

# Wind Turbines Theory - The Betz Equation and Optimal Rotor Tip Speed Ratio

Magdi Ragheb<sup>1</sup> and Adam M. Ragheb<sup>2</sup>

<sup>1</sup>*Department of Nuclear, Plasma and Radiological Engineering*

<sup>2</sup>*Department of Aerospace Engineering*

*University of Illinois at Urbana-Champaign, 216 Talbot Laboratory, USA*

## 1. Introduction

The fundamental theory of design and operation of wind turbines is derived based on a first principles approach using conservation of mass and conservation of energy in a wind stream. A detailed derivation of the “Betz Equation” and the “Betz Criterion” or “Betz Limit” is presented, and its subtleties, insights as well as the pitfalls in its derivation and application are discussed. This fundamental equation was first introduced by the German engineer Albert Betz in 1919 and published in his book “Wind Energie und ihre Ausnutzung durch Windmühlen,” or “Wind Energy and its Extraction through Wind Mills” in 1926. The theory that is developed applies to both horizontal and vertical axis wind turbines.

The power coefficient of a wind turbine is defined and is related to the Betz Limit. A description of the optimal rotor tip speed ratio of a wind turbine is also presented. This is compared with a description based on Schmitz whirlpool ratios accounting for the different losses and efficiencies encountered in the operation of wind energy conversion systems. The theoretical and a corrected graph of the different wind turbine operational regimes and configurations, relating the power coefficient to the rotor tip speed ratio are shown. The general common principles underlying wind, hydroelectric and thermal energy conversion are discussed.

## 2. Betz equation and criterion, performance coefficient $C_p$

The Betz Equation is analogous to the Carnot cycle efficiency in thermodynamics suggesting that a heat engine cannot extract all the energy from a given source of energy and must reject part of its heat input back to the environment. Whereas the Carnot cycle efficiency can be expressed in terms of the Kelvin isothermal heat input temperature  $T_1$  and the Kelvin isothermal heat rejection temperature  $T_2$ :

$$\eta_{Carnot} = \frac{T_1 - T_2}{T_1} = 1 - \frac{T_2}{T_1}, \quad (1)$$

the Betz Equation deals with the wind speed upstream of the turbine  $V_1$  and the downstream wind speed  $V_2$ .

The limited efficiency of a heat engine is caused by heat rejection to the environment. The limited efficiency of a wind turbine is caused by braking of the wind from its upstream speed  $V_1$  to its downstream speed  $V_2$ , while allowing a continuation of the flow regime. The additional losses in efficiency for a practical wind turbine are caused by the viscous and pressure drag on the rotor blades, the swirl imparted to the air flow by the rotor, and the power losses in the transmission and electrical system.

Betz developed the global theory of wind machines at the Göttingen Institute in Germany (Le Gourières Désiré, 1982). The wind rotor is assumed to be an ideal energy converter, meaning that:

1. It does not possess a hub,
2. It possesses an infinite number of rotor blades which do not result in any drag resistance to the wind flowing through them.

In addition, uniformity is assumed over the whole area swept by the rotor, and the speed of the air beyond the rotor is considered to be axial. The ideal wind rotor is taken at rest and is placed in a moving fluid atmosphere. Considering the ideal model shown in Fig. 1, the cross sectional area swept by the turbine blade is designated as  $S$ , with the air cross-section upwind from the rotor designated as  $S_1$ , and downwind as  $S_2$ .

The wind speed passing through the turbine rotor is considered uniform as  $V$ , with its value as  $V_1$  upwind, and as  $V_2$  downwind at a distance from the rotor. Extraction of mechanical energy by the rotor occurs by reducing the kinetic energy of the air stream from upwind to downwind, or simply applying a braking action on the wind. This implies that:

$$V_2 < V_1.$$

Consequently the air stream cross sectional area increases from upstream of the turbine to the downstream location, and:

$$S_2 > S_1.$$

If the air stream is considered as a case of incompressible flow, the conservation of mass or continuity equation can be written as:

$$\dot{m} = \rho S_1 V_1 = \rho S V = \rho S_2 V_2 = \text{constant} \quad (2)$$

This expresses the fact that the mass flow rate is a constant along the wind stream. Continuing with the derivation, Euler's Theorem gives the force exerted by the wind on the rotor as:

$$\begin{aligned} F &= ma \\ &= m \frac{dV}{dt} \\ &= \dot{m} \Delta V \\ &= \rho S V \cdot (V_1 - V_2) \end{aligned} \quad (3)$$

The incremental energy or the incremental work done in the wind stream is given by:

$$dE = F dx \quad (4)$$

From which the power content of the wind stream is:

$$P = \frac{dE}{dt} = F \frac{dx}{dt} = FV \tag{5}$$

Substituting for the force F from Eqn. 3, we get for the extractable power from the wind:

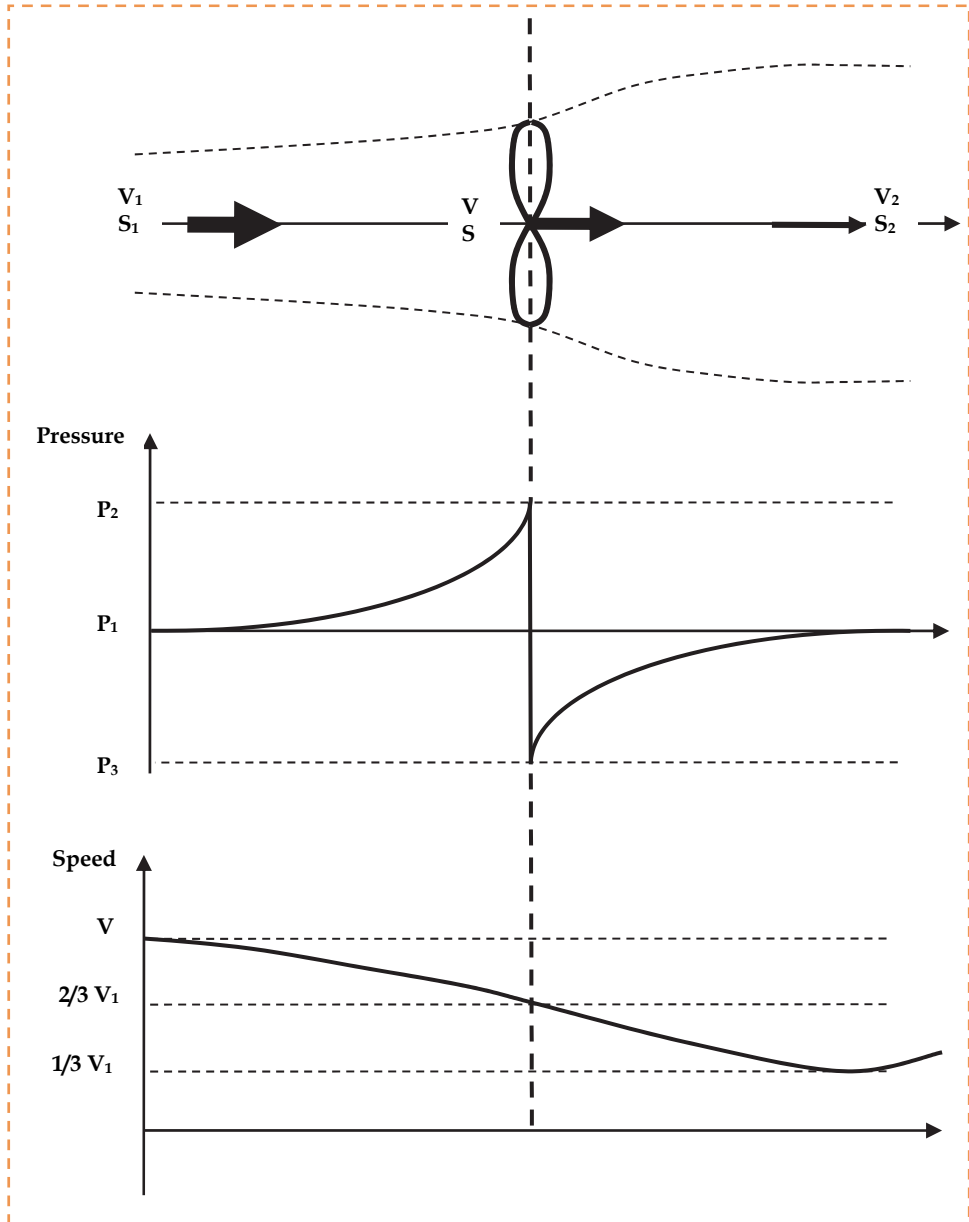


Fig. 1. Pressure and speed variation in an ideal model of a wind turbine.

$$P = \rho S V^2 \cdot (V_1 - V_2) \quad (6)$$

The power as the rate of change in kinetic energy from upstream to downstream is given by:

$$\begin{aligned} P &\approx \frac{\Delta E}{\Delta t} \\ &\approx \frac{\frac{1}{2} m V_1^2 - \frac{1}{2} m V_2^2}{\Delta t} \\ &= \frac{1}{2} \dot{m} (V_1^2 - V_2^2) \end{aligned} \quad (7)$$

Using the continuity equation (Eqn. 2), we can write:

$$P = \frac{1}{2} \rho S V (V_1^2 - V_2^2) \quad (8)$$

Equating the two expressions for the power P in Eqns. 6 and 8, we get:

$$P = \frac{1}{2} \rho S V (V_1^2 - V_2^2) = \rho S V^2 (V_1 - V_2)$$

The last expression implies that:

$$\begin{aligned} \frac{1}{2} (V_1^2 - V_2^2) &= \frac{1}{2} (V_1 - V_2)(V_1 + V_2) \\ &= V (V_1 - V_2), \quad \forall V, S, \rho \neq 0 \end{aligned}$$

or:

$$V = \frac{1}{2} (V_1 + V_2), \quad \forall (V_1 - V_2) \neq 0 \text{ or } V_1 \neq V_2 \quad (9)$$

This in turn suggests that the wind velocity at the rotor may be taken as the average of the upstream and downstream wind velocities. It also implies that the turbine must act as a brake, reducing the wind speed from  $V_1$  to  $V_2$ , but not totally reducing it to  $V = 0$ , at which point the equation is no longer valid. To extract energy from the wind stream, its flow must be maintained and not totally stopped.

The last result allows us to write new expressions for the force F and power P in terms of the upstream and downstream velocities by substituting for the value of V as:

$$\begin{aligned} F &= \rho S V (V_1 - V_2) \\ &= \frac{1}{2} \rho S (V_1^2 - V_2^2) \end{aligned} \quad (10)$$

$$\begin{aligned} P &= \rho S V^2 (V_1 - V_2) \\ &= \frac{1}{4} \rho S (V_1 + V_2)^2 (V_1 - V_2) \\ &= \frac{1}{4} \rho S (V_1^2 - V_2^2) (V_1 + V_2) \end{aligned} \quad (11)$$

We can introduce the “downstream velocity factor,” or “interference factor,”  $b$  as the ratio of the downstream speed  $V_2$  to the upstream speed  $V_1$  as:

$$b = \frac{V_2}{V_1} \quad (12)$$

From Eqn. 10 the force  $F$  can be expressed as:

$$F = \frac{1}{2} \rho S V_1^2 (1 - b^2) \quad (13)$$

The extractable power  $P$  in terms of the interference factor  $b$  can be expressed as:

$$\begin{aligned} P &= \frac{1}{4} \rho S (V_1^2 - V_2^2) (V_1 + V_2) \\ &= \frac{1}{4} \rho S V_1^3 (1 - b^2) (1 + b) \end{aligned} \quad (14)$$

The most important observation pertaining to wind power production is that the extractable power from the wind is proportional to the cube of the upstream wind speed  $V_1^3$  and is a function of the interference factor  $b$ .

The “power flux” or rate of energy flow per unit area, sometimes referred to as “power density” is defined using Eqn. 6 as:

$$\begin{aligned} P' &= \frac{P}{S} \\ &= \frac{\frac{1}{4} \rho S V_1^3 (1 - b^2) (1 + b)}{S} \\ &= \frac{1}{4} \rho V_1^3 \left[ \frac{\text{Joules}}{\text{m}^2 \cdot \text{s}} \right] \left[ \frac{\text{Watts}}{\text{m}^2} \right] \end{aligned} \quad (15)$$

The kinetic power content of the undisturbed upstream wind stream with  $V = V_1$  and over a cross sectional area  $S$  becomes:

$$W = \frac{1}{2} \rho S V_1^3 \left[ \frac{\text{Joules}}{\text{m}^2 \cdot \text{s}} \right] \left[ \text{Watts} \right] \quad (16)$$

The performance coefficient or efficiency is the dimensionless ratio of the extractable power  $P$  to the kinetic power  $W$  available in the undisturbed stream:

$$C_p = \frac{P}{W} \quad (17)$$

The performance coefficient is a dimensionless measure of the efficiency of a wind turbine in extracting the energy content of a wind stream. Substituting the expressions for  $P$  from Eqn. 14 and for  $W$  from Eqn. 16 we have:

$$\begin{aligned}
 C_p &= \frac{P}{W} \\
 &= \frac{\frac{1}{4}\rho S V_1^3 (1-b^2)(1+b)}{\frac{1}{2}\rho S V_1^3} \\
 &= \frac{1}{2}(1-b^2)(1+b)
 \end{aligned}
 \tag{18}$$

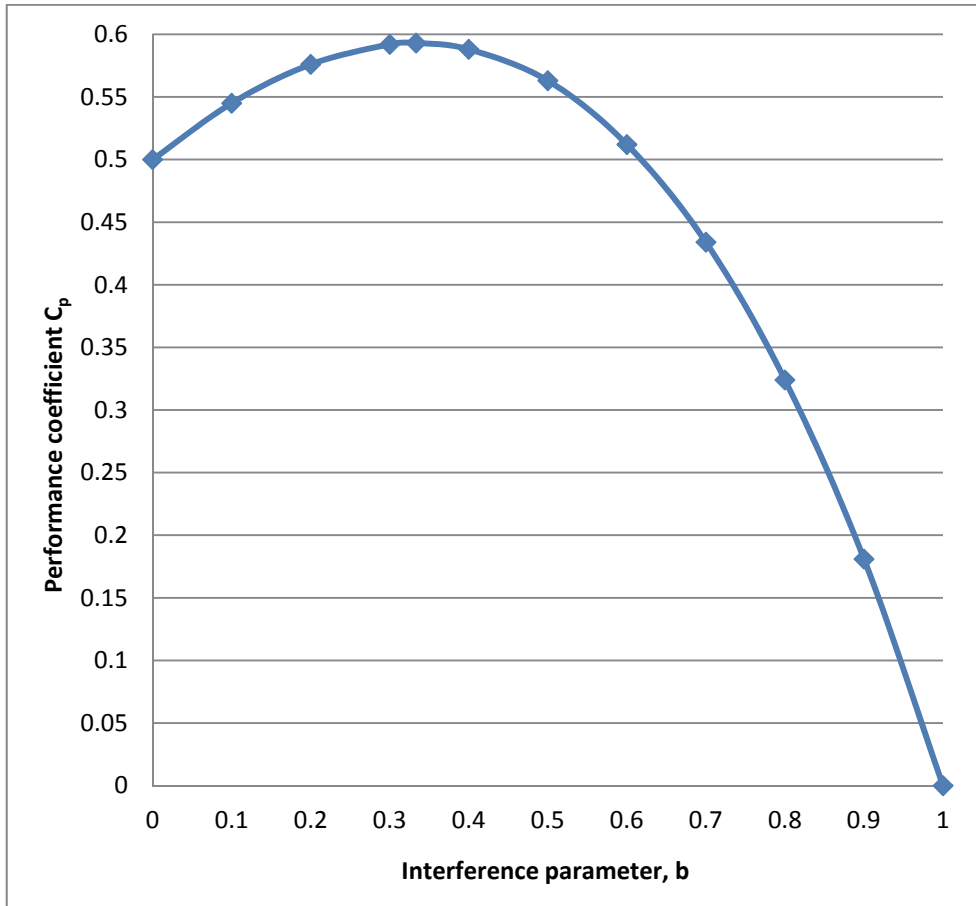


Fig. 2. The performance coefficient  $C_p$  as a function of the interference factor  $b$ .

When  $b = 1$ ,  $V_1 = V_2$  and the wind stream is undisturbed, leading to a performance coefficient of zero. When  $b = 0$ ,  $V_1 = 0$ , the turbine stops all the air flow and the performance coefficient is equal to 0.5. It can be noticed from the graph that the performance coefficient reaches a maximum around  $b = 1/3$ .

A condition for maximum performance can be obtained by differentiation of Eq. 18 with respect to the interference factor  $b$ . Applying the chain rule of differentiation (shown below) and setting the derivative equal to zero yields Eq. 19:

$$\begin{aligned} \frac{d}{dx}(uv) &= u \frac{dv}{dx} + v \frac{du}{dx} \\ \frac{dC_p}{db} &= \frac{1}{2} \frac{d}{db} [(1-b^2)(1+b)] \\ &= \frac{1}{2} [(1-b^2) - 2b(1+b)] \\ &= \frac{1}{2} (1-b^2 - 2b - 2b^2) \\ &= \frac{1}{2} (1 - 3b^2 - 2b) \\ &= \frac{1}{2} (1 - 3b)(1+b) \\ &= 0 \end{aligned} \tag{19}$$

Equation 19 has two solutions. The first is the trivial solution:

$$\begin{aligned} (1+b) &= 0 \\ b &= \frac{V_2}{V_1} = -1, \Rightarrow V_2 = -V_1 \end{aligned}$$

The second solution is the practical physical solution:

$$\begin{aligned} (1-3b) &= 0 \\ b &= \frac{V_2}{V_1} = \frac{1}{3}, \Rightarrow V_2 = \frac{1}{3}V_1 \end{aligned} \tag{20}$$

Equation 20 shows that for optimal operation, the downstream velocity  $V_2$  should be equal to one third of the upstream velocity  $V_1$ . Using Eqn. 18, the maximum or optimal value of the performance coefficient  $C_p$  becomes:

$$\begin{aligned} C_{p,opt} &= \frac{1}{2} (1-b^2)(1+b) \\ &= \frac{1}{2} \left(1 - \left(\frac{1}{3}\right)^2\right) \left(1 + \frac{1}{3}\right) \\ &= \frac{16}{27} \\ &= 0.59259 \\ &= 59.26 \text{ percent} \end{aligned} \tag{21}$$

This is referred to as the Betz Criterion or the Betz Limit. It was first formulated in 1919, and applies to all wind turbine designs. It is the theoretical power fraction that can be extracted from an ideal wind stream. Modern wind machines operate at a slightly lower practical non-ideal performance coefficient. It is generally reported to be in the range of:

$$C_{p,prac.} \approx \frac{2}{5} = 40 \text{ percent} \quad (22)$$

### Result I

From Eqns. 9 and 20, there results that:

$$\begin{aligned} V &= \frac{1}{2}(V_1 + V_2) \\ &= \frac{1}{2}\left(V_1 + \frac{V_1}{3}\right) \\ &= \frac{2}{3}V_1 \end{aligned} \quad (23)$$

### Result II

From the continuity Eqn. 2:

$$\begin{aligned} \dot{m} &= \rho S_1 V_1 = \rho S V = \rho S_2 V_2 = \text{constant} \\ S &= S_1 \frac{V_1}{V} = \frac{3}{2} S_1 \\ S_2 &= S_1 \frac{V_1}{V_2} = 3 S_1 \end{aligned} \quad (24)$$

This implies that the cross sectional area of the airstream downwind of the turbine expands to 3 times the area upwind of it.

Some pitfalls in the derivation of the previous equations could inadvertently occur and are worth pointing out. One can for instance try to define the power extraction from the wind in two different ways. In the first approach, one can define the power extraction by an ideal turbine from Eqns. 23, 24 as:

$$\begin{aligned} P_{ideal}^1 &= P_{upwind} - P_{downwind} \\ &= \frac{1}{2} \rho S_1 V_1^3 - \frac{1}{2} \rho S_2 V_2^3 \\ &= \frac{1}{2} \rho S_1 V_1^3 - \frac{1}{2} \rho 3 S_1 \left(\frac{1}{3} V_1\right)^3 \\ &= \frac{1}{2} \rho \left(\frac{8}{9} S_1 V_1^3\right) \\ &= \frac{8}{9} \frac{1}{2} \rho S_1 V_1^3 \end{aligned}$$

This suggests that fully 8/9 of the energy available in the upwind stream can be extracted by the turbine. That is a confusing result since the upwind wind stream has a cross sectional area that is smaller than the turbine intercepted area.



The second approach yields the correct result by redefining the power extraction at the wind turbine using the area of the turbine as  $S = 3/2 S_1$ :

$$\begin{aligned}
 P_{ideal} &= \frac{1}{2} \rho \left( \frac{8}{9} S_1 V_1^3 \right) \\
 &= \frac{1}{2} \rho \left( \frac{8}{9} \frac{2}{3} S V_1^3 \right) \\
 &= \frac{1}{2} \rho \left( \frac{16}{27} S V_1^3 \right) \\
 &= \frac{16}{27} \frac{1}{2} \rho S V_1^3
 \end{aligned} \tag{25}$$

The value of the Betz coefficient suggests that a wind turbine can extract **at most** 59.3 percent of the energy in an undisturbed wind stream.

$$\text{Betz coefficient} = \frac{16}{27} = 0.592593 = 59.26 \text{ percent} \tag{26}$$

Considering the frictional losses, blade surface roughness, and mechanical imperfections, between 35 to 40 percent of the power available in the wind is extractable under practical conditions.

Another important perspective can be obtained by estimating the maximum power content in a wind stream. For a constant upstream velocity, we can deduce an expression for the maximum power content for a constant upstream velocity  $V_1$  of the wind stream by differentiating the expression for the power  $P$  with respect to the downstream wind speed  $V_2$ , applying the chain rule of differentiation and equating the result to zero as:

$$\begin{aligned}
 \left. \frac{dP}{dV_2} \right|_{V_1} &= \frac{1}{4} \rho S \frac{d}{dV_2} [(V_1 + V_2)^2 (V_1 - V_2)] \\
 &= \frac{1}{4} \rho S \frac{d}{dV_2} [(V_1^2 - V_2^2)(V_1 + V_2)] \\
 &= \frac{1}{4} \rho S [(V_1^2 - V_2^2) - 2V_2(V_1 + V_2)] \\
 &= \frac{1}{4} \rho S (V_1^2 - V_2^2 - 2V_1V_2 - 2V_2^2) \\
 &= \frac{1}{4} \rho S (V_1^2 - 3V_2^2 - 2V_1V_2) \\
 &= 0
 \end{aligned} \tag{27}$$

Solving the resulting equation by factoring it yield Eqn. 28.

$$\begin{aligned}
 (V_1^2 - 3V_2^2 - 2V_1V_2) &= 0 \\
 (V_1 + V_2)(V_1 - 3V_2) &= 0
 \end{aligned} \tag{28}$$

Equation 28 once again has two solutions. The trivial solution is shown in Eqn. 29.

$$\begin{aligned}(V_1 + V_2) &= 0 \\ V_2 &= -V_1\end{aligned}\tag{29}$$

The second physically practical solution is shown in Eqn 30.

$$\begin{aligned}(V_1 - 3V_2) &= 0 \\ V_2 &= \frac{1}{3}V_1\end{aligned}\tag{30}$$

This implies the simple result that that the most efficient operation of a wind turbine occurs when the downstream speed  $V_2$  is one third of the upstream speed  $V_1$ . Adopting the second solution and substituting it in the expression for the power in Eqn. 16 we get the expression for the maximum power that could be extracted from a wind stream as:

$$\begin{aligned}P_{\max} &= \frac{1}{4}\rho S(V_1^2 - V_2^2)(V_1 + V_2) \\ &= \frac{1}{4}\rho S\left(V_1^2 - \frac{V_1^2}{9}\right)\left(V_1 + \frac{V_1}{3}\right) \\ &= \frac{1}{4}\rho S V_1^3\left(1 - \frac{1}{9}\right)\left(1 + \frac{1}{3}\right) \\ &= \frac{16}{27}\frac{\rho}{2}V_1^3 S \text{ [Watt]}\end{aligned}\tag{31}$$

This expression constitutes the formula originally derived by Betz where the swept rotor area  $S$  is:

$$S = \frac{\pi D^2}{4}\tag{32}$$

and the Betz Equation results as:

$$P_{\max} = \frac{16}{27}\frac{\rho}{2}V_1^3 \frac{\pi D^2}{4} \text{ [Watt]}\tag{33}$$

The most important implication from the Betz Equation is that there must be a wind speed change from the upstream to the downstream in order to extract energy from the wind; in fact by braking it using a wind turbine.

If no change in the wind speed occurs, energy cannot be efficiently extracted from the wind. Realistically, no wind machine can totally bring the air to a total rest, and for a rotating machine, there will always be some air flowing around it. Thus a wind machine can only extract a fraction of the kinetic energy of the wind. The wind speed on the rotors at which energy extraction is maximal has a magnitude lying between the upstream and downstream wind velocities.

The Betz Criterion reminds us of the Carnot cycle efficiency in Thermodynamics suggesting that a heat engine cannot extract all the energy from a given heat reservoir and must reject part of its heat input back to the environment.

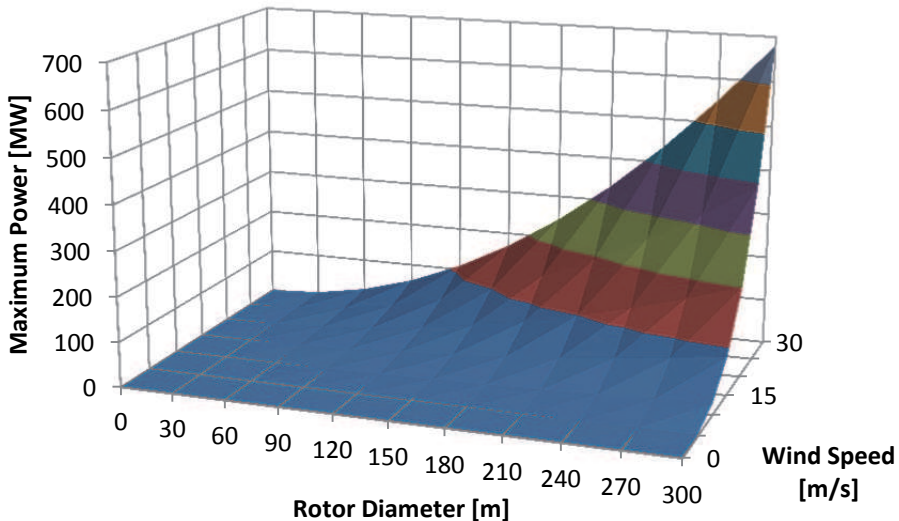


Fig. 3. Maximum power as a function of the rotor diameter and the wind speed. The power increases as the square of the rotor diameter and more significantly as the cube of the wind speed (Ragheb, M., 2011).

### 3. Rotor optimal Tip Speed Ratio, TSR

Another important concept relating to the power of wind turbines is the optimal tip speed ratio, which is defined as the ratio of the speed of the rotor tip to the free stream wind speed. If a rotor rotates too slowly, it allows too much wind to pass through undisturbed, and thus does not extract as much as energy as it could, within the limits of the Betz Criterion, of course.

On the other hand, if the rotor rotates too quickly, it appears to the wind as a large flat disc, which creates a large amount of drag. The rotor Tip Speed Ratio, TSR depends on the blade airfoil profile used, the number of blades, and the type of wind turbine. In general, three-bladed wind turbines operate at a TSR of between 6 and 8, with 7 being the most widely-reported value.

In addition to the factors mentioned above, other concerns dictate the TSR to which a wind turbine is designed. In general, a high TSR is desirable, since it results in a high shaft rotational speed that allows for efficient operation of an electrical generator. Disadvantages however of a high TSR include:

- Blade tips operating at 80 m/s or greater are subject to leading edge erosion from dust and sand particles, and would require special leading edge treatments like helicopter blades to mitigate such damage,
- Noise, both audible and inaudible, is generated,
- Vibration, especially in 2 or 1 blade rotors,

- d. Reduced rotor efficiency due to drag and tip losses,  
 e. Higher speed rotors require much larger braking systems to prevent the rotor from reaching a runaway condition that can cause disintegration of the turbine rotor blades.  
 The Tip Speed Ratio, TSR, is dimensionless factor defined in Eqn. 34

$$\text{TSR} = \lambda = \frac{\text{speed of rotor tip}}{\text{wind speed}} = \frac{v}{V} = \frac{\omega r}{V} \quad (34)$$

where:

$$\begin{aligned} V &= \text{wind speed [m/sec]} \\ v = \omega r &= \text{rotor tip speed [m/sec]} \\ r &= \text{rotor radius [m]} \\ \omega = 2\pi f &= \text{angular velocity [rad/sec]} \\ f &= \text{rotational frequency [Hz], [sec}^{-1}] \end{aligned}$$

### Example 1

At a wind speed of 15 m/sec, for a rotor blade radius of 10 m, rotating at 1 rotation per second:

$$\begin{aligned} f &= 1 \left[ \frac{\text{rotation}}{\text{sec}} \right], \\ \omega &= 2\pi f = 2\pi \left[ \frac{\text{radian}}{\text{sec}} \right] \\ v = \omega r &= 2\pi \cdot 10 = 20\pi \left[ \frac{\text{m}}{\text{sec}} \right] \\ \lambda = \frac{\omega r}{V} &= \frac{20\pi}{15} = \frac{62.83}{15} = 4 \end{aligned}$$

### Example 2

The Suzlon S.66/1250, 1.25 MW rated power at 12 m/s rated wind speed wind turbine design has a rotor diameter of 66 meters and a rotational speed of 13.9-20.8 rpm. Its angular speed range is:

$$\begin{aligned} \omega &= 2\pi f \\ &= 2\pi \frac{13.9 - 20.8}{60} \left[ \text{radian} \cdot \frac{\text{revolutions}}{\text{minute}} \cdot \frac{\text{minute}}{\text{second}} \right] \\ &= 1.46 - 2.18 \left[ \frac{\text{radian}}{\text{sec}} \right] \end{aligned}$$

The range of its rotor's tip speed can be estimated as:

$$\begin{aligned} v &= \omega r \\ &= (1.46 - 2.18) \frac{66}{2} \\ &= 48.18 - 71.94 \left[ \frac{\text{m}}{\text{sec}} \right] \end{aligned}$$

The range of its tip speed ratio is thus:

$$\begin{aligned}\lambda &= \frac{\omega r}{V} \\ &= \frac{48.18 - 71.94}{12} \\ &\approx 4 - 6\end{aligned}$$

The optimal TSR for maximum power extraction is inferred by relating the time taken for the disturbed wind to reestablish itself to the time required for the next blade to move into the location of the preceding blade. These times are  $t_w$  and  $t_b$ , respectively, and are shown below in Eqns. 35 and 36. In Eqns. 35 and 36,  $n$  is the number of blades,  $\omega$  is the rotational frequency of the rotor,  $s$  is the length of the disturbed wind stream, and  $V$  is the wind speed.

$$t_s = \frac{2\pi}{n\omega} [\text{sec}] \quad (35)$$

$$t_w = \frac{s}{V} [\text{sec}] \quad (36)$$

If  $t_s > t_w$ , some wind is unaffected. If  $t_w > t_s$ , some wind is not allowed to flow through the rotor. The maximum power extraction occurs when the two times are approximately equal. Setting  $t_w$  equal to  $t_s$  yields Eqn. 37 below, which is rearranged as:

$$\begin{aligned}t_s &\approx t_w \\ \frac{2\pi}{n\omega} &\approx \frac{s}{V} \Rightarrow \frac{n\omega}{V} \approx \frac{2\pi}{s}\end{aligned} \quad (37)$$

Equation 37 may then be used to define the optimal rotational frequency as shown in Eqn. 38:

$$\omega_{\text{optimal}} \approx \frac{2\pi V}{ns} \quad (38)$$

Consequently, for optimal power extraction, the rotor blade must rotate at a rotational frequency that is related to the speed of the oncoming wind. This rotor rotational frequency decreases as the radius of the rotor increases and can be characterized by calculating the optimal TSR,  $\lambda_{\text{optimal}}$  as shown in Eqn. 39.

$$\lambda_{\text{optimal}} \approx \frac{\omega_{\text{optimal}} r}{V} \approx \frac{2\pi}{n} \left( \frac{r}{s} \right) \quad (39)$$

#### 4. Effect of the number of rotor blades on the Tip Speed Ratio, TSR

The optimal TSR depends on the number of rotor blades,  $n$ , of the wind turbine. The smaller the number of rotor blades, the faster the wind turbine must rotate to extract the maximum power from the wind. For an  $n$ -bladed rotor, it has empirically been observed that  $s$  is approximately equal to 50 percent of the rotor radius. Thus by setting:

$$\frac{s}{r} \approx \frac{1}{2},$$

Eqn. 39 is modified into Eqn. 40:

$$\lambda_{optimal} \approx \frac{2\pi}{n} \left( \frac{r}{s} \right) \approx \frac{4\pi}{n} \quad (40)$$

For  $n = 2$ , the optimal TSR is calculated to be 6.28, while it is 4.19 for three-bladed rotor, and it reduces to 3.14 for a four-bladed rotor. With proper airfoil design, the optimal TSR values may be approximately 25 – 30 percent above these values. These highly-efficient rotor blade airfoils increase the rotational speed of the blade, and thus generate more power. Using this assumption, the optimal TSR for a three-bladed rotor would be in the range of 5.24 – 5.45. Poorly designed rotor blades that yield too low of a TSR would cause the wind turbine to exhibit a tendency to slow and stall. On the other hand, if the TSR is too high, the turbine will rotate very rapidly, and will experience larger stresses, which may lead to catastrophic failure in highly-turbulent wind conditions.

## 5. Power coefficient, $C_p$

The power generated by the kinetic energy of a free flowing wind stream is shown in Eqn. 41.

$$P = \frac{1}{2} \rho S V^3 \text{ [Watt]} \quad (41)$$

Defining the cross sectional area,  $S$ , of the wind turbine, in terms of the blade radius,  $r$ , Eqn. 41 becomes Eqn. 42.

$$P = \frac{1}{2} \rho \pi R^2 V^3 \quad (42)$$

The power coefficient (Jones, B., 1950), Eqn. 43, is defined as the ratio of the power extracted by the wind turbine relative to the energy available in the wind stream.

$$C_p = \frac{P_t}{P} = \frac{P_t}{\frac{1}{2} \rho \pi R^2 V^3} \quad (43)$$

As derived earlier in this chapter, the maximum achievable power coefficient is 59.26 percent, the Betz Limit. In practice however, obtainable values of the power coefficient center around 45 percent. This value below the theoretical limit is caused by the inefficiencies and losses attributed to different configurations, rotor blades profiles, finite wings, friction, and turbine designs. Figure 4 depicts the Betz, ideal constant, and actual wind turbine power coefficient as a function of the TSR.

As shown in Fig. 4, maximum power extraction occurs at the optimal TSR, where the difference between the actual TSR (blue curve) and the line defined by a constant TSR is the lowest. This difference represents the power in the wind that is not captured by the wind turbine. Frictional losses, finite wing size, and turbine design losses account for part of the

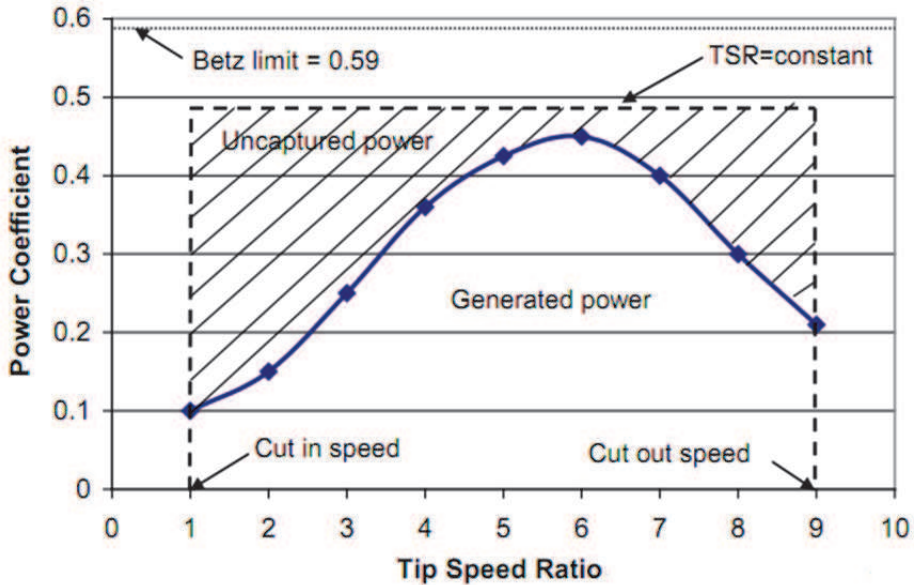


Fig. 4. Power coefficient as a function of TSR for a two-bladed rotor.

uncaptured wind power, and are supplemented by the fact that a wind turbine does not operate at the optimal TSR across its operating range of wind speeds.

**6. Inefficiencies and losses, Schmitz power coefficient**

The inefficiencies and losses encountered in the operation of wind turbines include the blade number losses, whirlpool losses, end losses and the airfoil profile losses (Çetin, N. S. et. al. 2005).

**Airfoil profile losses**

The slip or slide number *s* is the ratio of the uplift force coefficient of the airfoil profile used *C<sub>L</sub>* to the drag force coefficient *C<sub>D</sub>* is:

$$s = \frac{C_L}{C_D} \quad s = \frac{C_L}{C_D} \tag{44}$$

Accounting for the drag force can be achieved by using the profile efficiency that is a function of the slip number *s* and the tip speed ratio  $\lambda$  as:

$$\eta_{profile} = \frac{s - \lambda}{s} = 1 - \frac{\lambda}{s} \quad \eta_{profile} = \frac{s - \lambda}{s} = 1 - \frac{\lambda}{s} \tag{45}$$

**Rotor tip end losses**

At the tip of the rotor blade an air flow occurs from the lower side of the airfoil profile to the upper side. This air flow couples with the incoming air flow to the blade. The combined air flow results in a rotor tip end efficiency,  $\eta_{tip\ end}$ .

### Whirlpool Losses

In the idealized derivation of the Betz Equation, the wind does not change its direction after it encounters the turbine rotor blades. In fact, it does change its direction after the encounter.

This is accounted-for by a modified form of the power coefficient known as the Schmitz power coefficient  $C_{pSchmitz}$  if the same airfoil design is used throughout the rotor blade.

Tip speed ratio TSR $\lambda$	Whirlpool Schmitz Power Coefficient $C_{pSchmitz}$
0.0	0.000
0.5	0.238
1.0	0.400
1.5	0.475
2.0	0.515
2.5	0.531
3.0	0.537
3.5	0.538
4.0	0.541
4.5	0.544
5.0	0.547
5.5	0.550
6.0	0.553
6.5	0.556
7.0	0.559
7.5	0.562
8.0	0.565
8.5	0.568
9.0	0.570
9.5	0.572
10.0	0.574

Table 1. Whirlpool losses Schmitz power coefficient as a function of the tip speed ratio.

### Rotor blade number losses

A theory developed by Schmitz and Glauert applies to wind turbines with four or less rotor blades. In a turbine with more than four blades, the air movement becomes too complex for a strict theoretical treatment and an empirical approach is adopted. This can be accounted for by a rotor blades number efficiency  $\eta_{blades}$ .

In view of the associated losses and inefficiencies, the power coefficient can be expressed as:

$$C_p \approx C_{pSchmitz} \eta_{profile} \eta_{tip\ end} \eta_{blades} \quad (46)$$

There are still even more efficiencies involved:

1. Frictional losses in the bearings and gears:  $\eta_{friction}$



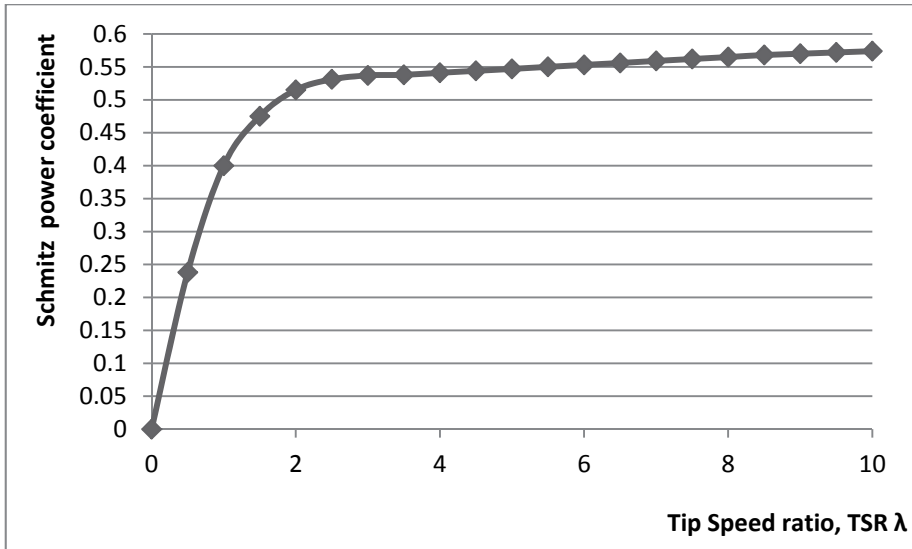


Fig. 5. Schmitz power coefficient as a function of the tip speed ratio, TSR.

2. Magnetic drag and electrical resistance losses in the generator or alternator:  $\eta_{\text{electrical}}$ .

$$C'_p \approx C_p \text{Schmitz} \eta_{\text{profile}} \eta_{\text{tip end}} \eta_{\text{blades}} \eta_{\text{friction}} \eta_{\text{electrical}} \quad (47)$$

In the end, the Betz Limit is an idealization and a design goal that designers try to reach in a real world turbine. A  $C_p$  value of between 0.35 – 0.40 is a realistic design goal for a workable wind turbine. This is still reduced by a capacity factor accounting for the periods of wind flow as the intermittency factor.

## 7. Power coefficient and tip speed ratio of different wind converters designs

The theoretical maximum efficiency of a wind turbine is given by the Betz Limit, and is around 59 percent. Practically, wind turbines operate below the Betz Limit. In Fig. 4 for a two-bladed turbine, if it is operated at the optimal tip speed ratio of 6, its power coefficient would be around 0.45. At the cut-in wind speed, the power coefficient is just 0.10, and at the cut-out wind speed it is 0.22. This suggests that for maximum power extraction a wind turbine should be operated around its optimal wind tip ratio.

Modern horizontal axis wind turbine rotors consist of two or three thin blades and are designated as low solidity rotors. This implies a low fraction of the area swept by the rotors being solid. Its configuration results in an optimum match to the frequency requirements of modern electricity generators and also minimizes the size and weight of the gearbox or transmission required, as well as increases efficiency.

Such an arrangement results in a relatively high tip speed ratio in comparison with rotors with a high number of blades such as the highly successful American wind mill used for water pumping in the American West and all over the world. The latter required a high starting torque.

The relationship between the rotor power coefficient  $C_p$  and the tip speed ratio is shown for different types of wind machines. It can be noticed that it reaches a maximum at different positions for different machine designs.

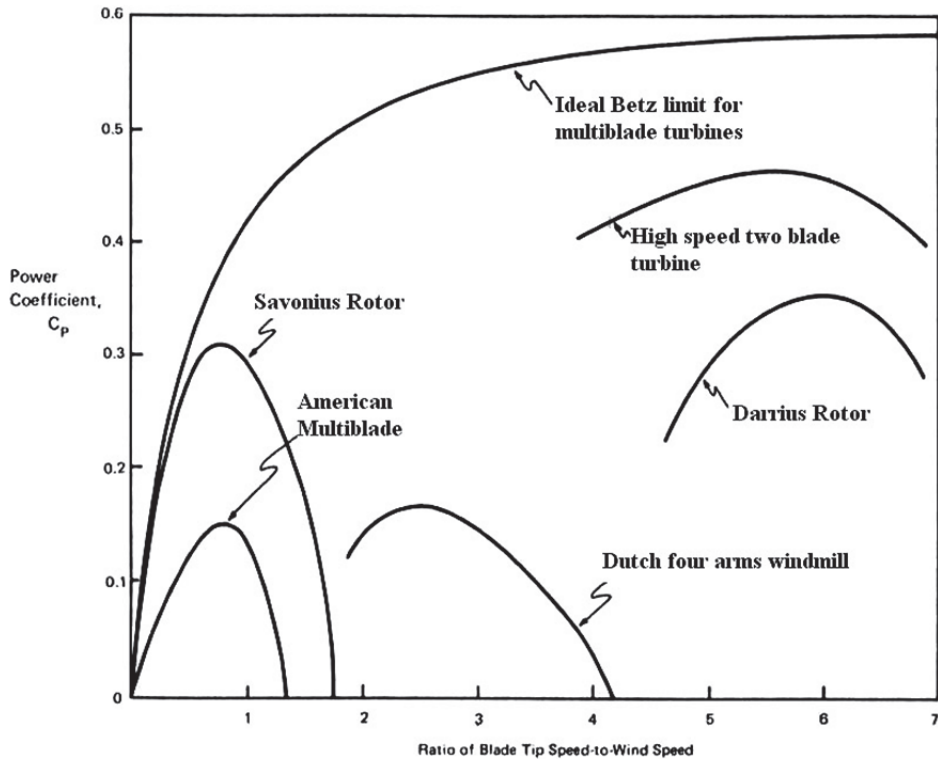


Fig. 6. The power coefficient  $C_p$  as a function of the tip speed ratio for different wind machines designs. Note that the efficiency curves of the Savonius and the American multiblade designs were inadvertently switched (Eldridge, F. R., 1980) in some previous publications, discouraging the study of the Savonius design.

The maximum efficiencies of the two bladed design, the Darrius concept and the Savonius reach levels above 30 percent but below the Betz Limit of 59 percent. The American multiblade design and the historical Dutch four bladed designs peak at 15 percent. These are not suited for electrical generation but are ideal for water pumping.

## 8. Discussion and conclusions

Wind turbines must be designed to operate at their optimal wind tip speed ratio in order to extract as much power as possible from the wind stream. When a rotor blade passes through the air stream it leaves a turbulent wake in its path. If the next blade in the rotating rotor arrives at the wake when the air is still turbulent, it will not be able to extract power from the wind efficiently, and will be subjected to high vibration stresses. If the rotor rotated slower, the air hitting each rotor blade would no longer be turbulent. This is another reason

for the tip speed ratio to be selected so that the rotor blades do not pass through turbulent air.

Wind power conversion is analogous to other methods of energy conversion such as hydroelectric generators and heat engines. Some common underlying basic principles can guide the design and operation of wind energy conversion systems in particular, and of other forms of energy conversion in general. A basic principle can be enunciated as:

*"Energy can be extracted or converted only from a flow system."*

In hydraulics, the potential energy of water blocked behind a dam cannot be extracted unless it is allowed to flow. In this case only a part of it can be extracted by a water turbine.

In a heat engine, the heat energy cannot be extracted from a totally insulated reservoir. Only when it is allowed to flow from the high temperature reservoir, to a low temperature one where it is rejected to the environment; can a fraction of this energy be extracted by a heat engine.

Totally blocking a wind stream does not allow any energy extraction. Only by allowing the wind stream to flow from a high speed region to a low speed region can energy be extracted by a wind turbine.

A second principle of energy conversion can be elucidated as:

*"Natural or artificial asymmetries in an aerodynamic, hydraulic, or thermodynamic system allow the extraction of only a fraction of the available energy at a specified efficiency."*

Ingenious minds conceptualized devices that take advantage of existing natural asymmetries, or created configurations or situations favoring the creation of these asymmetries, to extract energy from the environment.

A corollary ensues that the existence of a flow system necessitates that only a fraction of the available energy can be extracted at an efficiency characteristic of the energy extraction process with the rest returned back to the environment to maintain the flow process.

In thermodynamics, the ideal heat cycle efficiency is expressed by the Carnot cycle efficiency. In a wind stream, the ideal aerodynamic cycle efficiency is expressed by the Betz Equation.

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## **Fundamental and Advanced Topics in Wind Power**

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As the fastest growing source of energy in the world, wind has a very important role to play in the global energy mix. This text covers a spectrum of leading edge topics critical to the rapidly evolving wind power industry. The reader is introduced to the fundamentals of wind energy aerodynamics; then essential structural, mechanical, and electrical subjects are discussed. The book is composed of three sections that include the Aerodynamics and Environmental Loading of Wind Turbines, Structural and Electromechanical Elements of Wind Power Conversion, and Wind Turbine Control and System Integration. In addition to the fundamental rudiments illustrated, the reader will be exposed to specialized applied and advanced topics including magnetic suspension bearing systems, structural health monitoring, and the optimized integration of wind power into micro and smart grids.

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University Campus STeP Ri  
Slavka Krautzeka 83/A  
51000 Rijeka, Croatia  
Phone: +385 (51) 770 447  
Fax: +385 (51) 686 166  
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Unit 405, Office Block, Hotel Equatorial Shanghai  
No.65, Yan An Road (West), Shanghai, 200040, China  
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元  
Phone: +86-21-62489820  
Fax: +86-21-62489821

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