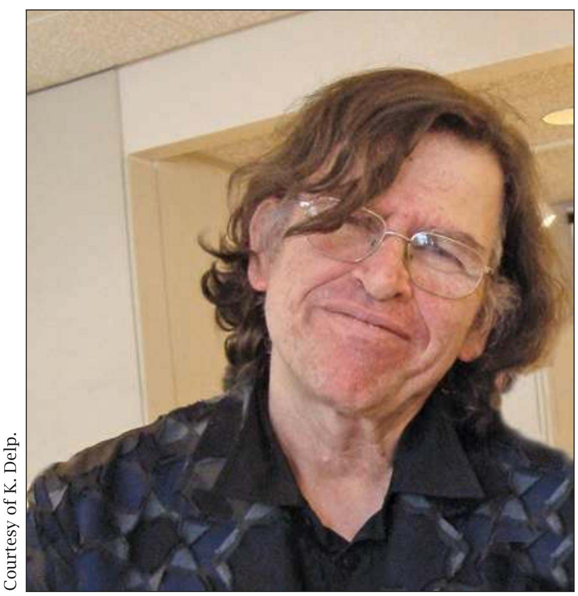
William P. Thurston, 1946–2012

David Gabai and Steve Kerckhoff, Coordinating Editors

William P. Thurston, a geometric visionary and one of the greatest mathematicians of the twentieth century, died on August 21, 2012, at the age of sixty-five. This obituary for Thurston contains reminiscences by some of his many colleagues and friends. What follows is the first part of the obituary; the second part will appear in the January 2016 issue of the *Notices*.

William Paul Thurston, known universally as Bill, was an extraordinary mathematician whose work and ideas revolutionized many fields of mathematics, including foliations, Teichmüller theory, automorphisms of surfaces, 3-manifold topology, contact structures, hyperbolic geometry, rational maps, circle packings, incompressible surfaces, and geometrization of 3-manifolds.

Bill's influence extended far beyond his incredible insights, theorems, and conjectures; he transformed the way people think about and view things. He shared openly his playful, ever curious, near magical and sometimes messy approach to mathematics. Indeed, in his MathOverflow profile he states, "Mathematics is a process of staring hard enough with enough perseverance at the fog of muddle and confusion to eventually break through to improved clarity. I'm happy when I can admit, at least to myself, that my thinking is muddled, and I try to overcome the embarrassment that I might reveal ignorance or confusion. Over the years, this has helped me develop clarity in some things, but I remain muddled in many others. I enjoy questions that seem honest, even when they admit or reveal confusion, in preference to questions that appear designed to project sophistication."



Courtesy of K. Delp.

William Paul Thurston

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Unless otherwise noted, all photographs are courtesy of Rachel Findley.

DOI: http://dx.doi.org/10.1090/noti1300

Margaret Thurston, with some of her creations.

His work was not only densely packed with wonderful ideas, it was immensely rich and deep. Studying an aspect of his work is often like opening a box to find two or more inside. Opening each of those reveals two or more boxes that often have little tunnels connecting to various other systems of boxes. With great effort one reaches an end, being simultaneously rewarded with both great illumination and the realization that one knows but a minuscule fraction of the whole. After a sufficient lapse of time, upon retracing the original trail, one catches further insights on topics that one once thought one completely understood.

In a similar manner, this article hardly does justice to the deep and complicated person that was Bill Thurston. It gives but a few facts and vignettes from various dimensions of his life and, at various spots, points readers to other resources where they might explore further.

Following this introduction are thirteen remembrances from mathematicians connected with different aspects of Thurston's professional life. It is difficult not to be overcome by how Bill affected people and institutions in so many different and profound ways.

Family

Bill's father, Paul Thurston, had a PhD in physics and worked at Bell Labs doing physics and engineering. He was an expert at building things and was bold, smart, imaginative, and energetic. Once he showed Bill how he could boil water with his bare hands. He took an ordinary basement vacuum pump and started it above the water so that the boiling point was just above the air temperature. Then he stuck his hands in the water and it started boiling!

Bill's mother, Margaret (nee Martt), was an expert seamstress who could sew intricate patterns that would baffle Paul and Bill. In later years, Bill's fascination with hyperbolic geometry inspired her to sew a hyperbolic hat-skirt, a seven-color torus, and a Klein quartic (genus-3 surface with a symmetry group of order 168) made from 24 heptagons that was designed by Bill and his sons, Nathaniel and Dylan. While at Ohio Wesleyan she wanted to be a math major but was told that women don't major in mathematics.

Bill was named after his father, Paul, and his mother's brother, William, who died in a hospital ship in the battle of Iwo Jima. He had an older brother, Bob, and sister, Jean, and a younger brother, George, who married Sarah, mathematician Hassler Whitney's daughter. He married his college sweetheart, Rachel Findley, and they had children Nathaniel, Dylan, and Emily. He had children Jade and Liam from his second marriage with Julian Thurston.

Bill Thurston, 1952, Wheaton, MD.

Childhood

Bill had a congenital case of strabismus and could not focus on an object with both eyes, eliminating his depth perception. He had to work hard to reconstruct a three-dimensional image from two two-dimensional ones. Margaret worked with him for hours when he was two, looking at special books with colors. His love for patterns dates at least to this time. As a first-grader he made the decision "to practice visualization every day." Asked how he saw in four or five dimensions he said it is the same as in three dimensions: reconstruct things from two-dimensional projections.

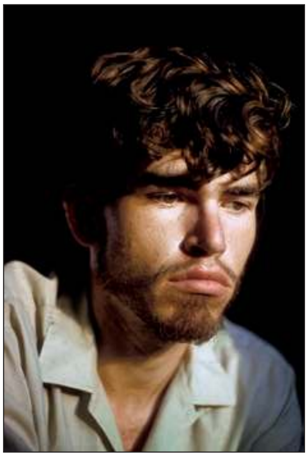
To stop sibling squabbling Paul would ask the kids math questions. While driving, Paul asked Bill (when he was five), "What is $1 + 2 + \cdots + 100$?" Bill said, "5,000." Paul said, "Almost right." Bill said, "Oh, I filled one square with 1, two squares for 2 and all the way up to 100, so that's half of $100 \times 100 = 10,000$, but I forgot that the middle squares are cut in two, so that's 5,050!"

Paul was a scoutmaster, and Bill was very involved with the Scouts. They did all sorts of things with ropes, e.g., making bridges. Bill was expert at making fires in the pouring rain. The family loved camping and took many long trips. Music was a big part of their lives.

New College, Florida

Bill was a member of the charter class starting in 1964. According to alumnus and mathematician John Smillie, the founders of New College decided that it was much easier to maintain quality than to bring it up, so a tremendous effort was made to recruit one hundred of the brightest young people for the inaugural class. This included Bill's future wife, Rachel Findley.

New College was a three-year program with classes eleven months of the year. There was tremendous freedom in how individual academic programs were structured and how students were allowed to live. At various times Bill lived in a tent in adjacent woods or slept in academic buildings, playing hide and seek with the janitor. The library was minimal, and the college would buy whatever math books Bill wanted. Smillie said that nearly all the books he took out had Bill's name in them.



Bill Thurston, New College.

In Bill's words: "I guess I was disenchanted with how high school was and I wanted to go to a place where I'd have some more freedom to work in my own way." New College gave him that and more. After the first year, half the faculty resigned and the library was on the brink of disaster. But Thurston says that he and many other students benefited from the turmoil. "It certainly taught us independence and how to think for ourselves."[1] According to Rachel, Bill could have left after two years for graduate school, but he really liked the independence the school offered and chose to stay.

Berkeley

Bill started Berkeley in 1967 in the thick of the Vietnam War. "Many of us were involved in student demonstrations and student strikes. We were sprayed with tear gas, whether or not we protested. We had friends who were killed, others who refused induction and were convicted as felons,…."[2]

His concern about military activities continued throughout his life, particularly as it pertained to his professional life. In 1984 he declined an invitation to become the first Fairchild Professor at Princeton University because of the donor's involvement in the business of military contracting.

Bill was a member of a committee that argued against mathematicians accepting military funding and that drafted five AMS resolutions related to this and related issues, each of which received majority votes from members of the AMS.[3,4] (See also Epstein's contribution.)

The early years in Berkeley also saw the expansion of his and Rachel's family. Rachel said they had a baby (Nathaniel) in part to keep Bill from being drafted. She went into labor the night before his qualifying exam, which couldn't be rescheduled. Although Bill passed, it was not a smooth process, as he gave answers that befuddled his examiners but provided hints at the remarkable originality of his thinking.

Thurston's graduate career contained an extraordinary collection of mathematical results. He proved the striking theorem that the Godbillon-Vey invariant takes on uncountably many values. This invariant comes from a de Rham cohomology class that he relates to the *helical wobble of the leaves of a foliation*. His results show, among other things, that there exist uncountably many noncobordant codimension-one foliations of the 3-sphere.

Bill's (unpublished) thesis, written under the direction of Moe Hirsch, provided a precise description of a large class of C^2 foliations on 3-manifolds that are circle bundles. It also gave a counterexample to a result of S. P. Novikov that a certain partial order associated to a foliation has a maximum. But, beyond any single theorem, this period provided the foundation for a collection of ideas that would help answer many fundamental questions about foliations and would ultimately lead to Thurston's revolutionary work on surfaces and 3-manifolds.

MIT and the Institute for Advanced Study

Bill was a member of the Institute for Advanced Study in 1972–73 and an assistant professor at MIT in 1973–74. There he continued his extraordinary work on foliations, for which he won the 1976 Veblen Prize (shared with James Simons).[5] In particular, it was during this period that he and Milnor began their work on the kneading invariant of piecewise monotone maps of the interval. His interest in one-dimensional (real and complex) dynamics would stretch throughout his career, influencing his work on two- and three-dimensional manifolds. He would return to the topic once again during the final years of his life.

Princeton

He arrived at Princeton as a full professor in 1974 and remained for almost twenty years. There he did his revolutionary and foundational work on Teichmüller theory, automorphisms of surfaces, hyperbolic geometry, 3-manifolds, contact structures, and rational maps.

One of the most striking aspects of this period was how Thurston intertwined so many types of mathematics that had previously been viewed as disparate and utilized them in shockingly original ways. His classification of mapping classes of automorphisms of surfaces utilized ideas from topology, hyperbolic geometry, complex analysis, Teichmüller theory, dynamics, and ergodic theory.

This work was followed by an explosion of results in three-dimensional topology, using hyperbolic geometry as a primary tool. These results

[1] Home News, *Lifestyle section, December 30, 1984.*

[2] *Military funding in mathematics,* Notices of the AMS ***34*** *(1987), 39–44.*

[3] *Commentary on defense spending,* Notices of the AMS ***35*** *(1988), 35–37.*

[4] *Referendum results and letter,* Notices of the AMS ***35*** *(1988), 554, 675.*

[5] `www.ams.org/profession/prizes-awards/pabrowse?purl=veblen-prize`

completely changed the landscape around the subject. The notes from his 1978 hyperbolic geometry course (in part written and edited by Bill Floyd and Steve Kerckhoff) were sent in installments (by regular mail) to over one thousand mathematicians. "The notes immediately circulated all over the world. It is probably the opinion of all the people working in low-dimensional topology that the ideas contained in these notes have been the most important and influential ideas ever written on the subject."[6]

The viewpoint represented by this work was encapsulated by his geometrization conjecture, which states that all closed, orientable 3-manifolds have a (classically known) decomposition into pieces, each of which has one of eight homogeneous metric geometries. Although this was a truly different way of viewing 3-manifolds, the conjecture subsumed many long-standing conjectures in the field, including the Poincaré Conjecture, the spherical space form problem, residual finiteness of their fundamental groups, and a classification of their universal covering spaces. He solved the conjecture for a large class of manifolds (*Haken manifolds*). His 1982 AMS *Bulletin* article, in which he described his view of 3-manifolds, included twenty-four problems. Although he did not view mathematics in terms of a list of problems to be solved, these received great attention and helped focus research in the field for the next three decades. See Otal ([7]) for an account of progress on these problems through 2013.

His Princeton years also included seminal contributions to the theory of rational maps of the 2-sphere, utilizing his ideas both from Kleinian groups and from the earlier work on surfaces. His interests in computing and group theory, both from a practical and theoretical point of view, led to his work (with Cannon, Epstein, and others) on automatic groups in *Word Processing in Groups*. The outpouring of ideas from this period was monumental and cannot be adequately captured in this space.

Bill was awarded the 1979 Waterman Prize "In recognition of his achievements in introducing revolutionary new geometrical methods in the theory of foliations, function theory and topology."[7] In 1982 he won the Fields Medal and in 1983 was elected to the National Academy of Sciences.

Generally very generous with his time and ideas, Thurston had twenty-nine PhD students from his Princeton period, who, in turn, as of this writing have had 151 finishing students. In addition he influenced multitudes of visitors to both the university and the Institute for Advanced Study. He was a social center too, often bringing out his volleyball net for an impromptu game after tea in Fine Hall.

Beyond the extraordinary scope of his mathematical results, Bill was equally influential in the way he thought about and did mathematics. He thought deeply about the process of doing mathematics and methods for communicating it. His enthusiasm was infectious, and his truly unique point of view provided many new avenues for including it in people's lives.

Computers

Bill was one of the first pure mathematicians to actively use computers in his research and was a strong proponent of all aspects of computing in the mathematical community. In the late 1970s he inspired Jeff Weeks to develop his SnapPea program to compute and visualize hyperbolic structures. This program and a later version, SnapPy (by Marc Culler and Nathan Dunfield), and the related Snap (Goodman, Hodgson, Neumann) are essential tools for anyone working in the area. Weeks himself discovered what is now known as the Weeks manifold with this program. (This manifold was independently discovered by Matveev-Fomenko and Józef Przytycki.)

According to Al Marden, Bill was the intellectual force behind the creation and activities of the Geometry Supercomputer Project and the Geometry Center, two NSF-funded programs centered at the University of Minnesota, with Marden as the founding director. Among many other things, these projects produced the wonderful videos *Not Knot* and *Outside In*. *Not Knot* is a computer-animated tour of hyperbolic space worlds that Bill discovered, portions of which have appeared in Grateful Dead concerts.[8] *Outside In* is an award-winning video of Bill's proof of sphere eversion, an amazing result first discovered by Steven Smale in 1957.

When a project by Gabai and Meyerhoff needed high-level computer expertise, they found it at the Geometry Center. This led to a computer-assisted proof of the log(3)/2-theorem with Nathaniel Thurston (Bill's son), a brilliant apprentice at the Geometry Center who made full use of their computer resources. This result plays a crucial role in the proof of the Smale conjecture for hyperbolic 3-manifolds and the proof that the Weeks manifold

[6] *A. Papadopoulos, MR1435975 (97m:57016) review of* Three-Dimensional Geometry and Topology.

[7] www.nsf.gov/od/waterman/waterman_recipients.jsp

[8] Scientific American, *October 1993, p. 101.*

Thurston and Ed Witten, Waterman Ceremony.

is the unique lowest-volume closed orientable hyperbolic 3-manifold.

Education

Bill was always interested in education. He and fellow New College students, including Rachel Findley, tutored disadvantaged students in math. At Berkeley he was part of a student committee that helped reform the TA program. He wrote, "I helped organize a program for our new teaching assistants, which involves discussion groups, and visiting of each others' classes. The group of TAs I observed seemed to me to retain much of the initial enthusiasm toward teaching which new TAs usually have, but older TAs frequently lose."[9]

Each year at the science day program at his kids' elementary school class (at a Princeton public school), he would teach a "thing or two." "It's really gratifying how open they are and how quickly they pick up things that seem to most adults far out and strange, the kinds of things that adults turn off as you're trying to explain it to them."[10]

At Princeton, with John Conway and Peter Doyle, he developed the innovative Geometry and the Imagination undergraduate course that was later given at the Geometry Center and UC Davis.

Thinking About and Doing Mathematics

Every student of mathematics needs to read Thurston's wide-ranging and well-thought-out essay "On proof and progress in mathematics".[11] There he asks, *How do mathematicians advance human understanding of mathematics?* which has the subquestions: *How do we understand and communicate mathematics? What motivates us to do it? What is a proof?* He closes with some personal experiences from his work on foliations, hyperbolic geometry, and geometrization. In addition, he discusses how experts transmit new results to other experts and how he, at least, thinks about things. This is far different from the way outsiders, or even many advanced students, think working mathematicians function.

Variants of ideas expressed in the BAMS article are given in other venues. He states, "Mathematics is an art of human understanding.…Mathematics sings when we feel it in our whole brain.…You only learn to sing by singing."[12] "The most important thing about mathematics is how it resides in the human brain. Mathematics is not something we sense directly: it lives in our imagination and we sense it only indirectly. The choices of how it flows in our brains are not standard and automatic, and can be very sensitive to cues and context. Our minds depend on many interconnected special-purpose but powerful modules. We allocate everyday tasks to these various modules instinctively and subconsciously."[13] Here are links to two other essays along related lines:

`mathoverflow.net/questions/43690/whats-a-mathematician-to-do/44213#44213`

`mathoverflow.net/questions/38639/thinking-and-explaining`

Building Things, Models, and Clothing Design

Bill enjoyed working with his hands and building things. He had a workshop in his Princeton home, and once, when guests came and there were not enough beds, he bought lumber and in an afternoon built a bunk bed. His mother, Margaret, was a master seamstress who sewed magnificent surfaces that he designed. He inspired Daina Taimina to crochet magnificent pieces and write the exquisite Euler Award-winning *Crocheting Adventures with Hyperbolic Planes.* With Kelly Delp he developed methods for building nearly smooth models by gluing Euclidean discs together. Characteristically, in just two days he built a workshop with a laminator, paper cutter, and riveter, and started constructing interesting models, eventually progressing to beautiful no-tape foam construction models that are marvels of engineering.

Fashion designer Dai Fujiwara contacted Thurston after reading about his eight geometries. Inspired by Bill, as well as by a multitude of geometric materials that Bill provided, including

[9] *Job application letter to Princeton while at Berkeley.*

[10] Home News*, Lifestyle section, Sunday, December 30, 1984.*

[11] Bull. Amer. Math. Soc. ***30** (1994), 161–177.*

[12] *From the forward to* Crocheting Adventures with Hyperbolic Planes*, by Daina Taimina.*

[13] *Excerpt from the forward to* Teichmüller Theory and Applications to Geometry, Topology, and Dynamics, Volume 1: Teichmüller Theory*, by John Hubbard.*

a set of links whose 2-fold covers represented orbifolds of all eight geometries, Fujiwara and his team at Issey Miyake, Inc., designed and created an array of beautiful new women's fashions that were presented at a March 2010 Issey Miyake fashion show.

Bill had expressed a connection between clothing design and manifold theory more than thirty-five years earlier. His 1974 ICM proceedings paper commences with "Given a large supply of some sort of fabric, what kinds of manifolds can be made from it, in a way that the patterns match up along the seams? This is a very general question, which has been studied by diverse means in differential topology and differential geometry." Quoting a front-page *Wall Street Journal* piece: "Mr. Thurston compares this discovery [the eight geometries] to finding eight apparel outfits that can fit anybody in the world, just as he hopes to prove some day that the eight geometric categories fit every three-manifold imaginable."[14]

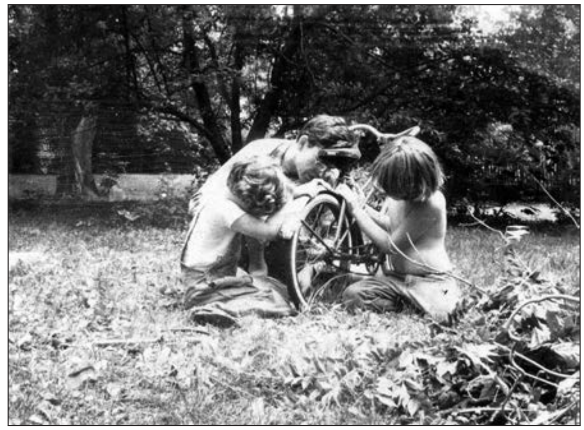
MSRI, Berkeley, and Davis

Bill moved to Berkeley in 1992, serving as director of MSRI from 1992 to 1997. He was on the faculty of UC Davis from 1996 to 2003. Carol Wood gives a detailed account of the MSRI years, pointing out that, during that period, MSRI introduced many innovative education and outreach initiatives that were revolutionary in their time but are standard for research institutes today. At Davis he published his long-awaited book, *Three-Dimensional Geometry and Topology* (edited by Silvio Levy), which grew out of a portion of his Princeton lectures and won the 2005 AMS Book Prize. He also wrote the influential monograph *Confoliations* with Yasha Eliashberg. At Davis he taught a wide array of courses, both for undergraduates and for graduate students. During his two-year postdoc at Davis, Ian Agol was both a co-teacher and a student in some of these classes. For family reasons, Bill was planning to return to Davis in the fall of 2012.

Cornell

Bill moved to Cornell in 2003. It was around this time that Grigori Perelman announced a proof of the geometrization conjecture. Although the techniques were primarily analytic and seemingly very different from his own work, Thurston felt that the proof was fully in keeping with the spirit of his vision, and he was genuinely pleased with its solution.[15]

In 2011 Bill was diagnosed with a melanoma, and in April he underwent surgery to remove a tumor, losing his right eye in the process.



Bicycle repairs with Dylan and Nathaniel Thurston.

Despite being under arduous medical treatments, he threw himself back into mathematics, attending conferences, inspiring young people, and proving fundamental results in the theory of rational maps that harkened back to his work with Milnor in the 1970s. He died on August 21, 2012, surrounded by his family.

In June 2014 a wide-ranging conference, What's Next? The Mathematical Legacy of Bill Thurston,[16] was held at Cornell. It was attended by his mother, brothers, and sister, Rachel and Julian, his children, and other family members, along with about three hundred mathematicians, including many students and recent PhDs. It was a true celebration of the tremendous future he left us.

Thurston Resources

1) William Thurston at the Mathematics Genealogy Project:
`www.genealogy.ams.org/id.php?id=11749`

2) Cornell tribute and remembrance page:
`www.math.cornell.edu/News/2012-2013/thurston.html`

3) New College obituary:
`www.ncf.edu/william-thurston`

4) AMS obituary:
`www.ams.org/news?news_id=1602`

5) Mathematics Meets Fashion: Thurston's Concepts Inspire Designer:
`www.ams.org/news/ams-news-releases/thurston-miyake`

6) AMS Feature column:
`www.ams.org/samplings/feature-column/fc-2012-10`

7) *New York Times* obituary:
`www.nytimes.com/2012/08/23/us/william-p-thurston-theoretical-mathematician-dies-at-65.html?_r=2`

[14] WSJ, *March 18, 1983.*

[15] *Perelman laudation at 2010 Clay Research Conference.*

[16] `www.math.cornell.edu/~thurston/index.php`

8) *The Atlantic* obituary: www.theatlantic.com/technology/archive/2012/08/remembering-bill-thurston-mathematician-who-helped-us-understand-the-shape-of-the-universe/261479/

9) AMS 2005 Book Prize citation for *Three-Dimensional Geometry and Topology*: www.ams.org/notices/200504/comm-book.pdf

10) AMS Steele Prize citation for "Seminal Contribution to Research": www.ams.org/notices/201204/rtx120400563p.pdf

11) W. Thurston and J-P. Bourguignon, Interview de William Thurston (in English), *Gaz. Math.* No. 65 (1995), 11–18.

12) J. Hubbard, Bill Thurston (1946–2012), *European Math. Soc. Newsletter*, June 2014, 36–37.

Expository Accounts of Thurston's Mathematical Work

[1] W. Browder and W-C. Hsiang, The work of William P. Thurston, *Notices Amer. Math. Soc.* **29** (1982), 501.

[2] S. Friedl, Thurston's vision and the virtual fibering theorem, *Jahresber. Deutsch. Math.-Verein.* **111** (2014), 223–241.

[3] D. Gabai and S. Kerckhoff, Thurston's Geometrization Conjecture, Clay Mathematics Institute, *2009 Annual Report*, pp. 32–38.

[4] M. Gromov, *Hyperbolic Manifolds (according to Thurston and Jørgensen)*, Bourbaki Seminar, Vol. 1979/80, Springer Lecture Notes in Math., 842, 1981, pp. 40–53.

[5] B. Lawson, Thurston's work on foliations, *Notices Amer. Math. Soc.* **26** (1979), 294–295.

[6] D. Margalit, Thurston's work on Surfaces, *Bulletin Amer. Math. Soc.* **51** (2014), 151-161.

[7] J. P. Otal, William P. Thurston: Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Jahresber. Deutsch. Math.-Verein.* **116** (2014), 3–20.

[8] F. Sergeraert, $B\Gamma$, *Seminaire Bourbaki* **524** (1977–78), 300–315.

[9] D. Sullivan, The new geometry of Thurston, *Notices Amer. Math. Soc.* **26** (1979), 295–296.

[10] W. Thurston, How to see 3-manifolds, *Class. Quantum Grav.* **15** (1998), 2545–2571.

[11] ______, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), no. 3, 357–381.

[12] C. T. C. Wall, *On the Work of W. Thurston*, Proceedings of the International Congress of Mathematicians, vol. 1, Warsaw, 1983, pp. 11–14 (PWN, Warsaw, 1984).

André Haefliger

I first met Thurston at a conference dedicated to foliation theory in March 1972 in Les Plans-sur-Bex in the Swiss mountains.

Let me first recall a few results in this area which played an important role. In 1969 Bott found a topological obstruction in terms of characteristic classes for deforming a plane field into an integrable one, i.e., tangent to a smooth foliation. It is this theorem that renewed my interest in foliation theory. In this year Gelfand and Fuks began to publish a series of articles on the cohomology of the Lie algebra of various vector fields. In a conference which took place in Mont-Aigual (near Montpellier) in 1969, I sketched the construction of a classifying space called $B\Gamma_q$ for Γ_q-structures, a notion I introduced in my thesis in 1958.

Here Γ_q stands for the topological groupoid of germs of smooth local diffeomorphisms of R^q. A Γ_q-structure on a topological space X can be defined by an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and a 1-cocycle over $\mathcal{U}$ with values in Γ_q; i.e., for each $i, j \in I$ a continuous map $\gamma_{ij} : U_i \cap U_j \to \Gamma_q$ is given such that

$$\gamma_{ik}(x) = \gamma_{ij}(x)\gamma_{jk}(x), \quad \forall x \in U_i \cap U_j \cap U_k.$$

So γ_{ii} is a continuous map $f_i : U_i \to R^q$ (identified with the subspace of units of Γ_q).

Two cocycles γ_{ij} and γ'_{ij} are equivalent if there exist continuous maps $\delta_i : U_i \to \Gamma_q$ such that $\gamma'_{ij}(x) = \delta_i(x)\gamma_{ij}(x)(\delta_j(x))^{-1}$ for all $x \in U_i \cap U_j$. A Γ_q-structure on X is an equivalence class of 1-cocycles after taking the limit over open covers of X. If X is a smooth manifold and if the maps f_i are submersions, then the 1-cocycle γ_{ij} defines a smooth foliation of codimension q on X. The construction above works as well if Γ_q is replaced by any topological groupoid, for instance, the groupoid of germs of local analytic diffeomorphisms of R^q or a Lie group G. The homotopy classes of Γ_q-structures on X are in bijective correspondence with the homotopy classes of continuous maps from X to the classifying space $B\Gamma_q$.

In 1971 Harold Rosenberg gave a course on foliation theory, and Thurston was among his students. Harold realized immediately that he was a brilliant student, and by the end of the course they wrote a paper together containing a lot of interesting geometric results. Meanwhile, in that year, Godbillon explained in a meeting at Oberwolfach the construction of the invariant GV of Godbillon and Vey, but at this time they did not know if it was trivial or not (but Roussarie, who was attending the meeting, almost immediately found a nontrivial example). When Lawson came back from this meeting he did not know the example of Roussarie; he explained the construction of the invariant to Bill (extract from an email): "I showed him the definition. The next day he knocked on my door, and said: 'The invariant is nontrivial.' The following day he knocked again, and this time said: 'The invariant can assume any real number.' WHAT???!!!! Who is this fellow? I said to myself."

Very quickly after that, connections of the

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Gelfand-Fuchs cohomology with the cohomology of various classifying spaces for foliations were understood theoretically and independently by many people.

Thurston got his PhD in Berkeley in early 1972. His thesis adviser was Moe Hirsch, and Blaine Lawson was the examiner. Bill submitted his thesis entitled "Foliations on 3-manifolds which are circle bundles" to *Inventiones*. The referee suggested that the author should give more explanations. As a consequence, Thurston, who was busy proving more theorems, decided not to publish it.

Meanwhile, John Mather became independently interested in discovering properties of classifying spaces for foliations. In 1971 he wrote several papers and preprints proving deep theorems on $B\Gamma_1$. His approach to foliation theory was not as geometric as Thurston's, but nevertheless was very efficient.

When Rosenberg came back to Paris after his stay in Berkeley, he invited Bill to Paris. When he heard that we organized, at the last moment, the meeting in Les Plans-sur-Bex, Rosenberg brought Bill with him. Also participating in this meeting were A'Campo, Hector, Herman, Moussu, Siebenmann, Roger, Roussarie, Tischler, Vey, Wood, my student Banyaga from Rwanda, and many others.

Under the suggestion of Lawson, Milnor invited Bill (as his "assistant") and me for the academic year 1972–73 to the IAS. Our two families were neighbors in the housing project of the institute. We organized a seminar which met once a week. Several people participated in it, and Bill gave several talks, always with new surprising results and ideas. In April 1973 I went back to Geneva.

On May 4, 1973, I wrote to William Browder a letter of recommendation for Bill for a position at Princeton University. In this letter I summarized the impressive list of results so far obtained by Thurston during this academic year: for instance, the deep connection between the homology of the group of diffeomorphisms of R^n with support in compact sets and the homology of $B\Gamma_n$; the fact that any field of 2-planes on a manifold (in codimension > 1) is homotopic to a smooth integrable one; an obstruction theory for deforming a field of p-planes on an n-manifold into a smooth integrable one (provided that $n - p > 1$), as a consequence that a field of p-planes whose normal bundle is trivial can be deformed to a smooth integrable one; also that in contrast there is no obstruction to deform a field of p-planes into one tangent to a C^0- foliation, etc. As a conclusion I wrote : "I am very much impressed by the way he understands mathematics. He has a very direct and original way of looking at geometrical questions and an immediate understanding. He has sometimes difficulties to fill in the details of a proof, but after a while you realize the few words that he has written are just the essential ones."

In a letter dated August 16, 1973, Thurston sent me, for publication in *Comment. Math. Helv.*, the manuscript of his paper "The theory of foliations of dimension greater than one." In the same letter, he sent me the preprint of a paper about volume-preserving diffeomorphisms of T^n and said that he would send me a paper on the construction of foliations on 3-manifolds.

On October 10 he sent me a letter of six pages from Cambridge, Massachusetts, indicating the changes he made in his manuscript to take account of my remarks. In the last five pages he explained to me in detail the proof of his generalization of the Reeb stability theorem. In addition, he sent three pages of corrections of his manuscript for CMH.

On December 7, 1973, he sent me the corrected version of his manuscript. In a letter of seven pages, he explained his generalization of the classifying theorem to codimension one.

During the summer of 1976 a big symposium was organized by the University of Warwick. Thurston, Milnor, Vey, Mather, and Hirsch were among the participants. With our two boys, at that time very much interested in folk music, we drove from Switzerland to Warwick, where we stayed for two weeks. The enclosed picture shows a huge pile of straw built up by Bill and the youngsters. On the top were sitting my two boys and the older daughter of Milnor; on the side one can see Bill and many others. Moe Hirsch played a lot of folk music with my boys, and Vey beautifully sang traditional French songs.

From August 25 to September 4, 1976, a summer school organized by the CIME took place in Varenna in Italy. Thurston, Mather, and I gave expository courses on foliations. In addition, many participants gave talks, like Sergeraert; Banyaga spoke on his thesis concerning the group of symplectic diffeomorphisms of a compact symplectic manifold. This was a generalization of a nonpublished preprint of Thurston on the group of volume-preserving diffeomorphisms. Vaughn Jones, then a student in Geneva, was among the participants.

Thurston gave a beautiful course, but he did not write it up. At that time he was not interested anymore in foliations, but was planning to teach at Princeton a course on hyperbolic geometry.

Photo includes: Rachel Findley (bottom right), Nathaniel Thurston (at Bill's feet), Dylan Thurston (peering out at top left).

David Epstein

Remembering Bill

In December 1970 I gave a lecture at UCBerkeley on my theorem that any foliation of a compact 3-manifold by circles is a Seifert fibration. After the lecture, Moe Hirsch, Bill's PhD supervisor, introduced us, and Bill told me rather diffidently that he knew how to decompose $\mathbb{R}^3$ into round planar circles. He further explained that his circles did not form a foliation. Forty-three years later, I'm still wondering how this is possible.

I next heard from Bill when he wrote me a long letter (airmail; email did not yet exist) about foliations, classifying spaces, and Haefliger structures, with lots of hand-drawn diagrams and spectral sequences—quite a bit more than I was able to digest. Here's one sentence to give you the flavor: *You'll see the point if you squint at this diagram.* I worked for a long time on Bill's letter, eventually coming up with a counterexample to one of the statements, thinking to myself with relief that now that I had found the error, I could stop struggling. However, by return post, Bill fixed the problem, at which point I had to reconcile myself to incomprehension.

In an interesting essay, "On proof and progress in mathematics," Bill distinguishes between the way he contributed to foliation theory and the way he contributed to 3-manifold theory. The huge and daunting advances he made in foliation theory were off-putting, and students stopped going into the area, resulting in an unfortunate premature arrest in the development of the subject while it was still in its prime. (If someone writes a book incorporating Bill's advances, it will take off again.) In contrast, Bill's huge and daunting contributions to low-dimensional manifold theory were buttressed by substantial notes written by Bill and his students (not formally published but very widely circulated) and by notes by others. These fleshed out the infrastructure of the subject, easing the task of those who wished to push Bill's work further. Later, there was also a beautiful book, *Three-Dimensional Geometry and Topology*, by Bill and Silvio Levy.

I knew Bill well during his marriage with Rachel Findley, and my remarks and reminiscences relate to that period, ending around 1993. I used to visit Princeton quite often, where I stayed in their large, untidy, warmly hospitable university-owned house, just a short walk from Fine Hall. Bill discovered or invented interesting mathematics all the time, in diverse situations, and there was never time to fully formalize and make public all or even a major part of what he thought about. As just one example, he developed probability distributions on the set of mazes made from a grid by marking certain edges as impenetrable walls, as found in children's magazines, and was delighted by his computer program that churned out random mazes of a prescribed level of difficulty.

Talking mathematics to Bill was interesting, inspiring, and frustrating. I often wished that I had a tape recorder: while listening to him, I was sure I understood. But when I tried to reconstruct the conversation, I almost always found difficulties. After some work, I would focus on one particular point where I asked urgently for clarification. Instead of answering, he would say, "Maybe you would like this other proof better," which he would then explain to me. And the process would repeat itself. On the other hand, reading, understanding, and helping to smooth the rough edges in Bill's notes was an enthralling and rewarding task from which I learned an enormous amount, and I believe the same was true for his many other helpers.

The difficulties that people had in understanding Bill's mathematics made him very alive to the problems of mathematical education. His views[17] continue to be widely quoted. Even more striking than his writings were his experiments into a different way of teaching. Al Marden, the director of the Geometry Center in Minneapolis, had made it easy for me to visit frequently, so I arrived there in the summer of 1991, to find an atmosphere

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[17]Mathematical Education, *arXiv: math/0503081*

of great intellectual ferment during a course entitled Geometry and the Imagination, given by Bill, John Conway, Peter Doyle, and Jane Gilman. The audience was a mixture of school children, undergraduates, school teachers, and college teachers of mathematics. Some of the material for this course is available online,[18] where I strongly recommend any reader interested in mathematics or in mathematical education to spend time.

Some of the school children in the audience excitedly explained to me that Peter Doyle had just smeared the tires of his bicycle with mud, that enormous blank posters had been spread out on the floor of the Geometry Center, and that Peter had then cycled around the huge room while members of the audience were excluded. The first task of the audience, working in small groups, was to demolish the reasoning of Sherlock Holmes, quoted in *The Adventure of the Priory School*: "The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school." The audience's second task was to correctly deduce from the mud-stained posters the direction of travel. Peter's ingenious solution is explained in "Which way did the bicycle go?" [19] by Stan Wagon and co-authors. Stan, from Macalester College, 'round the corner from the Geometry Center, was a contributor to this course and was, I am told, in the audience at the time.

Bill arranged many courses like this, at many different levels. Quite a few good mathematicians currently in post at universities in the US and around the world were inspired by one or more of his courses.

I cannot omit mention of the two videos, *Not Knot* and *Outside In*, in which some of Bill's ideas were concretely realized by Geometry Center staff. These winners of major worldwide prizes are still freely available for your enjoyment and amazement on YouTube. Each explains significant and difficult mathematical ideas without recourse to formal mathematics. You can use them in outreach to nonmathematicians, and your undergraduate and graduate students would also benefit. As you do so, you will appreciate more deeply Bill's imaginative spirit and the cultural power of constructive mathematics. (Each of the videos has an associated booklet to make deeper study more easily accessible.)

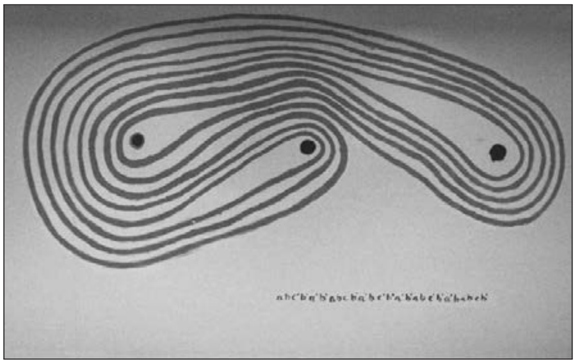
Bill was a pioneer in the use of computers as a tool in pure mathematics, an aspect of his strongly constructivist point of view. In fact, he wondered whether to study logic rather than topology for his PhD, but was dissuaded by Tarski, who told him that the logicians at Berkeley were not interested in intuitionism. Bill suggested to me the use of "hollow exist," $\exists$, to denote the result of a nonconstructive existence proof (like Cantor's proof of the existence of transcendental numbers). Bill wanted more than constructivism, something like "rapid constructivism," with programs that complete in a reasonable amount of time. Jeff Weeks, one of Bill's many talented students, developed SnapPea, a program that computes, if it exists, a complete hyperbolic structure on a simplicial 3-manifold. This program, based on Bill's ideas, has been used in many significant projects. It can be used to give a quick proof that two simplicial 3-manifolds are not homeomorphic or that two complicated knots are not equivalent.

During the early 1980s, Bill decided that the Fine Hall pure mathematicians should be introduced to computing. The funding that he raised covered the purchase of a computer (very expensive at that time) but not the cost of employing a person. I remember Bill going in to Fine Hall over the weekend to himself lay all necessary cables. He also became the local UNIX expert and an intrepid systems programmer. When the very large UNIX editing program *vi* didn't behave correctly, he decided to debug the code. Unsurprisingly, this venture failed. On another occasion, a graduate student invented a process that continually spawned clones of itself until the machine was full. It was a science fiction scenario: if one of the clones was killed, one of the other clones would immediately construct a replacement. Bill was determined to find a humane method of getting rid of the clones that would not entail switching off the machine, and he eventually succeeded where most professional systems programmers would have failed. Computer-related activity of this kind took up a good deal of his time, time that the mathematical community might have preferred him to devote to writing up his mathematical discoveries, but Bill placed a high priority on converting his colleagues to the use of computers.

Bill and Rachel grew up during the Vietnam War and were strongly opposed to US military activities. Rachel was particularly active. In the early 1980s, as the usefulness of computers to pure mathematicians increased (partly because of the spread of computer typesetting by LaTeX), the DOD (US Department of Defense) started to fund grants to pure mathematicians that were hard to refuse because alternative funding for computers was not readily available. Some of these grants dispensed with peer review and were funneled through the CIA. Mike Shub resigned from New York's City University over the issue and started a campaign against the acceptance of military funds by academics, in which Bill became involved. Those

[18] `www.geom.uiuc.edu/docs/education/institute91/handouts/handouts.html`

[19] *J. D. E. Kronhauser, D. J. Velleman, and Stan Wagon, MAA, 1996.*



Photographer: Professor Ken Ribet. Used with permission.

Wall painting, Berkeley math department.

with access to the *Notices of the AMS* for 1987 and 1988 will find interesting correspondence on this issue, with many different points of view represented, including letters from Bill. Eventually, in 1988, the AMS passed a resolution calling for a greater effort to decrease the proportion of funding for mathematics research coming from the DoD. At the same time, the AMS declared skepticism about SDI, Reagan's Star Wars program. Typically, Bill was not jubilant over the decision he had fought for and won, and was rather concerned at the dismay of applied mathematicians, who had a tradition of accepting military funds and who did not agree that this had a dangerous effect on the values of civil society.

During the period when I knew him well, Bill enjoyed life thoroughly. At conferences, and often at Fine Hall, he would organize and enthusiastically participate in games of volleyball. He had a great sense of humor. When my wife, staying with the Thurstons, asked about their clothes dryer, he puzzled her briefly by replying that it was solar-powered. He earned a stiff reprimand from Princeton township officials for tapping a fine maple tree on the street outside his home for its syrup, which he did carefully, with the proper equipment. He was a passionately involved father, and I have vivid memories of him with his children. He was an excellent cook who would regularly produce both everyday meals and special treats, such as his unique peach ice cream. He loved games, picnics, socializing, and talking at a deeper level than chit chat. He was a very considerate friend. Bill was a person of exceptional warmth, whose presence, once experienced, will never be forgotten.

Dennis P. Sullivan

A Decade of Thurston Stories

First Story

In December of 1971 a dynamics seminar ended at Berkeley with the solution to a thorny problem in the plane which had a nice application in dynamics. The solution purported to move N distinct points to a second set of "epsilon near" N distinct points by a motion which kept the points distinct and moved while staying always "epsilon prime near." The senior dynamicists in the front row were upbeat because the dynamics application up to then had only been possible in dimensions at least three, where this matching problem is obvious by general position, but now the dynamics theorem also worked in dimension two.

A heavily bearded, long-haired graduate student in the back of the room stood up and said he thought the algorithm of the proof didn't work. He, Bill Thurston, went shyly to the blackboard and drew two configurations of about seven points each and started applying to these the method at the end of the lecture. Little paths started emerging and getting in the way of other emerging paths, which, to avoid collision, had to get longer and longer. The algorithm didn't work at all for this quite involved diagrammatic reason. I had never seen such comprehension and such creative construction of a counterexample done so quickly. This combined with my awe at the sheer complexity of the geometry that emerged.

Second Story

A couple of days later, the grad students invited me (I was also heavily bearded with long hair) to paint math frescoes on the corridor wall separating their offices from the elevator foyer. While milling around before painting, that same graduate student came up to ask, "Do you think this is interesting to paint?" It was a complicated smooth one-dimensional object encircling three points in the plane. I asked, "What is it?" and was astonished to hear, "It is a simple closed curve." I said, "You bet it's interesting!". So we proceeded to spend several hours painting this curve on the wall. It was a great learning and bonding experience. For such a curve to look good it has to be drawn in sections of short parallel slightly curved strands (like the flow boxes of a foliation) which are subsequently smoothly spliced together. When I asked how he got such curves, he said, "By successively applying to a given simple curve a pair of Dehn twists along intersecting curves." The "wall curve painting," two meters high and four meters wide (see AMS *Notices* cover "Cave Drawings"), dated and signed "DPS and

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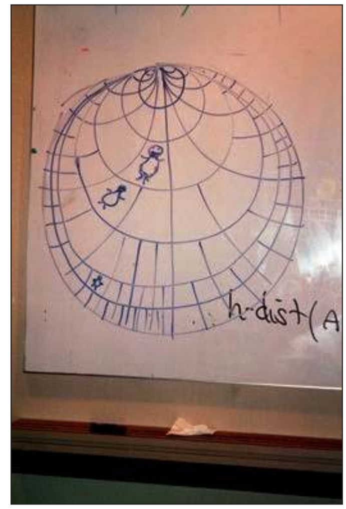
BT, December, 1971" lasted on that Berkeley wall with periodic restoration for almost four decades before finally being painted over a few years ago.

Third Story

That week in December 1971 I was visiting Berkeley from MIT to give a series of lectures on differential forms and the homotopy theory of manifolds. Since foliations and differential forms were appearing everywhere, I thought to use the one-forms that emerged in my story describing the lower central series of the fundamental group to construct foliations. Leaves of these foliations would cover graphs of maps of the manifold to the nilmanifolds associated to all the higher nilpotent quotients of the fundamental group. These would generalize Abel's map to the torus associated with the first homology torus. Being uninitiated in Lie theory, I had asked all the differential geometers at MIT and Harvard about this possibility but couldn't make myself understood. It was too vague, too algebraic. I presented the discussion in my first lecture at Berkeley, and to Bill privately, without much hope because of the weird algebra-geometry mixture. However, the next day Bill came up with a complete solution and a full explanation. For him it was elementary and really only involved actually understanding the basic geometric meaning of the Jacobi relation in the Elie Cartan $dd = 0$ dual form.

In between the times of the first two stories above I had spoken to old friend Moe Hirsch about Bill Thurston, who was working with Moe and was finishing in his fifth year after an apparently slow start. Moe or someone else told how Bill's oral exam was a slight problem, because when asked for an example of the universal cover of a space Bill chose the surface of genus two and started drawing awkward octagons with many (eight) coming together at each vertex. This exposition quickly became an unconvincing mess on the blackboard. I think Bill was the only one in that exam room who had ever thought about such a nontrivial universal cover. Moe then said, "Lately, Bill has started solving thesis-level problems at the rate of about one every month." Some years later, I heard from Bill that his first child, Nathaniel, didn't like to sleep at night, so Bill was sleep-deprived, "walking the floor with Nathaniel" for about a year of grad school.

That week of math at Berkeley was life-changing for me. I was very grateful to be able to seriously appreciate the Mozart-like phenomenon I had been observing, and I had a new friend. Upon returning to MIT after the week in Berkeley, I related my news to the colleagues there, but I think my enthusiasm was too intense to be believed: "I have just met the best graduate student I have ever seen or ever expect to see." It was arranged for Bill to give a talk at MIT, which evolved into a plan for him to come to MIT after going first to IAS in Princeton. It turned out that he did come to MIT, for just one year 1973–74. (That year I visited IHES, where I ultimately stayed for twenty-odd years, while Bill was invited back to Princeton, to the University.)



Horocycles.

Fourth Story
IAS Princeton, 1972–73

When I visited the environs of Princeton from MIT in 1972–73, I had chances to interact more with Bill. One day, walking outside towards lunch at IAS, I asked Bill what a horocycle was. He said, "You stay here," and he started walking away into the institute meadow. After some distance he turned and stood still, saying, "You are on the circumference of a circle with me as center." Then he turned, walked much farther away, turned back, and said something which I couldn't hear because of the distance. After shouting back and forth to the amusement of the members, we realized he was saying the same thing, "You are on the circumference of a circle with me as center." Then he walked even farther away, just a small figure in the distance and certainly out of hearing, whereupon he turned, and started shouting presumably the same thing again. We got the idea what a horocycle was.

Atiyah asked some of us topologists if we knew if flat vector bundles had a classifying space (he had constructed some new characteristic classes for such). We knew it existed from Brown's theorem but didn't know how to construct it explicitly. The next day, Atiyah said he asked Thurston this question, who did it by what was then a shocking construction: take the Lie structure group of the vector bundle as an abstract group with the discrete topology and form its classifying space. Later, I heard about Thurston drawing Jack Milnor a picture proving any dynamical pattern for any unimodal map appears in the quadratic family $x \to x^2 + c$. Since I was studying dynamics, I planned to spend a semester with Bill at Princeton to learn about the celebrated Milnor-Thurston universality paper that resulted from this drawing.

Fifth Story
Princeton University, Fall 1976

I expected to learn about one-dimensional dynamics upon arriving in Princeton in September 1976, but Thurston had already developed a new theory of surface transformations. In the first few days, he expounded on this in a wonderful three-hour extemporaneous lecture at the institute. Luckily for me, the main theorem about limiting foliations was intuitively clear because of the painstaking Berkeley wall curve painting described above. At the end of the stay that semester, Bill told me he believed the mapping torus of these carried hyperbolic metrics. When I asked why, he told me he couldn't explain it to me because I didn't understand enough differential geometry. A few weeks after I left Princeton, with more time to work without my distractions, Bill essentially understood the proof of the hyperbolic metric for appropriate Haken manifolds. The mapping torus case took two more years, as discussed below.

During the semester graduate course that Bill gave, the graduate students and I learned several key ideas:

(1) The quasi-analogue of "hyperbolic geometry at infinity becomes conformal geometry on the sphere at infinity." (A notable memory here is the feeling that Bill conveyed about really being inside hyperbolic space rather than being outside and looking at a particular model. For me this made a psychological difference.)

(2) We learned about the intrinsic geometry of convex surfaces outside the extreme points. (Bill came into class one day, and, for many minutes, he rolled a paper contraption he had made around and around on the lecturer's table without saying a word until we felt the flatness.)

(3) We learned about the thick-thin decomposition of hyperbolic surfaces. (I remember how Bill drew a fifty-meter-long thin part winding all around the blackboard near the common room, and suddenly everything was clear, including geometric convergence to the points of the celebrated DM compactification of the space of Riemann surfaces.)

During that fall 1976 semester stay at Princeton, Bill and I discussed understanding the Poincaré conjecture by trying to prove a general theorem about all closed 3-manifolds, based on the idea that three is a relatively small dimension. We included in our little paper on "Canonical coordinates…Commentarii" the sufficient for Poincaré conjecture possibility that all closed 3-manifolds carried conformally flat coordinates. (However, an undergrad, Bill Goldman, who was often around that fall, disproved this a few years later.) We decided to try to spend an academic year together in the future.

In the next period, Bill developed limits of quasi-Fuchsian Kleinian groups and pursued the mapping torus hyperbolic structure in Princeton, while I pursued the Ahlfors limit set measure problem in Paris. After about a year, Bill had made substantial positive progress (e.g., closing the cusp), and I had made substantial negative progress (showing all known ergodic methods coupled with all known Kleinian group information were inadequate: there was too much potential nonlinearity). We met in the Swiss Alps at the Plan-sur-Bex conference and compared notes. His mapping torus program was positively finished but very complicated, while my negative information had revealed a rigidity result extending Mostow's, which allowed a considerable simplification of Bill's fibring proof. (See the Bourbaki report on Thurston's work during the next year.)

Sixth Story
The Stonybrook Conference, Summer 1978

There was a big conference on Kleinian groups at Stonybrook, and Bill was in attendance but not as a speaker. Gromov and I got him to give a lengthy impromptu talk outside the schedule. It was a wonderful trip out into the end of a hyperbolic 3-manifold, combined with convex hulls, pleated surfaces, and ending laminations…. During the lecture, Gromov leaned over and said watching Bill made him feel like "this field hadn't officially started yet."

Seventh Story
Colorado, June 1980 to August 1981

Bill and I shared the Stanislaus Ulam Visiting Professorship at Boulder and ran two seminars: a big one drawing together all the threads for the full hyperbolic theorem and a smaller one on the dynamics of Kleinian groups and dynamics in general. All aspects of the hyperbolic proof passed in review with many grad students in attendance. One day in the other seminar, Bill was late. Dan Rudolph was very energetically explaining in just one hour a new, shorter version of an extremely complicated proof. The theorem promoted an orbit equivalence to a conjugacy between two ergodic measure-preserving transformations if the discrepancy of the orbit equivalence was controlled. The new proof was due to a subset of the triumvirate Katznelson, Ornstein, and Weiss and was notable because it could be explained in one hour, whereas the first proof took a minicourse to explain. Thurston at last came in and asked me to bring him up to speed, which I did. The lecture continued to the end with Bill wondering in loud whispers what the difficulty was and with me shushing him out of respect for the context. Finally, at the end, Bill said, just imagine a bi-infinite string of beads on a wire with finitely many missing spaces and (illustrated by full sweep of his extended arm) just slide them all to the left, say. Up to some standard bookkeeping this gave a new proof. Later that day an awe-struck Dan Rudolph said to me he never realized before then just how smart Bill Thurston really was.

Eighth Story
La Jolla and Paris, End of Summer 1981

The Colorado experience was very good, relaxing in the Thurston seminar with geometry (one day we worked out the eight geometries and another day we voted on terminology "manifolded" or "orbifold") and writing several papers of my own on Hausdorff dimension, dynamics, and measures on dynamical limit sets. Later that summer I was flying from Paris to La Jolla to give a series of AMS lectures on the dynamics stuff when I changed the plan and decided instead to try to expose the entire hyperbolic theorem "for the greater good" and as a self-imposed Boulder final exam for me. I managed to come up with a one-page sketch while on the plane. There were to be two lectures a day for four or five days. The first day would be okay, I thought: just survey things and then try to improvise for the rest, but I needed a stroke of luck. It came big time.

There is a nine-hour time difference between California and Paris, and the first day, awaking around midnight local time, I went to my assigned office to prepare. After a few hours I had generated many questions and fewer answers about the hyperbolic argument. I noticed a phone on the desk that miraculously allowed long-distance calls, and, by then, it was around 4:00 a.m. California time and 7:00 a.m. in Princeton. I called Bill's house, and he answered. I posed my questions. He gave quick responses, I took notes, and he said call back after he dropped the kids at school and got to his office. I gave my objections to his answers around 9:30 a.m. his time, and he responded more fully. We ended up with various alternate routes that all in all covered every point. By 8:00 a.m. my time, I had a pair of lectures prepared. The first day went well: lecture, lunch/beach/swim, second lecture, dinner, then goodbye to colleagues and back to bed. This took some discipline, but as viewers of the videos will see, the audience was formidable (Ahlfors, Bott, Chern, Kirby, Siebenmann, Edwards, Rosenberg, Freedman, Yau, Maskit, Kra, Keen, Dodziuk,…).

Bill and I repeated this each day, perfecting the back and forth, so that by 8:00 a.m. California time each day, I had my two lectures prepared, and they were getting the job done. The climax came when presenting Bill's delicious argument that controls the length of a geodesic representing the branching locus of a branched pleated surface by the dynamical rate of chaos or entropy created by the geodesic flow on the intrinsic surface. One knows that this is controlled by the area growth of the universal cover of the branched surface, which by negative curvature is controlled by the volume growth of the containing hyperbolic 3-space. QED. There was in addition Bill's beautiful example showing the estimate was qualitatively sharp. This splendid level of lecturing was too much for Harold Rosenberg, my astute friend from Paris, who was in the audience. He came to me afterwards and asked frustratingly, "Dennis, do you keep Thurston locked up in your office upstairs?" The lectures were taped by Michael Freedman, and I have kept my lips sealed until now. The taped (Thurston)-Sullivan lectures are available online.[20]

Ninth Story
Paris, Fall 1981

Bill visited me in Paris, and I bought a comfy sofa bed for my home office where he could sleep. He politely asked what would I have talked about had I not changed plans for the AMS lectures and, in particular, what had I been doing in detail in Colorado beyond the hyperbolic seminar. There were about six papers to tell him about. One of the most appealing ideas I had learned from him: namely, that the visual Hausdorff-dimensional measure of an appropriate set on the sphere at infinity, as viewed from a point inside, defines a positive eigenfunction for the hyperbolic 3-space Laplacian with eigenvalue $f(2-f)$. I started going through the ideas and statements. I made a statement, and he either immediately gave the proof or I gave the idea of my proof. We went through all the theorems in the six papers in one session, with either him or me giving the proof. There was one missing result: that the bottom eigenfunction when f was > 1 would be represented by a normalized eigenfunction whose square integral norm was estimated by the volume of the convex core. Bill lay back for a moment on his sofa bed, his eyes closed, and immediately proved the missing theorem. He produced the estimate by diffusing geodesics transversally and averaging. Then we went out to walk through Paris from Port d'Orleans to Port de Clignancourt. Of course, we spoke so much about mathematics that Paris was essentially forgotten, except maybe the simultaneous view of Notre Dame and the Conciergerie as we crossed over the Seine.

Tenth Story
Princeton-Manhattan, 1982–83

I began splitting time between IHES and the CUNY Grad Center, where I started a thirteen-year-long Einstein chair seminar on dynamics and quasi-conformal homeomorphisms (which changed then to quantum objects in topology), while Bill continued developing a cadre of young geometers to spread the beautiful ideas of negatively curved space. Bill delayed writing a definitive text on the hyperbolic proof in lieu of letting things develop along many opening avenues by his increasingly informed cadre of younger/older geometers. He wanted to avoid in hyperbolic geometry what had happened when his basic papers on foliations "tsunamied" the field in the early 1970s. Once,

[20] `www.math.sunysb.edu/Videos/Einstein`

Courtesy of Stony Brook University.

Thurston lecturing at "Jackfest", Banff, 2011.

we planned to meet in Manhattan to discuss holomorphic dynamics in one variable and the analogy with hyperbolic geometry and Kleinian groups that I had been preoccupied with. We were not disciplined and began talking about other things at the apartment; we finally got around to our agenda about thirty minutes before Bill had to leave for his train back to Princeton. I sketched the general analogy: Poincaré limit set, domain of discontinuity, deformations, rigidity, classification, Ahlfors finiteness theorem, the work of Ahlfors-Bers,…to be compared with Julia set, Fatou set, deformations, rigidity, classification, non-wandering domain theorem, the work of Hubbard-Douady…, which he perfectly and quickly absorbed until he had to leave for the train. Two weeks later we heard about his reformulation of a holomorphic dynamical system as a fixed point on Teichmüller space, analogous to part of his hyperbolic theorem. There were many new results, including those of Curt McMullen some years later, and the subject of holomorphic dynamics was raised to another higher level.

Postscript

Bill and I met again at Milnor's eightieth fest at Banff—Bill's "Jackfest" lecture is pictured above—after essentially thirty years and picked up where we had left off. (I admired his checked green shirt the second time it appeared, and he presented it to me the next day.) We promised to try to attack together a remaining big hole in the Kleinian group/holomorphic dynamics dictionary: "the invariant line field conjecture."

It was a good idea, but unfortunately turned out to be impossible.