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Windowed linear canonical transform: its relation to windowed Fourier transform and uncertainty principles

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Abstract

The windowed linear canonical transform is a natural extension of the classical windowed Fourier transform using the linear canonical transform. In the current work, we first remind the reader about the relation between the windowed linear canonical transform and windowed Fourier transform. It is shown that useful relation enables us to provide different proofs of some properties of the windowed linear canonical transform, such as the orthogonality relation, inversion theorem, and complex conjugation. Lastly, we demonstrate some new results concerning several generalizations of the uncertainty principles associated with this transformation.

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1 Introduction

As it is well known, the classical windowed Fourier transform (WFT) is a useful mathematical tool, which has been broadly studied in quantum physics, signal processing and many other fields of science and engineering. In recent years, a number of efforts have been made with an increasing interest in expanding various types of transformations in the context of the linear canonical transform (LCT), we refer the reader to the papers [1–4]. Some authors [5–7] have introduced an extension of the WFT in the LCT domain, the so-called windowed linear canonical transform (WLCT). The generalized transform is built by including the Fourier kernel with the LCT kernel in the definition of the windowed Fourier transform. They also have investigated its essential properties like linearity, orthogonality relation, inversion theorem, and the inequalities.

In [8], the author has discussed that the fractional Fourier transform is intimately related to the Fourier transformation. According to this idea, some properties of the fractional Fourier transform can be easily obtained using the basic connection between the fractional Fourier transform and Fourier transform. In [9], the authors have investigated the fundamental properties of the continuous shearlet transforms using the direct interaction between the Fourier transform and shearlet transform. In this work, we developed this

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approach within the framework of the linear canonical transform. We have provided different proofs of the WLCT properties like the orthogonality relation, inversion theorem, and complex conjugation using the direct interaction among the windowed linear canonical transform, the windowed Fourier transform and the Fourier transform, the proofs of which are simpler than those the authors proposed in [7]. As we know, the uncertainty principle is one of the fundamental results of the WLCT, which explains how an original function interacts with its WLCT. Therefore, we have proposed several versions of the uncertainty principles associated with this transformation, which are quite different from those investigated in [5, 7] as well as in [10].

The present work is structured in the following fashion. In Sect. 3, we provided a brief review of the linear canonical transform and basic notations that will be useful later. Section 4 is a part of the core of the article. This section presents the basic relation between the windowed linear canonical transform and windowed Fourier transform. In it, some famous properties for the windowed linear canonical transform are proved using this relation. Section 5 is also a part of the core of the article. This section is devoted to some generalizations of the uncertainty principles related to the windowed linear canonical transform. Lastly, the summary of this work is included in Sect. 6.

2 Generalities

In this segment, we state the definition of the linear canonical transform (LCT) and its useful properties, as well as the basic notations, which will be used in the derivation of the results of this work. For a detailed information on this transform, we refer to [11–15].

Definition 2.1 Let $B = (a, b, c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ be a matrix parameter such that $|B| = 1$. The LCT of a function $f \in L^1(\mathbb{R})$ is expressed as

$$
\mathcal{C}_B\{f\}(w) = \begin{cases} \int_{\mathbb{R}} f(x) \mathcal{K}_B(w, x) \, dx, & b \neq 0, \\ \sqrt{d} e^{i \frac{cd}{2} w^2} f(dw), & b = 0, \end{cases}
\tag{1}
$$

where $\mathcal{K}_B(w, x)$ is given by

$$
\mathcal{K}_B(w, x) = \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}(\frac{a}{b} x^2 - \frac{2}{b} xw + \frac{d}{b} w^2 - \frac{\pi}{2})}.
\tag{2}
$$

It is evident that relation (2) above fulfills

$$
\mathcal{K}_{B^{-1}}(x, w) = \overline{\mathcal{K}_B(w, x)} = \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{2}(\frac{a}{b} x^2 - \frac{2}{b} xw + \frac{d}{b} w^2 - \frac{\pi}{2})}.
$$

From equation (1), it can be observed that for $b = 0$ the LCT of a signal is a chirp product. Therefore, in the current work, we always consider the case $b > 0$.

It is worth noting that for $B = (a, b, c, d) = (0, 1, -1, 0)$, equation (1) can be expressed as

$$
F\{f\}(w) = \int_{\mathbb{R}} f(x) e^{-ixw} \, dx,
\tag{3}
$$

leading to the definition of the Fourier transform times $\frac{1}{\sqrt{2\pi i}}$ and providing that the infinite integral exists. The inverse LCT is given by

$$
\begin{aligned}
f(x) &= \int_{\mathbb{R}} \mathcal{C}_B\{f\}(w) \mathcal{K}_{B^{-1}}(x, w)\, dw \\
&= \int_{\mathbb{R}} \mathcal{C}_B\{f\}(w) \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{2}(\frac{a}{b}x^2 - \frac{2}{b}xw + \frac{d}{b}w^2 - \frac{\pi}{2})}\, dw.
\end{aligned}
\tag{4}
$$

The direct interaction between the LCT and the Fourier transform (FT) is described by

$$
\sqrt{2\pi b}\, e^{i\frac{\pi}{4}} e^{-\frac{id}{2b}w^2} \mathcal{C}_B\{f\}(w) = F\left\{ e^{\frac{ia}{2b}x^2} f \right\}\left(\frac{w}{b} \right).
\tag{5}
$$

Definition 2.2 Given f a measurable function on $\mathbb{R}$ and $1 \le r < \infty$, define

$$
\|f\|_{L^r(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^r\, dx \right)^{1/r} < \infty,
$$

$$
\|f\|_{L^\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty.
\tag{6}
$$

For $r = 2$, we get

$$
\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) \overline{g(x)}\, dx \quad \text{and} \quad \|f\|_{L^2(\mathbb{R})}^2 = \langle f, f \rangle_{L^2(\mathbb{R})}.
$$

Based on the above definition we state the following fact, which is known as Parseval's formula and Planchel's formula, respectively.

Lemma 2.1 *For every $f, g \in L^2(\mathbb{R})$, the following relation holds*:

$$
\langle f, g \rangle_{L^2(\mathbb{R})} = \left\langle \mathcal{C}_B\{f\}, \mathcal{C}_B\{g\} \right\rangle_{L^2(\mathbb{R})},
\tag{7}
$$

and

$$
\|f\|_{L^2(\mathbb{R})}^2 = \left\| \mathcal{C}_B\{f\} \right\|_{L^2(\mathbb{R})}^2.
\tag{8}
$$

The next result will be useful in this paper.

Theorem 2.2 ([7]) *Let $1 \le r \le 2$ and s be such that $\frac{1}{r} + \frac{1}{s} = 1$. Then for all $g \in L^r(\mathbb{R})$, it holds*

$$
\left\| \mathcal{C}_B\{g\} \right\|_{L^s(\mathbb{R})} \le \|g\|_{L^r(\mathbb{R})}.
\tag{9}
$$

It is straightforward to see that for $r = 1$, we get

$$
\left\| \mathcal{C}_B\{g\} \right\|_{L^\infty(\mathbb{R})} \le \|g\|_{L^1(\mathbb{R})}.
\tag{10}
$$

3 Windowed linear canonical transform (WLCT)

Below, we shortly introduce the windowed linear canonical transform (WLCT), which was studied in [5–7, 16].

Definition 3.1 Let $\phi \in L^2(\mathbb{R})$ be a non-zero window function. The WLCT of $f \in L^2(\mathbb{R})$ with respect to ϕ denoted by G_ϕ^B is given by

$$G_\phi^B f(w, v) = \frac{1}{\sqrt{2\pi b}} \int_{\mathbb{R}} f(x)\overline{\phi(x - v)} e^{\frac{i}{2}(\frac{a}{b}x^2 - \frac{2}{b}xw + \frac{d}{b}w^2 - \frac{\pi}{2})} \, dx \tag{11}$$

for $(x, v) \in \mathbb{R} \times \mathbb{R}$.

The relation of the WLCT to the Fourier transform takes the form

$$\begin{aligned}
G_\phi^B f(w, v) &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{i\frac{d}{2b}w^2} \int_{\mathbb{R}} f(x)\overline{\phi(x - v)} e^{i\frac{a}{2b}x^2} e^{-i\frac{xw}{b}} \, dx \\
&= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{i\frac{d}{2b}w^2} F\{e^{\frac{a}{b}x^2} f T_v \bar{\phi}\}\left(\frac{w}{b}\right),
\end{aligned} \tag{12}$$

where the shifting operator $T_v\bar{\phi}$ is expressed as

$$T_v\bar{\phi}(x) = \bar{\phi}(x - v).$$

The relation (12) above is equivalent to

$$\sqrt{2\pi b} e^{i\frac{\pi}{4}} e^{-i\frac{d}{2b}w^2} G_\phi^B f(w, v) = F\{e^{\frac{a}{b}x^2} f T_v \bar{\phi}\}\left(\frac{w}{b}\right). \tag{13}$$

Some useful consequences of the above definition are collected as the following:

- Especially, for $B = (a, b, c, d) = (0, 1, -1, 0)$, Definition 3.1 changes to the classical WFT definition, namely,

$$G_\phi^B f(w, v) = \frac{1}{\sqrt{2\pi i}} G_\phi f(w, v),$$

where

$$G_\phi f(w, v) = \int_{\mathbb{R}} f(x)\overline{\phi(x - v)} e^{-ixw} \, dx, \tag{14}$$

which means that

$$G_\phi f(w, v) = F\{f T_v \bar{\phi}\}(w). \tag{15}$$

- If we take the Gaussian signal as ϕ in (11), it is often called the Gabor linear canonical transformation.
- It is straightforward to verify that

$$G_\phi^B f(w, v) = \mathcal{C}_B\{f T_v \bar{\phi}\}(w), \tag{16}$$

which describes the connection between the windowed linear canonical transform and linear canonical transform.

4 Essential properties of windowed linear canonical transform

We need the following simple lemma, which will be useful in deriving results in this work. It demonstrates the interaction between the windowed linear canonical transform (WLCT) and windowed Fourier transform.

Lemma 4.1 ([7]) *The WLCT of a signal $f \in L^2(\mathbb{R})$ with $B = (a, b, c, d)$ can be changed to the WFT via*

$$e^{-\frac{id}{2b}w^2} G_\phi^B f(w, v) = G_\phi \check{f}\left(\frac{w}{b}, v\right), \tag{17}$$

where

$$\check{f}(x) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{\frac{ia}{2b}x^2} f(x). \tag{18}$$

Let us now build the orthogonality property and inversion theorem associated with the WLCT by applying the direct connection among the WLCT, WFT, and FT (in comparison with [7]).

Theorem 4.2 *Let ϕ, ψ be two window functions related to the LCT. For each $f, g \in L^2(\mathbb{R})$, one has*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^B f(w, v) \overline{G_\psi^B g(w, v)} \, dw \, dv = \langle \bar{\phi}, \bar{\psi} \rangle_{L^2(\mathbb{R})} \langle f, g \rangle_{L^2(\mathbb{R})}. \tag{19}$$

In particular,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) \right|^2 dw \, dv = \|\phi\|_{L^2(\mathbb{R})}^2 \|f\|_{L^2(\mathbb{R})}^2. \tag{20}$$

Proof With the help of the orthogonality relation for the WFT (see [17]), we obtain

$$\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi f(w, v) \overline{G_\psi g(w, v)} \, dw \, dv = \langle \bar{\phi}, \bar{\psi} \rangle_{L^2(\mathbb{R})} \langle f, g \rangle_{L^2(\mathbb{R})}. \tag{21}$$

With (15), the relation (21) may be expressed as

$$\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} F\{f T_v \bar{\phi}\}(w) \overline{F\{g T_v \bar{\psi}\}(w)} \, dw \, dv = \langle \bar{\phi}, \bar{\psi} \rangle_{L^2(\mathbb{R})} \langle f, g \rangle_{L^2(\mathbb{R})}. \tag{22}$$

Equation (22) may be rewritten as

$$\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}(w) \overline{F\{e^{\frac{ia}{2b}x^2} g T_v \bar{\psi}\}(w)} \, dw \, dv = \langle \bar{\phi}, \bar{\psi} \rangle_{L^2(\mathbb{R})} \langle e^{\frac{ia}{2b}x^2} f, e^{\frac{ia}{2b}x^2} g \rangle_{L^2(\mathbb{R})}.$$

Putting $w = \frac{w}{b}$, we can write the above identity in the form

$$\frac{1}{2\pi b} \int_{\mathbb{R}} \int_{\mathbb{R}} F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}\left(\frac{w}{b}\right) \overline{F\{e^{\frac{ia}{2b}x^2} g T_v \bar{\psi}\}\left(\frac{w}{b}\right)} \, dw \, dv = \langle \bar{\phi}, \bar{\psi} \rangle_{L^2(\mathbb{R})} \langle f, g \rangle_{L^2(\mathbb{R})}. \tag{23}$$

By virtue of (13), the left-hand side of (23) takes the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\frac{\pi}{4}} e^{-i\frac{d}{2b}w^2} G_\phi^B f(w,v) \overline{e^{i\frac{\pi}{4}} e^{-i\frac{d}{2b}w^2} G_\psi^B g(w,v)}\, dw\, dv = \langle \bar{\phi}, \bar{\psi} \rangle_{L^2(\mathbb{R})} \langle f, g \rangle_{L^2(\mathbb{R})}. \tag{24}$$

Hence,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^B f(w,v) \overline{G_\phi^B f(w,v)}\, dw\, dv = \langle \bar{\phi}, \bar{\psi} \rangle_{L^2(\mathbb{R})} \langle f, g \rangle_{L^2(\mathbb{R})},$$

and the proof is complete. □

Let us implement Lemma 4.1 to derive a basic property of the WLCT.

Theorem 4.3 *If a function $f \in L^2(\mathbb{R})$ and ϕ is real-valued, then we have*

$$G_\phi^B \bar{f}(w,v) = \overline{G_\phi^{B^{-1}} f(w,v)}. \tag{25}$$

Proof By including $\check{f}$ into the complex conjugate theorem for the windowed Fourier transform (see [17]) defined by (18), we see that

$$G_\phi \check{\bar{f}} \left(\frac{w}{b}, v \right) = \overline{G_\phi \check{f} \left(\frac{w}{-b}, v \right)}. \tag{26}$$

In view of (17), the expression on the left of (26) above can be expressed as

$$e^{-\frac{id}{2b}w^2} G_\phi^B \bar{f}(w,v) = G_\phi \check{\bar{f}} \left(\frac{w}{b}, v \right), \tag{27}$$

and thus, we obtain

$$e^{-\frac{id}{2b}w^2} G_\phi^B \bar{f}(w,v) = \overline{G_\phi \check{f} \left(\frac{w}{-b}, v \right)}. \tag{28}$$

From equations (14) and (28), we observe that

$$\begin{aligned}
G_\phi^B \bar{f}(w,v) &= e^{\frac{id}{2b}w^2} \overline{G_\phi \check{f} \left(\frac{w}{-b}, v \right)} \\
&= e^{\frac{id}{2b}w^2} \overline{\int_{\mathbb{R}} \frac{e^{-i\frac{\pi}{4}}}{\sqrt{-2\pi b}} e^{-\frac{ia}{2b}x^2} f(x) \overline{\phi(x-v)} e^{i\frac{xw}{b}}\, dx} \\
&= e^{\frac{id}{2b}w^2} \overline{\int_{\mathbb{R}} \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi b}} e^{-\frac{ia}{2b}x^2} f(x) \overline{\phi(x-v)} e^{i\frac{xw}{b}}\, dx} \\
&= \overline{\int_{\mathbb{R}} f(x) \overline{\phi(x-v)} \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{2}(\frac{a}{b}x^2 - \frac{2}{b}xw + \frac{d}{b}w^2 - \frac{\pi}{2})}\, dx} \\
&= \overline{G_\phi^{B^{-1}} f(w,v)},
\end{aligned}$$

which was to be proved. □

Theorem 4.4 *Let ϕ, ψ be two window functions related to the LCT. For any $f \in L^2(\mathbb{R})$, one has*

$$f(x) = \frac{1}{\langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^A f(w, v) K_{B^{-1}}(w, x) \psi(x - v) \, dw \, dv. \tag{29}$$

Proof Due to the inversion theorem for the WFT, we obtain

$$f(x) = \frac{1}{2\pi \langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi f(w, v) e^{iwx} \psi(x - v) \, dw \, dv$$

$$= \frac{1}{2\pi \langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} F\{f T_v \bar{\phi}\}(w) e^{iwx} \psi(x - v) \, dw \, dv. \tag{30}$$

According to (13), we see that

$$e^{\frac{ia}{2b}x^2} f(x) = \frac{1}{2\pi \langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}(w) e^{iwx} \psi(x - v) \, dw \, dv$$

$$= \frac{1}{2\pi b \langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}\left(\frac{w}{b}\right) e^{i\frac{wx}{b}} \psi(x - v) \, dw \, dv$$

$$= \frac{1}{2\pi b \langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \sqrt{2\pi b} e^{i\frac{\pi}{4}} e^{-i\frac{d}{2b}w^2} G_\phi^B f(w, v) e^{i\frac{wx}{b}} \psi(x - v) \, dw \, dv. \tag{31}$$

The above relation can be expressed in the form

$$f(x) = \frac{1}{\langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^A f(w, v) \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{2}(\frac{a}{b}x^2 - \frac{2}{b}xw + \frac{d}{b}w^2 - \frac{\pi}{2})} \psi(x - v) \, dw \, dv$$

$$= \frac{1}{\langle \phi, \psi \rangle_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^A f(w, v) K_{B^{-1}}(w, x) \psi(x - v) \, dw \, dv. \tag{32}$$

The proof is complete. □

5 Inequalities for windowed linear canonical transform
We first state the following result, which describes the Hausdorff–Young inequality related to the WLCT.

Theorem 5.1 (WLCT Hausdorff–Young) *For any $1 \le r \le 2$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Suppose that ϕ in $L^s(\mathbb{R})$ and f in $L^r(\mathbb{R})$, then we have*

$$\left\| G_\phi^B f(w, v) \right\|_{L^s(\mathbb{R})} \le \|\phi\|_{L^s(\mathbb{R})} \|f\|_{L^r(\mathbb{R})}. \tag{33}$$

Proof We assume that $\|\phi\|_{L^s(\mathbb{R})} = 1$. An application of the identity (10) together with the identity (16) will lead to

$$\left\| G_\phi^B f(w, v) \right\|_{L^\infty(\mathbb{R})} = \left\| \mathcal{C}_B\{f T_v \bar{\phi}\} \right\|_{L^\infty(\mathbb{R})}$$

$$\le \|f T_v \bar{\phi}\|_{L^1(\mathbb{R})}$$

$$\le \|f\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})}$$

$$= \|f\|_{L^1(\mathbb{R})}. \tag{34}$$

For $r = 2$, we obtain

$$\left\| G_\phi^B f(w, v) \right\|_{L^2(\mathbb{R})} = \|\phi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}, \tag{35}$$

and applying Riesz–Thorin interpolation theorem yields

$$\left\| G_\phi^B f(w, v) \right\|_{L^s(\mathbb{R})} \leq \|f\|_{L^r(\mathbb{R})}. \tag{36}$$

Let $\phi \in L^r(\mathbb{R})$ be a window function and putting $\psi = \frac{\phi}{\|\phi\|_{L^s(\mathbb{R})}}$, then we immediately get

$$G_\psi^B f = \frac{1}{\|\phi\|_{L^s(\mathbb{R})}} G_\phi^B f. \tag{37}$$

From (36), we conclude that

$$\left\| G_\psi^B f(w, v) \right\|_{L^s(\mathbb{R})} \leq \|f\|_{L^r(\mathbb{R})}. \tag{38}$$

This implies that

$$\left\| G_\phi^B f(w, v) \right\|_{L^s(\mathbb{R})} \leq \|\phi\|_{L^s(\mathbb{R})} \|f\|_{L^r(\mathbb{R})}, \tag{39}$$

and the proof is complete. □

We derive the following theorem, which is little different from those proposed in the paper [5].

Theorem 5.2 *Let ϕ be a window function belonging to $L^2(\mathbb{R})$ such that $\|\phi\|_{L^2(\mathbb{R})} = 1$. Let $f \in L^2(\mathbb{R})$ be a function with $\|f\|_{L^2(\mathbb{R})} = 1$. If*

$$\int\int_V \left| G_\phi^B f(w, v) \right|^2 dw\, dv \geq 1 - \epsilon, \tag{40}$$

then for every $\epsilon \geq 0$ one has

$$\mu(V) \geq \sqrt{2\pi b}(1 - \epsilon). \tag{41}$$

Here $V \subseteq \mathbb{R} \times \mathbb{R}$ is a measurable subset, and $\mu(V)$ is the Lebesgue measure of V.

Proof Applying the Cauchy–Schwarz inequality results in

$$\begin{aligned}
\left| G_\phi^B f(w, v) \right| &= \left| \frac{1}{\sqrt{2\pi b}} \int_{\mathbb{R}} f(x) \overline{\phi(x-v)} e^{\frac{i}{2}(\frac{a}{b}x^2 - \frac{2}{b}xw + \frac{d}{b}w^2 - \frac{\pi}{2})} dx \right| \\
&\leq \frac{1}{\sqrt{2\pi b}} \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\phi(x-v)|^2 dx \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi b}} \|f\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})},
\end{aligned} \tag{42}$$

which implies that

$$1 - \epsilon \leq \int \int_V \left| G_\phi^B f(w, v) \right|^2 dw \, dv \leq \left\| G_\phi^B f \right\|_{L^\infty(\mathbb{R})}^2 \mu(V) \leq \frac{1}{\sqrt{2\pi b}} \mu(V), \tag{43}$$

This is the desired result. $\qquad\square$

Theorem 5.3 *Let ϕ, ψ be two window functions, and $f \in L^2(\mathbb{R})$, then for all $r \in [1, \infty)$, we have*

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) G_\psi^B g(w, v) \right|^r dw \, dv \right)^{\frac{1}{r}}$$
$$\leq \left(\frac{1}{2\pi b} \right)^{\frac{r-1}{r}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}. \tag{44}$$

Proof According to the Cauchy–Schwarz inequality, we obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) G_\psi^B g(w, v) \right| dw \, dv$$
$$\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) \right|^2 dw \, dv \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\psi^B g(w, v) \right|^2 dw \, dv \right)^{\frac{1}{2}}. \tag{45}$$

By virtue of (20), we see that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) G_\psi^B g(w, v) \right| dw \, dv \leq \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}. \tag{46}$$

Thus,

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) \right| G_\psi^B g(w, v) \right|^r dw \, dv \right)^{\frac{1}{r}}$$
$$\leq \left\| G_\phi^B f G_\psi^B g \right\|_{L^\infty(\mathbb{R} \times \mathbb{R})}^{\frac{r-1}{r}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) G_\psi^B g(w, v) \right| dw \, dv \right)^{\frac{1}{r}}$$
$$\leq \left(\frac{1}{2\pi b} \|f\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \right)^{\frac{r-1}{r}} \left(\|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \right)^{\frac{1}{r}}$$
$$= \left(\frac{1}{2\pi b} \right)^{\frac{r-1}{r}} \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}, \tag{47}$$

which completes the proof of Theorem 5.3 as desired. $\qquad\square$

As an easy consequence of Theorem 5.3 mentioned above, we get the following result.

Corollary 5.4 *Let ϕ, ψ be two window functions. Then for every $f \in L^2(\mathbb{R})$ with $r \in [2, \infty)$, one has*

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(w, v) \right|^r dw \, dv \right)^{\frac{1}{r}} \leq \left(\frac{1}{2\pi b} \right)^{\frac{\frac{r}{2} - 1}{r}} \|f\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})}. \tag{48}$$

Proof For $r = \infty$, we have

$$\left| G_\phi^B f(w, v) \right| \le \frac{1}{2\pi b} \| f \|_{L^2(\mathbb{R})} \| \phi \|_{L^2(\mathbb{R})}. \tag{49}$$

For all $r \in [1, \infty)$, it holds

$$\left(\int_\mathbb{R} \int_\mathbb{R} \left| G_\phi^B f(w, v) \right|^{2r} dw \, dv \right)^{\frac{1}{r}} \le \left(\frac{1}{2\pi b} \right)^{\frac{r-1}{r}} \| f \|_{L^2(\mathbb{R})}^2 \| \phi \|_{L^2(\mathbb{R})}^2. \tag{50}$$

Now putting $s = 2r \in [2, \infty)$ yields

$$\left(\int_\mathbb{R} \int_\mathbb{R} \left| G_\phi^B f(w, v) \right|^{s} dw \, dv \right)^{\frac{2}{s}} \le \left(\frac{1}{2\pi b} \right)^{\frac{s-2}{s}} \| f \|_{L^2(\mathbb{R})}^2 \| \phi \|_{L^2(\mathbb{R})}^2. \tag{51}$$

Hence,

$$\left(\int_\mathbb{R} \int_\mathbb{R} \left| G_\phi^B f(w, v) \right|^{s} dw \, dv \right)^{\frac{1}{s}} \le \left(\frac{1}{2\pi b} \right)^{\frac{\frac{s}{2}-1}{s}} \| f \|_{L^2(\mathbb{R})} \| \phi \|_{L^2(\mathbb{R})}. \tag{52}$$

We thus finish the proof. □

Below, we obtain an immediate generalization of Theorem 5.2 in the following form.

Theorem 5.5 *With the notations of Theorem* 5.2. *For all* $r > 2$, *the following inequality holds*

$$\mu(V) \ge (1 - \epsilon)^{\frac{r}{(r-2)}} (2\pi b). \tag{53}$$

Proof Applying the Hölder inequality, we easily obtain

$$1 - \epsilon \le \int \int_V \left| G_\phi^B f(w, v) \right|^2 dw \, dv$$

$$\le \left(\int_\mathbb{R} \int_\mathbb{R} \left| G_\phi^B f(w, v) \right|^{2 \cdot \frac{r}{2}} dw \, dv \right)^{\frac{2}{r}} \left(\int_\mathbb{R} \int_\mathbb{R} \chi_V(w, v)^{\frac{r}{r-2}} dw \, dv \right)^{\frac{r-2}{r}}, \tag{54}$$

where χ_V denotes the indicator function of the set V. Substituting relation (48) into the first term in the right-hand side of the equation (54), we see that

$$1 - \epsilon \le \left(\left(\frac{1}{2\pi b} \right)^{\frac{\frac{r}{2}-1}{r}} \| f \|_{L^2(\mathbb{R})} \| \phi \|_{L^2(\mathbb{R})} \right)^2 \left(\mu(V) \right)^{\frac{r-2}{r}}. \tag{55}$$

For all $r > 2$, we get

$$1 - \epsilon \le \left(\frac{1}{2\pi b} \right)^{\frac{r-2}{r}} \left(\mu(V) \right)^{\frac{r-2}{r}}. \tag{56}$$

After simplifying the above relation, we get

$$\mu(V) \ge (1 - \epsilon)^{\frac{r}{(r-2)}} (2\pi b). \tag{57}$$

This proves the claim. □

Definition 5.1 A function $f \in L^2(\mathbb{R})$ is said to be the ϵ-concentrated on a measurable set $X \subseteq \mathbb{R}$ if for any $\epsilon > 0$, it holds

$$\left(\int_{\mathbb{R} \setminus X} \left| f(x) \right|^r dx \right)^{1/r} \leq \epsilon \|f\|_{L^r(\mathbb{R})}, \quad 0 < r \leq \infty. \tag{58}$$

Below, based on Definition 5.1 above, we immediately obtain the alternative form of Theorem 5.5.

Theorem 5.6 *Let ϕ be a complex window function, and $f \in L^2(\mathbb{R})$. If $G_\phi^B f$ is the ϵ-concentrated on L^2-norm on measurable set X of $\mathbb{R}$, then for all $r > 2$, one has*

$$\mu(V) \geq \left(1 - \epsilon^2 \right)^{\frac{r}{(r-2)}} (2\pi b). \tag{59}$$

Proof For simplicity, we first assume that $\|f\|_{L^2(\mathbb{R})}^2 = \|\phi\|_{L^2(\mathbb{R})}^2 = 1$. Invoking (58), we get

$$\int_{\mathbb{R} \setminus X} \int_{\mathbb{R} \setminus X} \left| G_\phi^B f(\omega, u) \right|^2 d\omega \, du \leq \epsilon^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(\omega, u) \right|^2 d\omega \, du. \tag{60}$$

An application of relation (8) together with relation (16) will lead to

$$\int_{\mathbb{R} \setminus X} \int_{\mathbb{R} \setminus X} \left| G_\phi^B f(\omega, u) \right|^2 d\omega \, du \leq \epsilon^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathcal{C}_B \{ f T_u \bar{\phi} \} \right|^2 d\omega \, du$$

$$= \epsilon^2 \|f\|_{L^2(\mathbb{R})}^2 \|\phi\|_{L^2(\mathbb{R})}^2$$

$$= \epsilon^2.$$

Hence,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(\omega, u) \right|^2 d\omega \, du \leq \int_X \int_X \left| G_\phi^A f(\omega, u) \right|^2 d\omega \, du + \epsilon^2. \tag{61}$$

Applying equation (20) and the Hölder inequality, we obtain

$$1 - \epsilon^2 \leq \int_X \int_X \left| G_\phi^B f(\omega, u) \right|^2 d\omega \, du$$

$$\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| G_\phi^B f(\omega, u) \right|^{2 \frac{r}{2}} d\omega \, du \right)^{\frac{2}{r}} \left(\mu(X) \right)^{\frac{r-2}{r}}$$

$$\leq \left(\frac{1}{2\pi b} \right)^{\frac{r-2}{r}} \left(\mu(X) \right)^{\frac{r-2}{r}},$$

which completes the proof of the theorem. $\qquad \square$

Next, we present a lemma, which describes the basic concept of Nazarovs's uncertainty principle for the Fourier transform (see [18]).

Lemma 5.7 *Suppose that X_1 and X_2 are two finite, measurable subsets of $\mathbb{R}$. Then for every $f \in L^2(\mathbb{R})$ there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}} |f(x)|^2 \, dx \leq C e^{C\mu(X_1)\mu(X_2)} \left(\int_{\mathbb{R}\setminus X_1} |f(x)|^2 \, dx + \int_{\mathbb{R}\setminus X_2} |F\{f\}(w)|^2 \, dw \right). \tag{62}$$

We are ready to obtain a straightforward generalization of Nazarovs's uncertainty principle in the framework of the WLCT.

Theorem 5.8 *With the notations of Lemma 5.7 above, if $\phi \in L^2(\mathbb{R})$, then*

$$\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} |f(x)|^2 \, dx$$

$$\leq C e^{C\mu(X_1)\mu(X_2)}$$

$$\times \left(\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}\setminus X_1} |f(x)|^2 \, dx + 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}\setminus bX_2} |G^B_\phi f(w,v)|^2 \, dw \, dv \right). \tag{63}$$

Proof Replacing $f(x)$ by $e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}(x)$ on both sides of (62), we obtain

$$\int_{\mathbb{R}} |e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}(x)|^2 \, dx$$

$$\leq C e^{C\mu(X_1)\mu(X_2)}$$

$$\times \left(\int_{\mathbb{R}\setminus X_1} |e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}(x)|^2 \, dx + \int_{\mathbb{R}\setminus X_2} |F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}(w)|^2 \, dw \right). \tag{64}$$

Hence,

$$\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} |f(x)|^2 \, dx$$

$$\leq C e^{C\mu(X_1)\mu(X_2)}$$

$$\times \left(\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}\setminus X_1} |f(x)|^2 \, dx + \int_{\mathbb{R}} \int_{\mathbb{R}\setminus X_2} |F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}(w)|^2 \, dw \, dv \right). \tag{65}$$

This further leads to

$$\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} |f(x)|^2 \, dx$$

$$\leq C e^{C\mu(X_1)\mu(X_2)}$$

$$\times \left(\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}\setminus X_1} |f(x)|^2 \, dx + \frac{1}{b} \int_{\mathbb{R}} \int_{\mathbb{R}\setminus bX_2} \left| F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}\left(\frac{w}{b}\right) \right|^2 \, dw \, dv \right). \tag{66}$$

Applying relation (13), we see that

$$\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} |f(x)|^2 \, dx$$

$$\leq C e^{C\mu(X_1)\mu(X_2)}$$

$$\times \left(\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}\setminus X_1} |f(x)|^2 \, dx \right.$$

$$\left. + \frac{1}{b} \int_{\mathbb{R}} \int_{\mathbb{R}\setminus bX_2} \left| \sqrt{2\pi b} e^{i\frac{\pi}{4}} e^{-i\frac{d}{2b}w^2} G^B_\phi f(w,v) \right|^2 \, dw \, dv \right), \tag{67}$$

and the required result follows. □

The following results concern some consequences of Nazarov's uncertainty principles described by equation (62).

Corollary 5.9 *Using the notations as in Lemma 5.7, we have*

$$\int_{\mathbb{R}} |f(x)|^2 \, dx \le C e^{C\mu(X_1)\mu(X_2)} \left(\int_{\mathbb{R}\setminus X_1} |f(x)|^2 \, dx + 2\pi \int_{\mathbb{R}\setminus bX_2} |\mathcal{C}_B\{f\}(w)|^2 \, dw \right), \tag{68}$$

which is Nazarovs's uncertainty principle in the context of the LCT.

Proof Including $f(x)$ as $e^{\frac{ia}{2b}x^2} f(x)$ on both sides of (62) and then implementing (5) will lead to the desired result. □

The above corollary is consistent with the result studied in [19], where, in this case, the LCT is a particular case of the offset linear canonical transform.

Corollary 5.10 *Under the assumptions of Lemma 5.7, if $\phi \in L^2(\mathbb{R})$, then one has*

$$\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} |f(x)|^2 \, dx$$

$$\le C e^{C\mu(X_1)\mu(X_2)}$$

$$\times \left(\|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}\setminus X_1} |f(x)|^2 \, dx + 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}\setminus X_2} |G_\phi f(w,v)|^2 \, dw \, dv \right), \tag{69}$$

which is Nazarovs's uncertainty principle associated with the WFT defined by (14).

Proof By (15), the proof is straightforward. □

Theorem 5.11 (WLCT local uncertainty principle) *Let $\phi \in L^2(\mathbb{R})$ be a window function. If $0 < \alpha < \frac{1}{2}$, then there exists some constant C_α such that for any $f \in L^2(\mathbb{R})$ and $X \subseteq \mathbb{R}$ measurable, one has*

$$\int_{\mathbb{R}} \int_{bX} |G^B_\phi f(w,v)|^2 \, dw \, dv \le \frac{C_\alpha \|\phi\|^2_{L^2(\mathbb{R})}}{2\pi} [\mu(X)]^{2\alpha} \int_{\mathbb{R}} |x|^{2\alpha} |f(x)|^2 \, dx, \tag{70}$$

and for $\alpha > \frac{1}{2}$, it holds

$$2\pi \int_{\mathbb{R}} \int_{bX} |G^B_\phi f(w,v)|^2 \, dw \, dv \le C_\alpha \mu(X) \|\phi\|^{\frac{2\alpha-1}{\alpha}}_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{\frac{2\alpha-1}{2\alpha}}$$

$$\times \|\phi\|^{\frac{1}{\alpha}}_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} ||x|^\alpha f(x)|^2 \, dx \right)^{\frac{1}{2\alpha}}, \tag{71}$$

Proof Let a set $X \subseteq \mathbb{R}$ with finite measure. The local uncertainty principle for the Fourier transform is expressed as (see [20])

$$\int_X \big|F\{f\}(w)\big|^2 \, dw \le C_\alpha \big[\mu(X)\big]^{2\alpha} \big\||x|^\alpha f\big\|_{L^2(\mathbb{R})}^2. \tag{72}$$

In fact, by inserting $f(x)$ as $fT_\nu\bar\phi(x)$ on both sides of the relation (72) above, we get

$$\int_X \big|F\{fT_\nu\bar\phi\}(w)\big|^2 \, dw \le C_\alpha \big[\mu(X)\big]^{2\alpha} \big\||x|^\alpha fT_\nu\bar\phi\big\|_{L^2(\mathbb{R})}^2. \tag{73}$$

Furthermore,

$$\int_\mathbb{R}\int_X \big|F\{e^{\frac{ia}{2b}x^2}fT_\nu\bar\phi\}(w)\big|^2 \, dw \, dv$$
$$\le C_\alpha \big[\mu(X)\big]^{2\alpha} \int_\mathbb{R}\int_\mathbb{R} \big||x|^\alpha e^{\frac{ia}{2b}x^2} f(x)\overline{\phi(x-v)}\big|^2 \, dx \, dv. \tag{74}$$

Setting $w = \frac{w}{b}$, we obtain

$$\frac{1}{b}\int_\mathbb{R}\int_{bX} \bigg|F\{e^{\frac{ia}{2b}x^2}fT_\nu\bar\phi\}\bigg(\frac{w}{b}\bigg)\bigg|^2 \, dw \, dv$$
$$\le C_\alpha \big[\mu(X)\big]^{2\alpha} \int_\mathbb{R}\int_\mathbb{R} \big||x|^\alpha f(x)\overline{\phi(x-v)}\big|^2 \, dx \, dv. \tag{75}$$

It follows from the expression (13) that

$$2\pi \int_\mathbb{R}\int_{bX} \big|G_\phi^B f(w,v)\big|^2 \, dw \, dv \le C_\alpha \big[\mu(X)\big]^{2\alpha} \|\phi\|_{L^2(\mathbb{R})}^2 \int_\mathbb{R} |x|^{2\alpha} \big|f(x)\big|^2 \, dx, \tag{76}$$

which gives the proof of (70). Based on the local uncertainty principle for the Fourier transform

$$\int_X \big|F\{f\}(w)\big|^2 \, dw \le C_\alpha \mu(X) \|f\|_{L^2(\mathbb{R})}^{2-\frac{1}{\alpha}} \big\||x|^\alpha f\big\|_{L^2(\mathbb{R})}^{\frac{1}{\alpha}}, \tag{77}$$

we apply the same arguments to get equation (71). $\qquad\square$

Recently, Kubo et al. [21] have proposed the logarithmic Sobolev-type uncertainty principle for the Fourier transform. Below, we search this uncertainty principle in the setting of the WLCT. For this purpose, we present the following.

Definition 5.2 For $1 \le r < \infty$ and $s > 0$ define the weighted Lebesgue space as

$$\mathcal{W}_s^r(\mathbb{R}) = \big\{f \in L^r(\mathbb{R}) : \langle x\rangle^s \in L^r(\mathbb{R})\big\}, \tag{78}$$

where $\langle x\rangle = (1+x^2)^{\frac{1}{2}}$ is the weight function.

We have the following.

Theorem 5.12 *Let $\phi \in L^2(\mathbb{R})$ be a window function. For every $f \in \mathcal{S}(\mathbb{R}) \cap \mathcal{W}_1^2(\mathbb{R})$, we have*

$$\|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \ln\left(\frac{1+|x|^2}{2}\right) dx + 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| |G_\phi^B f(w,v)|^2 \, dw \, dv$$
$$\geq \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)} + 2\pi \ln b\right) \|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \, dx. \tag{79}$$

In this case, $\mathcal{S}(\mathbb{R})$ is the Sobolev space on $\mathbb{R}$ defined by

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : Df \in L^2(\mathbb{R}) \right\},$$

where D stands for the differential operator, and Γ indicates the Gamma function.

Proof From the logarithmic Sobolev-type uncertainty principle for the Fourier transform, we obtain that (see [21])

$$\int_{\mathbb{R}} |f(x)|^2 \ln\left(\frac{1+|x|^2}{2}\right) dx + \int_{\mathbb{R}} \ln|w| |F\{f\}(w)|^2 \, dw \geq \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)}\right) \int_{\mathbb{R}} |f(x)|^2 \, dx. \tag{80}$$

Now replacing $f(x)$ by $e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}(x)$ on both sides of identity (80) yields

$$\int_{\mathbb{R}} |e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}(x)|^2 \ln\left(\frac{1+|x|^2}{2}\right) dx + \int_{\mathbb{R}} \ln|w| |F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}(w)|^2 \, dw$$
$$\geq \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)}\right) \int_{\mathbb{R}} |e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}(x)|^2 \, dx. \tag{81}$$

Hence,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f T_v \bar{\phi}(x)|^2 \ln\left(\frac{1+|x|^2}{2}\right) dx \, dv + \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| |F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}(w)|^2 \, dw \, dv$$
$$\geq \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)}\right) \int_{\mathbb{R}} \int_{\mathbb{R}} |f T_v \bar{\phi}(x)|^2 \, dx \, dv. \tag{82}$$

This implies that

$$\|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \ln\left(\frac{1+|x|^2}{2}\right) dx + \frac{1}{b} \int_{\mathbb{R}} \int_{\mathbb{R}} \ln\left|\frac{w}{b}\right| |F\{e^{\frac{ia}{2b}x^2} f T_v \bar{\phi}\}\left(\frac{w}{b}\right)|^2 \, dw \, dv$$
$$\geq \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)}\right) \|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \, dx. \tag{83}$$

Applying (13) gives

$$\|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \ln\left(\frac{1+|x|^2}{2}\right) dx + 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \ln\left|\frac{w}{b}\right| |G_\phi^B f(w,v)|^2 \, dw \, dv$$
$$\geq \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)}\right) \|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \, dx. \tag{84}$$

According to (20), we deduce that

$$\|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \ln\left(\frac{1+|x|^2}{2}\right) dx + 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| |G_\phi^B f(w,v)|^2 \, dw \, dv$$

$$\geq \left(\frac{\Gamma'(1/2)}{\Gamma(1/2)} + 2\pi \ln b\right) \|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \, dx,$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As an immediate consequence of the above theorem, we obtain the following (see [7]).

Corollary 5.13 *Let $\phi \in L^2(\mathbb{R})$ be a window function. For every $f \in \mathcal{S}(\mathbb{R})$, we have*

$$\|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} \ln|x| |f(x)|^2 \, dx + \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|w| |G_\phi^B f(w,v)|^2 \, dw \, dv$$

$$\geq (D + \ln b) \|\phi\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |f(x)|^2 \, dx, \tag{85}$$

where $D = \Phi(\frac{1}{2}) - \ln \pi$, $\Phi(x) = \frac{d}{dx} \ln[\Gamma(x)]$.

6 Conclusion

In this paper, we have investigated some properties of the windowed linear canonical transform like the orthogonality relation, inversion theorem, and conjugate function by utilizing the direct interaction among the windowed linear canonical transform, windowed Fourier transform and the Fourier transform. We have also discussed several generalizations of the uncertainty principles in the setting of the windowed linear canonical transform.

Declarations

Competing interests
The author declares that he has no competing interest.

Authors' contributions
The author read and approved the final manuscript.

References

1. Bai, R.F., Li, B.Z., Cheng, Q.Y.: Wigner–Ville distribution associated with the linear canonical transform. J. Appl. Math. **2012**, Article ID 740161 (2012)
2. Che, T.W., Li, B.Z., Xu, T.Z.: Ambiguity function associated with the linear canonical transform. EURASIP J. Adv. Signal Process. **138**, 1–14 (2012)
3. Bahri, M., Ashino, R.: Convolution and correlation theorems for Wigner–Ville distribution associated with linear canonical transform. In: 12th International Conference on Information Technology-New Generations, Las Vegas, NV, USA (2015)
4. Bahri, M., Karim, S.A.A.: On relationship between ambiguity function and linear canonical ambiguity function. In: Karim, S.A.A., Shafie, A. (eds.) Towards Intelligent Systems Modeling and Simulation. Springer, Cham (2022)
5. Kou, K.I., Xu, R.H., Zhang, Y.H.: Paley–Wiener theorems and uncertainty principles for the windowed linear canonical transform. Math. Methods Appl. Sci. **35**, 2122–2132 (2012)
6. Kou, K.I., Xu, R.H.: Windowed linear canonical transform and its applications. Signal Process. **92**(1), 179–188 (2012)
7. Bahri, M., Ashino, R.: Some properties of windowed linear canonical transform and its logarithmic uncertainty principle. Int. J. Wavelets Multiresolut. Inf. Process. **14**(3), Article ID 1650015 (2016)
8. Zayed, Z.I.: On the relationship between the Fourier and fractional Fourier transforms. IEEE Signal Process. Lett. **3**(12), 310–311 (1996)
9. Bahri, M., Shah, F.A., Tantary, A.Y.: Uncertainty principles for the continuous shearlet transforms in arbitrary space dimensions. Integral Transforms Spec. Funct. **31**(7), 538–555 (2020)
10. Gao, W.B., Li, B.Z.: Uncertainty principles for the short-time linear canonical transform of complex signals. Digit. Signal Process. **111**, 102953 (2021)
11. Xu, T.Z., Li, B.Z.: Linear Canonical Transform and Its Application. Science Press, Beijing (2013)
12. Guanlei, X., Xiaotong, W., Xiaogang, X.: New inequalities and uncertainty relations on linear canonical transform revisit. EURASIP J. Adv. Signal Process. **2009**, Article ID 563265 (2009). https://doi.org/10.1155/2009/563265
13. Tao, R., Li, Y.L., Wang, Y.: Uncertainty principles for linear canonical transforms. IEEE Trans. Signal Process. **57**(7), 2856–2858 (2009)
14. Bahri, M., Ashino, R.: Correlation formulation using relationship between convolution and correlation in linear canonical transform domain. In: International Conference on Wavelet Analysis and Pattern Recognition (ICWAPR), Ningbo China (2017)
15. Guanlei, X., Xiaotong, W., Xiaogang, X.: Uncertainty inequalities for linear canonical transform. IET Signal Process. **3**(5), 392–402 (2009)
16. Shah, F.A., Teali, A.A., Tantary, A.Y.: Windowed special affine Fourier transform. J. Pseudodiffer. Oper. Appl **11**, 1389–1420 (2020)
17. Gröchenig, K.: Foundation of Time-Frequency Analysis. Birkhäuser, Boston (2001)
18. Jaming, P.: Nazarov's uncertainty principles in higher dimension. J. Approx. Theory **149**(1), 30–41 (2007)
19. Huo, H.: Uncertainty principles for the offset linear canonical transform. Circuits Syst. Signal Process. **38**, 395–406 (2019). https://doi.org/10.1007/s00034-018-0863-z
20. Price, J.F.: Inequalities for local uncertainty principles. J. Math. Phys. **24**(7), 1711–1714 (1983)
21. Kubo, H., Ogawa, T., Suguro, T.: Beckner type of the logarithmic Sobolev and a new type of Shannon's inequalities and an application to the uncertainty principle. Proc. Am. Math. Soc. **147**, 1511–1518 (2019)