WINKLER'S SINGLE-PARAMETER SUBGRADE MODEL FROM THE PERSPECTIVE OF AN IMPROVED APPROACH OF CONTINUUM-BASED SUBGRADE MODELING

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ABSTRACT

Based on an isotropic elastic continuum of thickness H overlying a rigid stratum, a generalized formulation for the classical single-parameter Winkler's subgrade model is presented. In this formulation, all the normal components of the stress tensor are taken into consideration, whereas the shear stresses are intentionally dropped with the purpose of providing a useful perspective, with which Winkler's model and its associated coefficient of subgrade reaction can be viewed. The formulation takes into account the variation of the elasticity modulus with depth. It only demands specifying a relationship between the vertical and horizontal normal stresses. Accordingly, two such different assumptions are made to obtain two new Winkler-type subgrade models with the corresponding closed-form relations for the subgrade modulus. The models give consistently larger stiffness for the Winkler springs as compared to previously proposed similar continuum-based models that ignore the lateral stresses. It has also been pointed out that it is only if the shear stress components of the subgrade are taken into consideration that a multi-parameter model evolves regardless of whether the lateral normal stresses are included. Finally, the effective stiffness per unit area of the multiple beds of springs of such a higher order model is exactly the same as the subgrade modulus of the corresponding single-parameter Winkler model presented in this work.

Keywords: *Heterogeneous subgrade, Reissner's simplified continuum, Shear interaction, Simplified continuum, Winkler model, Winkler-type models.*

INTRODUCTION

The simplest representation of a foundation subgrade is in the form of the classical Winkler model, which replaces the subgrade by a mechanical analogy consisting of a single bed of closely spaced vertical springs acting independently of each other. Mathematically, Winkler's model translates into $p(x,y) = k_s w_0(x,y)$, in which p is the vertical contact pressure at an arbitrary point (x,y) in the foundation-soil interface area; w_0 is the corresponding vertical deformation; and k_s is a proportionality constant representing contact pressure per unit deformation - commonly referred to as the coefficient of subgrade reaction or simply as the subgrade modulus [1-7]. Thus, k_s is the only quantity characterizing the subgrade material.

Winkler's model has the well-known shortcoming of bringing about a vertical deformation of those springs alone that are located just under the loaded region. Because of this, the model entails a discontinuity of vertical deformation at the edges of the loaded area. Furthermore, the model implies that a point undergoes vertical deformation independently of other adjoining points [1, 3, 4, 5]. Both are consequences of the neglected vertical shear stresses that would have coupled the vertical deformations of neighboring points with each other so that continuity of displacement exists.

This shortcoming can be overcome by appropriately accounting for the shear stress components of the subgrade. A number of attempts have been made in the past to incorporate the shear stresses following two different approaches. In the first approach, the subgrade is idealized as an elastic continuum of finite thickness, and certain simplifying assumptions are made to reduce the mathematical work involved [3, 4, 8, 9]. In the second approach, mechanical models are developed that involve different combinations of spring beds and shear elements [3, 7]. Both approaches can be judiciously synthesized for the purpose of solving practical problems, whereby the mechanical-model parameters are quantified in terms of the elastic parameters of the continuum [3, 9].

However, mainly due to the simplicity of Winkler's model in practical applications and its long time familiarity among practical engineers, its usage has endured to this date. Many current commercial softwares continued incorporating the model as a major feature of their programs for the purpose of analysis and design of beams and plates on elastic foundations. A number of both analytical and empirical relationships have been suggested in the past for estimating k_s. As a pioneer, Terzaghi [10] identified the width of the foundation as the most important influencing factor of k_s in addition to the elastic properties of the soil. Accordingly, he suggested empirical relations for converting k_s values from field plate loading tests to k_s values of actual foundations that decrease with increasing width. For long beams, Vesić [11] later proposed a formula that depends on the rigidity of the beam itself in addition to its width and the elastic properties of the subgrade material. Based on a subgrade idealized as a simplified elastic continuum of finite thickness, in which only the vertical normal stress components are taken into consideration, Horvath [5] recently derived closed form relations for k_s for constant and varying elasticity modulus of the subgrade. Correlations with standard penetration blow counts were also suggested more recently [12].

Recognizing the enduring usage of Winkler's model in wide ranging applications of geotechnical engineering, this article attempts to provide some insight into this model and its associated coefficient of subgrade reaction from the perspective of continuum modeling. In this technique, the subgrade is idealized as an isotropic elastic continuum of finite thickness H. Heterogeneity with respect to subgrade rigidity is taken into consideration by assuming a variable elasticity modulus with depth. In order to clearly understand the influence of the soil shear stress on the form of the resulting mathematical model, the shear components are first intentionally omitted with the normal stress components alone accounted for in the formulation. Open functions of depth are introduced to relate the horizontal and vertical normal stress components. It is shown that the resulting model is a single-parameter Winkler-type model, for which the coefficient of subgrade reaction can be evaluated from an analytical relation obtained in form of a definite integral. The choice of the functions relating the normal stresses is at the discretion of the user. Two such functions are employed in this work to come up with two correspondingly different sets of closed-form relations for the coefficient of subgrade reaction for constant as well as variable elasticity modulus. These are compared with similar relations proposed in the past.

Finally, a brief account of ways of incorporating the shear stress components of the elastic subgrade is presented with the details being presented in the companion paper [9]. Here, it is pointed out that an additional assumption is needed regarding the variation of the vertical shear stress components with depth. It has been found that the resulting mathematical models are always second-order differential equations with constant coefficients regardless of the nature of this assumption. It is pointed out that a three-parameter mechanical model consisting of two beds of springs and a layer of shear element results also in a differential equation of similar form and order to that of the continuum. By taking advantage of this important analogy, it has been shown that the effective spring stiffness per unit area of the two beds of springs of the three-parameter mechanical model is nothing other than the coefficient of subgrade reaction of the single-parameter Winkler-type model established in the present work by excluding the shear stresses.

A GENERALIZED FORMULATION OF WINKLER'S SUBGRADE MODULUS

The subgrade is idealized as an isotropic elastic continuum of thickness H similar to Reissner's simplified continuum [8] as shown in Fig. 1.

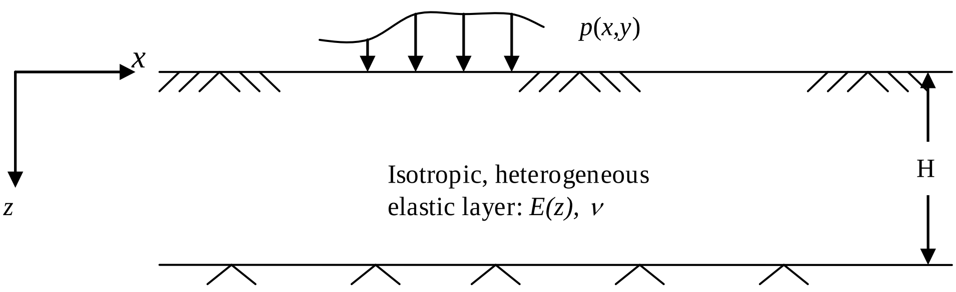

Figure 1 The subgrade idealized as an isotropic, heterogeneous elastic layer

The depth-wise heterogeneity is taken into account by assuming a variable elasticity modulus with depth. The Poisson ratio is assumed constant, because it is known that the problem is less sensitive to its variation.

The primary aim is to study the influence of the normal stress components alone by intentionally excluding the shear effect. Thus,

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \tag{1}$$

With this assumption, the equilibrium equation for the vertical direction becomes

$$\sigma_{z,z} = 0 \tag{2}$$

where the coma sign represents a derivative with respect to the symbol that follows.

Equation (2) implies that σ_z is constant with respect to depth. Applying the stress boundary condition at the surface and noting that compressive stresses are negative, it follows that

$$\sigma_z(x,y) = -p(x,y) \tag{3}$$

where, $-p(x,y)$ is the vertical contact pressure at the surface.

The lateral normal stresses, σ_x and σ_y, can be related to the vertical normal stress, σ_z, through appropriately selected functions, $g_x(z)$ and $g_y(z)$, respectively, so that

$$\sigma_x = g_x(z)\sigma_z; \qquad \sigma_y = g_y(z)\sigma_z \tag{4}$$

The generalized Hooke's law for the normal strain in the vertical direction is given by

$$w_{,z} = \frac{1}{E(z)}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \tag{5}$$

where $E(z)$ is the elasticity modulus that may generally vary with depth. Substituting Eqs. (3) and (4) in Eq. (5) and integrating, one obtains for the vertical displacement

$$w(x,y,z) = -p(x,y)\int \frac{g(z)}{E(z)}dz + f_1(x,y) + c_1 \tag{6}$$

where,

$$g(z) = 1 - \nu\left[g_x(z) + g_y(z)\right] \tag{7}$$

The function f_1 in Eq. (6) is a function of x and y, and c_1 is a constant of integration. Applying the zero-displacement boundary condition at the interface with the rigid base, these two unknowns are readily determined at once. Substituting the resulting expression back in Eq. (6) one obtains the vertical displacement function as

$$w(x,y,z) = p(x,y)\left\{\left[\int \frac{g(z)}{E(z)}dz\right]_{z=H} - \int \frac{g(z)}{E(z)}dz\right\} \tag{8}$$

The vertical displacement, w_0, of the surface in particular is obtained by evaluating Eq. (8) at $z=0$, so that

$$w_0(x,y) = p(x,y)\left[\int \frac{g(z)}{E(z)}dz\right]_{z=0}^{z=H} = p(x,y)\int_0^H \frac{g(z)}{E(z)}dz \tag{9}$$

This equation is the relationship sought between the vertical surface displacement and the vertical surface pressure that can be written as

$$p(x,y) = k_s w_0(x,y) \tag{10}$$

where, k_s is the coefficient of subgrade reaction given by

$$k_s = \frac{1}{\int_0^H \frac{g(z)}{E(z)}dz} \tag{11a}$$

It is evident from this result that it is always a Winkler-type single-parameter model that evolves for an elastic subgrade as far as the normal stress components alone are taken into account. Eq. (11a) provides a generalized analytical formulation for quantifying the coefficient of subgrade reaction, which depends only on the elastic properties of the subgrade and, implicitly through $g(z)$ of Eq. (7), on the size and shape of the loaded region on the surface as was correctly pointed out by Terzaghi [10].

If the lateral normal stress components are neglected in addition to the vertical shear stresses, $g(z)$ in Eq. (7) reduces to unity and Eq. (11a) takes the simplified form of

$$k_s = \frac{1}{\int_0^H \frac{1}{E(z)}dz} \tag{11b}$$

It will be shown in a later section that Eq. (11b) is a unified formulation of the closed-form relations proposed by Horvath [5] for estimating the subgrade modulus.

According to Eq. (11a), what it all demands to estimate the subgrade modulus is to select an appropriate function $g(z)$ that relates the vertical and horizontal normal stresses with each other and to employ a suitable function $E(z)$ for the variation of the elasticity modulus with depth if there is the need to do so. The following sections deal with these two functions.

THE ELASTICITY MODULUS FUNCTION, E(Z)

The depth-wise variation of $E(z)$ can be taken into consideration in one of two different forms: a power function or an exponential function of z. The most commonly employed option is the power function of z given by

$$E(z) = Bz^{\beta} \tag{12a}$$

where β is a positive dimensionless constant known as the non-homogeneity parameter, and B is a dimension-bound coefficient [6, 13, 14]. For $\beta = 0$, Eq. (12a) represents a homogenous elastic layer with a constant elasticity modulus, $B=E_0$. In this case, the coefficient B takes the dimension of a stress. For $\beta = 1$, Eq. (12a) represents a heterogeneous soil layer with a linearly varying elasticity modulus. The coefficient B takes in this case the dimensions of the coefficient of subgrade reaction. For other values of β, B assumes correspondingly different dimensions. Soils, the Young's moduli of which vary in accordance with Eq. (12a), are often referred to as Gibson soils [6, 14].

For the general case of a heterogeneous layer ($\beta \neq 0$), Eq. (12a) gives a zero value for the elasticity modulus at the surface ($z = 0$). This condition poses some difficulties in the evaluation of the integral in Eq. (11a) at $z = 0$ when $\beta = 1$. A more suitable variant for such a case is the form

$$E(z) = E_0 + Bz^{\beta} \tag{12b}$$

This alternative formulation enables the assignment of a non-zero elasticity modulus of $E = E_0$ at $z = 0$. In both Eqs. (12a) and (12b), it is common to use $\beta = 1$ for clayey soils and $\beta = 1/2$ for granular soils [13].

The other form of variation of the elasticity modulus that found some usage in the past is the exponential function given by [13]

$$E(z) = E_0 e^{\lambda z} \tag{13}$$

where E_0 is the elasticity modulus at the surface (for $z=0$) and λ is a non-negative quantity having the dimension of m^{-1}. Appropriate values of λ can be easily obtained by matching plots of Eqs. (13) and (12b). For reasons of mathematical convenience, only the relations in Eqs. (12b) and (13) are further used in this work.

Since the other important function in the estimation of k_s using Eq. (11a) is $g(z)$, two alternative forms of this function that lead to correspondingly two different Winkler-type models are discussed next.

WINKLER-TYPE CONTINUUM MODEL VARIANT I

In the development of this particular model, the functions g_x and g_y in Eq. (4) are assumed constant so that

$$\sigma_x = k_x \sigma_z ; \qquad \sigma_y = k_y \sigma_z \tag{14}$$

where k_x and k_y are constants that are presumed to be estimated from knowledge of lateral earth pressure theories. Substituting Eq. (14) in Eq. (7) and this in turn in Eq. (11a), one obtains

$$k_s = \frac{1}{\alpha \int_0^H \frac{dz}{E(z)}} \tag{15}$$

where α is a constant given by

$$\alpha = 1 - \nu (k_x + k_y) \tag{16}$$

If, as a special case, the assumption is made that $k_x = k_y = k_0$ (the coefficient of lateral pressure for at rest condition) and it is noted that this coefficient can be expressed as $k_0 = \nu/(1-\nu)$, then the factor α in Eq. (16) becomes dependent only on the Poisson and takes the form

$$\alpha = \frac{1 - \nu - 2\nu^2}{1 - \nu} \tag{17}$$

In plane-strain problems, the assumption of $k_x = k_0$ alone is sufficient, because it can be easily shown in this case that $k_x = k_y = k_0$. It may thus be expected that Eq. (17) can give reasonable results for strip foundations and for rectangular foundations with large aspect ratios.

Stratum with Constant E

The case of a homogenous stratum corresponds to $B = 0$ in Eq. (12b) and $\lambda = 0$ in Eq. (13), both of which give a constant modulus of elasticity, $E = E_0$. Evaluation of the integral in Eq. (15) then yields

$$k_s = \frac{E_0}{\alpha H} \tag{18}$$

Stratum with Variable E

Substituting Eqs. (12b) and (13) in Eq. (15) and performing the respective integrals, one obtains for the heterogeneous layer, depending on the type of $E(z)$ used,

For $E(z)=E_0+Bz^{\beta}$:

$$k_s = \begin{cases} \dfrac{B}{\alpha \ln\left(\dfrac{E_0 + BH}{E_0}\right)}; & \beta = 1 \\ \dfrac{B^2}{2\alpha\left[B\sqrt{H} - E_0 \ln\left(\dfrac{E_0 + B\sqrt{H}}{E_0}\right)\right]}; & \beta = \dfrac{1}{2} \end{cases} \tag{19}$$

For $E(z)=E_0 e^{\lambda z}$:

$$k_s = \frac{E_0 \lambda}{\alpha\left(1 - e^{-\lambda H}\right)} \tag{20}$$

WINKLER-TYPE CONTINUUM MODEL VARIANT II

This model is motivated by observations of plots of the depth-wise variation of the horizontal-to-vertical normal stress ratio underneath uniformly loaded circular and square regions on the surface of both a layered and non-layered elastic half space as presented in Fig. 2 [4,9, 15].

Accordingly, this ratio can be represented by a decaying exponential function of z as shown in the figure for a typical vertical plane through the loaded region. In this figure, n_z and n_r represent the normal stresses in the vertical and radial directions, respectively; b is the radius of the loaded circular region; s is the radial coordinate according to the cylindrical coordinate system, the origin of which is located at the center of the circle.

Based on observations of the trend in Fig. 2, the functions $g_x(z)$ and $g_y(z)$ are taken as

$$g_x(z) = r_x e^{-\zeta z}; \quad g_y(z) = r_y e^{-\zeta z} \tag{21}$$

where, the constants r_x and r_y, and the dimension-bound ζ can be established from best-fitting curves for the plots of the horizontal-to-vertical normal stress ratio. These constants take the values $r_x = r_y = 0.8$ and $\zeta = 3.96/H$ for a typical vertical plane and are indicated in Fig. 2 [9].

With Eq. (21) substituted in Eq. (7), this further in Eq. (11a), and noting that $r = r_x + r_y$ one obtains

$$k_s = \frac{1}{\displaystyle\int_0^H \frac{\left(1 - \nu r e^{-\zeta z}\right)dz}{E(z)}} \tag{22}$$

Stratum with Constant E

For a homogenous soil layer with a constant $E = E_0$, evaluation of the integral in Eq. (22) leads to

$$k_s = \frac{E_0/H}{1 - \dfrac{\nu r}{\zeta H}\left(1 - e^{-\zeta H}\right)} \tag{23}$$

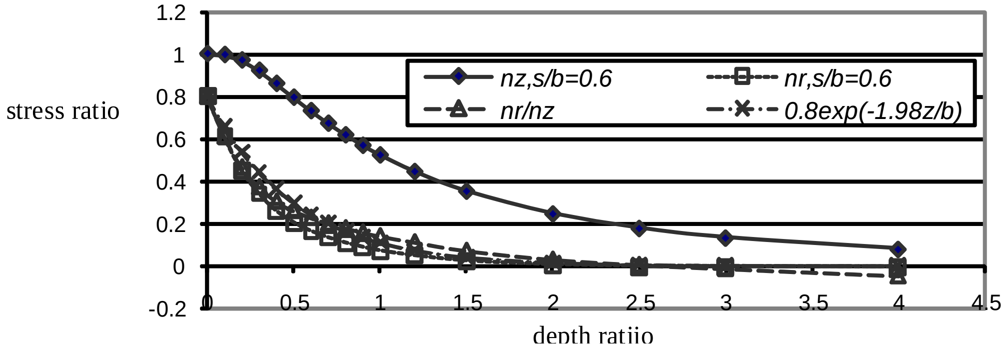

Figure 2 Plots of typical vertical and horizontal stresses and their ratios with depth for a circular region subjected to a uniformly distributed load together with the best-fitting curve for σ_r/σ_z [9]

Stratum with Variable E

For the general case of a heterogeneous stratum, the use of the exponential function of Eq. (13) for $E(z)$ is much more convenient to evaluate the integral of Eq. (22) than using Eq. (12b). This yields

$$k_s = \frac{E_0\lambda}{\left(1-e^{-\lambda H}\right) - \frac{\lambda \nu r}{(\zeta+\lambda)}\left[1-e^{-(\zeta+\lambda)H}\right]} \quad (24)$$

It is also possible to use Eq. (12b) for the variation of E. However, the evaluation of the integral in Eq. (22) leads to complicated expressions for the coefficient of subgrade reaction. For example, for a linearly varying E (i.e $\beta = 1$), k_s takes the form

$$k_s = \frac{B}{\ln\left(\frac{E_0+BH}{E_0}\right) + \nu r B\left[e^{-\zeta H}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}B^{n-1}}{\zeta^n(E_0+BH)^n} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B^{n-1}}{\zeta^n E_0^n}\right]} \quad (25)$$

The expression for k_s becomes even much more complicated in the case of $\beta = 1/2$. Obviously, Eq. (24) is much simpler to use than Eq. (25) and demands only selecting appropriate values of λ that give variations of $E(z)$ sufficiently closely matching with the power function, when β takes the respective values of *1* and1/2.

The foregoing two sections presented two different Winkler-type models together with the corresponding closed-form relations for k_s based on two different forms of assumed lateral-to-vertical normal stress ratio distribution with depth. It is important to note, however, that the approach enables to develop as many such models as the number of different assumptions made.

COMPARISON WITH SIMILAR PREVIOUS STUDIES

A similar study by Horvath [5] employed the same simplified-continuum idealization of the subgrade, but with only σ_z taken into account and all other stress components neglected. The cases of both constant and variable E were considered. The power function of z in Eq. (12b) with $\beta = 1$ and $\beta = 1/2$ was used to account for the variation of E. The subgrade models obtained were all Winkler-type similar to Eq. (10) and given by

For Constant E ($E = E_0$):

$$k_s = E_0/H \quad (26)$$

For Variable E ($E = E_0 + Bz^{\beta}$):

$$k_s = \begin{cases} \dfrac{B}{\ln\left(\dfrac{E_0+BH}{E_0}\right)} & \beta = 1 \\[2ex] \dfrac{B^2}{2\left[B\sqrt{H} - E_0\ln\left(\dfrac{E_0+B\sqrt{H}}{E_0}\right)\right]} & \beta = \dfrac{1}{2} \end{cases} \quad (27)$$

Equations (26) and (27) can be obtained directly from Eq. (11b) or be retrieved from Eqs. (18) and (19), respectively, of the models presented in this work, when $\alpha = 1$ - a case that corresponds to $k_x = k_y = 0$ or to a condition of zero lateral normal stresses. It is to be noted that Eqs. (18) and (19) are derived using the same power function of z of Eq. (12b).

If Eqs. (18) and (19) are now normalized with respect to the respective moduli of Eqs. (26) and (27), one obtains in both cases $1/\alpha$ - a parameter dependant on the soil Poisson ratio. Therefore, the model presented in this work based on the power function of z for E gives always a Winkler-type subgrade model with a spring stiffness that is $1/\alpha$ times the stiffness of the corresponding model of Horvath irrespective of how E varies with depth. The plot of $1/\alpha$ against ν is provided in Fig. 3 together with that of Horvath ($1/\alpha = 1$), where $k_0 = \nu/(1-\nu)$ is used for both k_x and k_y. This plot is referred to as New Variant I in the Fig. 3.

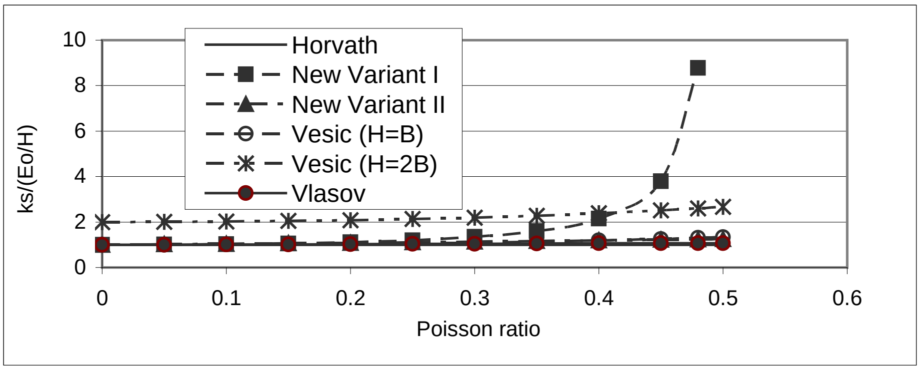

Figure 3 Plots of the normalized subgrade modulus according to different models

Figure 3 shows that the difference between the spring stiffness of the Winkler-type model New Variant I and the corresponding model of Horvath increases with increasing Poisson ratio for both the homogenous and heterogeneous cases, but becomes indeterminate at $\nu = 0.5$ - the case of an incompressible fluid. The indeterminacy is attributed to the nature of the definition of k_0 employed, which is expressed in terms of ν.

Equation (26) of Horvath can also be obtained from Eq. (23) by substituting $r = 0$ - a case that corresponds once again to zero lateral normal stresses. However, Eq. (27) cannot be retrieved from Eq. (24) by substituting $r = 0$, because the exponential function used to represent the variation of E in this case is different from the power function used by Horvath [5].

Similarly, Eq. (23) can be normalized with respect to Eq. (26) to get

$$\frac{k_s}{E_0/H} = \frac{1}{1-\frac{\nu r}{\zeta H}\left(1-e^{-\zeta H}\right)} \qquad (28a)$$

Using the values of r and ζ suggested in Eq. (21), this relation simplifies to

$$\frac{k_s}{E_0/H} = \frac{1}{1-0.4\nu} \qquad (28b)$$

The plot of this relation is also given in Fig. 3 as New Variant II, which shows that this second model presented based on a decaying exponential function for the lateral-to-vertical normal stress ratio also gives consistently larger spring stiffness that increases with increasing ν, but not as large as in the first model.

Vesić [11] proposed an analytical relation for computing the modulus of the homogenous, elastic subgrade in terms of the rigidity of the foundation element, its size, and the elastic subgrade properties. This relation, which is widely used in practice, is given by

$$k_s = 0.65\sqrt[12]{\frac{E_0 B^4}{E_f I_f}}\frac{E_0}{B\left(1-\nu^2\right)} \approx \frac{E_0}{B\left(1-\nu^2\right)} \qquad (29)$$

In this equation, $E_f I_f$ is the foundation rigidity. For relatively large-sized foundations like rafts, the value of the 12th root multiplied by 0.65 is close to 1, so that the simplified expression on the right hand side is commonly employed [18]. The plots of this expression normalized with respect to E_0/H are also given in Fig. 3 for the cases of $H = B$ and $H = 2B$, the latter case being applicable for a thick stratum or a half space. It can be seen from Fig.3 that the plot for the case of $H = B$ almost coincides with that of New Variant II, whereas the plot for the case of $H = 2B$ is double that of the case of $H = B$. Vesić's relation gives the highest estimate for the subgrade modulus for thick formations of most soils ($\nu \leq 0.42$), whereas New Variant I gives the highest values for soft cohesive soils ($\nu > 0.42$). Furthermore, Vesić's model gives at least double the magnitude of stiffness provided by the new Variant II model of this author for thick strata or a half space. This implies that Vesić's model underestimates deflections, especially for foundations on thick strata.

Vlasov and Leontiev presented [1, 16] a subgrade model based on the continuum approach, in which they introduced displacement constraints to simplify the continuum. Their model takes into account the shear interaction missing in the Winkler model. In the case of an assumed linear variation of the vertical displacement $w(x,z)$ with respect to depth that is deemed reasonable, according to the authors for relatively shallow subgrade, their model results in a subgrade modulus given by

$$k_s = \frac{E_0}{H\left(1-\nu^2+2\nu^3-\nu^4\right)} \tag{30}$$

The normalized plot of Eq. (30) is also included in Fig. 3. The plot shows k_s increasing with increasing Poisson ratio, though at a very low rate. For all practical purposes, Vlasov's subgrade modulus is the same as that of Horvath, whereas values of Vesić's subgrade modulus for $H = B$ are close to those of New Variant II. The largest estimates of k_s are obtained from Vesić's model for $H = 2B$ and from the New Variant I. These two models interchange positions at about $\nu = 0.42$

Following Schmertmann's semi-analytical approach [15,17] for estimating immediate (elastic) settlement, the thickness of the elastic subgrade stratum can be expressed as $H = I_z B$, where B is the width of the loaded area, and I_z is an influence factor dependant on the relative width of the foundation with respect to the thickness of the subgrade (For a half space $I_z = 2$ is taken). Using this approach, which was also employed by Horvath [5], the various relations for k_s presented in the preceding sections can be expressed in terms of the foundation width, B, and plotted. Such plots for two representative types of thick subgrade material with $I_k = 2$ are given in Fig. 4.

Figure 4(a) is for a medium dense coarse sand, for which $\nu = 0.25$ and $E = 40MPa$ are taken. Figure 4(b) is for a medium stiff clay with $\nu = 0.45$ and $E = 50MPa$. In the case of the medium dense sand (Fig. 4(a)), Horvath's model, the New Variant II, and Vlasov's model give practically identical values of k_s for all sizes of the plate. New Variant I gives a bit larger values of k_s in comparison, but the difference dwindles fast with increasing width of the plate. In contrast, Vesić's subgrade modulus values are consistently larger than those given by the other three models.

In the case of the medium stiff clay (Fig. 4(b)), Horvath's model, the New Variant II, and Vlasov's model give k_s values, which are close to each other for all sizes of the plates in the case of the medium dense sand. New Variant I gives the largest of k_s values for all foundation widths, with the difference from the rest increasing with decreasing plate width. Vesić's subgrade modulus fall in this case between values of New Variant I and II.

As could be observed from Eqs. (26) and (27) and Figs. 3 and 4, Horvath's model give the least values of subgrade modulus that are independent of Poisson ratio. This is a consequence of the omission of the lateral normal stresses in his highly simplified model.

Based on the above comparison, it can be concluded that Horvath's subgrade moduli form the lower bound for the subgrade modulus values that can be estimated using different assumed functions of $g(z)$ in Eq. (7), and Vlasov's model gives only slightly higher values of k_s, whereas Vesić's model gives consistently higher values of k_s only exceeded by the New Variant I model for soils with $\nu > 0.42$.

INTERPRETATION USING A CLUSTER OF CONTIGUOUS SOIL COLUMNS

Horvath's simplified subgrade [5] can be visualized as a medium made up of a cluster of contiguous short columns (no buckling) of each with a height of H and a cross sectional area of A that do not interact with each other at all and are behaving in a uniaxial state of strain. This can be easily verified as follows for both the constant and the variable E.

In the case of a constant E, it is evident that the stiffness of such a uniaxial member is given by $K = EA/H$ (Figs. 5(a) and 5(b)). On the other hand, the stiffness of the substitute spring responsible for the tributary area A in a Winkler's foundation is given by $K = k_s A$. Equating the two expressions then yields $k_s = E/H$, which is identical to Horvath's result of Eq. (26) for the case of constant E.

In the case of a variable E, the following relation is obtained using the uniaxial Hooke's law for the vertical deformation of a typical soil column at the location (x, y) (Fig. 6(b)):

$$w(x,y,z) = \int \frac{\sigma_z}{E(z)} dz + f_2(x,y) + c_2 \tag{31}$$

where, f_2 and c_2 are determined from the boundary conditions.

With $E(z) = E_0 + Bz^\beta$ substituted in Eq. (31) for the variation of E, noting that $\sigma_z = -p(x,y)$ is constant with respect to depth, and applying the boundary conditions at the two ends of the soil column, Eq. (31) gives expressions for k_s identical to Eq. (27) for $\beta = 1$ and 1/2. This shows that Horvath's subgrade can be idealized as a cluster of contiguous short soil columns that do not interact with each other in any manner.

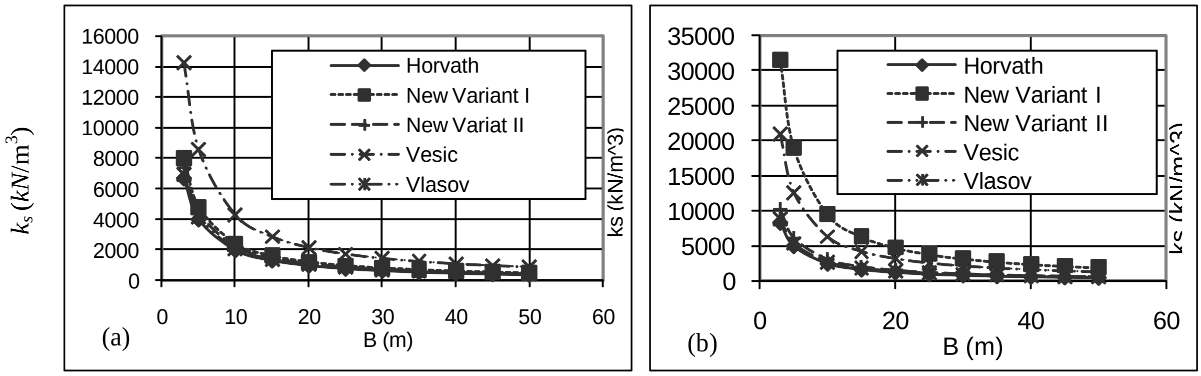

Figure 4 Plots of subgrade modulus against foundation width for: (a) a medium dense, coarse sand (ν = 0.25 and E = 40MPa); (b) a medium stiff clay (ν = 0.45 and E = 50MPa)

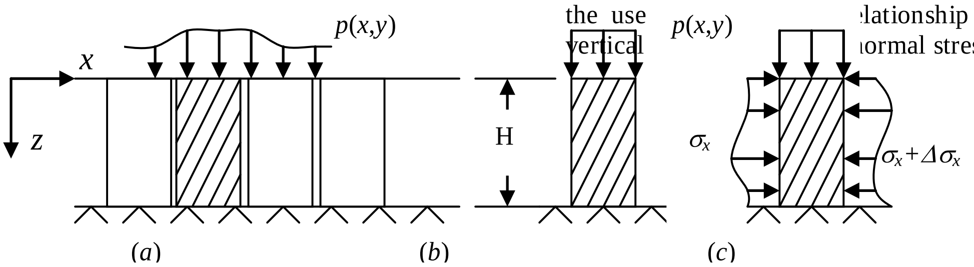

Figure 5 (*a*) The elastic subgrade idealized as a cluster of closely spaced soil columns; (*b*) a typical soil column without lateral normal stresses; (*c*) a typical soil column propped up by lateral normal stresses

In contrast to this, the subgrade represented by the models presented in this work is equivalent to the same cluster of contiguous soil columns, but with interaction through the lateral normal stresses as shown in Fig. 5(c). The mathematical proof goes analogously, the main difference being the use of the generalized three-dimensional Hooke's law because of the inclusion of the lateral normal stresses in these models. Thus,

$$w(x,y,z) = \int \frac{\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]}{E(z)} dz + f_3 + c_3 \qquad (32)$$

in which, f_3 and c_3 are dependant on the boundary conditions.

The integral in Eq. (32) is identical to Eq. (6) and its evaluation leads to the respective expressions for k_s presented in the preceding sections for the different cases considered. Due to the propping-up effect of the lateral normal stresses, the present model gives consistently stiffer springs.

One can finally draw the important conclusion that the use … lationship between the vertical … ormal stresses and any form of distribution of $E(z)$ will always result in a Winkler-type subgrade as far as the shear components of the stress tensor are not taken into account.

THE MISSING SHEAR INTERACTION

A more complete representation of the interaction among the soil columns or the Winkler springs can be achieved, only if the shear stress components in the stress tensor are taken into account in the formulation of the subgrade model. This can be achieved with relative ease, if simplifying, but reasonable, assumptions are made with regard to the depth-wise variation of the vertical shear stress components τ_{xz} and τ_{yz} in addition to the assumptions already made with respect to the lateral normal stresses. Such comprehensive considerations result in more complex mathematical models of higher order for the subgrade. Two such models have been proposed by the author in the accompanying paper that are counterparts of the two Winkler-type models presented in this paper. Both models are similar second order partial differential equations with constant coefficients given by

$$p(x,y) - c_1\nabla^2 p(x,y) = c_2 w_0(x,y) - c_3\nabla^2 w_0(x,y) \quad (33a)$$

where, ∇ is the Laplace operator. The constant coefficients c_1 to c_3 depend on the soil properties and are different for the two models:

Counterpart Model to Winkler-Type Variant I:

$$c_1 = \alpha\frac{GH^2}{12E};\quad c_2 = \frac{E}{\alpha H};\qquad c_3 = \frac{GH}{3} \qquad (33b)$$

Counterpart Model to Winkler-Type Variant II:

$$c_1 = a_1\frac{GH^2}{12E};\quad c_2 = a_2\frac{E}{H};\qquad c_3 = a_3\frac{GH}{3} \qquad (33c)$$

In Eqs. (33b) and (33c), E and G are the Young's modulus and the shear modulus of the homogenous subgrade, respectively, and H is the layer thickness. The coefficients a_1, a_2, and a_3 are dependent on the Poisson ratio of the soil, and α is the same as defined in Eq. (16).

Equation (33a) is similar in form and order to the three-parameter mechanical model proposed by Kerr [7]. This mechanical model consists of two beds of springs, one overlying the other, with corresponding subgrade moduli of k_u and k_l, separated by a shear layer of parameter g_k. Its governing differential equation is given by

$$p(x,y) - \frac{g_k}{(k_u + k_l)}\nabla^2 p(x,y) = \frac{k_l k_u}{(k_u + k_l)} w_0(x,y) - \frac{g_k k_u}{(k_u + k_l)}\nabla^2 w_0(x,y) \qquad (34)$$

Through comparison of coefficients in Eqs. (33a) and (34), one can easily express each of the three parameters of the Kerr mechanical model in terms of the known elastic soil properties and the layer thickness for both continuum-based models presented in Eq. (33). Of special interest here is the effective subgrade modulus, k_e. Since the two spring beds in Kerr model are arranged in series, the effective subgrade modulus can be expressed as

$$k_e = \frac{k_u k_l}{k_u + k_l} \qquad (35)$$

Inserting the relations for k_u and k_l in Eq. (35), one can easily obtain the following expressions for k_e for the two continuum-based models under consideration that take into account the shear interaction [9]:

Counterpart Model to Winkler-Type Variant I:

$$k_e = \frac{E}{\alpha H} \qquad (36)$$

Counterpart Model to Winkler-Type Variant II:

$$k_e = \frac{E}{(1-0.4\nu)H} \qquad (37)$$

Equations (36) and (37) for the effective subgrade modulus of the three-parameter Kerr mechanical model are identical to Eqs. (18) and (23), respectively, of the single-parameter Winkler's mechanical model. It is important to note that a complete subgrade model should take into account the shear interaction, which becomes more significant with increasing relative rigidity of the soil with respect to that of the foundation.

CONCLUSIONS

The presented work shows that a Winkler-type model will always evolve for an elastic subgrade as far as the shear components of the stress tensor are omitted regardless of whether all or part of the normal stress components are taken into consideration. A generalized analytical formulation in form of a definite integral for evaluating the coefficient of subgrade reaction is provided by accounting for all normal stresses. The variation with depth of the elasticity modulus is taken into consideration. It is only required too make a reasonable assumption on the function $g(z)$ relating the normal stresses, which is at the discretion of the user. With the introduction of two such functions, the paper provided closed-form relations for estimating the subgrade reaction for both constant and variable elasticity modulus.

A comparison with previously proposed simplified models shows some notable differences in the coefficient of subgrade reaction that generally decrease with decreasing Poisson ratio of the subgrade material and with increasing width of the foundation.

With the introduction of $H = I_z B$ in the expressions derived for k_s, it is possible to calibrate the Winkler-type models so that they give results that are in good agreement with finite-element based models by conducting a numerical study. Such a work has been completed recently and suggested the use of a calibrating factor $I_z = 2.8$ to 3 for beams on elastic foundations. Similar studies for plates are underway.

The work has also shown that the effective spring stiffness per unit area of higher order models remains the same as the subgrade modulus of the single-parameter Winkler's model as far as the way the normal stresses are considered remains the same regardless of the subgrade shear stresses.

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