major axis tilt, $\tan\psi$, in the x,y plane. The detailed procedure for solving for both σ_o and α_o, in terms of a computed WDF in the t,f plane, is discussed in the example in appendix D, especially (D-28) and (D-29).

When the exponential in (91) is down to 1/e of its peak value, the ellipse at that level has area π in the x,y plane. This may be seen by use of (D-1) and (D-20), with $A = 1 + \Theta_o^2$, $B = -2\Theta_o$, $C = 1$, $D = E = 0$, $F = -1$, for which $G = 1$ via (D-19). This corresponds to area 1/2 in the t,f plane, as seen by (78). Therefore, the peak height, 2E, times the "effective" area is again E, as it was for the simple Gaussian pulse of (71) and figure 4. Thus, although the volume of the WDF in (91) has been redistributed in the t,f plane, by virtue of linear frequency modulation, the effective area is maintained, although now located as a tilted ellipse.

A plot of the ellipse of (91) at the 1/e level, namely

$$x^2 + (y - x\Theta_o)^2 = x^2(1 + \Theta_o^2) - 2xy\Theta_o + y^2 = 1 \ , \tag{98}$$

is given in figure 5, when $\Theta_o = 1.5$. The instantaneous frequency line (92) as well as the major axis (95) are delineated, and are clearly seen not to overlie each other.

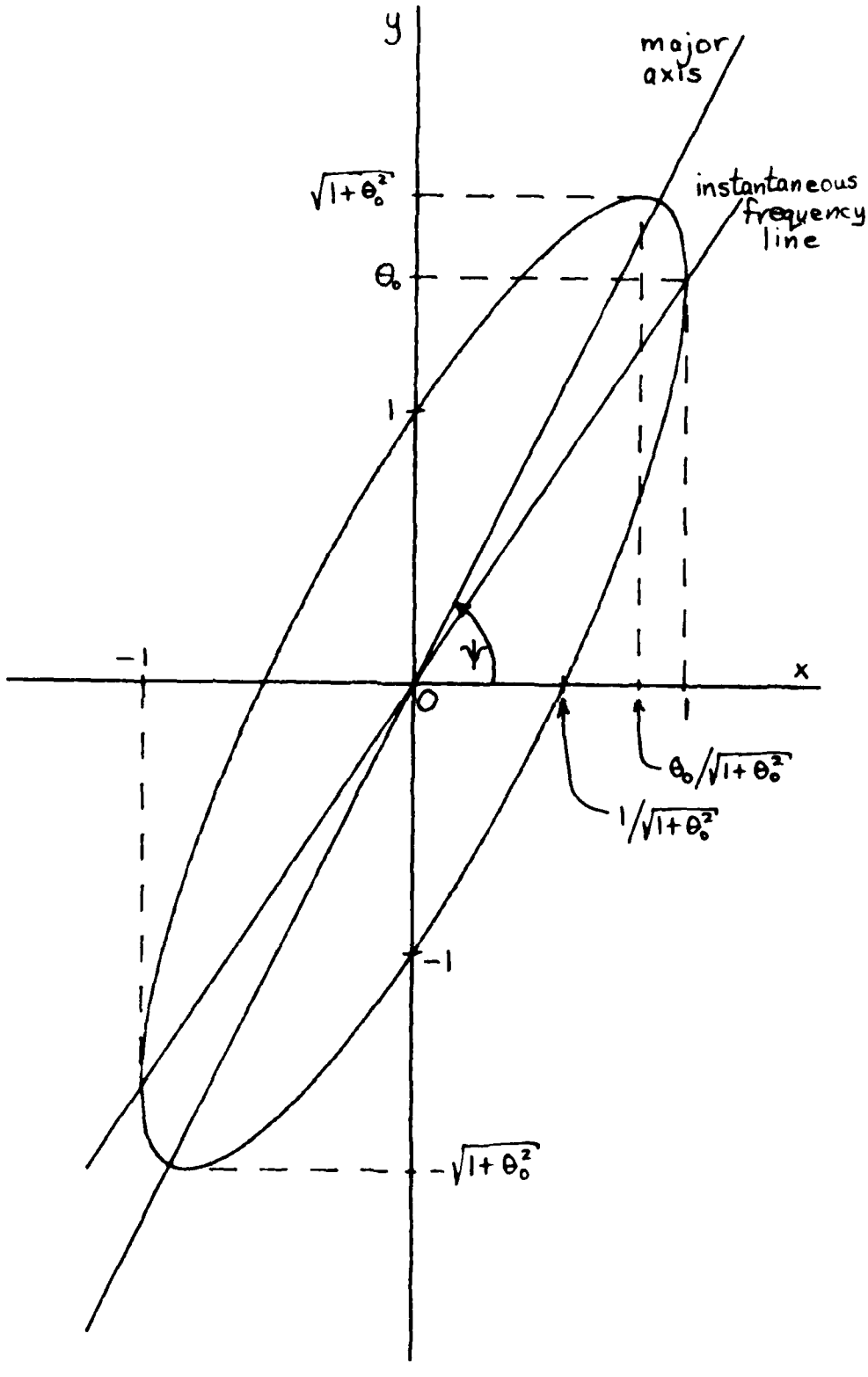


Figure 5. Contour of (91) for $\theta_0 = 1.5$

GATED LINEAR FREQUENCY MODULATION

All the previous examples in this section had Gaussian envelopes. We now consider a rectangularly gated waveform with linear frequency modulation:

$$s(t) = a_0 \exp\left[i \frac{\alpha_0}{2} t^2\right] \quad \text{for } |t| < \frac{T}{2}; \quad a_0 \text{ complex}. \tag{99}$$

Equation (3) yields directly WDF

$$W(t,f) = 2E \frac{\sin[(2\pi f - \alpha_0 t)(T - 2|t|)]}{(2\pi f - \alpha_0 t)T} \quad \text{for } |t| < \frac{T}{2}, \text{ all } f, \tag{100}$$

and zero otherwise. Along the instantaneous frequency line, (85), the WDF is $2E(1 - 2|t|/T)$ for $|t| < T/2$, which is nonoscillatory and positive. However, in other portions of the t,f plane, (100) does go negative, due to the sin term.

For a given value of t, the quantity W in (100) is maximized by choosing $f = \alpha_0 t/(2\pi)$, but, again, this is not the major axis of the contours of the WDF. In figure 6, these contours are plotted for $\alpha_0 T^2 = 1$ and $\alpha_0 T^2 = 10$. In fact, the contours are no longer ellipses, although they tend to resemble ellipses near the origin, when frequency modulation parameter $\alpha_0 T^2$ is large; see the bottom figure, where the instantaneous frequency line and the mountain ridge (curve of slowest descent) have been sketched.

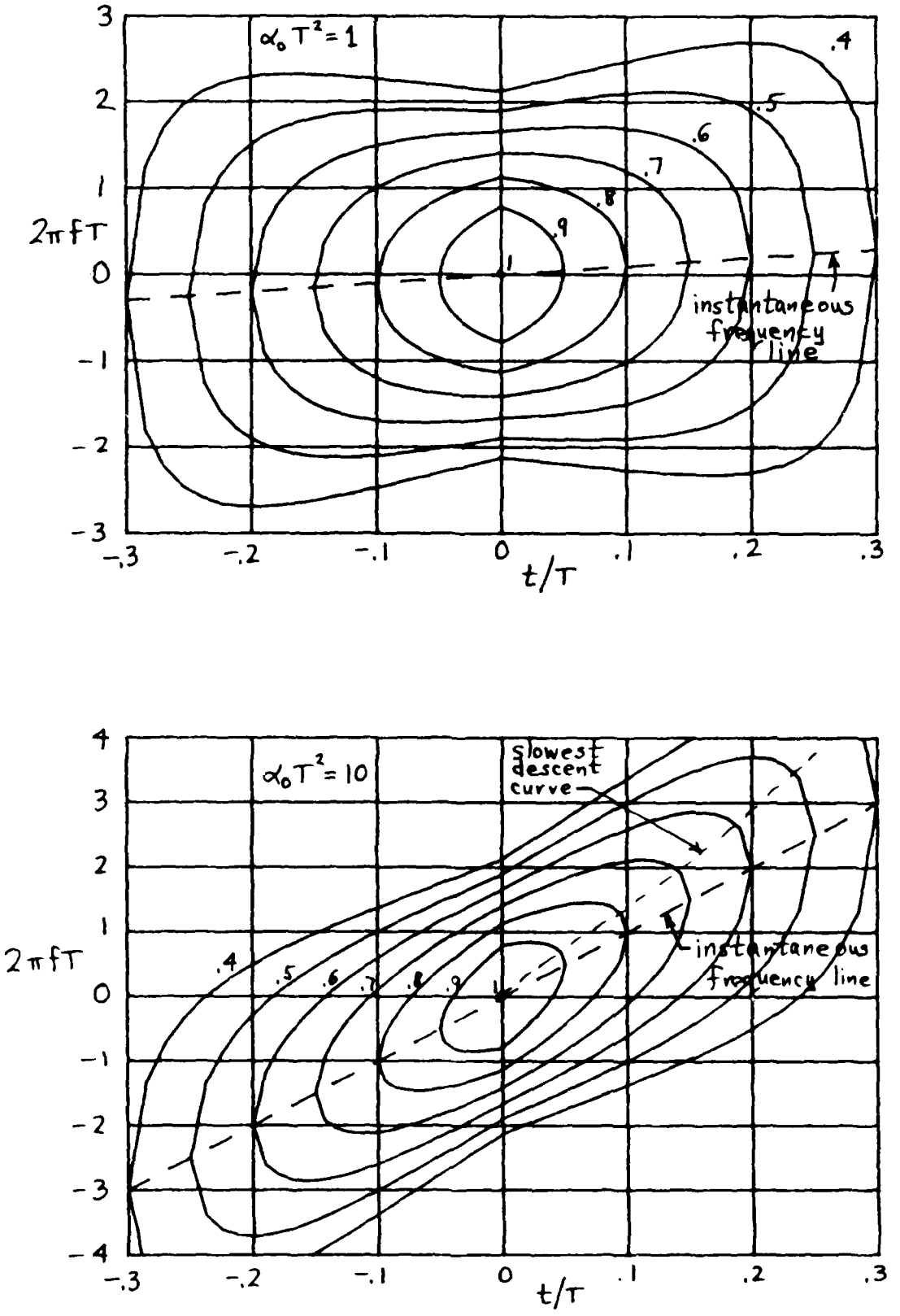


Figure 6. Contours of WDF for Gated LFM

The ambiguity function of waveform (99) is

$$\chi(\nu,\tau) = E\,\frac{\sin[\frac{1}{2}(2\pi\upsilon - \alpha_0\tau)(T - |\tau|)]}{\frac{1}{2}(2\pi\upsilon - \alpha_0\tau)T} \qquad \text{for } |\tau| < T\,, \text{ all } \upsilon\,, \qquad (101)$$

and zero otherwise. It is similar to the WDF in (100), but is spread out more in the υ,τ plane.

SHORT-TERM SPECTRAL ESTIMATION

Some advantageous features of the WDF have been brought out by earlier examples, such as the concentrated positive lobes in the t,f plane about locations corresponding to obvious bursts of energy. However, the WDF also goes negative in surrounding regions, causing difficulty in interpretation; see figure 1, (80), appendix B, or [7,8]. What is needed is some form of smoothing of the WDF so as to eliminate or suppress the oscillating components; however, this averaging must be two-dimensional, carried out in both time and frequency, for the reasons presented in the sequel to (83). We now present one method of smoothing the WDF, which guarantees a non-negative distribution in time-frequency.

WEIGHTED SPECTRAL ESTIMATE

The voltage density spectrum S(f), corresponding to waveform s(t), was defined in (9) as the Fourier transform over all time. In order to bring out properties which are local in time, a weighting must be applied before transformation. In particular, we generalize (9) to

$$S_u(t,f) = \int dt_1 \exp(-i2\pi f t_1)\, s(t_1)\, u^*(t - t_1) =$$

$$= \exp(-i2\pi f t) \int df_1 \exp(i2\pi t f_1)\, S(f_1)\, U^*(f - f_1) , \qquad (102)$$

where weighting u will tend to be a narrow function centered about its origin; thus the weighting in (102) will accent the behavior of waveform s in the neighborhood of time t. The function U is the Fourier transform of u. The short-term power spectral estimate (at time t and frequency f) of waveform s, relative to weighting u, is then defined as

$$\left|S_u(t,f)\right|^2 . \tag{103}$$

See also [2, p. 768].

The following symmetry properties of definition (102) follow:

$$S_u(t,f) = U_s^*(t,f)\ \exp(-i2\pi ft) ,$$

$$\left|S_u(t,f)\right|^2 = \left|U_s(t,f)\right|^2 , \tag{104}$$

where U_s is the spectrum of waveform u relative to weighting s. That is, U_s is the dual of S_u. Also, by use of (53), we can express

$$S_u(t,f) = \chi_{s\bar{u}}(f,t)\ \exp(-i\pi ft) , \tag{105}$$

in terms of the complex cross-ambiguity function of s and $\bar{u}$, where $\bar{u}$ is the mirror image of u: $\bar{u}(t) = u(-t)$. Also, the same shifting property, given in (66) and (67) for the WDF, holds as well for quantity (103).

RELATION TO WDFs

There is a very important relation between the short-term spectral estimate (STSE), (103), and the WDFs of s and u; namely, by use of (102), (2), and (3), we have

$$\left|S_u(t,f)\right|^2 = \iint dt_1\ dt_2\ \exp\left(-i2\pi f(t_1 - t_2)\right) s(t_1)s^*(t_2)u^*(t - t_1)u(t - t_2) =$$

$$= \iint d\tau\ dt'\ \exp(-i2\pi f\tau)\ s\left(t' + \frac{\tau}{2}\right) s^*\left(t' - \frac{\tau}{2}\right) u\left(t - t' + \frac{\tau}{2}\right) u^*\left(t - t' - \frac{\tau}{2}\right) =$$

$$= \iint d\tau\ dt'\ \exp(-i2\pi f\tau)\ R_s(t',\tau)\ R_u(t - t',\tau) =$$

$$= \iint dt'\ df'\ W_s(t',f')\ W_u(t - t',f - f') =$$

$$= W_s(t,f) \overset{tf}{\circledast} W_u(t,f)\ . \qquad (106)$$

This relation states that the STSE is a double convolution, in both t and f, of the WDFs of waveform s and weighting u. That is, the STSE $\left|S_u(t,f)\right|^2$ of waveform s, relative to weighting u, is a smoothed version of the WDF of waveform s, where the smoothing function is the WDF of weighting u. Furthermore, since the left-hand side of (106) can never be negative, and since s and u are arbitrary, (106) shows that the double convolution of <u>any</u> two WDFs is never negative for any values of t,f. This furnishes a possibility of accomplishing smoothing of a computed WDF of waveform s, with a guarantee of a nonnegative distribution resulting; of course, W_u must be a legal WDF, as discussed in (25) et seq., in order to guarantee this nonnegative property.

Since $\left|S_u\right|^2$ is a double convolution of WDFs W_s and W_u, it follows that the double Fourier transform of the STSE is given by

$$\iint dt\ df\ \exp(i2\pi f\tau - i2\pi t\upsilon)\ |S_u(t,f)|^2 = \chi_u(\upsilon,\tau)\ \chi_s(\upsilon,\tau)\ , \qquad (107)$$

where χ_u and χ_s are the complex ambiguity functions of u and s, respectively; see (23). This leads to an alternative expression for the STSE as

$$|S_u(t,f)|^2 = \iint d\tau\ d\upsilon\ \exp(-i2\pi f\tau + i2\pi t\upsilon)\ \chi_u(\upsilon,\tau)\ \chi_s(\upsilon,\tau)\ . \qquad (108)$$

Therefore, if the complex ambiguity function of s is computed, it can be multiplied by the ambiguity function of an arbitrary weighting function u, and followed by a two-dimensional Fourier transform. There is no need to calculate the WDF W_s via this route; also several different weighting functions could be utilized, each at the expense of a two-dimensional Fourier transform. The end result for the STSE is always nonnegative. Of course, the same result is obtainable directly by taking the magnitude-square of definition (102).

MARGINALS OF SPECTRAL ESTIMATE

There follows, from (106) and (12), the marginal relation

$$\int df\ |S_u(t,f)|^2 = \int dt'\ |s(t')|^2 |u(t - t')|^2 = |s(t)|^2 \overset{t}{\circledast} |u(t)|^2\ . \qquad (109)$$

Thus, the time marginal of $|S_u|^2$ is not directly $|s(t)|^2$, but is smeared by the weighting, according to $|u(t)|^2$. In a similar fashion, from (106) and (11), the frequency marginal is

$$\int dt \; |S_u(t,f)|^2 = |S(f)|^2 \overset{f}{\circledast} |U(f)|^2 \; . \tag{110}$$

Again, $|S(f)|^2$ is smeared by window $|U(f)|^2$.

Finally, completing either of the integrations in (109) or (110), over the remaining variable, yields

$$\iint dt \; df \; |S_u(t,f)|^2 = E_s \; E_u \; , \tag{111}$$

where E_s and E_u are the energies of s and u, respectively; see (14). Since weighting u is arbitrary and under our control, we can easily choose E_u to be 1, without loss of generality; then the volume under the STSE will be equal to the energy in waveform s being analyzed, just as for the WDF in (14).

MOMENTS OF SPECTRAL ESTIMATE

If we use (110) and (14), we find the following development:

$$\iint dt \; df \; f \; |S_u(t,f)|^2 = \int df \; f \int d\nu \; |S(\nu)|^2 |U(f-\nu)|^2 =$$

$$= \int d\nu \; |S(\nu)|^2 \int df \; (f - \nu + \nu) |U(f-\nu)|^2 =$$

$$= \int d\nu \; |S(\nu)|^2 \left[\int df_1 \; f_1 \; |U(f_1)|^2 + \nu E_u \right] =$$

$$= E_s \int df_1 \; f_1 |U(f_1)|^2 + E_u \int d\nu \; \nu \; |S(\nu)|^2 \; . \tag{112}$$

Combined with (111), there results

$$\frac{\iint dt\ df\ f\left|S_u(t,f)\right|^2}{\iint dt\ df\left|S_u(t,f)\right|^2} = \frac{\int df\ f\left|U(f)\right|^2}{\int df\ \left|U(f)\right|^2} + \frac{\int df\ f\left|S(f)\right|^2}{\int df\left|S(f)\right|^2} \quad . \tag{113}$$

That is, the first moment in f of the STSE is the sum of the frequency centers of $|U|^2$ and $|S|^2$. This should be compared with the corresponding result in (36) for the WDF, where only the last term in (113) is present. The presence of weighting u in definition (102) adds an additional term to the frequency center unless $|U(f)|$ is even about $f = 0$; in this latter case, (113) reduces to (36).

In a similar fashion, the first moment in t of the STSE is found to be

$$\frac{\iint dt\ df\ t\left|S_u(t,f)\right|^2}{\iint dt\ df\left|S_u(t,f)\right|^2} = \frac{\int dt\ t\left|u(t)\right|^2}{\int dt\ \left|u(t)\right|^2} + \frac{\int dt\ t\left|s(t)\right|^2}{\int dt\left|s(t)\right|^2} \quad . \tag{114}$$

Again, a sum of time centers results; but if weighting $|u(t)|$ is even about $t = 0$, then (114) reduces to the same result, (37), as for the WDF.

CONDITIONAL MOMENT

Just as in (28)-(35) for the WDF, conditional moments of the STSE can be defined. For example, directly from (105), we have

$$\int df\ f \left|S_u(t,f)\right|^2 = \int df\ f\ \left|\chi_{s\bar{u}}(f,t)\right|^2 , \tag{115}$$

in terms of the cross-ambiguity function of s and $\bar{u}$, where $\bar{u}(t) = u(-t)$.

An alternative time-domain expression is possible for the frequency moment in (115): define

$$g(t,t_1) = s(t_1)\ u^*(t - t_1) . \tag{116}$$

Then from (102) and (29),

$$\int df\ f \left|S_u(t,f)\right|^2 = \int df\ f \iint dt_1\ dt_2\ \exp\left(-i2\pi f(t_1 - t_2)\right) g(t,t_1)\ g^*(t,t_2) =$$

$$= \iint dt_1\ dt_2\ g(t,t_1)\ g^*(t,t_2)\ \frac{i}{2\pi}\ \delta'(t_1 - t_2) =$$

$$= \frac{-i}{2\pi} \int dt_1\ g'(t,t_1)\ g^*(t,t_1) , \tag{117}$$

where

$$g'(t,t_1) \equiv \frac{\partial}{\partial t_1}\ g(t,t_1) = \frac{\partial}{\partial t_1} \left\{s(t_1)\ u^*(t - t_1)\right\} . \tag{118}$$

If we represent waveform s in terms of its magnitude and phase according to (31), and do likewise for weighting u as

$$u(t) = E(t) \exp[iP(t)] , \tag{119}$$

then substitution in (117) results in the simplified form

$$\int df\ f \left|S_u(t,f)\right|^2 = \frac{1}{2\pi} \int dt_1 \left[\Theta'(t_1) - P'(t - t_1)\right] M^2(t_1)\ E^2(t - t_1) . \tag{120}$$

When this result is combined with (109), the normalized conditional first moment is

$$\frac{\int df\ f \left|S_u(t,f)\right|^2}{\int df\ \left|S_u(t,f)\right|^2} = \frac{1}{2\pi} \frac{\int dt_1 \left[\Theta'(t_1) - P'(t - t_1)\right] M^2(t_1)\ E^2(t - t_1)}{\int dt_1\ M^2(t_1)\ E^2(t - t_1)} . \tag{121}$$

(This reduces to (38) when $E(t) = 1$, $P(t) = 0$, that is, $u(t) = 1$, in which case $S_u(t,f) = S(f)$.) Generally, (121) is an average of $\Theta'(t_1) - P'(t - t_1)$, weighted according to the instantaneous powers of s and u.

EXAMPLES OF SHORT-TERM SPECTRAL ESTIMATION

Here we will reconsider many of the examples presented earlier for the WDF investigation. The particular example of weighting u adopted here in spectral definition (102) will be, for the time being,

$$u(t) = (\pi\,\sigma^2)^{-1/4} \exp\left(-\frac{t^2}{2\sigma^2}\right), \tag{122}$$

where duration measure σ is under our control. The energy E_u of this waveform is unity, in keeping with the discussion in (111) et seq., which guarantees that the volume under the SISE will be the energy $E_s = E$ of the waveform s being analyzed.

GAUSSIAN WAVEFORM

The waveform s was given in (71); its transform $S_u(t,f)$ is obtained by substituting (71) and (122) in (102) and using (68):

$$S_u(t,f) = \left(E\,\frac{\sigma_h}{\sigma_a}\right)^{1/2} \exp\left[-\frac{t^2}{4\sigma_a^2} - \pi^2 f^2 \sigma_h^2 - i\pi f t\,\frac{\sigma_o^2}{\sigma_a^2} + i\,\arg(a_o)\right], \tag{123}$$

where E is the energy of s and

$$\sigma_a^2 = \frac{1}{2}\left(\sigma^2 + \sigma_o^2\right),\quad \frac{1}{\sigma_h^2} = \frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_o^2}\right),\quad \frac{\sigma_a}{\sigma_h} = \frac{1}{2}\left(\frac{\sigma}{\sigma_o} + \frac{\sigma_o}{\sigma}\right). \tag{124}$$

The quantity σ_a^2 is the arithmetic average of σ^2 and σ_o^2, while σ_h^2 is the harmonic average. The STSE then follows immediately (or by use of (70) directly) as

$$\left|S_u(t,f)\right|^2 = E\,\frac{\sigma_h}{\sigma_a}\exp\left[-\frac{1}{2}\left(\frac{t^2}{\sigma_a^2}+4\pi^2f^2\sigma_h^2\right)\right]. \qquad (125)$$

The volume under STSE (125) is readily verified to be E, as it must be. The half widths of the ellipse at the 1/e relative level are $\sqrt{2}\,\sigma_a$, $\sqrt{2}\,(2\pi\sigma_h)^{-1}$, respectively, in the t,f plane. The area of this ellipse in the t,f plane is

$$\frac{\sigma_a}{\sigma_h} = \frac{1}{2}\left(\frac{\sigma}{\sigma_o}+\frac{\sigma_o}{\sigma}\right) \geq 1 . \qquad (126)$$

This area is at least <u>twice</u> as great as that for the corresponding WDF in figure 4, and even then, only when the proper guess is used for the weighting u, namely $\sigma = \sigma_o$. Since waveform duration σ_o will likely be unknown in practice, the mismatch factor in (126) will smear the concentration of the STSE somewhat. For example, if σ is off by a factor of 2 from σ_o (either double or half), (126) is 1.25 instead of its minimum value of 1.

The area enlargement factor σ_a/σ_h in (126) is also the same factor by which the peak of the STSE in (125) is down from its best value of E. Thus, the STSE has a decreased peak and enlarged effective area relative to the WDF, the relative factor being at least 2, and being $\sigma/\sigma_o + \sigma_o/\sigma$ in general. Both distributions contain volume E, independent of σ.

This example demonstrates the presence of "window effects" in the STSE that are not seen in the WDF. That is, whereas the effective ellipse in figure 4 depended only on waveform parameter σ_o, the ellipse here depends additionally on weighting parameter σ, in such a fashion as to always smear the concentration of energy in the t,f plane by at least a factor of 2. In trade, we always have the guarantee that the STSE (103) will be nonnegative, and that it will not contain the large interference phenomena inherent in the WDF; see (80)-(81).

MULTIPLE MODULATED TONES

The waveform of interest is given in (79). The transform $S_u(t,f)$ is found by use of (122), (102), and (68):

$$S_u(t,f) = \sum_k \left(E_k \frac{\sigma_{hk}}{\sigma_{ak}} \right)^{1/2} \exp\left[- \frac{(t - t_k)^2}{4\,\sigma_{ak}^2} - \pi^2 (f - f_k)^2\, \sigma_{hk}^2 - \right.$$

$$\left. - i\pi (f - f_k) \frac{\sigma_k^2\, t + \sigma^2\, t_k}{\sigma_{ak}^2} + i \arg(a_k) \right] , \qquad (127)$$

where energy $E_k = \sqrt{\pi}\,|a_k|^2\, \sigma_k$, and averages

$$\sigma_{ak}^2 = \frac{1}{2}\left(\sigma^2 + \sigma_k^2 \right) , \quad \frac{1}{\sigma_{hk}^2} = \frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_k^2} \right) . \qquad (128)$$

The STSE is the magnitude squared value of (127); the resulting double sum has diagonal terms

$$\sum_k E_k \frac{\sigma_{hk}}{\sigma_{ak}} \exp\left[-\frac{1}{2}\left\{\frac{(t - t_k)^2}{\sigma_{ak}^2} + 4\pi^2(f - f_k)^2\sigma_{hk}^2\right\}\right], \tag{129}$$

which are identical to (125), except for the indexing by k and the shift to center t_k, f_k in the t,f plane, as expected. If one value of weighting parameter σ is used to evaluate the SISE for all t of interest, it cannot simultaneously match all the different possible values of $\{\sigma_k\}$ for the various pulses. This will cause some of the components in (129) to be more severely degraded than others, in terms of decreased peaks and spread effective areas; the pertinent factor is again

$$\frac{\sigma_{ak}}{\sigma_{hk}} = \frac{1}{2}\left(\frac{\sigma}{\sigma_k} + \frac{\sigma_k}{\sigma}\right) \tag{130}$$

for the k-th component lobe. If some apriori knowledge of the values of $\{\sigma_k\}$ is available, this suggests using <u>different</u> values of σ for those values of t near the corresponding values of $\{t_k\}$, in an effort to minimize the factors (130) for different k.

As for the off-diagonal terms of $|S_u(t,f)|^2$ in (127), the k,ℓ term is proportional to

$$\exp\left[-\frac{(t - t_k)^2}{4\,\sigma_{ak}^2} - \frac{(t - t_\ell)^2}{4\,\sigma_{a\ell}^2} - \pi^2(f - f_k)^2\,\sigma_{hk}^2 - \pi^2(f - f_\ell)^2\,\sigma_{h\ell}^2\right]. \tag{131}$$

If t is not near one of the time centers $\{t_k\}$, or if f is not near one of the $\{f_k\}$, this term will be very small, due to the exponential decay. In particular, halfway between the dominant desired peaks of (129) at $\{t_k, f_k\}$, the quantity (131) will be essentially zero. This is in distinction to the WDF result in (80) et seq.

LINEAR FREQUENCY MODULATION

This waveform is specified in (84). Its STSE is, upon use of (70),

$$|S_u(t,f)|^2 = \frac{2E}{\sqrt{H_2}} \exp\left[- \frac{1}{H_2} \left\{ x^2(1 + r + r\,\theta_o^2) + \right.\right.$$

$$\left.\left. + y^2(1 + r)/r - 2xy\,\theta_o \right\}\right], \qquad (132)$$

where

$$\theta_o = \alpha_o\,\sigma_o^2 , \quad r = \frac{\sigma^2}{\sigma_o^2} , \quad H_2 = \frac{1}{r} + 2 + r + r\,\theta_o^2 , \qquad (133)$$

and where we define, here,

$$x = \frac{t}{\sigma} , \quad y = 2\pi f \sigma . \qquad (134)$$

By means of the results in appendix D, the area of the contour ellipse in (132), at the 1/e relative level, is found to be

$$\text{area} = \frac{1}{2}\sqrt{H_2} \quad \text{in the t,f plane.} \qquad (135)$$

Thus, the product of the peak height of the STSE in (132) and the effective area is again E, the volume under the STSE, regardless of the values of σ, σ_0, Θ_0. (For $\Theta_0 = 0$, (135) reduces to (126), as it must.)

To minimize the effective area and to maximize the peak value of the STSE, the common quantity H_2 in (133) should be minimized. This is accomplished by choosing the weighting parameter σ in (122) as

$$\sigma_{opt} = \frac{\sigma_0}{(1 + \Theta_0^2)^{1/4}} , \qquad r_{opt} = \frac{1}{(1 + \Theta_0^2)^{1/2}} , \tag{136}$$

which would require knowledge of both the duration σ_0 and the amount of frequency modulation Θ_0 in waveform s. Even if that information were available, the minimum area in (135) becomes

$$\text{mininum area} = \left(\frac{1 + \sqrt{1 + \Theta_0^2}}{2} \right)^{1/2} \quad \text{in the t,f plane} , \tag{137}$$

which still increases as $\sqrt{\Theta_0/2}$ for large Θ_0. Thus, even the best choice of σ for the weighting results in considerable spreading of the concentration ellipse and in peak reduction of the STSE; searching in σ is not overly helpful because the simple weighting pulse (122) is a poor facsimile to the linear frequency modulation waveform (84), especially for large amounts of frequency modulation, as measured by parameter Θ_0.

MORE GENERAL WEIGHTING

There is no need to restrict the weighting u in SISE (102) and (103) to be the simple Gaussian pulse in (122). In this section, we generalize it to allow for some linear frequency modulation:

$$u(t) = (\pi\,\sigma^2)^{-1/4} \exp\left[-\frac{t^2}{2\sigma^2} + i\,\frac{\alpha}{2}\,t^2\right]; \quad E_u = 1 . \tag{138}$$

The waveform of interest here is again the linear frequency modulation example

$$s(t) = a_o \exp\left[-\frac{t^2}{2\sigma_o^2} + i\,\frac{\alpha_o}{2}\,t^2\right], \quad \alpha_o \geq 0 , \tag{139}$$

as in (84).

The SISE follows from (102), (103), and (70), after a considerable amount of manipulations, as

$$\left|S_u(t,f)\right|^2 = \frac{2E}{\sqrt{H_3}} \exp\left[-\frac{1}{H_3}\left\{x^2(1 + r + r\theta_o^2 + r^2q^2\theta_o^2) + \right.\right.$$

$$\left.\left. + y^2(1 + r)/r - 2xy(1 + rq)\theta_o\right\}\right], \tag{140}$$

where

$$r = \frac{\sigma^2}{\sigma_o^2}, \quad q = \frac{\alpha}{\alpha_o}, \quad H_3 = \frac{1}{r} + 2 + r + r(q - 1)^2\,\theta_o^2 , \tag{141}$$

in addition to (134). The quantities r and q are mismatch factors, reflecting the lack of knowledge of weighting (138) about the waveform (139).

The area of the contour ellipse of (140) at the 1/e relative level is (by means of appendix D)

$$\text{area} = \frac{1}{2}\sqrt{H_3} \quad \text{in the t,f plane .} \tag{142}$$

This is also the same factor by which the peak of (140) is down from E. Thus, a minimum value for H_3 is desired. This can be achieved by choosing $r = 1$, $q = 1$ in (141), for which the minimum $H_3 = 4$ and the minimum area = 1; however, this requires that we choose $\sigma = \sigma_0$ and $\alpha = \alpha_0$, which is not a likely situation in practice, without some apriori knowledge about the waveform s. If this fortuitous situation of perfect match of the parameters does occur, the STSE in (140) reduces to

$$\left|S_u(t,f)\right|^2 = E \exp\left[-\frac{1}{2}\left\{x^2(1 + \Theta_0^2) - 2xy\,\Theta_0 + y^2\right\}\right], \tag{143}$$

which is identical to the corresponding WDF in (91), except for a factor of 2 outside and inside the exponential. Thus, the effective area is doubled and the peak is halved.

As special cases of weighting (138), if $\alpha = 0$ (no frequency modulation in the weighting), then H_3 in (141) reduces to H_2 in (133). Alternatively, if $\alpha_0 \to 0$ (no frequency modulation in the waveform s), then $\Theta_0(q - 1) = \alpha_0\,\sigma_0^2\,(\alpha/\alpha_0 - 1) \sim \alpha\,\sigma_0^2$ and

$$H_3 \sim \frac{1}{r} + 2 + r + r\,\alpha^2\sigma_0^4 = \left(\frac{\sigma}{\sigma_0} + \frac{\sigma_0}{\sigma}\right)^2 + (\alpha\,\sigma\,\sigma_0)^2 \ . \qquad (144)$$

This is minimized by choosing $\alpha = 0$ and $\sigma = \sigma_0$, giving value 4 as usual. Finally, for given q, H_3 in (141) is minimized by choosing $r = (1 + (q - 1)^2\,\theta_0^2)^{-1/2}$, for which the minimum $H_3 = 2 + 2(1 + (q - 1)^2\,\theta_0^2)^{1/2}$; however, again, this increases as $2|q - 1|\theta_0$ as θ_0 increases.

SMOOTHING THE WDF

It was demonstrated in (106) that the double convolution of any two WDFs is always nonnegative, and is in fact equal to the SISE of one waveform relative to the other:

$$W_s(t,f) \overset{tf}{\circledast} W_u(t,f) = \left|S_u(t,f)\right|^2 . \qquad (145)$$

This suggests that one should choose a (legal) WDF for weighting u which is as narrow as possible (least area or spread) in the t,f plane, in order to minimize the inherent spreading that (145) implies. The simple examples in the previous section demonstrated that, for the best choices of duration and linear frequency modulation parameters in the Gaussian weighting, an increase of .5 in the effective area in the t,f plane of the SISE, relative to the WDF, resulted.

PHILOSOPHY AND APPROACH

Since fine detail of the WDF $W_s(t,f)$ will likely vary in different portions of the t,f plane, this suggests the following possible procedure for analysis: For a given waveform s(t), compute and plot the WDF $W_s(t,f)$ according to (3) or (61). Locate a t,f region of interest in the plane, where large (perhaps oscillatory) values of W_s occur; denote the center of the region as t_c, f_c. Estimate the duration, σ_c, and linear frequency

modulation index, α_c, of this particular region in the t,f plane. Perform the SISE of waveform s(t) according to (102)-(103), with weighting

$$u(t) = (\pi\,\sigma_c^2)^{-1/4} \exp\left[-\frac{t^2}{2\sigma_c^2} + i\,\frac{\alpha_c}{2}\,t^2\right], \quad \theta_c = \alpha_c\,\sigma_c^2, \tag{146}$$

(for reasons to be given below), but only for locations t,f in the plane near $t = t_c$, $f = f_c$.

The WDF of weighting (146) is (with $\alpha_c = 2\pi\beta_c$)

$$W_u(t,f) = 2\exp\left[-\frac{t^2}{\sigma_c^2} - 4\pi^2\sigma_c^2(f - \beta_c t)^2\right] =$$

$$= 2\exp\left[-\frac{t^2}{\sigma_c^2}(1 + \theta_c^2) + 4\pi f t\theta_c - 4\pi^2 f^2\sigma_c^2\right], \tag{147}$$

which has a contour ellipse, at the 1/e relative level, of area 1/2 in the t,f plane, regardless of σ_c and α_c. This STSE procedure is equivalent to smoothing the WDF W_s of waveform s with the WDF in (147), for values near t_c, f_c in the t,f plane. Thus we have two alternative procedures for conducting the smoothing of a calculated WDF W_s, the first via direct evaluation of double convolution (145) for values of t,f near regions of interest, and the second via the STSE in (102) and (103). Which one to adopt will likely depend on the number of points that must be closely investigated in the t,f plane.

For other regions of interest in the t,f plane of the original WDF W_s, different values of t_c, f_c, σ_c, α_c must be extracted and the smoothing procedure repeated. Although tedious, this procedure will minimally spread the WDF W_s (by area .5) and it will guarantee a nonnegative distribution. This procedure is similar to that given in [2]; however, the information required to implement [2] is not easily available, and the current approach is not limited to constant-magnitude waveforms. A fine-grained analysis of a given general waveform s, for various t,f values and yielding nonnegative distribution values, is not going to be achieved without the expenditure of considerable effort and interaction between a user and preliminary analysis results.

This two-stage procedure, of observing the raw WDF and then computing different smoother versions in different regions, avoids the arbitrary pre-selection of time duration and frequency modulation content of the weighting in the STSE, which would overly smear the modified WDF for improper matches of parameter values. It also guarantees nonnegative estimates. In trade, there is approximately an increase of .5 in the effective area of the distribution in the t,f plane that must be accepted, in addition to a decreased peak value. For WDFs, W_s, with lobes which already occupy portions of the t,f plane with areas significantly greater than .5, this additional spreading (by area .5) is not very damaging, provided that σ_c and α_c are chosen correctly. Perhaps simultaneous plots of WDF $W_s(t,f)$ and STSE $\left|S_u(t,f)\right|^2$ would yield maximum information about waveform s.

In actual practice, where the integral definition in (102) is replaced by a numerical summation of samples taken at increment Δ, the quantity (145) is necessarily approximated. This problem is addressed in appendix E, where it is shown that the dominant term in the numerical approach is approximately the desired quantity (145). Furthermore, since the definition in (E-1) involves a magnitude-square, the approximation is guaranteed to be nonnegative. This need not be the case if the double convolution of WDFs W_s and W_u in (145) is approximated by sampling directly in the t,f plane and performing a double summation. However, for small enough increments in both t and f, this nonnegative aspect should be small and probably negligible; this latter approach was used in [1], although the smoothing function was not a legal WDF.

ALTERNATIVE AVERAGING PROCEDURES

Instead of using $R(t,\tau) = s(t + \frac{\tau}{2})\, s^*(t - \frac{\tau}{2})$ in (2) as the instantaneous correlation at time t and separation τ, one could use a local average, in hopes of improving the correlation and distribution functions. That is, consider correlation definition

$$\hat{R}(t,\tau) = v_1(t) \overset{t}{\circledast} R(t,\tau) = \int dt'\, v_1(t - t')\, s(t' + \frac{\tau}{2})\, s^*(t' - \frac{\tau}{2}) \ , \qquad (148)$$

where v_1 is a fairly sharp, even, real function centered at the origin. The corresponding "locally averaged" WDF is

$$\hat{W}(t,f) = \int d\tau \exp(-i2\pi f\tau)\ \hat{R}(t,\tau) =$$

$$= \int dt'\ v_1(t - t')\ W(t',f) = v_1(t) \overset{t}{\circledast} W(t,f)\ . \qquad (149)$$

This is a convolution, in time only, of the WDF of s with weighting v_1. Reference to the discussion following (83) reveals that this form of averaging is inadequate, since it does not average additionally on frequency. Also, (149) need not remain positive, as would be desired of a smoothed WDF.

Furthermore, the Fourier transform in (149) (as well as (3)) is over <u>all</u> τ, thereby involving argument values of waveform s in (148) which are very distant from the time point, t, of interest. If $\hat{W}(t,f)$ or $W(t,f)$ is to be considered as the "spectrum at time t," it is hard to justify why arbitrarily distant time points from location t should enter into their evaluations. Therefore, in addition to the local average in (148) for stability purposes, there should be a weighting in τ in (149) to better confine the Fourier transform to local values of waveform s about time instant t of interest.

To this aim, consider the more general form of average given by

$$\hat{R}(t,\tau) = v_2(t,\tau) \overset{t}{\circledast} R(t,\tau) = \int dt'\ v_2(t - t',\tau)\ R(t',\tau) =$$

$$= \int dt'\ v_2(t - t',\tau) \int df'\ \exp(i2\pi\tau f')\ W(t',f')\ , \qquad (150)$$

where weighting v_2 depends additionally on τ. Define its transform

$$V_2(t,f) = \int d\tau \exp(-i2\pi f\tau)\ v_2(t,\tau) \ . \tag{151}$$

Then the modified WDF corresponding to $\hat{R}$ in (150) is

$$\hat{W}(t,f) = \iint dt'\ df'\ W(t',f')\ V_2(t - t',\ f - f') =$$

$$= W(t,f) \overset{tf}{\circledast} V_2(t,f) \ , \tag{152}$$

which is a double convolution of W with V_2, on both t and f. However, since V_2 need not be a WDF, $\hat{W}$ in (152) can become negative for some t,f values. This form of smoothing was considered previously in [9; (1.5)] and [10; (2.1)].

An additional justification of two-dimensional smoothing, from the frequency domain alternative viewpoint, is given in appendix F. Also, a generalization of the Gaussian WDF (147), with arbitrary area and linear frequency modulation content and which guarantees a positive distribution $\hat{W}$, is given in (F-7)-(F-19); this result generalizes that in [11] for no frequency modulation.

If we specialize weighting v_2 in (150) to the form

$$v_2(t,\tau) = u(t + \tfrac{\tau}{2})\ u^*(t - \tfrac{\tau}{2}) \ , \tag{153}$$

then (151) yields

$$V_2(t,f) = W_u(t,f) \ , \tag{154}$$

and the general result in (152) specializes to (106), which is guaranteed positive. Thus the special case of weighting v_2 in (153) leads to the STSE of s, relative to u.

EFFICIENT CALCULATION OF SHORT-TERM SPECTRAL ESTIMATE

If we employ the weighting u in (146) with linear frequency modulation parameter α_c, the spectrum in (102) becomes

$$S_u(t,f) = \int dt_1 \exp(-i2\pi f t_1)\, s(t_1)\, u^*(t - t_1) =$$

$$= \exp(-i2\pi f t) \int dt_2 \exp(-i2\pi f t_2)\, s(t + t_2)\, u^*(- t_2) =$$

$$= (\pi\, \sigma_c^2)^{-1/4} \exp(-i2\pi f t) \int dt_2 \exp(-i2\pi f t_2)\, s(t + t_2) \exp\left(- \frac{t_2^2}{2\sigma_c^2} - i \frac{\alpha_c}{2} t_2^2\right). \tag{155}$$

The $\exp\left(-t_2^2/(2\sigma_c^2)\right)$ term gates out the portion of $s(t + t_2)$ near the origin in t_2, while the $\exp(-i\, \alpha_c t_2^2/2)$ term cancels linear frequency modulation in waveform s.

An approximation to (155) is obtained by sampling at increment Δ and using the Trapezoidal rule:

$$\tilde{S}_u(t,f) \approx (\pi\,\sigma_c^2)^{-1/4} \exp(-i2\pi ft)\,\Delta \sum_k \exp(-i2\pi f\Delta k)^*$$

$$^* s(t + k\Delta) \exp\left[-\frac{1}{2} k^2\Delta^2\left(\frac{1}{\sigma_c^2} + i\,\alpha_c\right)\right], \qquad (156)$$

which has period $1/\Delta$ in f. In particular, the approximation to the STSE, at selected points, is

$$\left|\tilde{S}_u\left(m\Delta\ ,\ \frac{n}{N\Delta}\right)\right|^2 = \frac{\Delta^2}{\sqrt{\pi}\,\sigma_c} \left|\sum_k \exp(-i2\pi nk/N)^*\right.$$

$$\left.^* s(m\Delta + k\Delta) \exp\left[-\frac{1}{2} k^2\Delta^2\left(\frac{1}{\sigma_c^2} + i\,\alpha_c\right)\right]\right|^2, \qquad (157)$$

which is an N-point discrete Fourier transform; m, n, N are integers.

The procedure for analysis is as follows: for a region of interest centered at t_c, f_c in the t,f plane, choose time values $m\Delta$ near t_c. Then for each m, sweep out n such that frequency $n/(N\Delta)$ is near f_c; an FFT will give all f values in $(0, 1/\Delta)$. Plots of (157) give a fine-tuned STSE near t_c, f_c for the particular choices of σ_c, α_c. Additional estimates with different parameters will be required in other regions; there is no globally optimum smoothing that will yield high-quality positive spectral estimates for all t,f values.

The numerical evaluation of the exponential quantities

$$Q_2(k) \equiv \exp\left[-\frac{1}{2}k^2\Delta^2\left(\frac{1}{\sigma_c^2} + i\alpha_c\right)\right] = Q_2(-k) \tag{158}$$

in (157) can be effected very efficiently by the methods given in [12]. They are given by recurrences (which need to be evaluated only once for each σ_c, α_c)

$$\left.\begin{aligned} Q_1(k) &= Q_1(k-1)\exp(2c_2) \\ Q_2(k) &= Q_2(k-1)\,Q_1(k) \end{aligned}\right\} \text{ for } k \geq 1 , \tag{159}$$

with starting values

$$Q_1(0) = \exp(-c_2) , \quad Q_2(0) = 1 ,$$

$$c_2 = -\frac{1}{2}\Delta^2\left(\frac{1}{\sigma_c^2} + i\,\alpha_c\right) . \tag{160}$$

Only two complex multiplications per stage are required in (159). Furthermore, since

$$\exp(-c_2) = \exp\left(\frac{\Delta^2}{2\sigma_c^2}\right)\exp\left(\frac{i\alpha_c\Delta^2}{2}\right) \equiv E\,(C + iS) , \tag{161}$$

and

$$\exp(2c_2) = \frac{C^2 - S^2 - i2SC}{E^2} , \tag{162}$$

only one exp, cos, and sin must be evaluated to accomplish (159) for all k.

WDF WITH MINIMUM SPREAD

The virtues of smoothing WDF W_s of waveform s with the WDF W_u, (147), of weighting u, (146), were discussed earlier in this section. At that time, the selection of form (146) for the weighting was seemingly arbitrary. However, it is shown in appendix G that the weighting, u, which has a minimally spread WDF, is precisely that given in (146). The measure of spread is

$$I = \iint dt\, df\, W_u(t,f)\, (f - \beta_c t)^2 , \tag{163}$$

where β_c is a specified (observed) slope of interest in the t,f plane, and $\alpha_c = 2\pi\beta_c$. This measure of spread concentrates the WDF about the specified slope; see (147). The actual *minimum* value of spread (163) is given in (G-24) as

$$\text{minimum } I = \frac{1}{8\pi^2\sigma_c^2} , \tag{164A}$$

when weighting u is constrained to have mean square duration

$$\int dt\, t^2 |u(t)|^2 = \frac{\sigma_c^2}{2} , \tag{164B}$$

in addition to unit energy. Without these two constraints, the minimization of spread (163) is ill-posed; see appendix G.

PERFORMANCE IN NOISE

In this section, we investigate the bias and stability of a WDF estimate obtained from a noisy waveform. In particular, the given waveform x is

$$x(t) = s(t) + n(t) \ , \tag{165}$$

where s is a deterministic signal of interest, and n is an additive zero-mean stationary noise. In fact, we have

$$\overline{n(t)} = 0 \ , \qquad \overline{n(t_1)n^*(t_2)} = C_n(t_1 - t_2) \ ,$$

$$G_n(f) = \int d\tau \ \exp(-i2\pi f\tau) \ C_n(\tau) \ , \tag{166}$$

where C_n and G_n are the noise covariance and power density spectrum, respectively.

WAVEFORM WEIGHTING

If the WDF of given waveform x in (165) were directly evaluated via definition (3), the result would be infinite, since the NxN (noise-cross-noise) terms do not decay for large arguments. Also, since the signal s will be assumed to be transient and decay to zero for large arguments, some gating or weighting of given waveform x is appropriate, in order to

concentrate on the time regions where signal s is largest. Accordingly, we consider the weighted waveform

$$y(t) = v(t)\, x(t) = v(t)\, [s(t) + n(t)] \,, \tag{167}$$

where v(t) is a deterministic function under our control.

The WDF that will be calculated is therefore

$$W_{yy}(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, y(t + \tfrac{\tau}{2}) \, y^*(t - \tfrac{\tau}{2}) =$$

$$= a + b + c + d \,, \tag{168}$$

where

$$a = \int d\tau \, \exp(-i2\pi f\tau) \, R_{vv}(t,\tau) \, s(t + \tfrac{\tau}{2}) \, s^*(t - \tfrac{\tau}{2}) \,,$$

$$b = \int d\tau \, \exp(-i2\pi f\tau) \, R_{vv}(t,\tau) \, n(t + \tfrac{\tau}{2}) \, n^*(t - \tfrac{\tau}{2}) \,,$$

$$c = \int d\tau \, \exp(-i2\pi f\tau) \, R_{vv}(t,\tau) \, s(t + \tfrac{\tau}{2}) \, n^*(t - \tfrac{\tau}{2}) \,,$$

$$d = \int d\tau \, \exp(-i2\pi f\tau) \, R_{vv}(t,\tau) \, n(t + \tfrac{\tau}{2}) \, s^*(t - \tfrac{\tau}{2}) \,, \tag{169}$$

and

$$R_{vv}(t,\tau) = v(t + \tfrac{\tau}{2}) \, v^*(t - \tfrac{\tau}{2}) \,. \tag{170}$$

The first two quantities in (168) are, respectively, SxS and NxN terms, while the last two are SXN terms; here, S denotes signal, while N denotes noise. The SxS term, a, in (169) is real and non-random, while NxN term, b, is real and random. On the other hand, the SxN terms, c and d, are complex random, with $d^* = c$.

MEAN VALUES

An alternative expression for the SxS term in (169) is

$$a = \int d\tau \exp(-i2\pi f\tau)\ R_{vv}(t,\tau)\ R_{ss}(t,\tau) =$$

$$= W_{vv}(t,f) \overset{f}{\circledast} W_{ss}(t,f)\ , \tag{171}$$

in terms of the WDFs of weighting v and signal s.

The mean of the NxN term is

$$\bar{b} = \int d\tau \exp(-i2\pi f\tau)\ R_{v'v'}(t,\tau)\ C_n(\tau) =$$

$$= W_{vv}(t,f) \overset{f}{\circledast} G_n(f)\ , \tag{172}$$

where we made use of (166). And since noise n has zero mean, there follows, for the SxN terms, $\bar{c} = \bar{d} = 0$. Collecting these results together, the mean of WDF estimate (168) is

$$\overline{W_{yy}(t,f)} = W_{vv}(t,f) \overset{f}{\circledast} [W_{ss}(t,f) + G_n(f)]\ . \tag{173}$$

No additional statistical properties on the noise n, such as a Gaussian process, are needed for result (173); this holds for an arbitrary stationary noise process. The difference in mean outputs, for signal present versus signal absent, is just a, as given by (171).

VARIANCE OF WDF ESTIMATE

In order to determine the variance of estimate (168), we need to assume that noise n is Gaussian. Furthermore, in addition to properties (166), we will presume that

$$\overline{n(t_1)\ n(t_2)} = 0\ , \tag{174}$$

as is true when n is an analytic process or a complex envelope [13, ch.2]. Then (168) yields

$$\overline{W_{yy}^2(t,f)} = \overline{\left|W_{yy}(t,f)\right|^2} = a^2 + \overline{b^2} + \overline{|c|^2} + \overline{|d|^2} +$$

$$+ 2\,a\bar{b} + \overline{cd^*} + \overline{c^*d}\ , \tag{175}$$

the other terms being zero due to n being zero-mean Gaussian noise.

The second term on the right-hand side of (175) can be developed from (169) as

$$\bar{b}^2 = \overline{|b|^2} = \iint d\tau_1\ d\tau_2\ \exp\left(-i2\pi f(\tau_1 - \tau_2)\right)\ R_{vv}(t,\tau_1)*$$

$$*R_{vv}^*(t,\tau_2)\left[C_n(\tau_1)\ C_n^*(\tau_2) + C_n^2\left(\frac{\tau_1 - \tau_2}{2}\right)\right]\ , \tag{176}$$

where we used (166) and (174). Referring to (172), we have

$$\overline{b^2} = \bar{b}^2 + \iint d\tau_1\, d\tau_2 \exp\left(-i2\pi f(\tau_1 - \tau_2)\right) R_{vv}(t,\tau_1)^*$$

$$^*R^*_{vv}(t,\tau_2)\; C_n^2\left(\frac{\tau_1 - \tau_2}{2}\right) . \qquad (177)$$

At this point, it is convenient to define

$$G_n^{(2)}(f) = \int d\tau \exp(-i4\pi f\tau)\; C_n^2(\tau) . \qquad (178)$$

Then

$$G_n^{(2)}(f/2) = \int d\tau \exp(-i2\pi f\tau)\; C_n^2(\tau) = G_n(f) \circledast G_n(f) \qquad (179)$$

and

$$C_n^2(\tau) = \int df \exp(i2\pi f\tau)\; G_n^{(2)}(f/2) =$$

$$= 2 \int d\upsilon \exp(i4\pi\upsilon\tau)\; G_n^{(2)}(\upsilon) . \qquad (180)$$

When this result is substituted in (177), there follows

$$\overline{b^2} = \overline{b}^2 + 2\int d\upsilon\, W_{vv}^2(t,f-\upsilon)\; G_n^{(2)}(\upsilon) =$$

$$= \overline{b}^2 + 2\, W_{vv}^2(t,f) \overset{f}{\circledast} G_n^{(2)}(f) . \qquad (181)$$

The third and fourth terms in (175) are

$$\overline{|d|^2} = \overline{|c|^2} = \iint d\tau_1 \, d\tau_2 \, \exp\left(-i2\pi f(\tau_1 - \tau_2)\right) \, R_{vv}(t,\tau_1)^*$$

$$* \, R^*_{vv}(t,\tau_2) \, s\left(t + \frac{\tau_1}{2}\right) s^*\left(t + \frac{\tau_2}{2}\right) C_n\left(\frac{\tau_1 - \tau_2}{2}\right) =$$

$$= \int d\upsilon \, G_n(\upsilon) \left| B\left(t, f - \frac{\upsilon}{2}\right)\right|^2 , \qquad (182)$$

where

$$B(t,f) \equiv \int d\tau \, \exp(-i2\pi f\tau) \, R_{vv}(t,\tau) \, s\left(t + \frac{\tau}{2}\right) =$$

$$= \int d\upsilon \, \exp(i2\pi\upsilon t) \, W_{vv}\left(t, f - \frac{\upsilon}{2}\right) S(\upsilon) \, . \qquad (183)$$

(As special cases, if weighting $v(t) = 1$ for all t, then $R_{vv}(t,\tau) = 1$ and

$$\overline{|c|^2} = 4 \int d\upsilon \, G_n(\upsilon) \left|S(2f - \upsilon)\right|^2 ; \qquad (184)$$

while, instead, if $G_n(f) = \hat{N}_d$ for all f, then $C_n(\tau) = \hat{N}_d \, \delta(\tau)$ and

$$\overline{|c|^2} = 2 \, \hat{N}_d \int d\tau \, R^2_{vv}(t,\tau) \left| s\left(t + \frac{\tau}{2}\right)\right|^2 . \qquad (185)$$

If both conditions above hold, then

$$\overline{|c|^2} = 4 \, \hat{N}_d \, E \, . \;) \qquad (186)$$

Returning to the general case again, the fifth term in (175) is given by combining (171) and (172), while the sixth term is

$$\overline{cd^*} = \iint d\tau_1\, d\tau_2\, \exp\left(-i2\pi f(\tau_1 - \tau_2)\right)\, R_{vv}(t,\tau_1)\, *$$

$$* R^*_{vv}(t,\tau_2)\, s\left(t + \frac{\tau_1}{2}\right) s^*\left(t - \frac{\tau_2}{2}\right) \overline{n^*\left(t - \frac{\tau_1}{2}\right) n^*\left(t + \frac{\tau_2}{2}\right)} = 0 , \qquad (187)$$

by use of (174).

Combining the above results, we have, for the variance of the WDF estimate,

$$\begin{aligned} \mathrm{Var}\{W_{yy}(t,f)\} &= a^2 + \overline{b^2} + 2\overline{|c|^2} + 2a\bar{b} - (a + \bar{b})^2 = \\ &= \overline{b^2} - \overline{b}^2 + 2\overline{|c|^2} = \\ &= 2\, W^2_{vv}(t,f) \overset{f}{\circledast} G^{(2)}_n(f) + \qquad \text{(NxN)} \\ &\quad + 2 \int d\upsilon\, G_n(\upsilon)\, \left| B\left(t, f - \frac{\upsilon}{2}\right)\right|^2 . \qquad \text{(SxN)} \qquad (188) \end{aligned}$$

This result holds for arbitrary signal s, weighting v, and noise spectrum G_n. The quantities $G^{(2)}_n$ and B are defined in (178) and (183), respectively.

If we do not weight waveform x(t), that is, choose v(t) = 1 for all t in (167), then $R_{vv}(t,\tau) = 1$, $W_{vv}(t,f) = \delta(f)$, and the NxN term in (188) becomes infinite; that is, the WDF estimate (168) has infinite variance if we do not weight in time, regardless of what the actual noise spectrum, G_n, is.

On the other hand, if the noise n is white, then $G_n(f) = \hat{N}_d$ for all f, $C_n(\tau) = \hat{N}_d\ \delta(\tau)$, and $G_n^{(2)}$ in (178) is infinite, which makes the NxN term in (188) infinite. Thus, if we do not filter out the noise which is out of the band of the signal, the WDF estimate has infinite variance, regardless of what time weighting v is employed.

WDF PROCESSOR

In view of the above observations, we now consider the general WDF processor depicted in figure 7. The only new element here is the

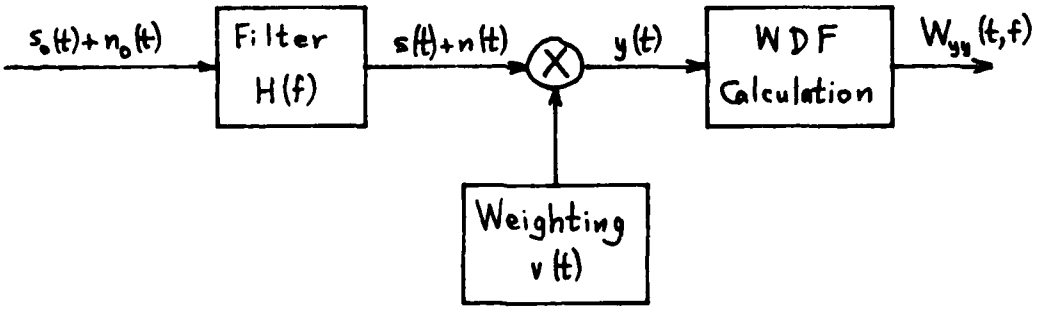


Figure 7. WDF Processor

time-invariant linear filter with transfer function H. The input noise n_o is presumed to be white over the band of the input signal s_o; mathematically, this is handled by letting

$$G_{n_o}(f) = N_d \quad \text{for all } f \ , \tag{189}$$

where N_d is the double-sided noise spectral density level in watts/Hz. The linear filter H(f) approximately matches the bandwidth of the input signal and passes $S_o(f)$ essentially unaltered, while filtering out undesired noise spectral components. The actual filter output signal s is given by

$$s(t) = h(t) \circledast s_o(t) = \int df \ \exp(i2\pi ft)\ H(f)\ S_o(f) \ . \tag{190}$$

The weighting v(t) approximately matches the duration of the signal and passes s(t) essentially unaltered, while gating out undesired noise temporal components. Representative plots of the various quantities in figure 7 are given in figure 8.

A numerical example of the WDF processor in figure 7 is carried out in complete detail in appendix H, including the mean and variance results given earlier in this section. In particular, the input signal s_o is a linear frequency modulation waveform with Gaussian amplitude modulation.

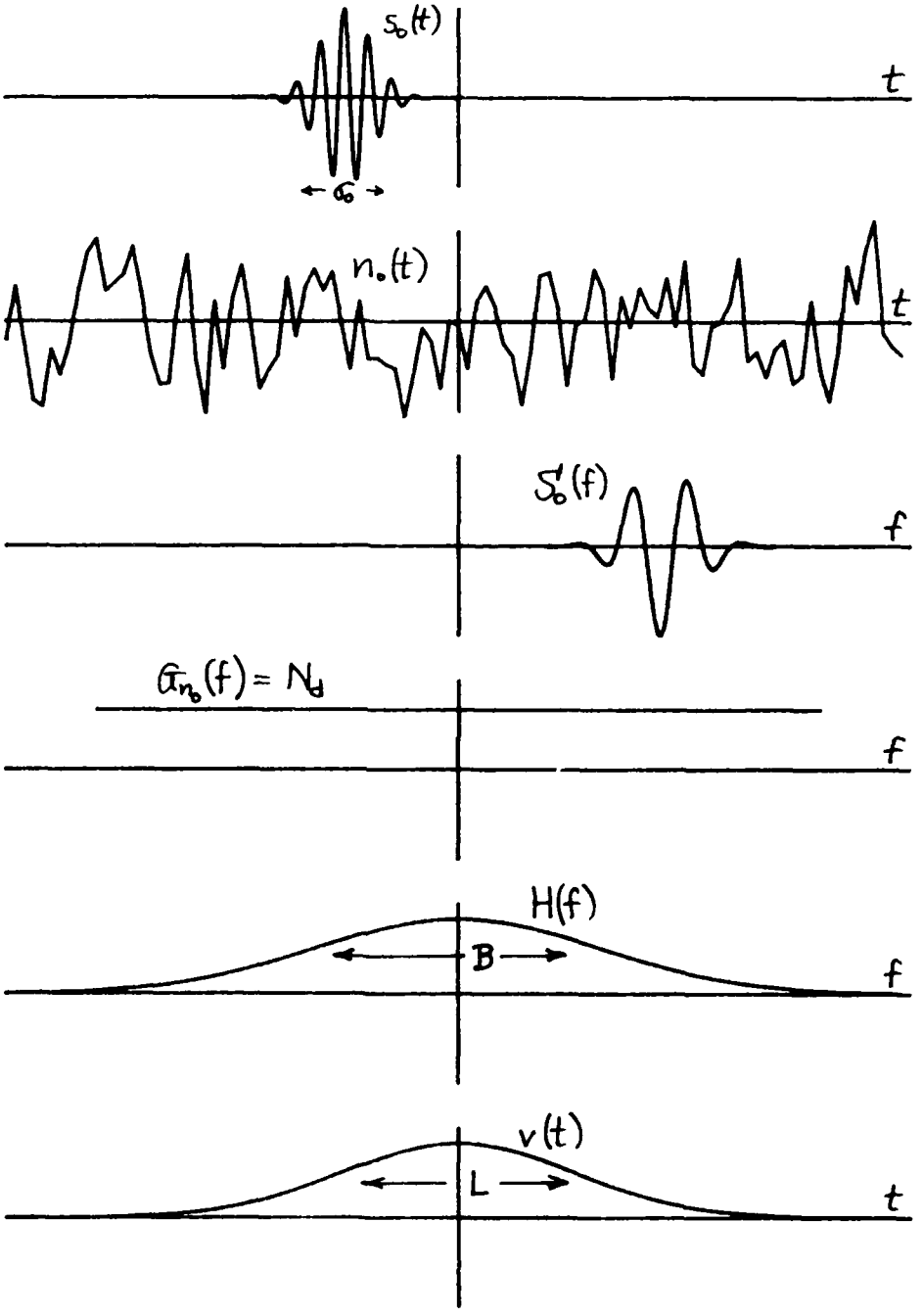


Figure 8. Time and Frequency Characteristics of Figure 7

SUMMARY

When a segment of a stationary random process is available, the method of Blackman and Tukey [14] tells us that, to estimate the correlation function at delay τ, we should average the product of waveform values separated in time by τ seconds, and that we should carry out this averaging over the total available data record, in order to reduce the effect of random fluctuations. For a nonstationary process, the averaging interval is further limited to that in which a significant change is statistics does not occur.

After obtaining the estimated correlation, the Blackman-Tukey method further directs us to weight the correlation values in the neighborhood of $\tau = 0$ more heavily than those for larger τ, and to Fourier transform the weighted correlation estimate. The weighting should taper off to zero for larger τ, so as to suppress these more noisy estimates, and the taper should be gradual so as not to create significant positive and negative sidelobes in the frequency domain.

These two operations, averaging in time and weighting in delay, are both totally absent in the WDF, as may be seen from (2) and (3). In fact, (2) and (3) might be viewed as the ultimate in greediness of a spectral estimate, since they include no averaging and no weighting. Viewed in this light, it is not surprising that the WDF has some very debilitating behavior in terms of negative distribution values and large interference terms.

The inclusion of averaging and weighting in the spectral estimate, as typified by (150)-(152) and (F-1), results in a modified distribution function which is a double convolution with a smoothing function in the t,f plane. Furthermore, the averaging and weighting in (F-1) takes place both in the f and υ domain (line 4) just as well as in the t and τ domain (line 3). Alternatively, line 2 indicates that the complex ambiguity function may be weighted in two dimensions and doubly Fourier transformed. However, the resultant modified WDF need not be positive.

The identity of this double convolution with a positive STSE, when the smoothing function is a legal WDF, allows for an alternative approach that is very attractive computationally and is easy to interpret. The preliminary calculation of the WDF serves to point out regions of interest in the t,f plane and to quantify the time and frequency extents, as well as the amount of linear frequency modulation, to utilize in weighting u in the STSE. This procedure is illustrated in appendix I for the waveform $s(t) = t \exp(-t^2/2)$ and shown to yield a physically meaningful smoothed distribution function, whereas the WDF is very difficult to justify and interpret on any physical grounds.

It was pointed out earlier that double convolution of a given WDF with a Gaussian WDF increases the spread of the smoothed function by area .5 in the t,f plane, since the effective area of a Gaussian WDF is .5. Strictly speaking, this is only true when the Gaussian WDF contour ellipse has the same tilt and the same ratio of major-to-minor axes as the given WDF (assumed Gaussian in the region of interest in the t,f plane). More

generally, if there is a mismatch in tilt or ratio of major-to-minor axes, the effective area is increased by more than .5, thereby leading to additional spreading in the t,f plane. The detailed derivations are presented in appendix J.

The performance of an estimator of the WDF of a signal in the presence of noise depends on the amount of filtering and weighting employed to suppress noise components in frequency and time. Exact relations for the mean output, the bias, and the variance of the WDF estimate are given.

APPENDIX A. SLICES IN TIME OF THE WDF

The voltage density spectrum of waveform s(t) was given in (9). If s(t) is sampled at increment Δ_t, an approximation is afforded according to

$$S(f) = \int dt \exp(-i2\pi ft)\, s(t) =$$

$$\cong \Delta_t \sum_k \exp(-i2\pi fk\Delta_t)\, s(k\Delta_t) \equiv \tilde{S}(f) \quad \text{for all } f. \tag{A-1}$$

The summation on k runs over the range of nonzero summand. Since

$$\tilde{S}\left(f + \frac{1}{\Delta_t}\right) = \tilde{S}(f)\ , \tag{A-2}$$

then $\tilde{S}(f)$ has period $1/\Delta_t$ in f. We limit the evaluation of $\tilde{S}(f)$ to the values

$$\tilde{S}\left(\frac{n}{N_f\Delta_t}\right) = \Delta_t \sum_k \exp(-i2\pi nk/N_f)\, s(k\Delta_t) \qquad \text{for } 0 \le n \le N_f - 1\ , \tag{A-3}$$

where n and N_f are integers, and thereby cover a full period of $\tilde{S}(f)$. A representative plot of $|\tilde{S}(f)|$ and its sampled values appears in figure A-1.

For the low-pass case of $\tilde{S}(f)$ depicted in figure A-1, it is necessary to choose

$$\Delta_t < (2f_H)^{-1} \tag{A-4}$$

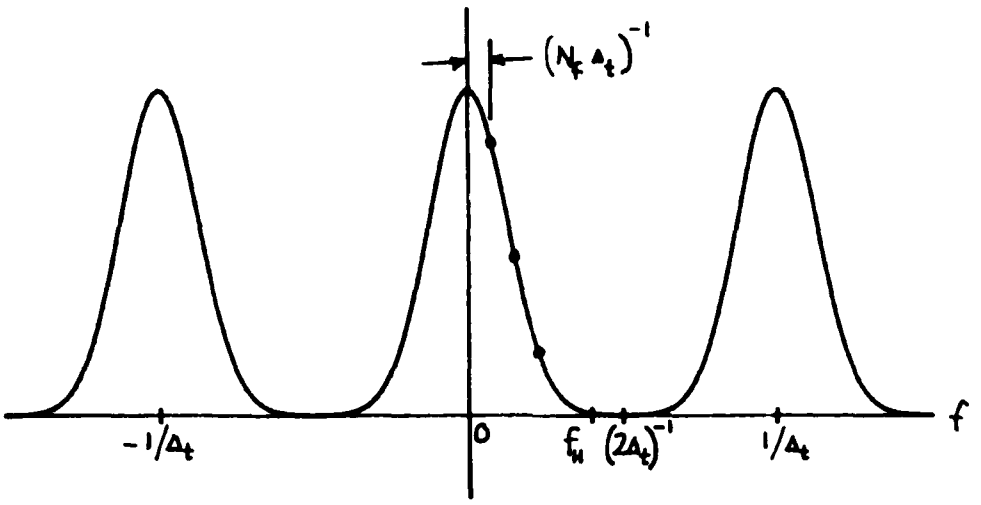


Figure A-1. Low-Pass Spectrum $\tilde{S}(f)$

in order to avoid aliasing. We will also need frequency spacing

$$(N_f \Delta_t)^{-1} < (2T)^{-1} , \quad \text{that is, } N_f > 2T/\Delta_t \tag{A-5}$$

in order to track the wiggles of S(f) in frequency, where T is the effective time duration of s(t). In fact, we may need frequency spacing $(N_f \Delta_t)^{-1}$ to be very small if we are to do <u>further</u> accurate integrations on S(f). Thus we need

$$N_f > 4 f_H T, \tag{A-6}$$

and perhaps much larger for further manipulations.

The WDF can be written in terms of the spectrum S(f) according to (10):

$$W(t,f) = \int dv \exp(i2\pi vt) S(f + \tfrac{v}{2}) S^*(f - \tfrac{v}{2}) =$$

$$= 2\int du \exp(i4\pi ut) S(f + u) S^*(f - u) . \tag{A-7}$$

If we sample S(f) at increment Δ_f in frequency, the Trapezoidal approximation to the WDF is

$$\hat{W}(t,f) \equiv 2\Delta_f \sum_{\ell} \exp(i4\pi t\ell\Delta_f)\ S(f + \ell\Delta_f)\ S^*(f - \ell\Delta_f) \qquad (A\text{-}8)$$

for all t,f. Since

$$\hat{W}\left(t + \frac{1}{2\Delta_f}\ ,\ f\right) = \hat{W}(t,f)\ , \qquad (A\text{-}9)$$

then $\hat{W}(t,f)$ has period $(2\Delta_f)^{-1}$ in t. Accordingly, we evaluate only

$$\hat{W}\left(\frac{m}{2N_t\Delta_f}\ ,\ n\Delta_f\right) = 2\Delta_f \sum_{\ell} \exp(i2\pi m\ell/N_t)\ S\left((n+\ell)\Delta_f\right)\ S^*\left((n-\ell)\Delta_f\right) \qquad (A\text{-}10)$$

for $0 \le m \le N_t - 1$, where m, n, N_t are integers. In this manner, we get a slice of $\hat{W}(t,f)$ in time t (m) for fixed frequency f (n). The operation in (A-10) can be efficiently realized as an N_t-point FFT of collapsed samples when N_t is a power of 2.

Now the only information on S(f) that we have available are the samples of $\tilde{S}(f)$ given in (A-3). If we choose, without loss of generality,

$$\Delta_f = \left(N_f\Delta_t\right)^{-1}\ , \qquad (A\text{-}11)$$

then (A-10) becomes (exactly)

$$\hat{W}\left(\frac{m\Delta_t}{2}\ \frac{N_f}{N_t}\ ,\ \frac{n}{N_f\Delta_t}\right) = \frac{2}{N_f\Delta_t} \sum_{\ell} \exp(i2\pi m\ell/N_t)\ S\left(\frac{n+\ell}{N_f\Delta_t}\right)\ S^*\left(\frac{n-\ell}{N_f\Delta_t}\right) \qquad (A\text{-}12)$$

for $0 \le m \le N_t - 1$. We then adopt as our approximation to $\hat{W}$, which itself is an approximation to W, the quantity

$$\tilde{W}\left(\frac{m\Delta_t}{2}\,\frac{N_f}{N_t}\,,\,\frac{n}{N_f\Delta_t}\right) = \frac{2}{N_f\Delta_t}\sum_{\ell\in L_n} \exp(i2\pi m\ell/N_t)\;\tilde{S}\left(\frac{n}{N_f\Delta_t}+\ell\right)\;\tilde{S}^{*}\left(\frac{n}{N_f\Delta_t}-\ell\right) \qquad (A\text{-}13)$$

for $0 \le m \le N_t - 1$, where

$$\ell \in L_n \text{ denotes } |n \pm \ell| < N_f/2\ , \qquad (A\text{-}14)$$

and we presume that $\tilde{S}(f)$ has been calculated for $|f| < (2\Delta_t)^{-1}$; see figure A-1.

The quantity $N_f\Delta_t$ may have to be large, in order to sample $\tilde{S}(f)$ finely enough for an accurate WDF. Then since the spacing in t, applied to W(t,f), is $\frac{N_f\Delta_t}{2N_t}$, it may require a large value of N_t in order to keep track of the variations versus t. Also, n may not have to run through consecutive integer values, but may take on decimated values, so that $n/(N_f\Delta_t)$ tracks the f behavior adequately.

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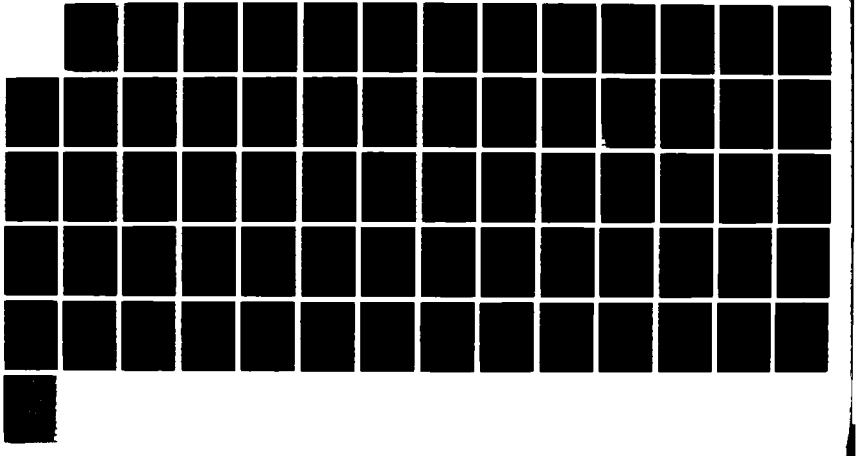
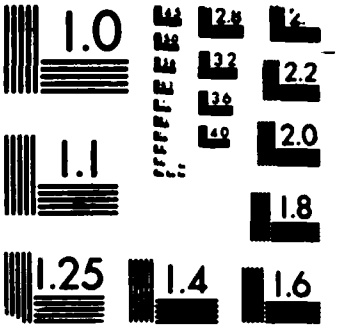




MICROCOPY RESOLUTION TEST CHART

APPENDIX B. OSCILLATING WDF FOR SEPARATED PULSES

Consider the waveform s(t) in figure B-1 consisting of two separated energy bursts of general shape. Let t_1 and t_2 represent the "center" of each pulse, and let T_1 and T_2 be some measure of their durations. Define

$$t_c = \frac{1}{2}(t_1 + t_2) \ . \tag{B-1}$$

Let us investigate the WDF of s(t) for t near t_c, that is, near the center of the two pulses. In particular, let time

$$t = t_c + \Delta \ , \tag{B-2}$$

where Δ is small. Then from (3),

$$W(t_c + \Delta, f) = \int d\tau \exp(-i2\pi f\tau)\ s(t_c + \Delta + \tfrac{\tau}{2})\ s^*(t_c + \Delta - \tfrac{\tau}{2}) =$$

$$= \exp\left(-i2\pi f(t_1 - t_2)\right) \int dx \exp(-i2\pi fx)\ s(t_2 + \Delta + \tfrac{x}{2})\ s^*(t_1 + \Delta - \tfrac{x}{2}), \tag{B-3}$$

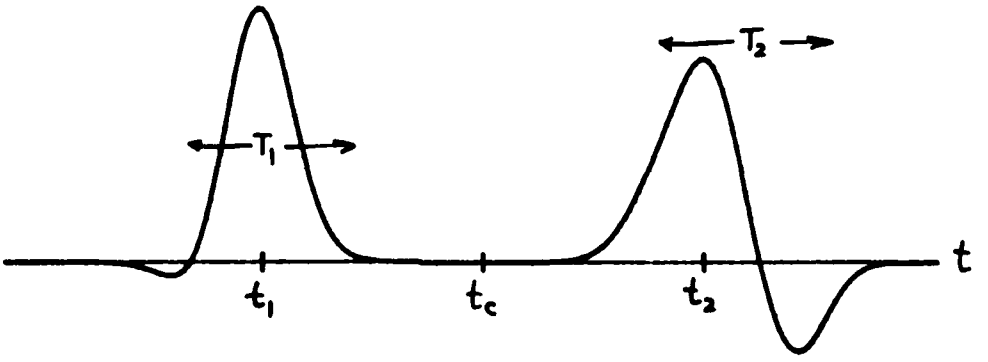


Figure B-1. Waveform s(t)

where we let $\tau = t_2 - t_1 + x$.

It can be seen from the integral expression in (B-3) and figure B-1, that for small Δ, only small values of x will contribute to the value of the WDF. In fact,

$$|x| < \min(T_1, T_2) \tag{B-4}$$

is the dominant range of contribution to the integral. Thus the variation with f of the integral component of (B-3) is slow. By contrast, the leading exponential term in (B-3) varies much faster with f, since

$$t_2 - t_1 > \max(T_1, T_2) > \min(T_1, T_2) \ . \tag{B-5}$$

Since these faster varying oscillations of the exponential term cannot be cancelled by the slower integral contribution of (B-3), the WDF will oscillate in f, for times t near $t_c = (t_1 + t_2)/2$. Thus separated time pulses will lead to oscillations (in f) of the WDF, near times midway between the pulses, regardless of their detailed shapes. An analogous argument can be presented for spectral components, based upon form (10) of the WDF.

Notice that as t_2 approaches t_1, and the two pulses become one, the oscillating exponential term in (B-3) disappears, allowing for the possibility of a slowly varying (hopefully positive) lobe in the WDF.

APPENDIX C. AMBIGUITY FUNCTION OF (79)

The complex ambiguity function of waveform s(t) in (79) is obtained by substitution into (22) and use of (68):

$$\chi(\nu,\tau) = \sqrt{\pi}\sum_{k\ell} a_k a_\ell^* \tilde{\sigma}_{k\ell} \exp\left[-\frac{1}{4}\frac{T_{k\ell}^2}{\bar{\sigma}_{k\ell}^2} - \frac{1}{4}\tilde{\sigma}_{k\ell}^2 V_{k\ell}^2 + \right.$$

$$\left. + i\frac{1}{2}\frac{\sigma_\ell^2 - \sigma_k^2}{\sigma_\ell^2 + \sigma_k^2} T_{k\ell} V_{k\ell} + i2\pi f_{k\ell} T_{k\ell} - i t_{k\ell} V_{k\ell} + i2\pi f_{k\ell}(t_k - t_\ell)\right], \qquad (C\text{-}1)$$

where

$$t_{k\ell} = \tfrac{1}{2}(t_k + t_\ell), \quad f_{k\ell} = \tfrac{1}{2}(f_k + f_\ell),$$

$$\bar{\sigma}_{k\ell}^2 = \tfrac{1}{2}(\sigma_k^2 + \sigma_\ell^2), \quad \frac{1}{\tilde{\sigma}_{k\ell}^2} = \frac{1}{2}\left(\frac{1}{\sigma_k^2} + \frac{1}{\sigma_\ell^2}\right),$$

$$T_{k\ell} = \tau + t_\ell - t_k, \quad V_{k\ell} = 2\pi(\nu + f_\ell - f_k). \qquad (C\text{-}2)$$

The diagonal terms of (C-1) are

$$\sum_k E_k \exp\left[-\frac{1}{4}\frac{\tau^2}{\sigma_k^2} - \frac{1}{4}\sigma_k^2 4\pi^2\nu^2 + i2\pi f_k\tau - i2\pi t_k\nu\right], \qquad (C\text{-}3)$$

which are complex and oscillate with τ and ν due to the imaginary terms. The contour at the 1/e relative level, of the magnitude of the k-th term, is an ellipse with axes twice as large as those depicted in figure 4. In addition, the peak amplitude is decreased by a factor of 2 below that for

the WDF in (80). Thus the ambiguity function is a more smeared function of time-frequency than the WDF.

The ambiguity function has peaks of value

$$\sqrt{\pi}\, a_k a_\ell^* \tilde{\sigma}_{k\ell} \exp\left[i\pi(f_k + f_\ell)(t_k - t_\ell)\right] \tag{C-4}$$

centered at

$$(\tau, \upsilon) = (t_k - t_\ell, f_k - f_\ell) \tag{C-5}$$

for all k, ℓ. The phases of (C-4) are virtually random relative to each other. A slice in υ, for fixed τ, varies (in addition to the Gaussian envelope) as

$$\exp\left[i2\pi\upsilon\left(\frac{1}{2}\,\frac{\sigma_\ell^2 - \sigma_k^2}{\sigma_\ell^2 + \sigma_k^2}\, T_{k\ell} - t_{k\ell}\right)\right], \tag{C-6}$$

which could be either a slow or fast variation, depending on the particular parameter values. All of these features make physical interpretation of the ambiguity function very difficult.

APPENDIX D. ROTATION OF AXES

Consider the general second-order curve described by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \ . \qquad (D\text{-}1)$$

If we rotate the x,y coordinate axes according to figure D-1, we have

$$x = x' \cos(\beta) - y' \sin(\beta) \ ,$$
$$y = x' \sin(\beta) + y' \cos(\beta) \ . \qquad (D\text{-}2)$$

Substitution in (D-1) yields

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F = 0 \ , \qquad (D\text{-}3)$$

where

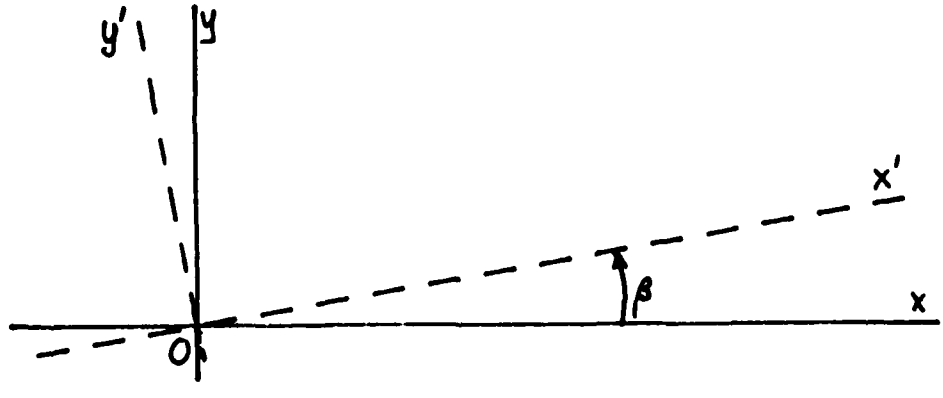


Figure D-1. Rotated Coordinate Axes

$$A' = \frac{1}{2}[A + C + (A - C)\cos(2\beta) + B\sin(2\beta)]$$

$$B' = B\cos(2\beta) + (C - A)\sin(2\beta)$$

$$C' = \frac{1}{2}[A + C - (A - C)\cos(2\beta) - B\sin(2\beta)]$$

$$D' = D\cos(\beta) + E\sin(\beta)$$

$$E' = -D\sin(\beta) + E\cos(\beta)\ . \qquad \text{(D-4)}$$

If we want to eliminate the cross-product term in (D-3), we must make $B' = 0$, that is, take

$$\tan(2\beta) = \frac{B}{A - C}\ . \qquad \text{(D-5)}$$

We will also choose 2β in the principal value range:

$$2\beta = \arctan\left(\frac{B}{A - C}\right); \qquad \text{(D-6)}$$

that is

$$-\frac{\pi}{2} \leq 2\beta \leq \frac{\pi}{2}\ , \qquad -\frac{\pi}{4} \leq \beta \leq \frac{\pi}{4}\ . \qquad \text{(D-7)}$$

All other solutions for 2β differ by $n\pi$; that is, β differs by $n\pi/2$. These are the major and minor axes of the curve described by (D-1).

If we now define

$$R = \sqrt{(A - C)^2 + B^2}\ , \qquad P = \mathrm{sgn}(A - C)\ , \qquad \text{(D-8)}$$

where $\sqrt{\ }$ denotes the positive square root, we find

$$\cos(2\beta) = \frac{|A - C|}{R} = \frac{A - C}{R} P \; ; \qquad (D\text{-}9)$$

see figure D-2. And since sin(2β) has the same polarity as tan(2β) in the principal value range,

$$\sin(2\beta) = \frac{|B|}{R} \operatorname{sgn}\left(\frac{B}{A - C}\right) = \frac{B}{R} P \; . \qquad (D\text{-}10)$$

Also, since

$$\cos^2(\beta) = \frac{1}{2}(1 + \cos(2\beta)) = \frac{1}{2}\left(1 + \frac{|A - C|}{R}\right), \qquad (D\text{-}11)$$

then

$$\cos(\beta) = \sqrt{\frac{R + |A - C|}{2R}} \; ,$$

$$\sin(\beta) = \sqrt{\frac{R - |A - C|}{2R}} \operatorname{sgn}\left(\frac{B}{A - C}\right) \; . \qquad (D\text{-}12)$$

It then follows that the coefficients in (D-4) simplify to

$$A' = \frac{1}{2}(A + C + RP)$$

$$B' = 0$$

$$C' = \frac{1}{2}(A + C - RP) \; , \qquad (D\text{-}13)$$

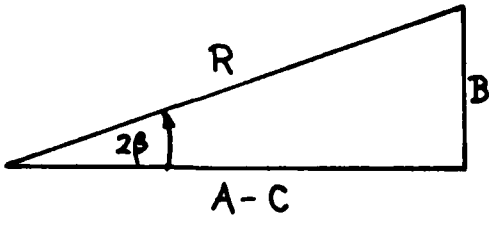


Figure D-2. Triangle Interpretation of (D-5)

from which there follows

$$A'C' = AC - \frac{1}{4}B^2 \quad . \tag{D-14}$$

Additionally, we have

$$\tan(\beta) = \frac{PR - (A - C)}{B} \quad ,$$

$$\cot(\beta) = \frac{PR + (A - C)}{B} \quad . \tag{D-15}$$

As a result of the above, the general equation in (D-3) can be written as

$$A'\left(x' + \frac{D'}{2A'}\right)^2 + C'\left(y' + \frac{E'}{2C'}\right)^2 = \frac{D'^2}{4A'} + \frac{E'^2}{4C'} - F \quad , \tag{D-16}$$

where

$$\frac{1}{2A'} = \frac{A + C - RP}{4AC - B^2} \quad , \quad \frac{1}{2C'} = \frac{A + C + RP}{4AC - B^2} \quad . \tag{D-17}$$

The simplest expressions for D' and E' appear to be those given in (D-4), in conjunction with (D-12). However, D'^2 and E'^2 can be simplified, resulting in expression

$$A'\left(x' + \frac{D'}{2A'}\right)^2 + C'\left(y' + \frac{E'}{2C'}\right)^2 = G \quad , \tag{D-18}$$

where

$$G = \frac{AE^2 + CD^2 - BDE}{4AC - B^2} - F \quad . \tag{D-19}$$

Now suppose that A and C in (D-1) are positive and that $4AC > B^2$. Then $A' > 0$ and $C' > 0$, meaning that (D-18) is an ellipse if $G > 0$.

Furthermore, if $A < C$, then $A' < C'$ and x' is the major axis. On the other hand, if $A > C$, then $A' > C'$ and x' is the minor axis. See figure D-3. The area of this ellipse is

$$\text{Area} = \frac{\pi G}{\sqrt{A'C'}} = \frac{2\pi G}{\sqrt{4AC - B^2}}, \tag{D-20}$$

by use of (D-14), where G is given by (D-19).

It follows directly from (D-18) that the curve is a circle if and only if $A' = C'$, that is, $R = 0$ via (D-13), which in turn means $A = C$ and $B = 0$ from (D-8).

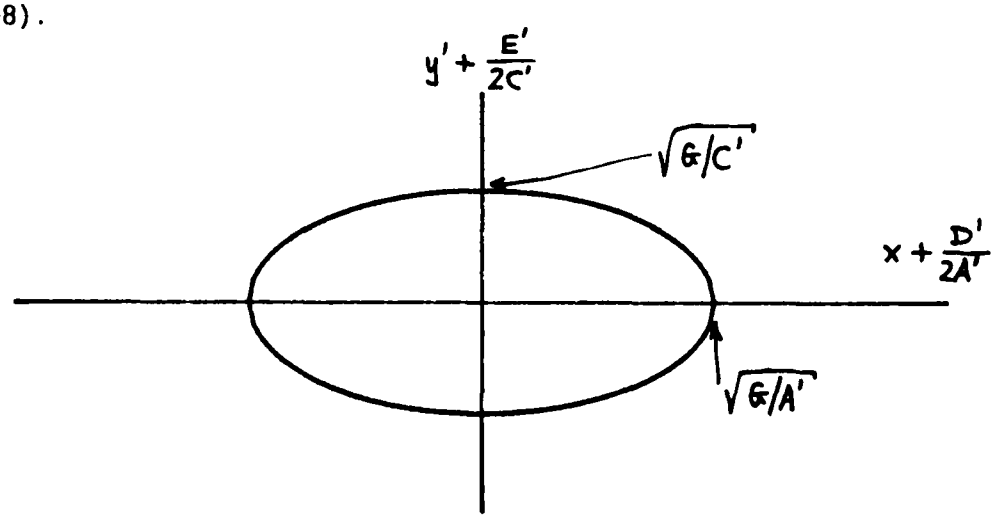


Figure D-3. Ellipse in Rotated Coordinates

EXAMPLE

Consider the ellipse in (91), for which

$$A = 1 + \Theta_o^2 \ , \quad B = -2\Theta_o \ , \quad C = 1; \quad \Theta_o \geq 0 \ . \tag{D-21}$$

Then (D-5) yields

$$\tan(2\beta) = -2/\Theta_o \ , \tag{D-22}$$

from which there follows

$$\tan\left(2\beta + \frac{\pi}{2}\right) = -1/\tan(2\beta) = \Theta_o/2 \ ,$$

$$\beta = \frac{1}{2} \arctan\left(\frac{\Theta_o}{2}\right) - \frac{\pi}{4} \ . \tag{D-23}$$

As Θ_o varies from 0 to $+\infty$, β varies from $-\pi/4$ to 0; thus β always lies in the principal value range, as required by (D-7).

However, since $A > C$ in (D-21), then β is the angle in the x,y plane of the <u>minor</u> axis of ellipse (91). The major axis angle in the x,y plane is

$$\psi = \beta + \frac{\pi}{2} \ , \tag{D-24}$$

which varies from $\pi/4$ to $\pi/2$. There follows

$$\tan(2\psi) = \tan(2\beta + \pi) = \tan(2\beta) = -2/\Theta_o \tag{D-25}$$

from (D-22), whereupon the slope of the major axis in the x,y plane can be obtained as

$$\tan\psi = \frac{1}{2}\left(\theta_0 + \sqrt{\theta_0^2 + 4}\right). \qquad \text{(D-26)}$$

This slope varies from 1 to $+\infty$ as θ_0 varies from 0 to $+\infty$. Conversely, given a measured slope, $\tan\psi$, of the major axis of a WDF contour in the x,y plane, the corresponding amount of linear frequency modulation can be determined from (D-25) or (D-26) as

$$\theta_0 = \tan\psi - 1/\tan\psi . \qquad \text{(D-27)}$$

The final determination of frequency modulation parameter α_0 in (84) requires the additional knowledge of σ_0 in (90).

In practice, where both σ_0 and α_0 are unknown apriori, the WDF will likely be plotted directly on the t,f plane. According to (78) and (D-24), the major axis will then lie on the line

$$f = t\,\frac{\tan\psi}{2\pi\sigma_0^2} , \qquad \text{(D-28)}$$

which can be observed and measured. But σ_0 can be determined separately from a slice in f (at fixed t) of the WDF, since the variation in f in (91) is proportional to

$$\exp\left[-\sigma_0^2\left(2\pi f - \alpha_0 t\right)^2\right]. \qquad \text{(D-29)}$$

Thus the distance, between frequency values that are down by 1/e from the peak on this frequency slice, is $(\pi\sigma_0)^{-1}$, and can be used for direct calculation of σ_0. Then (D-28) and (D-27) yield $\tan\psi$ and θ_0, respectively.

APPENDIX E. DISCRETE APPROXIMATION TO SHORT-TERM SPECTRAL ESTIMATE

The STSE is given by (102) and (103). A discrete approximation, by means of the Trapezoidal rule, is furnished by

$$\mathcal{S}(t,f) \equiv \left| \Delta \sum_{k} \exp(-i2\pi f \Delta k)\ s(k\Delta)\ u^*(t - k\Delta) \right|^2 =$$

$$= \Delta^2 \sum_{k\ell} \exp\left(-i2\pi f \Delta (k - \ell)\right)\ s(k\Delta)\ s^*(\ell\Delta)\ u^*(t - k\Delta)\ u(t - \ell\Delta) =$$

$$= \Delta^2 \sum_{k\ell} \exp\left(-i2\pi f \Delta (k - \ell)\right) R_s\left(\frac{k+\ell}{2}\Delta, k\Delta - \ell\Delta\right) R_u\left(t - \frac{k+\ell}{2}\Delta,\ k\Delta - \lambda\Delta\right), \tag{E-1}$$

where Δ is the sampling increment in time. Let

$$m = k - \ell\,, \quad n = k + \ell\,, \tag{E-2}$$

to get

$$\mathcal{S}(t,f) = \Delta^2 \sum_{\substack{m+n \\ \text{even}}} \exp(-i2\pi f \Delta m)\ R_s\left(\frac{n\Delta}{2}, m\Delta\right) R_u\left(t - \frac{n\Delta}{2},\ m\Delta\right). \tag{E-3}$$

Define, for use below, the function

$$C(t_1,t,f) = W_s(t_1,f) \overset{f}{\circledast} W_u(t - t_1,f) =$$

$$= \int df_1\ W_s(t_1,f_1)\ W_u(t - t_1, f - f_1)\,, \tag{E-4}$$

which is the convolution, on f, of WDFs W_s and W_u. When the integral on t_1 is effected, it yields the desired double convolution:

$$\int dt_1\ C(t_1,t,f) = W_s(t,f) \overset{tf}{\circledast} W_u(t,f)\ . \qquad \text{(E-5)}$$

We now express R_s and R_u in (E-3) in terms of the inverse transforms of W_s and W_u, respectively, according to (24), interchange summations and integrations, and use the facts that [4, chapter 2]

$$\sum_{m\ \text{even}} \exp(-i2\pi xm) = \frac{1}{2}\sum_{\ell}\delta(x - \tfrac{1}{2}\ell)\ ,$$

$$\sum_{m\ \text{odd}} \exp(-i2\pi xm) = \frac{1}{2}\sum_{\ell}(-1)^{\ell}\,\delta(x - \tfrac{1}{2}\ell)\ , \qquad \text{(E-6)}$$

to get approximation (E-1) in the form

$$\mathcal{J}(t,f) = \frac{\Delta}{2}\sum_{\ell}\sum_{n\ \text{even}} C\left(\frac{n\Delta}{2},t,f - \frac{\ell}{2\Delta}\right) +$$

$$+ \frac{\Delta}{2}\sum_{\ell}(-1)^{\ell}\sum_{n\ \text{odd}} C\left(\frac{n\Delta}{2},t,f - \frac{\ell}{2\Delta}\right)\ . \qquad \text{(E-7)}$$

The $\ell = 0$ terms together give, where the sum is now over all n,

$$\frac{\Delta}{2}\sum_{n} C(\frac{n\Delta}{2},t,f) \ , \qquad \text{(E-8)}$$

which is a discrete approximation to desired quantity (E-5).

The $\ell = 1$ terms in (E-7) yield

$$\frac{\Delta}{2}\sum_{n}(-1)^n \, C(\frac{n\Delta}{2},t,f - \frac{1}{2\Delta}) \ , \qquad \text{(E-9)}$$

which is approximately zero. A similar result holds for $\ell = -1$.

The $\ell = 2$ terms in (E-7) are

$$\frac{\Delta}{2}\sum_{n} C(\frac{n\Delta}{2},t,f - \frac{1}{\Delta}) \ , \qquad \text{(E-10)}$$

which is a discrete approximation to convolution (E-5), but shifted by frequency $1/\Delta$. For small sampling increment Δ in approximation (E-1), the quantity (E-10) will be small in the fundamental region centered at $f = 0$, and can be neglected. Thus (E-8) is the dominant term, giving

$$\ell(t,f) \cong \frac{\Delta}{2}\sum_{n} C(\frac{n\Delta}{2},t,f) =$$

$$\cong \int dt_1 \, C(t_1,t,f) = W_s(t,f) \overset{tf}{\bullet} W_u(t,f) \ . \qquad \text{(E-11)}$$

APPENDIX F. SOME SMOOTHING CONSIDERATIONS

This discussion complements and extends that given in (148)-(152) regarding two-dimensional smoothing of the WDF. For easy reference, we repeat the diagram under (24) and furnish an additional one for the smoothing functions that will be employed here. An arrow denotes a Fourier transform.

WAVEFORM FUNCTIONS

$$s\left(t+\frac{\tau}{2}\right)s^*\left(t-\frac{\tau}{2}\right) = R(t,\tau) \xleftrightarrow[f]{\tau} W(t,f)$$

$$t \updownarrow \upsilon \qquad\qquad t \updownarrow \upsilon$$

$$\chi(\upsilon,\tau) \xleftrightarrow[f]{\tau} A(\upsilon,f) = S\left(f+\frac{\upsilon}{2}\right)S^*\left(f-\frac{\upsilon}{2}\right)$$

SMOOTHING FUNCTIONS

$$v_2(t,\tau) \xleftrightarrow[f]{\tau} V_2(t,f)$$

$$t \updownarrow \upsilon \qquad\qquad t \updownarrow \upsilon$$

$$q_2(\upsilon,\tau) \xleftrightarrow[f]{\tau} Q_2(\upsilon,f)$$

By using the basic Fourier transform relations above, we may readily show that two-dimensional smoothing of WDF W with general function V_2 may be written in several alternative forms:

$$W(t,f) \overset{tf}{\circledcirc} V_2(t,f) =$$

$$= \iint d\upsilon\, d\tau\, \exp[i2\pi(\upsilon t - f\tau)]\, \chi(\upsilon,\tau)\, q_2(\upsilon,\tau) =$$

$$= \int d\tau \exp(-i2\pi f\tau)\, [R(t,\tau) \overset{t}{\circledcirc} v_2(t,\tau)] =$$

$$= \int d\upsilon\, \exp(i2\pi\upsilon t)\, [A(\upsilon,f) \overset{f}{\circledcirc} Q_2(\upsilon,f)] \; . \qquad \text{(F-1)}$$

The second line says that the ambiguity function χ of waveform s should be weighted by q_2 and the product then double Fourier-transformed into the t,f plane. The third line indicates smoothing of R on t, followed by transformation of a weighted function of τ. The last line performs smoothing of A on f, followed by transformation of a weighted function of υ. These relations extend those given in (150)-(152). The function V_2 above need <u>not</u> be a legal WDF. The volume under the smoothed distribution (F-1) is the product of the volumes under W and V_2; if the latter volume is unity, the energy of waveform s results again, as is desired.

INADEQUACY OF TIME SMOOTHING ALONG

Consider the special case where smoothing function V_2 is a delta function of f; then

$$V_2(t,f) = v_1(t)\, \delta(f)$$

$$v_2(t,\tau) = v_1(t)$$

$$q_2(\upsilon,\tau) = V_1(\upsilon)$$

$$Q_2(\upsilon,f) = V_1(\upsilon)\, \delta(f) \; . \qquad \text{(F-2)}$$

Equation (F-1) then simplifies to an averaging of the WDF solely in time:

$$W(t,f) \overset{t}{\circledast} v_1(t) =$$

$$= \iint d\upsilon\, d\tau\, \exp[i2\pi(\upsilon t - f\tau)]\, \chi(\upsilon,\tau)\, V_1(\upsilon) =$$

$$= \int d\tau\, \exp(-i2\pi f\tau)\, [R(t,\tau) \overset{t}{\circledast} v_1(t)] =$$

$$= \int d\upsilon\, \exp(i2\pi\upsilon t)\, A(\upsilon,f)\, V_1(\upsilon)\ , \qquad \text{(F-3)}$$

which is an extension of (148)-(149).

The advantageous feature of locally averaging the instantaneous correlation R in time, indicated in line 3, is equivalent to weighting the "local spectrum"

$$A(\upsilon,f) = S(f + \tfrac{\upsilon}{2})\, S^*(f - \tfrac{\upsilon}{2}) \qquad \text{(F-4)}$$

in line 4 by function $V_1(\upsilon)$, prior to Fourier transforming back into the t domain. This weighting on υ is sensible, since if WDF W(t,f) or some modified version is to represent the spectrum at f, the transform on υ in line 4 of (F-3) ought not to involve arbitrarily distant values of υ; otherwise, waveform spectrum S in (F-4) will then be utilized at argument values very different from the frequency f of interest and would be nonrepresentative. However, there is no weighting on τ in line 3 of (F-3), thereby allowing arbitrarily distant argument values of signal s, from the

time instant t of interest, to be considered; this unrealistic feature of the WDF is one of the reasons for its undesirable properties.

INADEQUACY OF FREQUENCY SMOOTHING ALONE

Now consider the alternative special case where V_2 in (F-1) is a delta function of t; then

$$V_2(t,f) = \delta(t)\ V_1(f)$$

$$v_2(t,\tau) = \delta(t)\ v_1(\tau)$$

$$q_2(v,\tau) = v_1(\tau)$$

$$Q_2(v,f) = V_1(f)\ . \qquad \text{(F-5)}$$

Equation (F-1) then simplifies to an averaging of the WDF solely in frequency:

$$W(t,f) \overset{f}{\bullet} V_1(f) =$$

$$= \iint dv\ d\tau\ \exp[i2\pi(vt - f\tau)]\ \chi(v,\tau)\ v_1(\tau) =$$

$$= \int d\tau\ \exp(-i2\pi f\tau)\ R(t,\tau)\ v_1(\tau) =$$

$$= \int dv\ \exp(i2\pi vt)\ [A(v,f) \overset{f}{\bullet} V_1(f)]\ . \qquad \text{(F-6)}$$

The advantageous feature of averaging the "local spectrum" A in frequency, indicated in line 4, is equivalent to weighting the instantaneous correlation $R(t,\tau)$ in line 3 by function $v_1(\tau)$, prior to Fourier transforming back into the f domain. This weighting on τ is sensible, since if WDF $W(t,f)$ or some modified version is to represent the time behavior at t, the transform on τ in line 3 of (F-6) ought not to involve arbitrarily distant values of τ; otherwise, waveform s will then be utilized at argument values very different from the time t of interest and would be non-representative. However, there is no weighting on υ in line 4 of (F-6), thereby allowing arbitrarily distant argument values of spectrum S, from the frequency f of interest, to be considered; this unrealistic feature of the WDF is an additional reason for its drawbacks.

SEPARABLE SMOOTHING

If smoothing function V_2 is separable, then

$$V_2(t,f) = v_a(t)\, V_b(f)$$

$$v_2(t,\tau) = v_a(t)\, v_b(\tau)$$

$$q_2(\upsilon,\tau) = V_a(\upsilon)\, v_b(\tau)$$

$$Q_2(\upsilon,f) = V_a(\upsilon)\, V_b(f)\ .$$

Then (F-1) gives, for example,

$$W(t,f) \overset{t}{\circledast} v_a(t) \overset{f}{\circledast} V_b(f) =$$

$$= \int d\tau \exp(-i2\pi f\tau)\ v_b(\tau) \int dt'\ R(t - t',\tau)\ v_a(t') \ .$$

This has both the desirable features of locally averaging the correlation and suppressing large-τ contributions. However, it restricts the form of averaging in the t,f plane and specifically disallows tilted smoothing regions which are not parallel to the t or f axes.

GENERAL GAUSSIAN TWO-DIMENSIONAL SMOOTHING

The inadequacies of smoothing in time alone or frequency alone suggest consideration of the general two-dimensional result in (F-1):

$$\hat{W}(t,f) \equiv W(t,f) \overset{tf}{\circledast} V_2(t,f) =$$

$$= \int d\tau \exp(-i2\pi f\tau) \int dt'\ R(t - t',\tau)\ v_2(t',\tau) =$$

$$= \int d\tau \exp(-i2\pi f\tau) \int dt'\ s(t - t' + \tfrac{\tau}{2})\ s^*(t - t' - \tfrac{\tau}{2})\ v_2(t',\tau) \ . \qquad \text{(F-7)}$$

If we let $t_1 = t' + \frac{\tau}{2}$, $t_2 = t' - \frac{\tau}{2}$, this becomes

$$\hat{W}(t,f) = \iint dt_1\ dt_2 \exp(-i2\pi f(t_1 - t_2))\ s(t - t_2)\ s^*(t - t_1)\ v_2\left(\frac{t_1 + t_2}{2}\ ,\ t_1 - t_2\right) . \qquad \text{(F-8)}$$

Now let two-dimensional smoothing function V_2 have the general Gaussian form

$$V_2(t,f) = 2\sqrt{Q}\, \exp(-a^2t^2 - 4\pi^2b^2f^2 - 4\pi ctf) , \qquad (F-9)$$

where a,b,c are real constants and

$$Q = a^2b^2 - c^2 . \qquad (F-10)$$

The scale factor, $2\sqrt{Q}$, is chosen so that the volume under V_2 in the t,f plane is 1; this keeps the volume under the smoothed distribution in (F-1) or (F-7) at E, the energy of waveform s. In order that V_2 tend to zero at infinity in the t,f plane, we must have $Q > 0$. The area in the t,f plane of the contour ellipse of (F-9), at the 1/e relative level, is (appendix D)

$$\text{Area} = \frac{1/2}{\sqrt{a^2b^2 - c^2}} = \frac{1/2}{\sqrt{Q}} . \qquad (F-11)$$

The transform on f of V_2 in (F-9) is

$$v_2(t,\tau) = \frac{\sqrt{Q/\pi}}{b} \exp\left[-\frac{1}{b^2}\left\{Q\,t^2 + \frac{\tau^2}{4} + ict\tau\right\}\right] . \qquad (F-12)$$

For completeness, the two remaining smoothing functions in (F-1) are

$$q_2(v,\tau) = \exp\left[-\frac{1}{4Q}\left(a^2\tau^2 + 4\pi^2b^2v^2 + 4\pi c\tau v\right)\right] ,$$

$$Q_2(v,f) = \frac{2}{a}\sqrt{Q\pi}\, \exp\left[-\frac{4\pi^2}{a^2}\left(Qf^2 + \frac{v^2}{4} - icfv\right)\right] .$$

We can now determine the quantity necessary for evaluation of (F-8), namely

$$V_2\left(\frac{t_1 + t_2}{2}\,,\ t_1 - t_2\right) = \frac{\sqrt{Q/\pi}}{b} \exp\left[-\frac{1}{4b^2}\left\{(Q + 1 + i2c)\ t_1^2 + \right.\right.$$

$$\left.\left. + (Q + 1 - i2c)\ t_2^2 + 2(Q - 1)\ t_1 t_2\right\}\right]. \qquad \text{(F-13)}$$

By the discussion in (25) et seq., this function in (F-13) is separable in t_1 and t_2 if and only if

$$Q = 1, \quad \text{that is,} \quad a^2b^2 - c^2 = 1. \qquad \text{(F-14)}$$

Then V_2 in (F-9) is a legal WDF and the area in (F-11) becomes 1/2. Also, smoothing V_2 in (F-9) is then exactly equivalent to the Gaussian smoothing considered previously in (147).

We are interested here, however, in the more general case of V_2 where Q is not necessarily 1, and therefore V_2 is not a legal WDF. If we substitute (F-13) in (F-8), the smoothed WDF becomes

$$\hat{W}(t,f) = \frac{\sqrt{Q/\pi}}{b} \iint dt_1\ dt_2\ x(t_1)\ x^*(t_2) \exp\left[-\frac{Q-1}{2b^2}\ t_1 t_2\right], \qquad \text{(F-15)}$$

where

$$x(t_1) = s^*(t - t_1) \exp\left[-i2\pi f t_1 - (Q + 1 + i2c)\ \frac{t_1^2}{4b^2}\right]. \qquad \text{(F-16)}$$

By expanding the exp in (F-15) in a power series, there follows

$$\hat{W}(t,f) = \frac{\sqrt{Q/\pi}}{b} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1 - Q}{2b^2}\right)^n \left|\int dt_1\ x(t_1)\ t_1^n\right|^2. \qquad \text{(F-17)}$$

It is obvious from (F-17) that a sufficient condition for smoothed WDF $\hat{W}$ to be non-negative is $Q \leq 1$, that is

$$a^2b^2 - c^2 \leq 1 \ . \qquad \text{(F-18)}$$

(The special case of $c = 0$ was given in [11, (5)].) When this condition is used in (F-11), we see that the area of the concentration ellipse of (F-9) is

$$\text{Area} \geq 1/2 \ . \qquad \text{(F-19)}$$

Thus, smoothing with the Gaussian two-dimensional function V_2 in (F-9) always results in a non-negative distribution, provided that the area of the ellipse at the 1/e relative level is greater than or equal to 1/2. It is not necessary that V_2 in (F-9) be a legal WDF; that is, the area of the ellipse need not be precisely 1/2. However, the most concentrated V_2 in (F-9), that guarantees nonnegative results, has area 1/2. The only restrictions on parameters a,b,c are given by $0 < Q \leq 1$, that is, $0 < a^2b^2 - c^2 \leq 1$.

APPENDIX G. DERIVATION OF MINIMUM-SPREAD WDF

The short-term spectral estimate of waveform s, relative to weighting u, is given by the double convolution (106),

$$\left|S_u(t,f)\right|^2 = W_s(t,f) \overset{tf}{\circledast} W_u(t,f) \ , \tag{G-1}$$

of the WDFs of s and u. It is therefore important to use, for weighting u, a function which has as narrow a WDF as possible, so that the smearing implied by (G-1) is minimized. In particular, since we are interested in analyzing waveforms with linear frequency modulation, we are interested in minimizing the spread of WDF W_u, as measured by the quadratic quantity

$$I = \iint dt\ df\ W_u(t,f)\ (f - \beta_c t)^2 \ , \tag{G-2}$$

where β_c is a specified (observed) slope in the t,f plane.

By expanding the quadratic in (G-2), we obtain spread

$$I = I_o + I_1 + I_2 \ , \tag{G-3}$$

where

$$I_o = \iint dt\ df\ f^2\ W_u(t,f) \ ,$$

$$I_1 = -2\beta_c \iint dt\ df\ t\ f\ W_u(t,f) \ ,$$

$$I_2 = \beta_c^2 \iint dt\ df\ t^2\ W_u(t,f) \ . \tag{G-4}$$

Reference to (43), (28), and (12), respectively, allows the terms in (G-4) to be simplified and expressed solely in the time domain as

$$I_0 = \frac{1}{4\pi^2} \int dt \, |u'(t)|^2 ,$$

$$I_1 = - \frac{\beta_c}{\pi} \int dt \, t \, \mathrm{Im}\{u'(t) \, u^*(t)\} ,$$

$$I_2 = \beta_c^2 \int dt \, t^2 |u(t)|^2 . \qquad \text{(G-5)}$$

Adding these results together, the spread in (G-3) becomes

$$I = \frac{1}{4\pi^2} \int dt \, |u'(t) - i\alpha_c t \, u(t)|^2 , \qquad \text{(G-6)}$$

where we define

$$\alpha_c = 2\pi\beta_c . \qquad \text{(G-7)}$$

Observe that the spread I in (G-6) is nonnegative, for all weightings u.

The function that minimizes I in (G-6) is

$$u(t) = a_0 \exp\left(i \frac{\alpha_c}{2} t^2\right) , \quad a_0 \text{ complex}, \qquad \text{(G-8)}$$

for which

$$I = 0 , \quad W_u(t,f) = |a_o|^2 \, \delta(f - \beta_c t) . \tag{G-9}$$

That is, the WDF is concentrated on the $f = \beta_c t$ line in the t,f plane. However, the energy of (G-8) is

$$E_u = \infty , \tag{G-10}$$

which is unacceptable.

If we attempt to approximate (G-8) by unit energy weighting

$$u(t) = (\pi\sigma^2)^{-1/4} \exp\left(-\frac{t^2}{2\sigma^2} + i\frac{\alpha}{2}t^2\right) , \tag{G-11}$$

the spread of u, as given by (G-6), turns out to be

$$I = \frac{1}{8\pi^2}\left[\frac{1}{\sigma^2} + \sigma^2(\alpha - \alpha_c)^2\right] . \tag{G-12}$$

Now if $\alpha = \alpha_c$, that is, the linear frequency modulation parameter α in weighting u is exactly equal to given quantity α_c (from (G-7) and (G-2)), the spread is

$$I = \frac{1}{8\pi^2\sigma^2} > 0 . \tag{G-13}$$

However, as duration σ of weighting (G-11) gets larger, the spread I tends to zero, even though the weighting has finite (unit) energy. Also, then (G-11) tends to a scaled version of (G 8) at each fixed t.

In order to eliminate these undesirable features of the weighting, it will not be sufficient to minimize spread 1, subject only to a constraint on the energy of u. Rather, it will also be necessary to constrict the time duration of the weighting u. Accordingly, we will minimize spread I in (G-2)-(G-6), under the two constraints that

$$\int dt\ |u(t)|^2 = 1 = \iint dt\ df\ W_u(t,f)\ ,$$

$$\int dt\ t^2 |u(t)|^2 = \frac{\sigma_c^2}{2} = \iint dt\ df\ t^2\ W_u(t,f)\ ; \qquad \text{(G-14)}$$

see (14) and (12).

Thus consider

$$Q = \int dt\ |u'(t) - i\alpha_c t\ u(t)|^2 + \lambda \int dt\ |u(t)|^2 + \mu \int dt\ t^2\ |u(t)|^2\ , \qquad \text{(G-15)}$$

where λ and μ are real Lagrange multipliers. Replacing u(t) by $u(t) + \epsilon n(t)$, where n(t) is an allowed variation, we have

$$Q + \delta Q = \int dt\ [u'(t) + \epsilon n'(t) - i\alpha_c t\ (u(t) + \epsilon n(t))]\ [u'^*(t) + \epsilon^* n'^*(t) +$$

$$+ i\alpha_c t\ (u^*(t) + \epsilon^* n^*(t))] + \int dt\ [u(t) + \epsilon n(t)]\ [u^*(t) + \epsilon^* n^*(t)]\ (\lambda + \mu t^2)\ . \qquad \text{(G-16)}$$

Since the coefficient of ε^* must be zero, in order for u(t) to be the optimum [15], we require the following:

$$\int dt\ \eta'^*(t)\ [u'(t) - i\alpha_c t\ u(t)] + \int dt\ \eta^*(t)\ [u'(t) - i\alpha_c t\ u(t)]\ i\alpha_c t +$$

$$+ \int dt\ \eta^*(t)\ u(t)\ (\lambda + \mu t^2) = 0 \quad \text{for all } \eta(t)\ . \tag{G-17}$$

We now integrate by parts on the first term of (G-17), and presume that η, u, u' all decay to zero at $\pm\infty$. Since η is arbitrary, its coefficient under the integral must be zero; namely, we find that u must satisfy the following differential equation:

$$-u''(t) + i2\alpha_c t\ u'(t) + (i\alpha_c + \alpha_c^2 t^2 + \lambda + \mu t^2)\ u(t) = 0 \quad \text{for all } t\ . \tag{G-18}$$

If we try solution

$$u(t) = a \exp\left(\frac{1}{2} ct^2\right)\ , \quad a,c \text{ complex}\ , \tag{G-19}$$

in (G-18), we find that

$$t^2(-c^2 + i2\alpha_c c + \alpha_c^2 + \mu) - c + i\alpha_c + \lambda = 0 \quad \text{for all } t\ . \tag{G-20}$$

Then a solution of the form (G-19) exists with the choices

$$c = i\alpha_c \pm \sqrt{\mu}\ , \quad \lambda = \pm\sqrt{\mu}\ . \tag{G-21}$$

To determine u(t) explicitly, we substitute (G-21) into (G-19) to get

$$u(t) = a \exp\left(i \frac{1}{2} \alpha_c t^2 \pm \frac{1}{2}\sqrt{\mu}\, t^2\right) . \qquad \text{(G-22)}$$

When the two constraints in (G-14) are satisfied, there follows, for the optimum weighting,

$$u(t) = \left(\pi \sigma_c^2\right)^{-1/4} \exp\left(-\frac{t^2}{2\sigma_c^2} + i \frac{\alpha_c}{2} t^2\right) . \qquad \text{(G-23)}$$

This is linear frequency modulation with a Gaussian envelope.

The minimum value of the spread I in (G-6) for the optimum weighting (G-23) is

$$\text{minimum } I = \frac{1}{8\pi^2\sigma_c^2} , \qquad \text{(G-24)}$$

and the corresponding WDF is

$$W_u(t,f) = 2 \exp\left[-\frac{t^2}{\sigma_c^2} - 4\pi^2\sigma_c^2(f - \beta_c t)^2\right] =$$

$$= 2 \exp\left[-\frac{t^2}{\sigma_c^2}\left(1 + \theta_c^2\right) + 4\pi f t \theta_c - 4\pi^2 f^2 \sigma_c^2\right] , \qquad \text{(G-25)}$$

where $\theta_c = \alpha_c \sigma_c^2$. The area of the contour ellipse at the 1/e relative level is 1/2 in the t,f plane.

The mean-square time extent of the optimum weighting u in (G-23) is $\sigma_c^2/2$, as required by constraint (G-14). The mean-square frequency extent is obtained from voltage spectrum

$$U(f) = \pi^{1/4}\left(\frac{2\sigma_c}{1 - i\theta_c}\right)^{1/2} \exp\left(\frac{-2\pi^2 f^2 \sigma_c^2}{1 - i\theta_c}\right) \qquad \text{(G-26)}$$

as

$$\int df\ f^2 |U(f)|^2 = \frac{1 + \theta_c^2}{8\pi^2 \sigma_c^2}\ . \qquad \text{(G-27)}$$

APPENDIX H. EXAMPLE OF WDF PROCESSOR

The processor of interest here is depicted in figure 7, while representative characteristics of the waveforms and devices are sketched in figure 8. The mean output is given by (173) and the variance is given by (188). We will use the definitions and results in (165)-(189) freely in the following.

INPUT INFORMATION

The input signal waveform to figure 7 is Gaussian-modulated linear frequency modulation:

$$s_0(t) = a_0 \exp\left[-\frac{t^2}{2\sigma_0^2} + i\,\frac{\alpha_0}{2}\,t^2\right], \quad E_0 = \sqrt{\pi}\,|a_0|^2\sigma_0 , \tag{H-1}$$

where a_0 can be complex. The instantaneous input signal power,

$$|s_0(t)|^2 = |a_0|^2 \exp(-t^2/\sigma_0^2) , \tag{H-2}$$

peaks at $t = 0$ and has effective duration σ_0.

The corresponding signal voltage density spectrum is

$$S_0(f) = a_0\sigma_0\left(\frac{2\pi}{1 - i\theta_0}\right)^{1/2} \exp\left(-\frac{2\pi^2\sigma_0^2 f^2}{1 - i\theta_0}\right), \quad \theta_0 = \alpha_0\sigma_0^2 . \tag{H-3}$$

The energy density spectrum is

$$\left|S_o(f)\right|^2 = \frac{2\pi |a_o|^2 \sigma_o^2}{\sqrt{1 + \theta_o^2}} \exp\left(-\frac{4\pi^2 \sigma_o^2 f^2}{1 + \theta_o^2}\right), \tag{H-4}$$

which peaks at f = 0. When

$$f = \pm f_o = \pm \frac{\sqrt{1 + \theta_o^2}}{2\pi\sigma_o}, \tag{H-5}$$

the energy density spectrum is reduced to 1/e of its peak value; hence f_o is a measure of the bandwidth of the linear frequency modulation waveform.

The input noise n_o to figure 7 has a white spectrum

$$G_{n_o}(f) = N_d \quad \text{for all } f . \tag{H-6}$$

The filter transfer function is

$$H(f) = \exp\left(-\frac{f^2}{2B^2}\right), \tag{H-7}$$

which peaks at f = 0; this coincides with the signal spectrum peak, which means that we are considering the most fortuitous situation. The weighting in figure 7 is taken to be

$$v(t) = \exp\left(-\frac{t^2}{2L^2}\right), \tag{H-8}$$

which peaks at t = 0; again, this coincides with the signal peak and is most favorable. The maximum values of H and v, being equal to 1, are chosen for convenience, without loss of generality; absolute level does not influence the performance of the processor in figure 7.

CALCULATIONS OF BASIC FUNCTIONS

We will make frequent use of (68)-(70) in evaluating the following quantities which are needed in (165)-(189); the choices for Gaussian functions for s_o,H,v, above, were made for analytic simplicity, since the various integrals can be conducted in closed form. More general cases would require numerical integrations.

The noise spectrum at the output of the filter is

$$G_n(f) = G_{n_o}(f) \left|H(f)\right|^2 = N_d \exp(-f^2/B^2) \quad . \tag{H-9}$$

The corresponding noise correlation is

$$C_n(\tau) = \int df \exp(i2\pi f\tau)\, G_n(f) = \sqrt{\pi}\, N_d\, B \exp(-\pi^2 B^2 \tau^2) \quad . \tag{H-10}$$

The auxiliary spectrum in (178) is

$$G_n^{(2)}(f) = \int d\tau \exp(-i4\pi f\tau)\, C_n^2(\tau) = \sqrt{\frac{\pi}{2}}\, N_d^2\, B \exp(-2f^2/B^2) \quad . \tag{H-11}$$

The instantaneous correlation of the weighting is

$$R_v(t,\tau) = v\left(t + \frac{\tau}{2}\right) v^*\left(t - \frac{\tau}{2}\right) = \exp\left(-\frac{t^2}{L^2} - \frac{\tau^2}{4L^2}\right), \qquad (H\text{-}12)$$

and its corresponding WDF is

$$W_v(t,f) = \int d\tau \exp(-i2\pi f\tau)\, R_v(t,\tau) = 2\sqrt{\pi}\, L \exp\left(-\frac{t^2}{L^2} - 4\pi^2 L^2 f^2\right). \qquad (H\text{-}13)$$

The filter impulse response is

$$h(\tau) = \int df \exp(i2\pi f\tau)\, H(f) = \sqrt{2\pi}\, B \exp(-2\pi^2 B^2 \tau^2), \qquad (H\text{-}14)$$

leading to filter output signal

$$s(t) = \int d\tau\, h(\tau)\, s_0(t - \tau) =$$

$$= a_0 \left(\frac{D}{1 + D - i\theta_0}\right)^{1/2} \exp\left[-2\pi^2 B^2 \frac{1 - i\theta_0}{1 + D - i\theta_0} t^2\right], \qquad (H\text{-}15)$$

where

$$\theta_0 = \alpha_0 \sigma_0^2, \quad D = (2\pi B\sigma_0)^2. \qquad (H\text{-}16)$$

This filter output signal is again a Gaussian-modulated linear frequency modulation waveform.

In general, for signal

$$s(t) = c_0 \exp(-ct^2/2), \quad c_0, c \text{ complex}, \quad E = \sqrt{\pi}\,|c_0|^2/\sqrt{c_r}, \qquad (H\text{-}17)$$

the WDF is

$$W_s(t,f) = 2\sqrt{\pi}\,\frac{|c_o|^2}{\sqrt{c_r}}\,\exp\left[-\frac{t^2|c|^2 + 4\pi^2 f^2 + 4\pi f t c_i}{c_r}\right]. \qquad \text{(H-18)}$$

When applied to example (H-15), we identify

$$c_o = a_o\left(\frac{D}{1 + D - i\theta_o}\right)^{1/2}, \quad c = 4\pi^2 B^2\,\frac{1 - i\theta_o}{1 + D - i\theta_o}, \qquad \text{(H-19)}$$

to obtain

$$|c_o|^2 = |a_o|^2\,\frac{D}{\sqrt{D_2}}, \quad |c|^2 = (2\pi B)^4\,\frac{1 + \theta_o^2}{D_2},$$

$$c_r = 4\pi^2 B^2\,\frac{D_1}{D_2}, \quad c_i = -4\pi^2 B^2\,\frac{D\theta_o}{D_2}, \qquad \text{(H-20)}$$

where

$$D_1 = 1 + D + \theta_o^2, \quad D_2 = (1 + D)^2 + \theta_o^2. \qquad \text{(H-21)}$$

When substituted in (H-18), there follows, for the WDF of s in (H-15),

$$W_s(t,f) = 2E_o\left(\frac{D}{D_1}\right)^{1/2}\exp\left[-\frac{1}{D_1}\left\{(1 + \theta_o^2)(2\pi Bt)^2 + D_2\left(\frac{f}{B}\right)^2 - 2D\theta_o(2\pi Bt)\left(\frac{f}{B}\right)\right\}\right]. \qquad \text{(H-22)}$$

As bandwidth B of filter H in (H-7) tends to infinity, then $D \to \infty$, $D_1 \to D$, $D_2 \to D^2$, and (H-22) yields

$$W_s(t,f) \to 2E_o\exp\left[-(1 + \theta_o^2)\,\frac{t^2}{\sigma_o^2} - (2\pi\sigma_o f)^2 + 4\pi\theta_o f t\right] \text{as } B \to \infty. \qquad \text{(H-23)}$$

This agrees with (91) and (78). More generally, in order to keep the scale factor $(D/D_1)^{1/2}$ in (H-22) near 1, we need

$$D > 1 + \theta_0^2 , \quad \text{that is, } B > \frac{\sqrt{1 + \theta_0^2}}{2\pi\sigma_0} , \tag{H-24}$$

according to (H-21) and (H-16). But this latter quantity is just the bandwidth of signal s_0; see (H-5). Thus condition (H-24) yields the physically intuitive statement that the filter passband should be wider than the input signal bandwidth, in order not to decrease the peak value of the WDF of the filter output signal.

The area of the elliptical contour of the general WDF in (H-22), at the 1/e relative level, is 1/2 in the t,f plane, regardless of the values of any of the parameters of the input signal and filter; this follows by the direct use of (D-1), (D-19), and (D-20). It is also consistent with the general fact that this is true for any signal of the form of (H-17), as may be seen by application of appendix D directly to (H-18), where c and c_0 are arbitrary complex constants.

The peak height of the signal WDF in (H-22) is $2E_0\sqrt{D/D_1}$; hence, the product of peak height and effective area is $E_0\sqrt{D/D_1}$, which is just the energy of s:

$$E = \int dt\, |s(t)|^2 = E_0\sqrt{\frac{D}{D_1}} = \iint dt\, df\, W_s(t,f) . \tag{H-25}$$

This follows directly from (H-15) or (H-17).

The tilt of the major axis of the elliptical contours of (H-22) is given by β radians in the $(2\pi Bt, f/B)$ plane, where

$$\tan(2\beta) = \frac{\Theta_o}{1 + D/2} , \tag{H-26}$$

according to (D-5) and (H-21).

MEAN SIGNAL OUTPUT

The mean signal output of the WDF processor in figure 7 is given by (171) as

$$a = \int d\tau \exp(-i2\pi f\tau) R_v(t,\tau) R_s(t,\tau) =$$

$$= \int d\tau \exp(-i2\pi f\tau) R_{\tilde{s}}(t,\tau) = W_{\tilde{s}}(t,f) , \tag{H-27}$$

where

$$\tilde{s}(t) = v(t) s(t) = c_o \exp(-\tilde{c} t^2/2) \tag{H-28}$$

and

$$\tilde{c} = c + \frac{1}{L^2} \tag{H-29}$$

according to (H-8) and (H-15)-(H-19). By analogy to (H-18), we have

$$a = W_{\tilde{s}}(t,f) = 2\sqrt{\pi}\,\frac{|c_0|^2}{\sqrt{\tilde{c}_r}}\,\exp\left[-\frac{t^2|\tilde{c}|^2 + 4\pi^2 f^2 + 4\pi f t\,\tilde{c}_i}{\tilde{c}_r}\right] =$$

$$= 2E_0\left(\frac{DR}{H_1}\right)^{1/2}\exp\left[-\frac{1}{H_1}\left\{H_2\,x^2 + D_2 y^2 - 2RD\Theta_0\,xy\right\}\right], \qquad \text{(H-30)}$$

where we defined

$$R = (2\pi BL)^2\ ,\quad x = t/L\ ,\quad y = 2\pi fL\ ,$$

$$H_1 = D_2 + RD_1 = (1 + D)(1 + D + R) + (1 + R)\,\Theta_0^2\ ,$$

$$H_2 = D_2 + 2RD_1 + R^2(1 + \Theta_0^2) = (1 + D + R)^2 + (1 + R)^2\Theta_0^2\ , \qquad \text{(H-31)}$$

and used

$$\tilde{c}_r = \frac{H_1}{L^2 D_2}\ ,\quad \tilde{c}_i = -\frac{RD\Theta_0}{L^2 D_2}\ ,\quad |\tilde{c}|^2 = \frac{H_2}{L^4 D_2}\ . \qquad \text{(H-32)}$$

The area of the concentration ellipse of $W_{\tilde{s}}$ in (H-30) at the 1/e relative level is 1/2 in the t,f plane, regardless of the sizes of D and R; so the signal WDF is not spread by the filtering and weighting operations in figure 7, at least for this example of (H-1) coupled with (H-7) and (H-8).

The peak height of WDF $W_{\tilde{s}}$ is given by the leading factor in (H-30). Since the effective area of this WDF is 1/2, the product of peak height and effective area is $E_0\sqrt{DR/H_1}$, which is just the energy of $\tilde{s}$:

$$\tilde{E} = \int dt\ |\tilde{s}(t)|^2 = E_0\left(\frac{DR}{(1 + D)(1 + D + R) + (1 + R)\,\Theta_0^2}\right)^{1/2}\ . \qquad \text{(H-33)}$$

The parameters Θ_o,D,R are given in (H-16) and (H-31).

We now define

$$L_b = \frac{L}{\sqrt{1+R}} = \frac{L}{\sqrt{1+(2\pi BL)^2}} ; \quad \frac{1}{L_b^2} = \frac{1}{L^2} + (2\pi B)^2 . \tag{H-34}$$

Then (H-30) and (H-31) become

$$W_{\tilde{s}}(t,f) = 2E_o\left(\frac{DR}{H_1}\right)^{1/2} \exp\left[-\frac{1}{H_1}\left\{\frac{H_2}{1+R}\left(\frac{t}{L_b}\right)^2 + D_2(1+R)(2\pi L_b f)^2 - 2RD\Theta_o(2\pi L_b f)\left(\frac{t}{L_b}\right)\right\}\right]. \tag{H-35}$$

The major axis of the elliptical contours is at angle $\tilde{\beta}$ radians in the $(t/L_b, 2\pi L_b f)$ plane, where

$$\tan(2\tilde{\beta}) = \frac{\Theta_o(1+R)}{1+R+D+RD/2} . \tag{H-36}$$

Given measurements or observations $\tilde{\beta}$ and σ_o, this can be immediately solved for Θ_o, where it is presumed that B and L are known since they are under our control.

As alternative fundamental parameter is more useful than the above; we introduce

$$M = (L/\sigma_o)^2 , \tag{H-37}$$

which is the square of the ratio of weighting time to input signal duration. Then (H-31) yields

$$R = DM , \tag{H-38}$$

where $D = (2\pi B\sigma_0)^2$ just as defined in (H-16). Eliminating R in favor of D and M, the peak height of W_ξ in (H-30) becomes

$$\text{Peak} = 2E_0\left(\frac{D^2M}{(1+D)(1+D+DM)+(1+DM)\theta_0^2}\right)^{1/2} . \qquad \text{(H-39)}$$

As checks on this quantity, observe that as $D \to \infty$, the factor of $2E_0$ in (H-39) approaches $(M/(M+1))^{1/2}$; in order to keep this latter quantity near 1, we need $M > 1$, that is, $L > \sigma_0$. This is consistent with physical reasoning on figure 7. Alternatively, as $M \to \infty$, the factor of $2E_0$ in (H-39) approaches $(D/(1+D+\theta_0^2))^{1/2}$; in order to keep this near 1, we need (H-24) to be observed, just as before.

More generally, in order to keep the signal factor in (H-39) near unity, we need to choose the combination of D and M large enough. For example, to keep the factor at value F, that is, maintain

$$\left(\frac{D^2M}{(1+D)(1+D+DM)+(1+DM)\theta_0^2}\right)^{1/2} = F \quad (<1) , \qquad \text{(H-40)}$$

we need to choose L in (H-37) such that

$$M = \frac{F^2[(1+D)^2+\theta_0^2]}{D[D-F^2(1+D+\theta_0^2)]} . \qquad \text{(H-41)}$$

However, this relationship is useful only for

$$D > \frac{F^2}{1-F^2}(1+\theta_0^2) . \qquad \text{(H-42)}$$

A representative sketch of (H-40) is displayed in figure H-1. Small values of parameters M or D are not realizable without the other parameter tending to infinity, in order to maintain the factor in (H-40) at value F. Three numerical examples, for Θ_o = 0,1,5 are given in figures H-2, H-3, and H-4, respectively. The larger F values can only be achieved through rather large D and/or M values.

There is, however, a minimum value of the product, MD, required to realize a specified value F for the factor in (H-40), when Θ_o is specified. In fact, we find from (H-41) that, for given F and Θ_o, the product MD is minimized by the choices

$$D_{opt} = \frac{1 + \Theta_o^2}{1 - F^2}\left[F^2 + \sqrt{\frac{1 + F^4\Theta_o^2}{1 + \Theta_o^2}}\right],$$

$$M_{opt} = \frac{2F^2}{1 - F^4}\left[1 + \sqrt{\frac{1 + F^4\Theta_o^2}{1 + \Theta_o^2}}\right]. \qquad \text{(H-43)}$$

The value M_{opt} is relatively insensitive to Θ_o; in fact, it varies from $4F^2/(1 - F^4)$ to $2F^2(1 - F^2)$ as Θ_o varies from 0 to∞, which is less than a 2:1 variation.

The corresponding minimum product is

$$(MD)_{min} = \frac{2F^2}{(1 - F^2)^2}\left[1 + F^2\Theta_o^2 + \sqrt{(1 + \Theta_o^2)(1 + F^4\Theta_o^2)}\right]. \qquad \text{(H-44)}$$

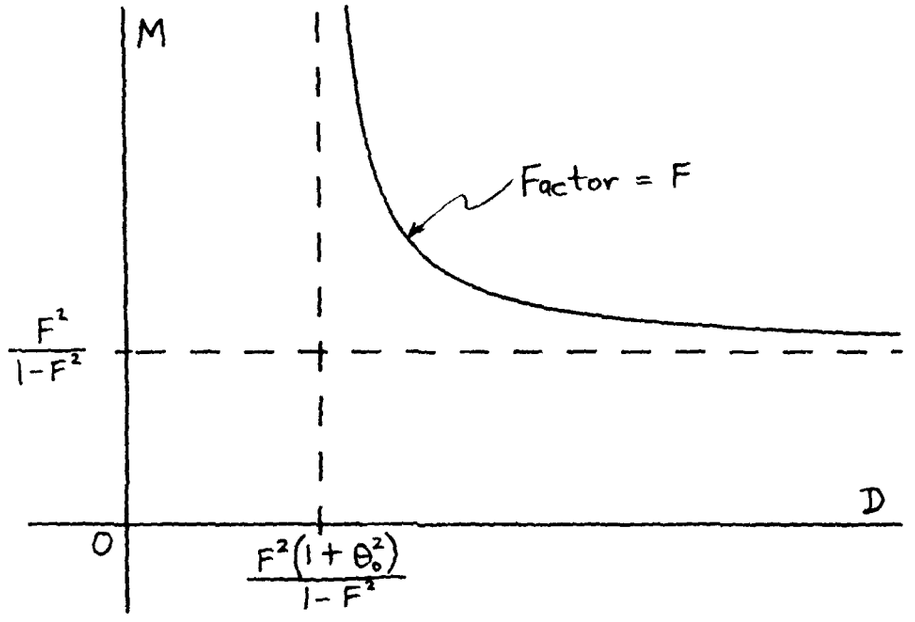


Figure H-1. Plot of (H-40) or (H-41) for Fixed θ_0

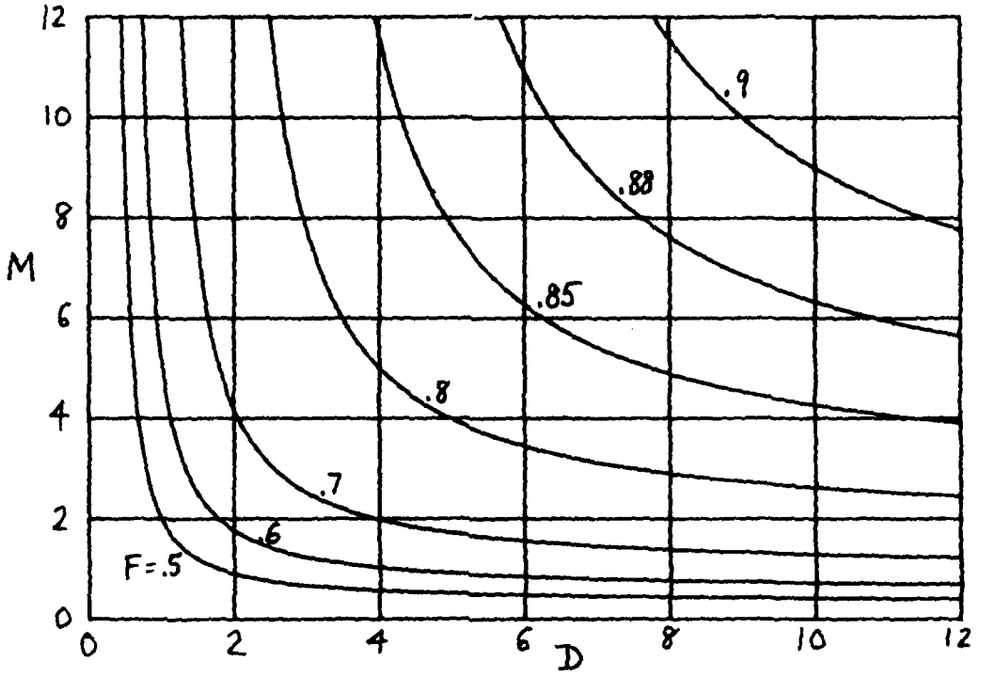


Figure H-2. Plot for $\theta_0 = 0$

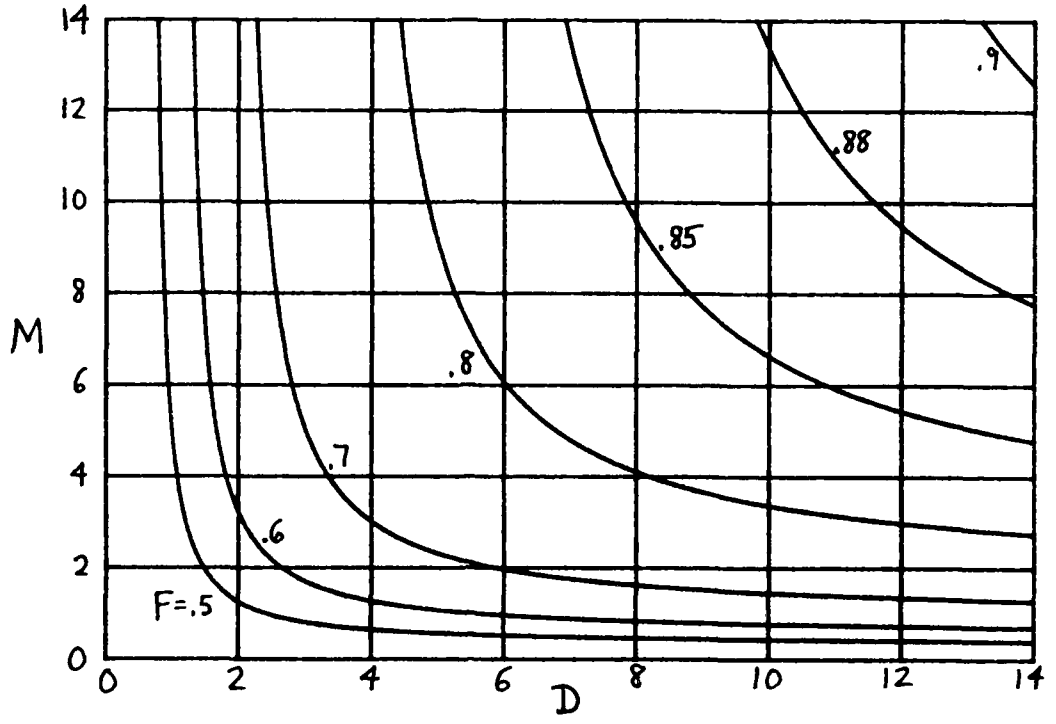


Figure H-3. Plot for θ_o = 1

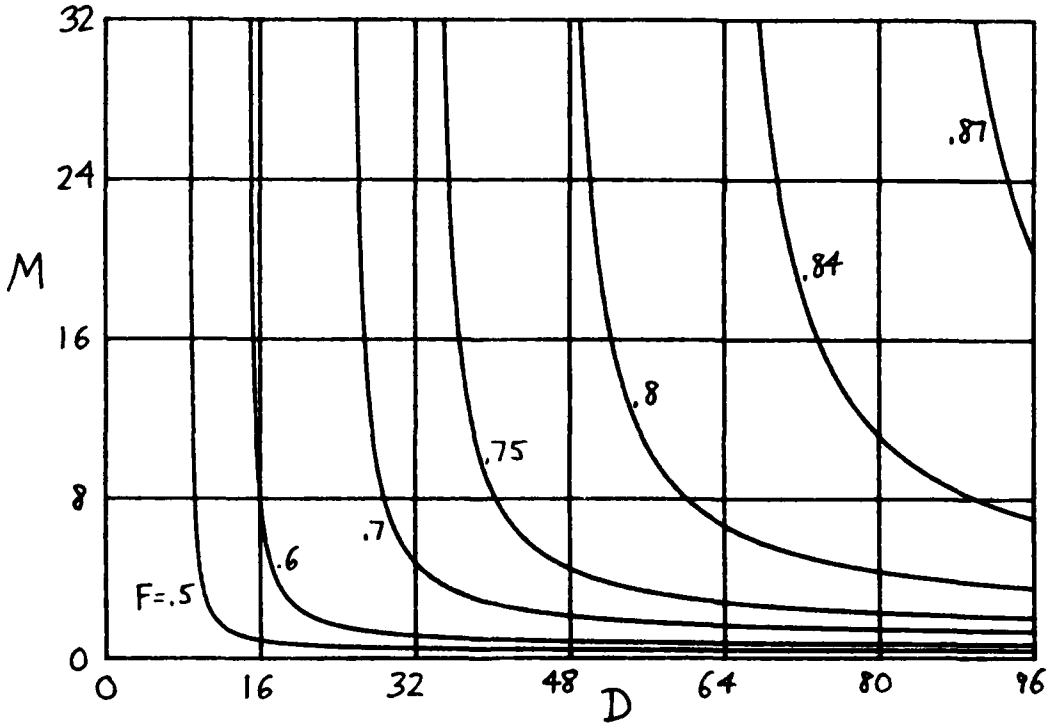


Figure H-4. Plot for θ_o = 5

In particular,

$$(MD)_{min} \sim \frac{4F^2}{(1 - F^2)^2} (1 + F^2\theta_0^2) + 1 \quad \text{as } \theta_0 \to \infty ; \tag{H-45}$$

in fact, this is a good approximation except near $\theta_0 = 0$. Thus, large amounts of linear frequency modulation, or values of F near 1, require very large MD.

At the other extreme,

$$(MD)_{min} = \frac{4F^2}{(1 - F^2)^2} \quad \text{for } \theta_0 = 0 . \tag{H-46}$$

For example, if $F = 1/\sqrt{2}$, this product is 8; thus relatively large values of the product are required, even at the low end where there is no linear frequency modulation. A plot of (H-44) is given in figure H-5, for various specified values of factor F.

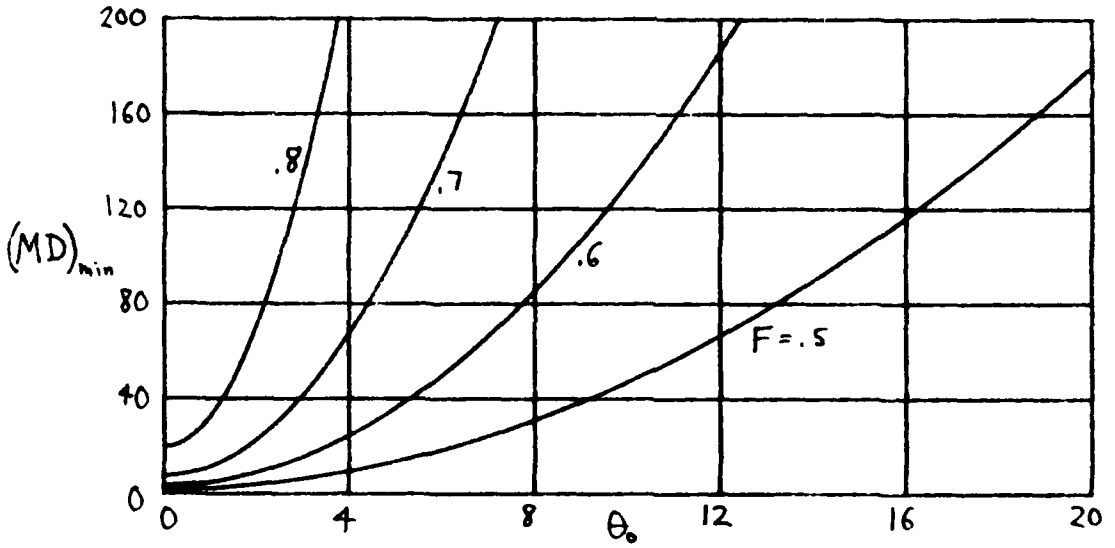


Figure H-5. Minimum MD Product

As a particular numerical example, for $\theta_o = 0$, $F = 1/\sqrt{2}$, we find

$$D_{opt} = 3 \ , \quad M_{opt} = \frac{8}{3} \ , \quad (MD)_{min} = 8 \ . \tag{H-47}$$

So both D_{opt} and M_{opt} are somewhat larger than unity, even for F equal to $1/\sqrt{2}$. All these conclusions are drawn relative to the mean signal output alone; we now consider the noise output contributions.

MEAN NOISE OUTPUT

The mean noise output of the WDF processor in figure 7 is given by (172), (H-9), (H-13), and (H-31) as

$$\bar{b} = W_v(t,f) \overset{f}{\circledast} G_n(f) =$$

$$= N_d \left(\frac{R}{1+R}\right)^{1/2} \exp\left[-\frac{t^2}{L^2} - \frac{f^2}{B^2}\frac{R}{1+R}\right] . \tag{H-48}$$

(As $L \to \infty$, that is, no weighting, then $R \to \infty$, and $\bar{b} \to N_d \exp(-f^2/B^2) = G_n(f)$, as expected.)

The noise factor in (H-48), namely

$$\left(\frac{R}{1+R}\right)^{1/2} = \left(\frac{DM}{1+DM}\right)^{1/2} , \tag{H-49}$$

is virtually unity when the mean signal degradation is small, according to the results of figure H-5. Thus the ratio of peak signal-to-noise means is approximately $2E_o/N_d$, according to (H-30) and (H-48). These latter

quantities, E_o and N_d, are directly the input parameters to the WDF processor in figure 7; see (H-1) and (H-6).

VARIANCE OF NxN TERM

The variance of the NxN term at the WDF processor output is given by the third line of (188) as

$$V_{NN} = 2\, W_v^2(t,f) \overset{f}{\circledast} G_n^{(2)}(f) =$$

$$= N_d^2 \frac{R}{\sqrt{1+R}} \exp\left[- \frac{2t^2}{L^2} - \frac{2f^2}{B^2} \frac{R}{1+R} \right]. \qquad \text{(H-50)}$$

Here, we also used (H-11) and (H-13). As $L \to \infty$, then $M \to \infty$, $R \to \infty$, and $V_{NN} \to \infty$. Alternatively, as $B \to \infty$, then $D \to \infty$, $R \to \infty$, and $V_{NN} \to \infty$. These results for this particular example confirm the general observations in the sequel to (188).

The standard deviation of the NxN term, namely $\sqrt{V_{NN}}$, is precisely equal to the noise mean output in (H-48) for all t,f, except for a constant factor $(1+R)^{1/4} = (1+DM)^{1/4}$. Also, the axes of the elliptical contours of (H-48) and (H-50) are parallel to the t and f axes and are independent of Θ_o, the amount of linear frequency modulation in the input signal.

VARIANCE OF SxN TERM

The variance of the SxN terms at the WDF processor output is given by the last line of (188) or by double the results in (182). Upon substitution of (H-10), (H-12), and (H-15) in (182) and an extreme amount of manipulations, there follows variance

$$V_{SN} = 4E_o N_d \frac{R\sqrt{D}}{\sqrt{H_3}} \exp[-\xi(t,f)] , \qquad (H-51)$$

where

$$H_3 = (1 + D + R/2)(1 + D + R + DR/2) + (1 + R)(1 + R/2)\,\theta_o^2 \qquad (H-52)$$

and $\xi(t,f)$ is an elliptical function with minimum value at $t = f = 0$. Namely,

$$\xi(t,f) = \frac{2t^2}{L^2} + c_r t^2 - \frac{\alpha^* \mu^2 + \alpha \mu^{*2} + 2\gamma|\mu|^2}{2(|\alpha|^2 - \gamma^2)} , \qquad (H-53)$$

where

$$c = 4\pi^2 B^2 \frac{1 - i\theta_o}{1 + D - i\theta_o} ,$$

$$\alpha = \frac{1}{4}\left(c + \frac{2}{L^2} + 2\pi^2 B^2\right) ,$$

$$\mu = -\left(\frac{c}{2} t + i2\pi f\right) ,$$

$$\gamma = \frac{1}{2}\pi^2 B^2 . \qquad (H-54)$$

The quantity in the denominator of (H-53) can be simplified to

$$2(|\alpha|^2 - \gamma^2) = \frac{H_3}{2L^4 D_2} \; ; \tag{H-55}$$

however, $\xi(t,f)$ has not been reduced to its most compact form,

$$b_1 t^2 + b_2 f^2 + 2b_3 tf \; , \tag{H-56}$$

due to the excessive amount of labor required to simplify and obtain b_1, b_2, b_3.

QUALITY MEASURE OF PERFORMANCE

We define a quality measure for the WDF processor output in figure 7 as

$$Q = \frac{\text{Difference of mean outputs}}{\text{Standard deviation of output}} = \frac{a}{\left(V_{SN} + V_{NN}\right)^{1/2}} \; . \tag{H-57}$$

The relevant quantities are given by (171) and (188) generally. For the specific example in this appendix, the quality measure, at peak signal location $t = f = 0$, is obtained by combining (H-30),(H-50), and (H-51):

$$Q = \left(\frac{E_o}{N_d}\right)^{1/2} \left(\frac{\frac{4E_o}{N_d} \frac{D\sqrt{1+R}}{H_1}}{1 + \frac{4E_o}{N_d}\sqrt{\frac{D(1+R)}{H_3}}} \right)^{1/2} \; . \tag{H-58}$$

For convenience, we repeat the parameter definitions here:

$$\Theta_o = \alpha_o \sigma_o^2 \ , \ D = (2\pi B\sigma_o)^2 \ , \quad M = (L/\sigma_o)^2 \ ,$$

$$R = DM \ ,$$

$$H_1 = (1 + D)(1 + D + R) + (1 + R)\Theta_o^2 \ ,$$

$$H_3 = (1 + D + R/2)(1 + D + R + DR/2) + (1 + R)(1 + R/2)\Theta_o^2 \ . \qquad \text{(H-59)}$$

It has already been observed in (H-24) and in the sequel to (H-39) that $D > 1 + \Theta_o^2$ and $M > 1$ are desirable, in so far as the mean signal output is concerned. However, if filter bandwidth B (D) is made too large, then too much noise is allowed through; alternatively, if weighting duration L (M) is made too large, a noise degradation also results. Thus, it is expected that the quality ratio Q will peak for D in the neighborhood of $1 + \Theta_o^2$ and for M near 1.

It should be observed from (H-58) that even if input signal-to-noise measure E_o/N_d gets extremely large, the quality measure Q behaves according to $\sqrt{E_o/N_d}$, and not E_o/N_d. This is due to the saturation effects caused by the SxN term in the denominator of definition (H-57); it can also be seen directly from the quantitative result in (H-51), where variance V_{SN} is directly proportional to input signal energy E_o as well as the noise density level N_d.

The quality ratio Q in (H-58) is plotted versus M in figures H-6, H-7, H-8 for Θ_o = 0,1,5, respectively. The input ratio E_o/N_d is kept at value 20 in all cases; the only other fundamental parameter, D, is varied over a range wide enough to encompass the maximum of Q. However, for ease of plotting the results, the values of D which are less than the critical value, which leads to the peak Q, are separated from those that are greater than the critical value. For example, in figure H-6, D = 1 leads to the maximum value of Q that can be achieved for any value of M; thus, the upper part of figure H-6 contains results for $D \leq 1$, while the lower part contains the remainder for $D \geq 1$. The corresponding critical values of D are 8 and 80 in figures H-7 and H-8, when Θ_o = 1 and 5, respectively.

One important observation that is made apparent by these figures is that near the maximum, the quality ratio Q is not too sensitive to M and D; that is, the maximum is broad in the neighborhood of the best parameter pair M,D. It should also be observed that as Θ_o increases, the peak value of Q decreases, although the decrease is not very significant, at least over the range Θ_o = 0,1,5 used here. Finally, the values of the peaks in these figures are slightly less than $\sqrt{E_o/N_d}$, as anticipated above.

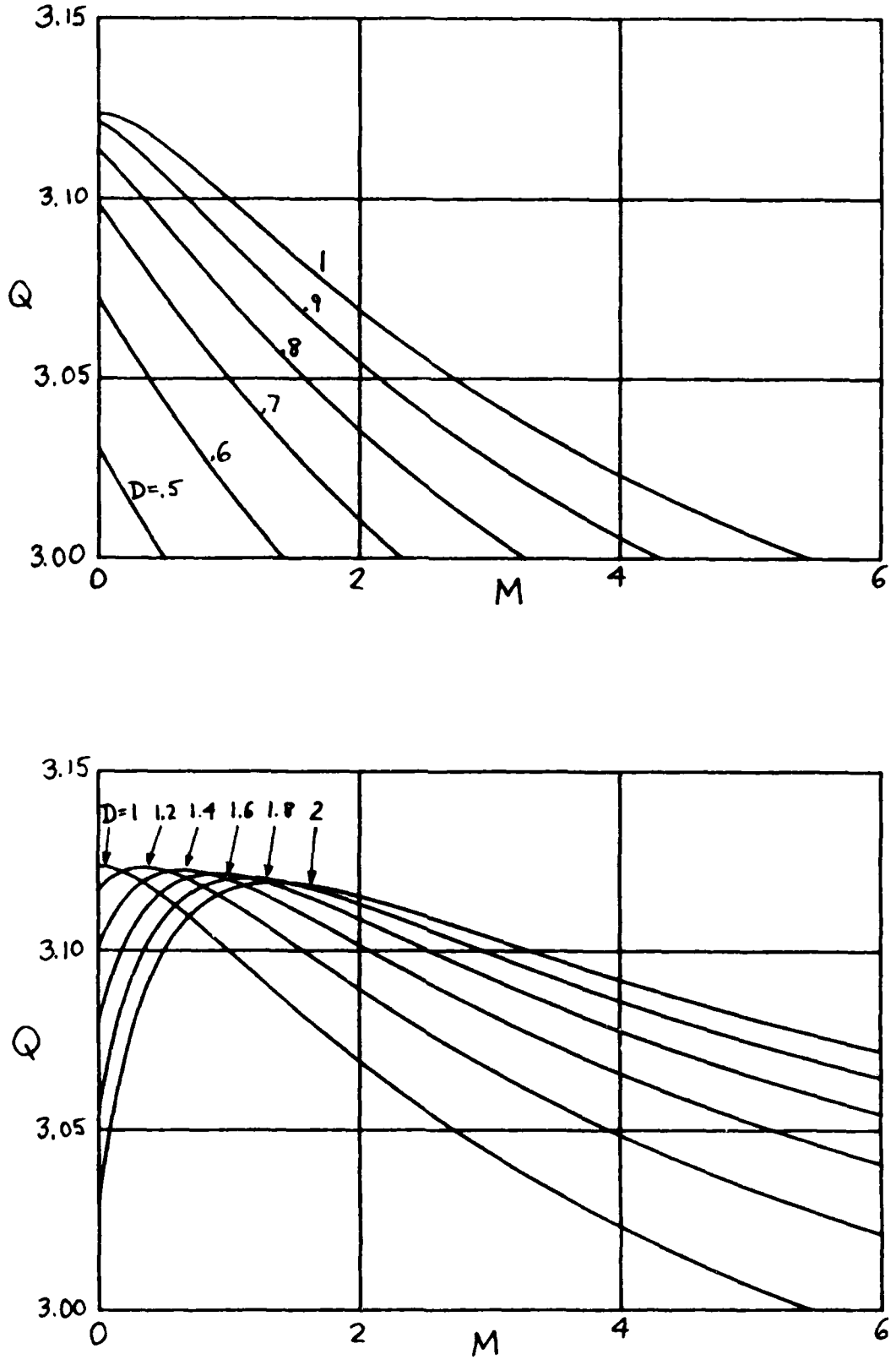


Figure H-6. Quality Ratio for $\theta_o = 0$, $E_o/N_d = 20$

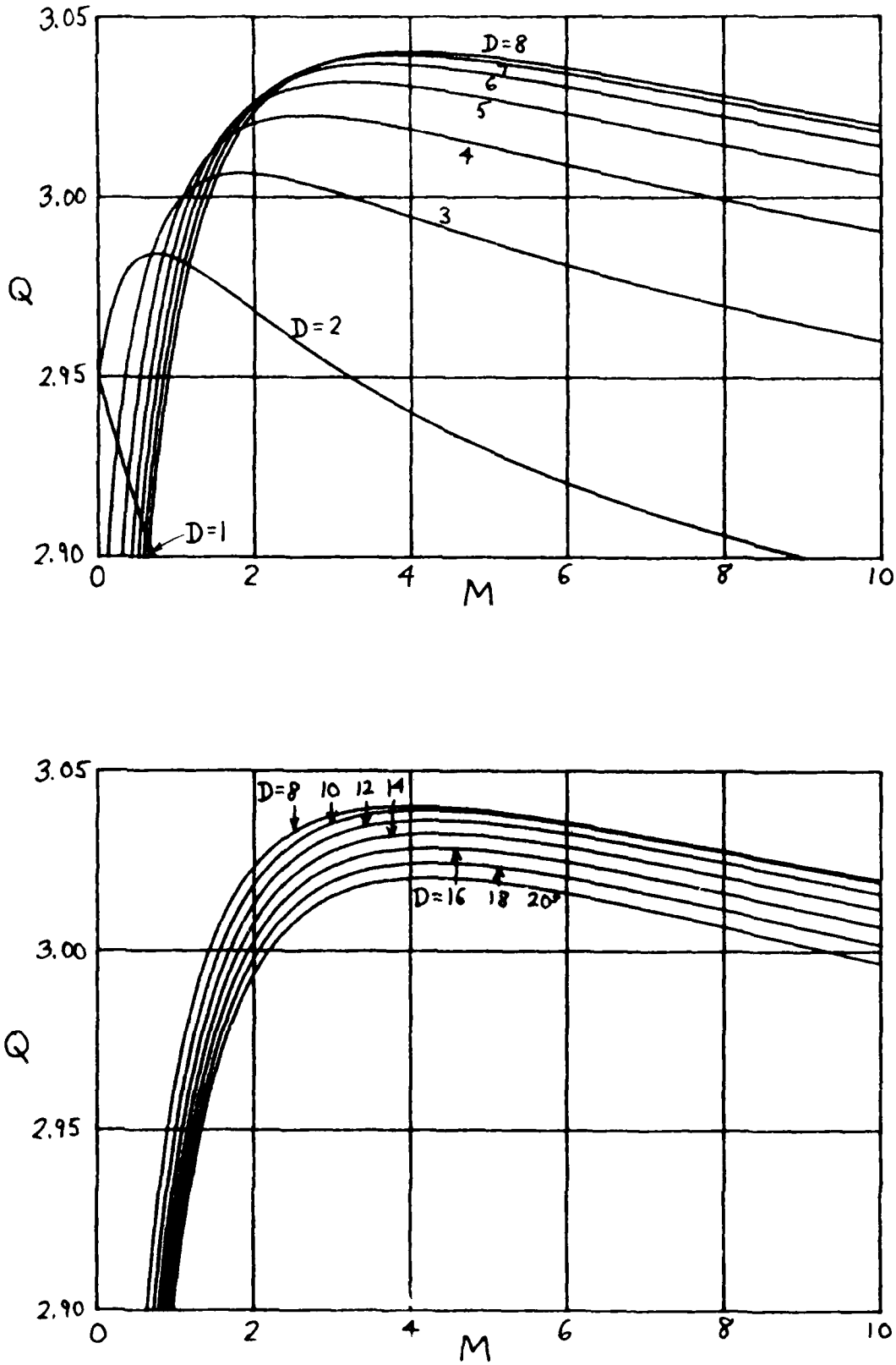
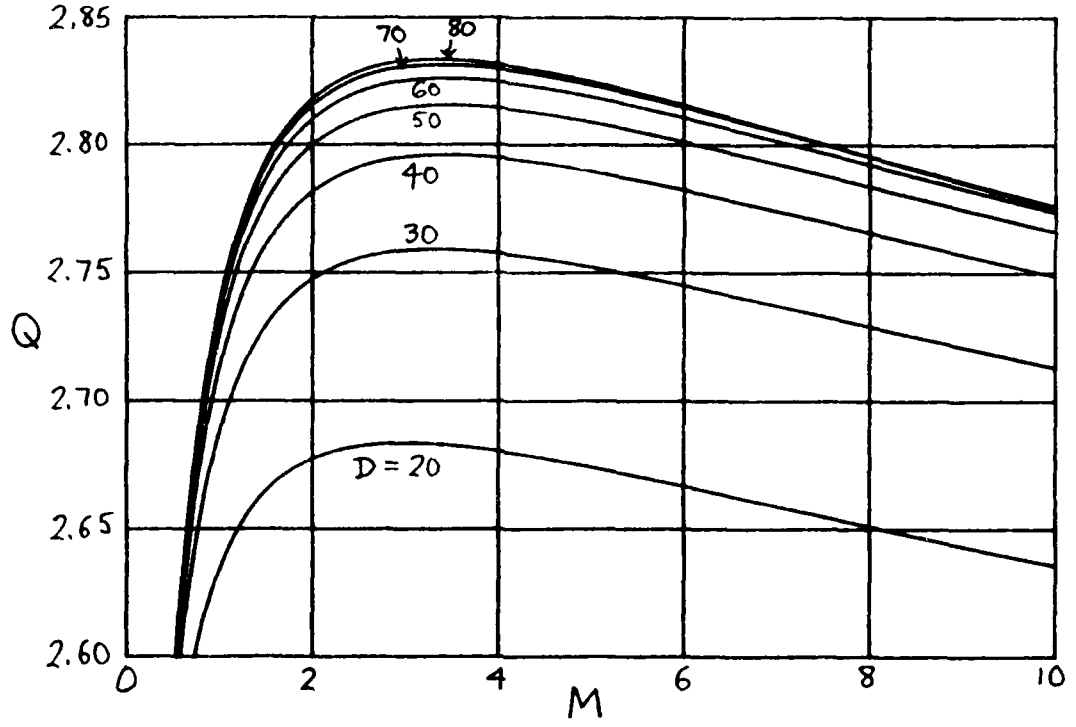


Figure H-7. Quality Ratio for θ_o =1, E_o/N_d = 20



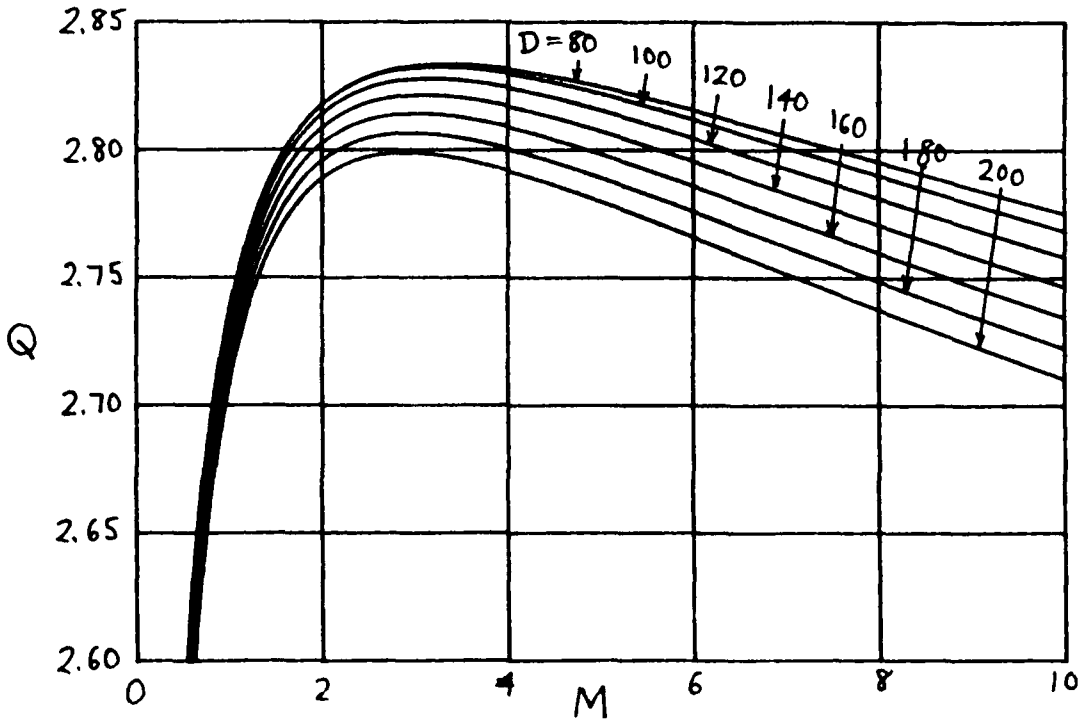


Figure H-8. Quality Ratio for θ_o = 5, E_o/N_d = 20

APPENDIX I. SMOOTHED WDF FOR $s(t) = t \exp(-t^2/2)$

For the waveform

$$s(t) = t \exp(-t^2/2) \quad \text{for all } t \,, \tag{I-1}$$

the WDF is

$$W_s(t,f) = 2\sqrt{\pi} \exp(-t^2 - 4\pi^2 f^2)(t^2 + 4\pi^2 f^2 - \tfrac{1}{2}) = 2\sqrt{\pi} \exp(-r^2)(r^2 - \tfrac{1}{2}) \,, \tag{I-2}$$

with energy

$$E = \int dt \, |s(t)|^2 = \sqrt{\pi}/2 \,. \tag{I-3}$$

Contour plots of the WDF in (I-2) are concentric circles in the $(t,2\pi f)$ plane; in fact, (I-2) is a function only of $r^2 = t^2 + (2\pi f)^2$. The origin value of W_s is $-2E = -\sqrt{\pi}$, and the WDF is negative for $r < 1/\sqrt{2}$, while it is positive for $r > 1/\sqrt{2}$.

Let us smooth this WDF with the most compact WDF; namely, use the Gaussian weighting function in (G-23) with WDF (G-25) with $\theta_c = 0$, $\sigma_c = 1$:

$$W_u(t,f) = 2 \exp(-t^2 - 4\pi^2 f^2) = 2 \exp(-r^2) \,. \tag{I-4}$$

The reason for these parameter choices of θ_c and σ_c is that the contours of (I-4) are also circles in the $(t,2\pi f)$ plane and exactly match those of the waveform WDF in (I-2); this should lead to minimal spreading.

The result of smoothing (I-2) by (I-4) is

$$\left|S_u(t,f)\right|^2 = W_s(t,f) \overset{tf}{\circledast} W_u(t,f) =$$

$$= \frac{1}{4}\sqrt{\pi}\,(t^2 + 4\pi^2 f^2)\,\exp\left(-\frac{1}{2}(t^2 + 4\pi^2 f^2)\right) = \frac{1}{4}\sqrt{\pi}\,r^2\,\exp(-r^2/2)\ , \qquad (I\text{-}5)$$

which has volume E as given by (I-3). Again, this is a function only of r^2, but it is never negative. This smoothed distribution is zero at r = 0, and peaks at $r = \sqrt{2}$ with value .326. By contrast, the WDF in (I-2) is -1.77 at the origin, a large negative value. However, the waveform WDF in (I-2) does decay faster than the short-term spectral estimate in (I-5); this is an example of the tradeoffs that must be accepted when using short-term spectral estimation versus the WDF.

To lend credence to (I-5) as a better measure of the time-frequency content of s(t), we observe that at t = 0, the center of gravity of (I-5) is

$$\bar{f}_o = \frac{\int_0^\infty df\,\left|S_u(0,f)\right|^2 f}{\int_0^\infty df\,\left|S_u(0,f)\right|^2} = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\ . \qquad (I\text{-}6)$$

Then we expect that

$$A\,\sin(2\pi\bar{f}_o t) \qquad (I\text{-}7)$$

ought to be a good fit to s(t) of (I-1) for t near zero. In fact, plots of (I-1) and (I-7) for A = exp(-.5) overlap for -1 < t < 1.

If we attempt this same procedure for WDF W_s in (I-2), the denominator is

$$\int_0^\infty df\ W_s(0,f) = 0\ , \tag{I-8}$$

giving rise to $\bar{f}_0 = \infty$, which is useless.

With respect to $t = 1$ instead, we find center of gravity

$$\bar{f}_1 = \frac{\int_0^\infty df\ |S_u(1,f)|^2\ f}{\int_0^\infty df\ |S_u(1,f)|^2} = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\ \frac{3}{4}\ . \tag{I-9}$$

Since $s(t)$ in (I-1) peaks at $t = 1$, we expect that

$$A\cos(2\pi\bar{f}_1(t-1)) \tag{I-10}$$

ought to be a good fit to $s(t)$ for t near 1. In fact, plots of (I-1) and (I-10) for $A = \exp(-.5)$ show very good agreement for $.8 < t < 1.7$.

Thus, smoothing of the WDF W_s in (I-2) by means of WDF W_u in (I-4), for this example, results in a very meaningful distribution function.

APPENDIX J. DOUBLE CONVOLUTION OF TWO GAUSSIAN FUNCTIONS

By means of the double integral result

$$\iint dx\, dy\, \exp\left[-\frac{1}{2}\alpha x^2 - \frac{1}{2}\beta y^2 + \gamma xy + \mu x + \nu y\right] =$$

$$= \frac{2\pi}{\left(\alpha\beta - \gamma^2\right)^{1/2}} \exp\left[\frac{\beta\mu^2 + \alpha\nu^2 + 2\gamma\mu\nu}{2(\alpha\beta - \gamma^2)}\right] \qquad \text{(J-1)}$$

for $\alpha_r > 0$, $\beta_r > 0$, $\alpha_r\beta_r > \gamma_r^2$, it is readily shown that the double convolution of two general Gaussian functions is given by

$$\exp\left[-\frac{1}{2}ax^2 - \frac{1}{2}by^2 - \sqrt{ab}\,\rho xy\right] \overset{xy}{\circledast} \exp\left[-\frac{1}{2}cx^2 - \frac{1}{2}dy^2 - \sqrt{cd}\,\lambda xy\right] = \qquad \text{(J-2)}$$

$$= \frac{2\pi}{D^{1/2}} \exp\left[-\frac{N_1x^2 + N_2y^2 + 2N_3xy}{2D}\right] \qquad \text{(J-3)}$$

for $a,b,c,d > 0$, $|\rho| < 1$, $|\lambda| < 1$, where

$$D = ab(1 - \rho^2) + cd(1 - \lambda^2) + ad + bc - 2\sqrt{abcd}\,\rho\,\lambda$$

$$N_1 = ac[b(1 - \rho^2) + d(1 - \lambda^2)]$$

$$N_2 = bd[a(1 - \rho^2) + c(1 - \lambda^2)]$$

$$N_3 = \sqrt{abcd}\,[\sqrt{ab}\,\lambda(1 - \rho^2) + \sqrt{cd}\,\rho(1 - \lambda^2)]\ . \qquad \text{(J-4)}$$

Also, a useful auxiliary relation is

$$N_1 N_2 - N_3^2 = D\ abcd\ (1 - \rho^2)(1 - \lambda^2) \ . \qquad (J\text{-}5)$$

Now let

$$\rho = \sin(\theta), \quad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \ ,$$

$$\lambda = \sin(\phi), \quad \text{where } -\frac{\pi}{2} < \phi < \frac{\pi}{2} \ . \qquad (J\text{-}6)$$

Then the area of the contour ellipse at the 1/e relative level of the first exp in (J-2) is

$$A_1 = \frac{2\pi}{\sqrt{ab}\cos(\theta)} \qquad (J\text{-}7)$$

in the x,y plane, where we used (D-1), (D-19), and (D-20). Similarly, the area of the second exp in (J-2) is

$$A_2 = \frac{2\pi}{\sqrt{cd}\cos(\phi)} \ . \qquad (J\text{-}8)$$

The sum of these two effective areas is

$$A_1 + A_2 = 2\pi \frac{\sqrt{ab}\cos(\theta) + \sqrt{cd}\cos(\phi)}{\sqrt{abcd}\cos(\theta)\cos(\phi)} \ . \qquad (J\text{-}9)$$

On the other hand, the area of the contour ellipse at the 1/e relative level of the smoothed exp in (J-3) is

$$A_3 = 2\pi \frac{\sqrt{D}}{\sqrt{abcd}\cos(\theta)\cos(\phi)} \qquad (J\text{-}10)$$

in the x,y plane, where we can express D from (J-4) as

$$D = ab\cos^2(\Theta) + cd\cos^2(\phi) + ad + bc - 2\sqrt{abcd}\sin(\Theta)\sin(\phi) =$$

$$= [\sqrt{ab}\cos(\Theta) + \sqrt{cd}\cos(\phi)]^2 + \left|\sqrt{ad}\exp(i\Theta) - \sqrt{bc}\exp(i\phi)\right|^2 . \qquad (J\text{-}11)$$

Comparison of the square root of (J-11) with the numerator of (J-9) reveals that

$$A_3 \geq A_1 + A_2 , \qquad (J\text{-}12)$$

with equality occuring if and only if

$$\sqrt{ad} = \sqrt{bc} \quad \text{and} \quad \Theta = \phi . \qquad (J\text{-}13)$$

That is, in order for $A_3 = A_1 + A_2$, we must have

$$\frac{d}{c} = \frac{b}{a} \quad \text{and} \quad \lambda = \rho . \qquad (J\text{-}14)$$

Physically, this requirement states that the contour ellipses of the two exp terms in (J-2) must have the same ratio of major-to-minor axes and they must have the same tilt. If either condition is violated, then $A_3 > A_1 + A_2$, the exact amount depending on the second term in (J-11).

EFFICIENT CALCULATION OF GAUSSIAN FUNCTION

If the general two-dimensional Gaussian function in (J-2) is sampled on an equi-spaced grid, for purposes of convolution, it will be necessary to compute the quantity

$$Q_2(m,n) = \exp(-am^2 - bn^2 - cmn) \qquad (J\text{-}15)$$

for integers $-M \leq m \leq M$, $-N \leq n \leq N$. The following efficient procedure is based upon the general method given in [12].

We observe first that

$$Q_2(-m,-n) = Q_2(m,n) \ , \tag{J-16}$$

which cuts the effort by one-half. There also follows

$$Q_2(m,n) = Q_2(m,n-1)\, Q_1(m,n) \ , \tag{J-17}$$

where

$$Q_1(m,n) = \exp[-b(2n-1) - cm] = Q_1(m,n-1)\exp(-2b) \ . \tag{J-18}$$

These recurrences can be started with

$$Q_1(m,0) = \exp(b - cm) \ ,$$

$$Q_2(m,0) = \exp(-am^2) = Q_2(-m,0) \ . \tag{J-19}$$

Furthermore, these latter two quantities are available through the recurrence

$$\left.\begin{aligned} Q_1(m,0) &= Q_1(m-1,0)\exp(-c) \\ Q_1(m-1,0) &= Q_1(m,0)\exp(+c) \end{aligned}\right\} \quad \text{for } m \geq 1 \ , \tag{J-20}$$

with

$$Q_1(0,0) = \exp(b) \ , \tag{J-21}$$

and the recurrence

$$\left.\begin{aligned} Q_2(m,0) &= Q_2(m-1,0)\ E(m) \\ E(m) &= E(m-1)\ \exp(-2a) \end{aligned}\right\} \quad \text{for } m \geq 1 \ , \tag{J-22}$$

with

$$Q_2(0,0) = 1, \quad E(0) = \exp(a) \ . \tag{J-23}$$

The only case not covered by the above recurrences is for $m = 0$; then

$$\left.\begin{aligned} Q_2(0,n) &= Q_2(0,n-1)\ F(n) \\ F(n) &= F(n-1)\ \exp(-2b) \end{aligned}\right\} \quad \text{for } n \geq 1 \ , \tag{J-24}$$

with

$$F(0) = \exp(b) \ . \tag{J-25}$$

A program for the evaluation of (J-15) is given below. Only three exponentials, in lines 90-110, need to be evaluated. Also, the only storage required is for the final quantity $Q_2(m,n)$ in lines 60-70. The auxiliary variables $Q_1(m,n)$, $E(m)$, $F(n)$ introduced above need never be stored. The check on accuracy in lines 390-470 would be discarded, of course, in any

practical application; it is appended as a check on any typographical errors in entering the program into another computer.

```
 10    A=.037                          !  exp(-A m^2 - B n^2 - C m n)
 20    B=.051                          !  for -M<=m<=M, -N<=n<=N
 30    C=.044
 40    M=5
 50    N=7
 60    REDIM Q2(-M:M,-N:N)
 70    DIM Q2(50,50)
 80    DOUBLE M,N,Ms,Ns                !  INTEGERS
 90    Ea=EXP(A)
100    Eb=EXP(B)
110    Ec=EXP(C)
120    Q2(0,0)=1.
130    E=Eb
140    E2b=Eb*Eb
150    FOR Ns=1 TO N
160    E=E/E2b
170    Q2(0,Ns)=Q2(0,Ns-1)*E
180    NEXT Ns
190    E=Ea
200    E2a=Ea*Ea
210    Q1po=Q1mo=Eb
220    FOR Ms=1 TO M
230    Q1p=Q1po=Q1po/Ec
240    Q1m=Q1mo=Q1mo*Ec
250    E=E/E2a
260    Q2(-Ms,0)=Q2(Ms,0)=Q2(Ms-1,0)*E
270    FOR Ns=1 TO N
280    Q1p=Q1p/E2b
290    Q1m=Q1m/E2b
300    Q2(Ms,Ns)=Q2(Ms,Ns-1)*Q1p
310    Q2(-Ms,Ns)=Q2(-Ms,Ns-1)*Q1m
320    NEXT Ns
330    NEXT Ms
340    FOR Ms=-M TO M
350    FOR Ns=1 TO N
360    Q2(-Ms,-Ns)=Q2(Ms,Ns)
370    NEXT Ns
380    NEXT Ms
390      Big=0.                        !  MAXIMUM ERROR CHECK
400      FOR Ms=-M TO M
410      FOR Ns=-N TO N
420      E=EXP(-A*Ms*Ms-B*Ns*Ns-C*Ms*Ns)
430      Error=E-Q2(Ms,Ns)
440      Big=MAX(Big,ABS(Error))
450      NEXT Ns
460      NEXT Ms
470      PRINT Big
480    END
```

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