

$\mathcal{W}_{1+\infty}$ and $\mathcal{W}(gl_N)$ with Central Charge N

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Abstract: We study representations of the central extension of the Lie algebra of differential operators on the circle, the $\mathcal{W}_{1+\infty}$ algebra. We obtain complete and specialized character formulas for a large class of representations, which we call primitive; these include all quasi-finite irreducible unitary representations. We show that any primitive representation with central charge N has a canonical structure of an irreducible representation of the \mathcal{W} -algebra $\mathcal{W}(gl_N)$ with the same central charge and that all irreducible representations of $\mathcal{W}(gl_N)$ with central charge N arise in this way. We also establish a duality between “integral” modules of $\mathcal{W}(gl_N)$ and finite-dimensional irreducible modules of gl_N , and conjecture their fusion rules.

Introduction

The Lie algebra $\widehat{\mathcal{D}}$, which is the unique non-trivial central extension of the Lie algebra \mathcal{D} of differential operators on the circle [KP1], has appeared recently in various models of two-dimensional quantum field theory and integrable systems, cf., e.g., [BK, FKN, PRS, IKS, CTZ, ASvM]. A systematic study of representation theory of the Lie algebra $\widehat{\mathcal{D}}$, which is often referred to as $\mathcal{W}_{1+\infty}$ algebra, was initiated in [KR]. In that paper irreducible quasi-finite highest weight representations of $\widehat{\mathcal{D}}$ were classified and it was shown that they can be realized in terms of irreducible highest weight representations of the Lie algebra of infinite matrices.

In the first part of the present paper we recall some of the results of [KR] and, as an immediate corollary, obtain complete and specialized character formulas for an arbitrary *primitive* representation of $\widehat{\mathcal{D}}$. (A primitive representation of $\widehat{\mathcal{D}}$ is an “analytic continuation” of a quasi-finite irreducible unitary representation of $\widehat{\mathcal{D}}$.) The results of [KR] were used previously in [Mat, AFOQ] to derive character formulas of primitive representations of central charge $c = 1$.

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In the second part of the paper we exhibit a connection between $\widehat{\mathcal{D}}$ and the \mathcal{W} -algebra $\mathcal{W}(gl_N)$ at the central charge N . Our main result is that any primitive representation of $\widehat{\mathcal{D}}$ with central charge N has a canonical structure of an irreducible representation of $\mathcal{W}(gl_N)$ with the same central charge and that all irreducible representations of $\mathcal{W}(gl_N)$ with central charge N arise in this way. An immediate corollary is a character formula for these representations.

The vacuum module of $\widehat{\mathcal{D}}$ of central charge c is irreducible if and only if c is non-integral [KR]. If c is a positive integer N , then this vacuum module contains a unique singular vector of degree $N + 1$, and the quotient by the submodule generated from this singular vector is an irreducible $\widehat{\mathcal{D}}$ -module [KR]. We will show that this quotient is isomorphic to the vacuum module of the \mathcal{W} -algebra $\mathcal{W}(gl_N)$ with the same central charge. All these modules carry vertex operator algebra (or chiral algebra) structures, and this isomorphism holds at the level of vertex operator algebras. It follows that the Lie algebra of Fourier components of the fields from $\mathcal{W}(gl_N)$ with central charge N is a quotient of the local completion $U_N(\widehat{\mathcal{D}})_{loc}$ of the universal enveloping algebra of $\widehat{\mathcal{D}}$ with the same central charge, by a certain ideal. A similar statement was conjectured in [FKN], where it was used in the study of the so-called \mathcal{W} -constraints in two-dimensional quantum gravity.

We recall that the \mathcal{W} -algebra $\mathcal{W}(gl_N)$ can be defined as the kernel of certain “screening” operators acting on bosonic Fock spaces, cf. [FF2]. These operators depend on a complex parameter β , and for “generic” values of β , this kernel is finitely generated as a chiral algebra. We show that $\beta = 1$ is a generic value; this value corresponds to central charge N , cf. also [F2, Bo, BS]. We then use the realization of $\widehat{\mathcal{D}}$ in terms of N free bosonic fields to construct an explicit map from the chiral algebra of $\widehat{\mathcal{D}}$ to the chiral algebra $\mathcal{W}(gl_N)$, and to show that this map is an isomorphism. It follows that under this map of chiral algebras, the first N generating fields $J^0(z), \dots, J^{N-1}(z)$ of $\widehat{\mathcal{D}}$ map to the generating fields $W^0(z), \dots, W^{N-1}(z)$ of $\mathcal{W}(gl_N)$, and the remaining fields $J^m(z), m \geq N$, map to certain normally ordered combinations of $W^0(z), \dots, W^{N-1}(z)$ and their derivatives. This happens so because by taking the quotient by a submodule generated from the singular vector (this is often referred to as decoupling of a singular vector), we effectively set to zero a field of the form

$$J^N(z) - : P(J^0(z), \dots, J^{N-1}(z)) :,$$

where the second term is a normally ordered polynomial in $J^0(z), \dots, J^{N-1}(z)$ and their derivatives. This allows one to express $J^m(z), m \geq N$, as a combination of $J^0(z), \dots, J^{N-1}(z)$ and their derivatives. Since such combinations are non-linear, the resulting commutation relations between $W^0(z), \dots, W^{N-1}(z)$ also become non-linear, which is what we expect in $\mathcal{W}(gl_N)$.

One could try to prove this correspondence between $\widehat{\mathcal{D}}$ and $\mathcal{W}(gl_N)$ by using an explicit formula for the singular vector in the vacuum module of $\widehat{\mathcal{D}}$. Although we know a simple formula for this vector in the Verma module over $\widehat{\mathcal{D}}$ [KR, Sect. 5.2], for large N it is difficult to derive from it a formula for such a vector in the vacuum module in the PBW basis. But even if we knew a precise formula for it, it would still be unclear how to show that decoupling of this vector gives $\mathcal{W}(gl_N)$. That is why we prefer a less direct, but simpler and more transparent proof, which uses free field realizations of $\widehat{\mathcal{D}}$ and $\mathcal{W}(gl_N)$.

Our result implies that any irreducible representation of $\mathcal{W}(gl_N)$ with central charge N gives rise to a quasi-finite irreducible representation of $\widehat{\mathcal{D}}$ with the same

central charge. Irreducible representations of $\mathcal{W}(gl_N)$ can be constructed as submodules of the Fock modules over the Heisenberg algebra of N scalar fields. They yield primitive representations of $\widehat{\mathcal{D}}$ and all of them can be constructed in this way.

In the third part of the paper we establish a remarkable duality between “integral” irreducible representations of $\mathcal{W}(gl_N)$ and finite-dimensional irreducible representations of $GL_N(\mathbb{C})$, cf. also [K, F1, F2, KP1]. This leads us to the conjecture that the fusion algebra of integral representations of $\mathcal{W}(gl_N)$ is isomorphic to the representation algebra of the group $GL_N(\mathbb{C})$.

Throughout the paper we systematically use vertex operator algebra methods. In particular, we give a general existence theorem (Proposition 3.1) which allows one to check easily axioms of a vertex operator algebra.

The paper is organized as follows. In Sect. 1 and the first part of Sect. 2 we set notation and recall some of the results of [KR]. In the second part of Sect. 2 we establish character formulas for primitive representations of $\widehat{\mathcal{D}}$ with central charge N . In Sects. 3 and 4 we study the vertex operator algebra structure on the vacuum module of $\widehat{\mathcal{D}}$ and on $\mathcal{W}(gl_N)$. In Sect. 5 we construct a surjective homomorphism between them and derive consequences of this fact. In Sect. 6 we establish the duality between $\mathcal{W}(gl_N)$ and $GL_N(\mathbb{C})$.

1. The Lie Algebra $\widehat{\mathcal{D}}$

Let \mathcal{D} be the Lie algebra of complex regular differential operators on \mathbb{C}^x with the usual bracket, in an indeterminate t . The elements

$$J_k^l = -t^{l+k}(\partial_t)^l, \quad (k \in \mathbb{Z}, l \in \mathbb{Z}_+) \tag{1.1}$$

form a basis of \mathcal{D} . The Lie algebra \mathcal{D} has the following 2-cocycle with values in \mathbb{C} [KP1, p.3310]:

$$\Psi(f(t)(\partial_t)^m, g(t)(\partial_t)^n) = \frac{m!n!}{(m+n+1)!} \text{Res}_{t=0} f^{(n+1)}(t)g^{(m)}(t) dt, \tag{1.2}$$

where $f^{(m)}(t) = \partial_t^m f(t)$. We denote by $\widehat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}C$, where C is the central element, the corresponding central extension of the Lie algebra \mathcal{D} .

Another important basis of \mathcal{D} is

$$L_k^l = -t^k D^l \quad (k \in \mathbb{Z}, l \in \mathbb{Z}_+), \tag{1.3}$$

where $D = t\partial_t$. These two bases are related by the formula [KR]:

$$J_k^l = -t^k [D]_l. \tag{1.4}$$

Here and further we use the usual notation $[x]_l = x(x-1)\dots(x-l+1)$. One has another formula for this cocycle [KR]

$$\Psi(t^r f(D), t^s g(D)) = \begin{cases} \sum_{-r \leq j \leq -1} f(j)g(j+r), & \text{if } r = -s \geq 0, \\ 0, & \text{if } r + s \neq 0. \end{cases} \tag{1.5}$$

The bracket in $\widehat{\mathcal{D}}$ may be conveniently calculated by the following formula:

$$[t^r f(D), t^s g(D)] = t^{r+s}(f(D+s)g(D) - f(D)g(D+r)) + \Psi(t^r f(D), t^s g(D))C. \tag{1.6}$$

The Lie algebra $\widehat{\mathcal{D}}$ contains two 1-parameter families of Virasoro subalgebras $Vir^\pm(\beta), \beta \in \mathbb{C}$, defined by

$$\begin{aligned} L_k^+(\beta) &= L_k^1 + \beta(k + 1)L_k^0, \\ L_k^-(\beta) &= L_k^1 + (k + \beta(-k + 1))L_k^0, \end{aligned} \tag{1.7}$$

so that

$$[L_m^\pm(\beta), L_n^\pm(\beta)] = (m - n)L_{m+n}^\pm(\beta) + \delta_{m,-n} \frac{m^3 - m}{12} C_\beta, \tag{1.8}$$

where $C_\beta = -(12\beta^2 - 12\beta + 2)C$.¹ Note that these two families intersect at $\beta = \frac{1}{2}$ and that $C_{\frac{1}{2}} = C$.

As in [KR], define an anti-linear anti-involution σ of $\widehat{\mathcal{D}}$ by:

$$\sigma(t^k f(D)) = t^{-k} \bar{f}(D - k), \quad \sigma(C) = C, \tag{1.9}$$

where for $f(D) = \sum_i f_i D^i, f_i \in \mathbb{C}$, we let $\bar{f}(D) = \sum_i \bar{f}_i D^i$. Then we have $\sigma L_k^+(\beta) = L_{-k}^-(\beta)$. In particular, $Vir(\frac{1}{2}) := Vir^\pm(\frac{1}{2})$ is the only σ -stable subalgebra among the $Vir^\pm(\beta)$.

Define a \mathbb{Z} -gradation $\widehat{\mathcal{D}} = \bigoplus_{j \in \mathbb{Z}} \widehat{\mathcal{D}}_j$ by letting

$$\text{wt } L_k^1 = \text{wt } J_k^1 = k, \quad \text{wt } C = 0.$$

This gives us the triangular decomposition of $\widehat{\mathcal{D}}$:

$$\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_+ \oplus \widehat{\mathcal{D}}_0 \oplus \widehat{\mathcal{D}}_-, \tag{1.10}$$

where $\widehat{\mathcal{D}}_\pm = \bigoplus_{j \in \pm \mathbb{N}} \widehat{\mathcal{D}}_j, \widehat{\mathcal{D}}_0 = \mathcal{D}_0 \oplus \mathbb{C}C$. Note that $\sigma(\widehat{\mathcal{D}}_j) = \widehat{\mathcal{D}}_{-j}, \sigma(\widehat{\mathcal{D}}_+) = \widehat{\mathcal{D}}_-, \sigma(\widehat{\mathcal{D}}_0) = \widehat{\mathcal{D}}_0$.

Fix $c \in \mathbb{C}$. Given $\lambda \in \mathcal{D}_0^*$, we define in a standard way the *Verma module* with central charge c over $\widehat{\mathcal{D}}$:

$$M_c(\widehat{\mathcal{D}}, \lambda) = U(\widehat{\mathcal{D}}) \otimes_{U(\widehat{\mathcal{D}}_0 \oplus \widehat{\mathcal{D}}_+)} \mathbb{C}_\lambda,$$

where \mathbb{C}_λ is the 1-dimensional $\widehat{\mathcal{D}}_0 \oplus \widehat{\mathcal{D}}_+$ -module, on which C acts as multiplication by $c, h \in \widehat{\mathcal{D}}_0$ acts as multiplication by $\lambda(h)$, and $\widehat{\mathcal{D}}_+$ acts by 0. Here and further we denote by $U(\mathfrak{g})$ the universal enveloping algebra of a Lie algebra \mathfrak{g} . In general, we shall say that a $\widehat{\mathcal{D}}$ -module has central charge $c \in \mathbb{C}$ if C acts on it by multiplication by c .

Denote by \mathcal{P} the subalgebra of \mathcal{D} , which consists of the operators that can be extended to regular differential operators on \mathbb{C} . We have:

$$\mathcal{P} = \text{linear span of } \{J_k^l \mid l + k \geq 0\}.$$

It follows from (1.2) that \mathcal{P} is a subalgebra of $\widehat{\mathcal{D}}$. Let $\widehat{\mathcal{P}} = \mathcal{P} \oplus \mathbb{C}C$. Note that $\widehat{\mathcal{D}}_0 \oplus \widehat{\mathcal{D}}_+ \subset \widehat{\mathcal{P}}$ and that the $\widehat{\mathcal{D}}_0 \oplus \widehat{\mathcal{D}}_+$ -module \mathbb{C}_0 can be extended to be a $\widehat{\mathcal{P}}$ -module by letting $\mathcal{P} \mapsto 0$. The induced $\widehat{\mathcal{D}}$ -module

$$M_c(\widehat{\mathcal{D}}, \widehat{\mathcal{P}}) = U(\widehat{\mathcal{D}}) \otimes_{U(\widehat{\mathcal{P}})} \mathbb{C}_0,$$

¹ The value of C_β given in [KR] should be corrected.

which is a quotient of the Verma module $M_c(\widehat{\mathcal{D}}, 0)$, is called the *vacuum $\widehat{\mathcal{D}}$ -module* with central charge c .

There exists a unique irreducible quotient of the Verma module $M_c(\widehat{\mathcal{D}}, \lambda)$, denoted by $V_c(\widehat{\mathcal{D}}, \lambda)$. The module $V_c(\widehat{\mathcal{D}}, \lambda)$ is called *quasi-finite* if all eigenspaces of D are finite-dimensional (note that D is diagonalizable). It was proved in [KR, Theorem 4.2] that $V_c(\widehat{\mathcal{D}}, \lambda)$ is a quasi-finite module if and only if the generating series

$$\Delta_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \lambda(L_0^n)$$

has the form:

$$\Delta_\lambda(x) = \frac{\phi(x)}{e^x - 1}, \tag{1.11}$$

where $\phi(x)$ is a quasi-polynomial (i.e. a linear combination of functions of the form $x^n e^{\alpha x}$, where $n \in \mathbb{Z}_+$ and $\alpha \in \mathbb{C}$) such that $\phi(0) = 0$.

Furthermore, it was shown in [KR, Theorem 5.2] that $V_c(\widehat{\mathcal{D}}, \lambda)$ is a non-trivial unitary module with respect to the anti-involution σ if and only if c is a positive integer and

$$\Delta_\lambda(x) = \sum_{i=1}^c \frac{e^{r_i x} - 1}{e^x - 1} \tag{1.12}$$

for some $r_1, \dots, r_c \in \mathbb{R}$.

Definition 1.1. *The $\widehat{\mathcal{D}}$ -module $V_c(\widehat{\mathcal{D}}, \lambda)$ with c a positive integer and $\Delta_\lambda(x)$ of the form (1.12) with $r_1, \dots, r_c \in \mathbb{C}$ is called a primitive $\widehat{\mathcal{D}}$ -module. The numbers r_1, \dots, r_c are called the exponents of this module.*

2. Characters of Primitive $\widehat{\mathcal{D}}$ -Modules

Let \widetilde{gl} be the Lie algebra of all matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with only finitely many nonzero diagonals. Letting $\text{wt } E_{ij} = j - i$ defines a \mathbb{Z} -gradation $\widetilde{gl} = \bigoplus_{j \in \mathbb{Z}} \widetilde{gl}_j$. Given $s \in \mathbb{C}$, we may consider the natural action of $\widehat{\mathcal{D}}$ on the space $t^s \mathbb{C}[t, t^{-1}]$. Choosing the basis $v_j = t^{-j+s}$ ($j \in \mathbb{Z}$) of this space gives us a homomorphism of Lie algebras $\phi_s : \widehat{\mathcal{D}} \rightarrow \widetilde{gl}$:

$$\phi_s(t^k f(D)) = \sum_{j \in \mathbb{Z}} f(-j + s) E_{j-k, j}. \tag{2.1}$$

Denote by $\widehat{gl} = \widetilde{gl} \oplus \mathbb{C}K$ the central extension given by the 2-cocycle [KP1]

$$C(A, B) = \text{tr}([J, A]B), \quad \text{where } J = \sum_{i \leq 0} E_{ii}.$$

The \mathbb{Z} -gradation of \widetilde{gl} extends to \widehat{gl} by letting $\text{wt } K = 0$. The Lie algebra \widehat{gl} has the following antilinear anti-involution:

$$A \mapsto {}^t \bar{A}, \quad K \mapsto K,$$

where ${}^t \bar{A}$ stands for the hermitian conjugate of a matrix A .

The map $\widehat{\phi}_s : \widehat{\mathcal{D}} \mapsto \widehat{gl}$ defined by

$$\begin{aligned} \widehat{\phi}_s|_{\widehat{\mathcal{D}}_j} &= \phi_s|_{\widehat{\mathcal{D}}_j} \text{ if } j \neq 0, \\ \widehat{\phi}_s(e^{xD}) &= \phi_s(e^{xD}) - \frac{e^{sx} - 1}{e^x - 1}K, \quad \widehat{\phi}_s(C) = K \end{aligned} \tag{2.2}$$

is an injective homomorphism compatible with the \mathbb{Z} -gradations and the involutions [KR]. Let \widehat{gl}^m be the direct sum of m copies of \widehat{gl} . Given $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{C}^m$ we have a homomorphism $\widehat{\phi}_s = \oplus_i \widehat{\phi}_{s_i} : \widehat{\mathcal{D}} \mapsto \widehat{gl}^m$.

Given $\lambda \in \widetilde{gl}_0^*$ and $c \in \mathbb{C}$, there exists a unique irreducible \widehat{gl} -module $V_c(\widehat{gl}, \lambda)$ with $K = cI$, which admits a non-zero vector $|\lambda\rangle$ such that

$$\begin{aligned} E_{ij}|\lambda\rangle &= 0 \quad \text{for } i < j, \\ E_{ii}|\lambda\rangle &= \lambda(E_{ii})|\lambda\rangle \quad \text{for } i \in \mathbb{Z}. \end{aligned}$$

All the modules $V_c(\widehat{gl}, \lambda)$ are quasi-finite in the sense that all the eigenspaces of $\widehat{\phi}_0(D)$ (and hence of $\widehat{\phi}_s(D), s \in \mathbb{C}$) are finite-dimensional.

Define $A_j \in \widetilde{gl}_0^* (j \in \mathbb{Z})$ as follows:

$$A_j(E_{ii}) = \begin{cases} 1 & \text{for } 0 < i \leq j, \\ -1 & \text{for } j < i \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Then a \widehat{gl} -module $V_c(\widehat{gl}, \lambda)$ is a non-trivial unitary module if and only if c is a positive integer and

$$\lambda = A_{n_1} + A_{n_2} + \dots + A_{n_c}, \quad \text{where } n_1 \geq n_2 \geq \dots \geq n_c. \tag{2.4}$$

One has the following ‘‘specialized’’ character formula for these modules [KP2]:

$$\text{tr}_{V_c(\widehat{gl}, \lambda)} q^{\widehat{\phi}_s(L_0^1)} = q^{a(\lambda)} \prod_{1 \leq i < j \leq c} (1 - q^{n_i - n_j + j - i}) / \varphi(q)^c, \tag{2.5}$$

where λ is given by (2.4), $a(\lambda) = \sum_k (n_k - s)(n_k - s + 1)/2$, and $\varphi(q) = \prod_{j=1}^\infty (1 - q^j)$ is the Euler product.

It is proved in [KR, Theorem 4.5] that an irreducible quasi-finite \widehat{gl}^m -module remains irreducible when restricted to $\widehat{\phi}_s(\widehat{\mathcal{D}})$, provided that $s_i - s_j \notin \mathbb{Z}$ for $i \neq j$. This allows one to describe the primitive $\widehat{\mathcal{D}}$ -modules.

Proposition 2.1 (KR, Theorem 4.6). *Let V be a primitive $\widehat{\mathcal{D}}$ -module with exponents r_1, \dots, r_c . Break the set $\{r_1, \dots, r_c\}$ into a disjoint union of congruent mod \mathbb{Z} classes, i.e.*

$$\{r_1, \dots, r_c\} = S_1 \cup \dots \cup S_m,$$

where $S_i = \{s_i + n_1^{(i)}, \dots, s_i + n_{c_i}^{(i)}\}, n_j^{(i)} \in \mathbb{Z}$ and $s_i - s_j \notin \mathbb{Z}$. Let $\mathbf{s} = (s_1, \dots, s_m)$ and $A^{(i)}(V) = A_{n_1^{(i)}} + \dots + A_{n_{c_i}^{(i)}}$. Then the $\widehat{\mathcal{D}}$ -module V is obtained from the \widehat{gl}^m -module

$$\bigotimes_{i=1}^m V_{c_i}(\widehat{gl}, A^{(i)}(V))$$

by restricting to $\widehat{\phi}_s(\widehat{\mathcal{D}})$. In particular the specialized character $\text{tr}_V q^{L_0^1}$ is equal to the product of the corresponding characters of the irreducible \widehat{gl} -modules (given by the right-hand side of (2.5)).

Let $H_i = E_{i,i} - E_{i+1,i+1} + \delta_{i0}K (i \in \mathbb{Z})$ be the simple coroots of \widehat{gl} . We define $\widehat{A}_0 \in \widehat{gl}_0^*$ by $\widehat{A}_0(K) = 1, \widehat{A}_0(E_{ii}) = 0$ for all i and extend A_j from \widehat{gl}_0^* to \widehat{gl}_0^* by letting $A_j(K) = 0$. Then $\widehat{A}_j = A_j + \widehat{A}_0 (j \in \mathbb{Z})$ become the fundamental weights, i.e. $\widehat{A}_j(H_i) = \delta_{ij}$.

The highest weight of a unitary module $V_c(\widehat{gl}, \lambda)$ is defined as $\widehat{\lambda} = \lambda + c\widehat{A}_0$. We have

$$\widehat{\lambda} = \sum_i k_i \widehat{A}_i, \quad \text{where } k_i \in \mathbb{Z}_+ \quad \text{and} \quad \sum_i k_i = c. \tag{2.6}$$

We shall often write $V(\widehat{gl}, \widehat{\lambda})$ in place of $V_c(\widehat{gl}, \lambda)$.

Let $p(\widehat{\lambda}) = \{p_i\}$ be the sequence associated to $\widehat{\lambda}$ which is non-increasing and contains k_j integers equal $j (j \in \mathbb{Z})$. Because of the obvious stabilization properties, the classical character formula for $gl(N, \mathbb{C})$, cf. [M, Chap. 1,(2.9')], still holds for \widehat{gl} and can be stated as follows. Define the complete character of the module $V_c(\widehat{gl}, \lambda)$ by

$$\text{ch}_{c,\lambda}(h) (= \text{ch}_{\widehat{\lambda}}(h)) = \text{tr}_{V_c(\widehat{gl}, \lambda)} e^h, \quad h \in \widehat{gl}_0.$$

Let $S_n = \text{ch}_{\widehat{A}_n}(n \in \mathbb{Z})$ be the character of the n^{th} fundamental module. Then we have:

$$\text{ch}_{c,\lambda} = \det(S_{p_i - i + j}),_{i,j=1,\dots,c}. \tag{2.7}$$

Example.

$$\text{ch}_{2\widehat{A}_n} = S_n^2 - S_{n-1}S_{n+1}.$$

We can now define the *complete character* of a $\widehat{\mathcal{D}}$ -module V by

$$\text{ch} V = \text{tr}_V \prod_{n=0}^{\infty} x_n^{L_0^n}.$$

Due to Proposition 2.1 and formula (2.7), the calculation of the characters of primitive $\widehat{\mathcal{D}}$ -modules reduces to the calculation of characters of fundamental \widehat{gl} -modules restricted to $\widehat{\phi}_s(\widehat{\mathcal{D}})$.

Let $\mathcal{F} = \bigoplus_{i \in \mathbb{Z}} V(\widehat{gl}, \widehat{A}_i)$ denote the direct sum of all the fundamental \widehat{gl} -modules. The following construction of \mathcal{F} is well known (see e.g., [K2, Chap. 14]). Fix $s \in \mathbb{C}$ and consider the Clifford algebra Cl over \mathbb{C} on generators ψ_j and $\psi_j^* (j \in \mathbb{Z})$ with defining relations

$$[\psi_i, \psi_j^*]_+ = \delta_{i,-j}, \quad [\psi_i, \psi_j]_+ = 0, \quad [\psi_i^*, \psi_j^*]_+ = 0. \tag{2.8}$$

Then \mathcal{F} is identified with the space of the unique irreducible representation of the algebra Cl which admits a non-zero vacuum vector $|0\rangle$ such that

$$\psi_j |0\rangle = 0 \text{ if } j \geq 0, \quad \psi_j^* |0\rangle = 0 \text{ if } j > 0.$$

The basis element E_{ij} of \widehat{gl} is represented by the operator $:\psi_{-i}\psi_j^*:$ ($= \psi_{-i}\psi_j^*$ if $j > 0$ and $= -\psi_j^*\psi_{-i}$ otherwise) and K by the identity operator. The decomposition

of \mathcal{F} into irreducibles with respect to \widehat{gl} coincides with the charge decomposition $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}^{(m)}$, where charge $|0\rangle = 0$, charge $\psi_j = -\text{charge } \psi_j^* = 1$. Due to (2.1), we have

$$\left[\widehat{\phi}_s(L_0), \psi_r \right] = -(r+s)\psi_r, \quad \left[\widehat{\phi}_s(L_0), \psi_r^* \right] = (-r+s)\psi_r^* .$$

Since the vectors $\psi_{-i_1} \dots \psi_{-i_a} \psi_{-j_1}^* \dots \psi_{-j_b}^* |0\rangle$ with $0 < i_1 < i_2 < \dots$, and $0 \leq j_1 < j_2 < \dots$, form a basis of \mathcal{F} , we obtain the following formula for $\text{ch}\mathcal{F}$ restricted to $\widehat{\phi}_s(\widehat{\mathcal{D}})$ (cf. [AFOQ]):

$$\text{ch}\mathcal{F} = \prod_{r \in \mathbb{Z}_+} \left(1 + \prod_{n \in \mathbb{Z}_+} x_n^{-(r+1+s)^n} \right) \left(1 + \prod_{n \in \mathbb{Z}_+} x_n^{(-r+s)^n} \right) . \tag{2.9}$$

Then the character of m^{th} fundamental \widehat{gl} -module restricted to $\widehat{\phi}_s(\widehat{\mathcal{D}})$ is equal to

$$S_m(x) = x_0^{-m} \text{Res}_{x_0=0} x_0^{m-1} \text{ch}\mathcal{F} dx_0 . \tag{2.10}$$

Summarizing, we obtain the following result:

Theorem 2.1. *Let V be a primitive $\widehat{\mathcal{D}}$ -module with exponents r_1, \dots, r_c . We keep notation of Proposition 2.1. Let $\widehat{\Lambda}^{(k)}(V) = \Lambda^{(k)}(V) + c_k \widehat{\Lambda}_0, k = 1, \dots, m$. Then the complete character of V is given by*

$$\text{ch}V = \prod_{k=1}^m \text{ch}_{\widehat{\Lambda}^{(k)}(V)}(x) ,$$

where $\text{ch}_{\widehat{\Lambda}}(x) = \det(S_{p_i - i+j}(x))_{i,j=1,\dots,c}, \{p_i\} = p(\widehat{\Lambda})$ and $S_n(x) (n \in \mathbb{Z})$ are given by formulas (2.9) and (2.10).

3. VOA Structure on the Vacuum Module of $\widehat{\mathcal{D}}$

In this section we will define the structure of a vertex operator algebra (VOA) on the vacuum module M_c and hence on the irreducible quotient module V_c over $\widehat{\mathcal{D}}$. The general definition of VOA was given in [B, FLM]. We will however use a slightly different approach, inspired by [G]. This approach will allow us to give a simple proof that this structure indeed satisfies all axioms of VOA.

Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a \mathbb{Z}_+ -graded vector space, where $\dim V_n < \infty$ for all n , called the *space of states*. A *field* on V of conformal dimension $\Delta \in \mathbb{Z}$ is a power series $\phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^{-j-\Delta}$, where $\phi_j \in \text{End } V$ and $\phi_j V_n \subset V_{n-j}$. Note that if $\phi(z)$ is a field of conformal dimension Δ , then the power series $\partial_z \phi(z) = \sum_{j \in \mathbb{Z}} (-j - \Delta) \phi_j z^{-j-\Delta-1}$ is a field of conformal dimension $\Delta + 1$. Let

$$\phi_+(z) = \sum_{j > -\Delta} \phi_j z^{-j-\Delta}, \quad \phi_-(z) = \sum_{j \leq -\Delta} \phi_j z^{-j-\Delta} . \tag{3.1}$$

Given two fields $\phi(z)$ and $\psi(z)$ of conformal dimensions Δ_ϕ and Δ_ψ one defines their *normally ordered product* as the field

$$: \phi(z)\psi(z) := \phi_-(z)\psi(z) + \psi(z)\phi_+(z) \tag{3.2}$$

of conformal dimension $\Delta_\phi + \Delta_\psi$. The Leibniz rule holds for the normally ordered product:

$$\partial_z : \phi(z)\psi(z) :=: \partial_z \phi(z)\psi(z) : + : \phi(z)\partial_z \psi(z) : . \tag{3.3}$$

Two fields $\phi(z)$ and $\psi(z)$ are called *local* with respect to each other, if for any $v \in V_n$ and $v^* \in V_m^*$ both matrix coefficients $\langle v^* | \phi(z)\psi(w) | v \rangle$ for $|z| > |w|$ and $\langle v^* | \psi(w)\phi(z) | v \rangle$ for $|z| < |w|$ converge to the same rational function in z and w which has no poles outside the hyperplanes $z = 0, w = 0$ and $z = w$.

A VOA structure on V is a linear map (the state-field correspondence) $Y(\cdot, z) : V \rightarrow \text{End } V[[z, z^{-1}]]$ which associates to each $a \in V_n$ a field of conformal dimension n (also called a *vertex operator*) $Y(a, z) = \sum_{j \in \mathbb{Z}} a_j z^{-j-n}$, such that the following axioms hold:

(A1) (vacuum axiom) There exists an element $|0\rangle \in V_0$ such that $Y(|0\rangle, z) = \text{Id}$ and $\lim_{z \rightarrow 0} Y(a, z)|0\rangle = a$.

(A2) (translation invariance) There exists an operator $T \in \text{End } V$ such that

$$\partial_z Y(a, z) = [T, Y(a, z)], \quad \text{and} \quad T|0\rangle = 0 .$$

(A3) (locality) All fields $Y(a, z)$ are local with respect to each other.

A VOA V is called *conformal of rank* $c \in \mathbb{C}$ if there exists an element $\omega \in V_2$ (called the *Virasoro element*), such that the corresponding vertex operator $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies the following properties:

(C) $L_{-1} = T, L_0|_{V_n} = n \cdot \text{Id}$, and $L_2\omega = \frac{1}{2}c|0\rangle$.

One can show that a VOA automatically satisfies the associativity property (cf. [G]):

$$Y(a, z)Y(b, w) = Y(Y(a, z - w)b, w) . \tag{3.4}$$

Here the left-hand (resp. right-hand) side is the analytic continuation from the domain $|z| > |w|$ (resp. $|w| > |z - w|$). Formula (3.4) gives the operator product expansions. In particular, one easily derives from **(C)** that the L_n form a Virasoro algebra with central charge c .

Let us call two fields $\phi(z)$ and $\psi(z)$ *ultralocal* with respect to each other if there exists an integer N , such that for any $v \in V_n$ and $v^* \in V_m^*$, both series $\langle v^* | \phi(z)\psi(w) | v \rangle (z - w)^N$ and $\langle v^* | \psi(w)\phi(z) | v \rangle (z - w)^N$ are equal to the same finite polynomial in $z^{\pm 1}$ and $w^{\pm 1}$. Clearly, ultralocality implies locality. Moreover, in a vertex operator algebra any two vertex operators are automatically ultralocal with respect to each other according to formula (3.4) and the fact that the \mathbb{Z} -gradation on V is bounded from below.

The following proposition allows one to check easily the axioms of a VOA.

Proposition 3.1. *Let V be a \mathbb{Z}_+ -graded vector space. Suppose that to some vectors $a^{(0)} = |0\rangle \in V_0, a^{(1)} \in V_{\Delta_1}, \dots$, one associates fields $Y(|0\rangle, z) = \text{Id}, Y(a^{(1)}, z) = \sum_j a_j^{(1)} z^{-j-\Delta_1}, \dots$ of conformal dimensions $0, \Delta_1, \dots$, such that the following properties hold:*

- (1)** all fields $Y(a^{(i)}, z)$ are ultralocal with respect to each other ;
- (2)** $\lim_{z \rightarrow 0} Y(a^{(i)}, z)|0\rangle = a^{(i)}$;

(3) the space V is spanned by the vectors

$$a_{-j_s - \Delta_{k_s}}^{(k_s)} \cdots a_{-j_1 - \Delta_{k_1}}^{(k_1)} |0\rangle, \quad j_1, \dots, j_s \in \mathbb{Z}_+; \tag{3.5}$$

(4) there exists an endomorphism T of V such that

$$[T, Y(a^{(i)}, z)] = \partial_z Y(a^{(i)}, z), \quad T(|0\rangle) = 0. \tag{3.6}$$

Then letting

$$\begin{aligned} & Y(a_{-j_s - \Delta_{k_s}}^{(k_s)} \cdots a_{-j_2 - \Delta_{k_2}}^{(k_2)} a_{-j_1 - \Delta_{k_1}}^{(k_1)} |0\rangle, z) \\ &= (j_1! \cdot \dots \cdot j_s!)^{-1} \cdot \partial_z^{j_s} Y(a^{(k_s)}, z) \cdots \partial_z^{j_2} Y(a^{(k_2)}, z) \partial_z^{j_1} Y(a^{(k_1)}, z); \end{aligned} \tag{3.7}$$

(where the normal ordering of more than two fields is from right to left as usual), gives a well-defined VOA structure on V .

Proof. Choose a basis of monomials (3.5) and construct the map $Y(\cdot, z)$ by formula (3.7). Then it is clear that axiom (A1) holds. Given two fields $\phi(z)$ and $\psi(z)$, if $[T, \phi(z)] = \partial_z \phi(z)$ and $[T, \psi(z)] = \partial_z \psi(z)$, then from (3.2) and (3.3) it follows that

$$[T, : \phi(z)\psi(z) :] = \partial_z : \phi(z)\psi(z) :. \tag{3.8}$$

Hence the axiom (A2) follows inductively from (3.7) and (3.8).

Using an argument of Dong (cf. [L, Proposition 3.2.7]), one can show that if three fields $\chi(z)$, $\phi(z)$ and $\psi(z)$ are ultralocal with respect to each other, then $: \phi(z)\psi(z) :$ and $\chi(z)$ are ultralocal. It is also clear that if $\phi(z)$ and $\psi(z)$ are ultralocal, then $\partial_z \phi(z)$ and $\psi(z)$ are ultralocal. This implies axiom (A3).

Finally, from the uniqueness theorem of [G] it follows that the map $Y(\cdot, z)$ is independent of the choice of the basis. \square

We shall say that the VOA constructed in Proposition 3.1 is *generated* by the fields $Y(a^{(i)}, z), i > 0$.

Now fix $c \in \mathbb{C}$, and consider the vacuum $\widehat{\mathcal{D}}$ -module $M_c = M_c(\widehat{\mathcal{D}}, \mathcal{P})$. The space M_c is \mathbb{Z}_+ -graded by eigenspaces of the operator $-D : M_c = \bigoplus_{j \in \mathbb{Z}_+} M_{c,j}$, so that $M_{c,0} = \mathbb{C}|0\rangle$, where $|0\rangle = 1 \otimes 1$. Recall that $J_k^l |0\rangle = 0$ for $l \in \mathbb{Z}_+$ and $k + l \geq 0$. Note that vectors of the form

$$J_{-k_n - l_{n-1}}^{l_n} \cdots J_{-k_1 - l_{1-1}}^{l_1} |0\rangle,$$

where $(l_i, k_i) \in \mathbb{Z}_+^2$ span M_c . It follows that the generating fields $J^l(z) = \sum_{k \in \mathbb{Z}} J_k^l z^{-k-l-1}$ satisfy conditions (2) and (3) of Proposition 3.1. Condition (4) clearly holds. Condition (1) follows from the operator product expansion [R]:

$$\begin{aligned} J^m(z)J^n(w) &\sim \sum_{a=1}^{m+n} ([n]_a J^{m+n-a}(w) - (-1)^a [m]_a J^{m+n-a}(z)) / (z-w)^{a+1} \\ &+ (-1)^m m!n!c / (z-w)^{m+n+2}. \end{aligned} \tag{3.9}$$

Hence the vacuum $\widehat{\mathcal{D}}$ -module M_c is a VOA. It follows (by skewsymmetry of vertex operators) that any quotient of the $\widehat{\mathcal{D}}$ -module M_c is a VOA. In particular, the irreducible vacuum module $V_c = V_c(\widehat{\mathcal{D}}, 0)$ is a VOA.

Let now $\omega(\beta) = (J_{-2}^1 - \beta J_{-2}^0)|0\rangle$. It is easy to see that $Y(\omega(\beta), z) = \sum_{k \in \mathbb{Z}} L_k^+(\beta) z^{-k-2}$, where $L_k^+(\beta)$ is defined in (1.7). We know from (1.8) that $L_k^+(\beta)$ generate the Virasoro algebra $Vir(\beta)$ with central element C_β . Furthermore from (1.6) and (1.7) it is easy to see that the axiom (C) of the Virasoro element holds. Thus, we have established the following result:

Theorem 3.1. *For any c and β , the quadruple $(M_c, |0\rangle, \omega(\beta), Y(\cdot, z))$ is a conformal VOA of rank $c_\beta = -(12\beta^2 - 12\beta + 2)c$ generated by the fields $J^l(z), l = 0, 1, 2, \dots$, of conformal dimension $l + 1$. The same holds for V_c .*

Remark. 3.1. Recall that a field $\phi(z)$ is called *primary* of conformal dimension Δ with respect to a Virasoro element ω , if the following operator product expansion holds:

$$Y(\omega, z)\phi(w) \sim \frac{\Delta\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w}.$$

The field $J^0(z)$ is primary with respect to $Y(\omega(\beta), z)$ if and only if $\beta = \frac{1}{2}$ (in this case the rank of M_c equals c). For $l > 0$ one can always add to the field $J^l(z), l \geq 0$, a normally ordered combination of the fields $J^k(z), 0 \leq k < l$, so that the resulting field is primary of conformal dimension $l + 1$ with respect to $\omega(\beta)$ for all but a finite number of values of β .

4. Vertex Operator Algebra of $\mathcal{W}_\beta(gl_N)$

Let \mathfrak{g} be the Lie algebra $gl_N(\mathbb{C})$ or $sl_N(\mathbb{C})$. Let \mathfrak{h} be the corresponding subalgebra of diagonal matrices. We denote by $\Delta \subset \mathfrak{h}^*$ the set of roots, Δ_+ the set of positive roots corresponding to upper triangular matrices, and let $\alpha_1, \dots, \alpha_{N-1}$ be the simple roots. We identify \mathfrak{h}^* with \mathfrak{h} using the trace form $(a, b) = \text{tr } ab$ on gl_N , so that $(\alpha, \alpha) = 2$ for $\alpha \in \Delta$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ be the affine algebra associated to $(\mathfrak{g}, (\cdot, \cdot))$ and let $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ be the homogeneous Heisenberg subalgebra of $\widehat{\mathfrak{g}}$. The Lie algebra $\widehat{\mathfrak{h}}$ has generators $u(n), u \in \mathfrak{h}, n \in \mathbb{Z}$ with commutation relations

$$[u(m), v(n)] = m\delta_{m, -n}(u, v)\mathbf{k}.$$

Given $\gamma \in \mathfrak{h}^*$, denote by $\pi_\gamma = \pi_\gamma(\widehat{\mathfrak{h}})$ the space of the irreducible representation of the Lie algebra $\widehat{\mathfrak{h}}$ which admits a non-zero vector $|\gamma\rangle$ such that

$$u(n)|\gamma\rangle = \delta_{n,0}\gamma(u)|\gamma\rangle, \quad \text{for } n \geq 0, \quad \mathbf{k}|\gamma\rangle = |\gamma\rangle.$$

By Proposition 3.1, the space π_0 has a structure of a VOA generated by the fields $u(z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1}, u \in \mathfrak{h}$. This VOA has the well-known family of conformal structures given by the Virasoro element (in the sl_N case the linear term should be dropped):

$$\omega_a = \frac{1}{2} \sum_i u_i(-1)u^i(-1)|0\rangle + aI(-2)|0\rangle, \quad (a \in \mathbb{C}),$$

where $\{u_i\}$ and $\{u^i\}$ are dual bases of \mathfrak{h} and I is the identity matrix in gl_N . The central charge of the corresponding Virasoro field $Y(\omega_a, z)$ for sl_N (resp. gl_N) is $N - 1$ (resp. $N(1 - 12a^2)$). The corresponding \mathbb{Z}_+ -gradation $\pi_0 = \bigoplus_{j \in \mathbb{Z}_+} \pi_{0,j}$ is given by $\text{wt } |0\rangle = 0, \text{ wt } u(n) = -n$.

There exists a unique operator $e^\gamma : \pi_0 \rightarrow \pi_\gamma$ which maps $|0\rangle$ to $|\gamma\rangle$ and which commutes with all operators $u(n)$ with $u \in \mathfrak{h}, n \neq 0$. Let

$$X_\gamma(z) = e^\gamma \exp\left(-\sum_{n < 0} \frac{\gamma(n)z^{-n}}{n}\right) \exp\left(-\sum_{n > 0} \frac{\gamma(n)z^{-n}}{n}\right),$$

and let $X_\gamma(z) = \sum X_\gamma(n)z^{-n}$ be its Fourier expansion where $X_\gamma(n)$ are linear operators from π_0 to π_γ .

Given $\beta \in \mathbb{C}$, let

$$\overline{\mathcal{W}}_\beta(\mathfrak{g}) = \bigcap_{i=1}^{N-1} \text{Ker}_{\pi_0} X_{\beta\alpha_i}(1).$$

This is a vertex operator subalgebra of the VOA π_0 (cf. [FF2, Lemma 4.2.8]).

The VOA $\overline{\mathcal{W}}_\beta(\mathfrak{g})$ is a \mathbb{Z}_+ -graded subspace of π_0 , i.e., $\overline{\mathcal{W}}_\beta(\mathfrak{g}) = \bigoplus_{j \in \mathbb{Z}_+} \overline{\mathcal{W}}_\beta(\mathfrak{g})_j$, where $\overline{\mathcal{W}}_\beta(\mathfrak{g})_j$ is a subspace of the (finite-dimensional) vector space $\pi_{0,j}$. It is clear that, given j , for all but finitely many $\beta \in \mathbb{C}$ the dimension of $\overline{\mathcal{W}}_\beta(\mathfrak{g})_j$ is the same (say a_j) and is minimal. Such β is called j -generic. The value $\beta \in \mathbb{C}$ which is j -generic for all $j \in \mathbb{Z}_+$ is called generic. Thus for each j we have a rational map from \mathbb{C} to the Grassmannian of a_j -dimensional subspaces in $\pi_{0,j}$ given by $\beta \mapsto \overline{\mathcal{W}}_\beta(\mathfrak{g})_j$. This allows us to define for an arbitrary $\beta_0 \in \mathbb{C}, \beta_0 \neq 0$, the analytic continuation: $\mathcal{W}_{\beta_0}(\mathfrak{g}) = \bigoplus_{j \in \mathbb{Z}_+} \mathcal{W}_{\beta_0}(\mathfrak{g})_j$, where $\mathcal{W}_{\beta_0}(\mathfrak{g})_j = \lim_{\beta \rightarrow \beta_0} \overline{\mathcal{W}}_\beta(\mathfrak{g})_j$ and the limit is taken over the set of generic β . Thus $\mathcal{W}_{\beta_0}(\mathfrak{g})$ is a family of vertex operator subalgebras of π_0 , which depends on β . This family of VOA is called the family of \mathcal{W} -algebras of \mathfrak{g} .

Remark. 4.1. On a more formal level, consider β as a formal variable, i.e., consider π_0 and $\pi_{\beta\alpha_i}$ as free modules over the ring $\mathbb{C}[\beta]$. Then the intersection of the kernels of operators $X_{\beta\alpha_i}(1) : \pi_0 \rightarrow \pi_{\beta\alpha_i}, i = 1, \dots, l$, is also a free $\mathbb{C}[\beta]$ -module. For any $\beta_0 \neq 0, \mathcal{W}_{\beta_0}(\mathfrak{g})$ is then defined by the specialization of this ring at $\beta = \beta_0$, i.e., as the quotient of this kernel by the submodule generated by $(\beta - \beta_0)$. Clearly, $\mathcal{W}_{\beta_0}(\mathfrak{g})$ is a vertex operator subalgebra of $\overline{\mathcal{W}}_\beta(\mathfrak{g})$, and it coincides with $\overline{\mathcal{W}}_\beta(\mathfrak{g})$ for generic β_0 .

Remark. 4.2. The VOA $\mathcal{W}_\beta(\mathfrak{gl}_N)$ (resp. $\overline{\mathcal{W}}_\beta(\mathfrak{gl}_N)$) is isomorphic to the tensor product of the VOA $\mathcal{W}_\beta(\mathfrak{sl}_N)$ (resp. $\overline{\mathcal{W}}_\beta(\mathfrak{sl}_N)$) and the VOA associated to a free bosonic field (of conformal dimension 1).

The VOA $\mathcal{W}_\beta(\mathfrak{sl}_N)$ coincides with the \mathcal{W} -algebra defined in [Z] for $N = 3$ and in [FL] for general N .

The following is a corollary of Theorem 4.5.9 from [FF2].

Theorem 4.1. *The VOA $\mathcal{W}_\beta(\mathfrak{gl}_N)$ is freely generated by fields $W^{(i),\beta}(z), i = 0, 1, \dots, N - 1$, of conformal dimensions $i + 1$.*

Remark. 4.3. Note that $\omega_a \in \mathcal{W}_1(\mathfrak{gl}_N)$; recall that the central charge is equal to $N(1 - 12a^2)$.

The following theorem was conjectured in [Bo] and its proof was indicated in [BS] (cf. also [BBSS]).

Theorem 4.2. $\mathcal{W}_i(\mathfrak{g}) = \overline{\mathcal{W}}_i(\mathfrak{g})$.

Proof. Due to Remark 4.2, it suffices to prove the theorem for $\mathfrak{g} = sl_N$. By definition, $\mathcal{W}_\beta(sl_N) \subset \overline{\mathcal{W}}_\beta(sl_N)$, for any β . The theorem now follows from the comparison of characters of $\mathcal{W}_i(sl_N)$ and $\overline{\mathcal{W}}_i(sl_N)$ given below.

It follows from Theorem 4.1 that for generic β , $\overline{\mathcal{W}}_\beta$ has a basis, which consists of lexicographically ordered monomials in the following Fourier components of the fields $W^{(i),\beta}(z) = \sum_{n \in \mathbb{Z}} W_n^{(i),\beta} z^{-n-i-1} : \{W_n^{(i),\beta}, n \leq -i-1, i = 1, \dots, N-1\}$. Therefore we have for a generic β :

$$\text{ch } \mathcal{W}_i(sl_N) = \text{ch } \overline{\mathcal{W}}_\beta(sl_N) = \text{ch } \overline{\mathcal{W}}_\beta(sl_N) = \prod_{i=1}^{N-1} \prod_{j=1}^{\infty} (1 - q^{i+j})^{-1}. \tag{4.1}$$

On the other hand, due to the vertex operator construction [FK] of the basic $\widehat{\mathfrak{g}}$ -module $L(A_0)$, we have:

$$\pi_0 = \{v \in L(A_0) | \mathfrak{h} \cdot v = 0\}, \tag{4.2}$$

$$E_{\alpha_i}(0)|_{\pi_0} = X_{\alpha_i}(1). \tag{4.3}$$

By the complete reducibility of the \mathfrak{g} -module $L(A_0)$, we conclude from (4.2) and (4.3) that

$$\overline{\mathcal{W}}_i(sl_N) = \{v \in L(A_0) | \mathfrak{g} \cdot v = 0\}. \tag{4.4}$$

But the character of the right-hand side of (4.4) is known [K1, Proposition 2], which gives us

$$\begin{aligned} \text{ch } \overline{\mathcal{W}}_i(sl_N) &= \prod_{\alpha \in A_+} (1 - q^{\rho(\alpha)}) / \varphi(q)^{N-1} \\ &= \prod_{1 \leq i \leq j \leq N-1} (1 - q^{j-i+1}) / \varphi(q)^{N-1} \\ &= \prod_{i=1}^{N-1} \prod_{j=1}^{\infty} (1 - q^{i+j})^{-1}, \end{aligned} \tag{4.5}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in A_+} \alpha$. Comparing (4.1) and (4.5) completes the proof. \square

Corollary 4.1. Let $\mathfrak{g} = gl_N$. Then the intersection of the kernels of the operators $X_{\alpha_i}(1)$ on π_0 coincides with $\mathcal{W}_i(gl_N)$. Furthermore one has:

$$\text{ch } \mathcal{W}_i(gl_N) = \prod_{i=0}^{N-1} \prod_{j=1}^{\infty} (1 - q^{i+j})^{-1}. \tag{4.6}$$

Remark. 4.4. As in [FF2], $\overline{\mathcal{W}}_\beta(\mathfrak{g})$ and $\mathcal{W}_\beta(\mathfrak{g})$ may be defined for an arbitrary simple Lie algebra in the same way as for $\mathfrak{g} = sl_N$. The same argument as above shows that for an arbitrary simply-laced simple Lie algebra \mathfrak{g} , $\mathcal{W}_i(\mathfrak{g}) = \overline{\mathcal{W}}_i(\mathfrak{g})$, cf. also [Bo, BS] (note that this is not true for non-simply laced \mathfrak{g}).

The Fourier components of vertex operators from the VOA π_0 span a Lie algebra $U(\widehat{\mathfrak{h}})_{loc}$, which lies in a completion of $U(\widehat{\mathfrak{h}}) / (\mathfrak{k} - 1)U(\widehat{\mathfrak{h}})$ [FF1]. The Fourier components of vertex operators from the VOA $\mathcal{W}_\beta(\mathfrak{g})$ span a Lie subalgebra of $U(\widehat{\mathfrak{h}})_{loc}$, which we denote by $U\mathcal{W}_\beta(\mathfrak{g})_{loc}$. This Lie algebra is also called \mathcal{W} -algebra of \mathfrak{g} .

Remark. 4.5. The Lie algebra $U\mathcal{W}_1(\mathfrak{g})_{loc}$ (for simply-laced \mathfrak{g}) was considered for the first time in [F2]. More precisely, in [F2] the Lie algebra \widehat{S}^W was defined, which is linearly spanned by all Fourier components of vertex operators from π_0 , which commute with the action of \mathfrak{g} and which are invariant with respect to the natural action of the Weyl group W of \mathfrak{g} on π_0 .

The Lie algebra \widehat{S}^W coincides with $U\mathcal{W}_1(\mathfrak{g})_{loc}$. Indeed, according to Theorem 4.2, the Lie algebra $U\mathcal{W}_1(\mathfrak{g})_{loc}$ consists of all Fourier components of vertex operators from π_0 , which commute with the operators $X_{\alpha_i}(1), i = 1, \dots, N - 1$. This is equivalent to commuting with \mathfrak{g} . On the other hand, it is easy to show that all elements of $U\mathcal{W}_1(\mathfrak{g})_{loc}$ are automatically W -invariant.

This is obvious in the case $\mathfrak{g} = sl_2$, because $U\mathcal{W}_1(sl_2)_{loc}$ is generated by the Fourier components of the vertex operator $\frac{1}{2} : u(z)^2 :$, which is invariant under the transformation $u(z) \rightarrow -u(z)$. This fact and Theorem 4.2 imply that all elements of $U\mathcal{W}_1(\mathfrak{g})_{loc}$ for an arbitrary simply-laced \mathfrak{g} are invariant under the simple reflections from W , and hence are W -invariant (note however that not all W -invariant elements of $U(\mathfrak{h})_{loc}$ belong to $U\mathcal{W}_1(\mathfrak{g})_{loc}$).

Moreover, it follows from Theorem 4.5.9 of [FF2] and Theorem 4.2 that if \mathfrak{g} is simply-laced, and $P_1(\mathbf{u}), \dots, P_l(\mathbf{u})$ is a set of generators of the ring $\mathbb{C}[\mathfrak{h}]^W$, then the fields $:P_1(\mathbf{u}(z)):, \dots, :P_l(\mathbf{u}(z)):$ freely generate the VOA $\mathcal{W}_1(\mathfrak{g})$.

5. Connection Between $\widehat{\mathcal{D}}$ and $\mathcal{W}_1(gl_N)$

Due to Proposition 3.1, \mathcal{F} is a (super) VOA with the generating (odd) fields

$$\psi(z) = \sum_{i \in \mathbb{Z}} \psi_i z^{-i-1}, \quad \psi^*(z) = \sum_{i \in \mathbb{Z}} \psi_i^* z^{-i}.$$

Recall that $\alpha(z) =: \psi(z)\psi^*(z) := \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}$ is a free bosonic field, i.e., $[\alpha_m, \alpha_n] = m\delta_{m,-n}$, and that each $\mathcal{F}^{(m)}$, the subspace of \mathcal{F} of charge m , is irreducible with respect to the α_n (this follows from formula (2.5) for $c = 1$). It follows that $\mathcal{F}^{(0)}$ is a vertex operator subalgebra isomorphic to the VOA $\pi_0(\mathbb{C})$.

Note that the homomorphism $\widehat{\phi}_0 : \widehat{\mathcal{D}} \rightarrow \widehat{gl}$ induces a $\widehat{\mathcal{D}}$ -module homomorphism

$$\varepsilon : M_1 = U(\widehat{\mathcal{D}}) \otimes_{U(\widehat{\mathcal{D}})} \mathbb{C}_0 \rightarrow \mathcal{F}^{(0)}.$$

Similarly, the homomorphism $\widehat{\phi}_0^N : \widehat{\mathcal{D}} \rightarrow \widehat{gl}^N$ induces the homomorphism of $\widehat{\mathcal{D}}$ -modules

$$\varepsilon_N : M_N \rightarrow (\mathcal{F}^{(0)})^{\otimes N}.$$

On the level of operators the map ε_N is given by

$$J^l(z) \mapsto \sum_{i=1}^N : \psi^i(z) \partial_z^l \psi^{*i}(z) : , \tag{5.1}$$

where $\psi^i(z)$ and $\psi^{*i}(z)$ denote the fermionic fields on the i^{th} copy of \mathcal{F} . Recall also the following formula of the well-known boson-fermion correspondence (see e.g., [K2, Chapter 14])

$$X_{\alpha_i}(z) =: \psi^i(z) \psi^{*i+1}(z) : . \tag{5.2}$$

Lemma 5.1. $\text{Im } \varepsilon_N \subset \mathcal{W}_1(gl_N)$.

Proof. Due to (5.2), $\mathcal{W}_1(gl_N)$ is the intersection of the kernels of operators $A_i = \int : \psi^i(z)\psi^{*i+1}(z) : dz$. Since $|0\rangle \in \widehat{\pi}_0$ is in the kernels of these operators, and M_N is generated by the Fourier components of $J^l(z), l \geq 0$, applied to $|0\rangle$, it suffices to check that $J^l(z)$ commutes with A_i . Due to (5.1), we have to check that

$$\left[\sum_{k=1}^N : \psi^k(z)\partial^m\psi^{*k}(z) :, \int : \psi^i(u)\psi^{*i+1}(u) : du \right] = 0$$

for any $m \geq 0$ and $i = 1, \dots, N - 1$. In fact, a much stronger statement holds:

$$\left[\sum_{k=1}^N : \psi^k(z)\psi^{*k}(w) :, \int : \psi^i(u)\psi^{*i}(u) : du \right] = 0. \tag{5.3}$$

In order to prove (5.3) we calculate the operator product expansions (OPE). We have

$$\psi^m(z)\psi^{*n}(w) \sim \frac{\delta_{m,n}}{z-w}, \quad \psi^{*m}(z)\psi^n(w) \sim \frac{\delta_{m,n}}{z-w}.$$

By the Wick theorem, we have

$$\left(\sum_{k=1}^N : \psi^k(z)\psi^{*k}(w) : \right) (: \psi^i(u)\psi^{*j}(u) :) \sim \frac{: \psi^i(z)\psi^{*j}(u) :}{w-u} + \frac{: \psi^{*j}(w)\psi^i(u) :}{z-u}.$$

But for local fields $a(z)$ and $b(z)$ with OPE $a(z)b(u) \sim \sum_j c_j(z)/(z-u)^j$ we have $[a(z), \int b(u) du] = c_1(z)$. Hence the left-hand side of (5.3) is equal to $: \psi^i(z)\psi^{*j}(w) : + : \psi^{*j}(w)\psi^i(z) := 0$. \square

By Lemma 5.1, we have a $\widehat{\mathcal{D}}$ -module homomorphism $\varepsilon_N : M_N \rightarrow \mathcal{W}_1(gl_N)$. We know that (see (4.1)):

$$\text{ch } \mathcal{W}_1(gl_N) = \prod_{i=0}^{N-1} \prod_{j=1}^{\infty} (1 - q^{i+j})^{-1} \equiv \sum_{n \geq 0} a_n q^n,$$

and clearly

$$\text{ch } M_N = \prod_{k=1}^{\infty} (1 - q^k)^{-k} \equiv \sum_{n \geq 0} b_n q^n.$$

Observe that $\text{ch } M_N$ and $\text{ch } \mathcal{W}_1(gl_N)$ coincide from weight 0 to weight N , i.e., $a_n = b_n$ for $n = 0, \dots, N$ and, moreover, $a_{N+1} = b_{N+1} + 1$. It follows that the lowest nonzero weight of the kernel of ε_N is $N + 1$. From [KR, Example 5.2] we know that there exists a unique singular vector v of weight $N + 1$ such that the submodule $\langle v \rangle$ generated by v is the maximal proper submodule in M_N , i.e., $V_N = M_N / \langle v \rangle$ is irreducible. Hence we see that the kernel of ε_N is nothing but the submodule $\langle v \rangle$. The homomorphism $\varepsilon_N : M_N \rightarrow \mathcal{W}_1(gl_N)$ therefore induces an injective $\widehat{\mathcal{D}}$ -module homomorphism $\eta_N : V_N \rightarrow \mathcal{W}_1(gl_N)$. By comparing specialized characters (see (2.5)), we have $\text{ch } V_N = \text{ch } \mathcal{W}_1(gl_N)$. Thus we have proved the following fact.

Theorem 5.1. *The map $\varepsilon_N : M_N \rightarrow \mathcal{W}_1(gl_N)$ induces a $\widehat{\mathcal{D}}$ -module isomorphism $\eta_N : V_N \xrightarrow{\sim} \mathcal{W}_1(gl_N)$, which is also an isomorphism of VOAs. One has:*

$$\varepsilon_N(\omega(\beta)) = \omega_{\beta-1/2} .$$

Remark 5.1. To any VOA V one can canonically associate a Lie algebra, which consists of all Fourier components of vertex operators from V . For the VOA M_N this Lie algebra, which we denote by $U_N(\widehat{\mathcal{D}})_{loc}$, lies in a certain topological completion of $U(\widehat{\mathcal{D}})/(C - N)U(\widehat{\mathcal{D}})$. We call $U_N(\widehat{\mathcal{D}})_{loc}$ the local completion of $U(\widehat{\mathcal{D}})$. Denote by $s_n, n \in \mathbb{Z}$, the Fourier components of the vertex operator $Y(S, z)$, where S is a singular vector of degree $N + 1$ in M_N . Now let $U\mathcal{W}_1(gl_N)_{loc}$ be the Lie algebra of all Fourier components of vertex operators from the VOA $\mathcal{W}_1(gl_N)$, which was defined in Remark 4.4. By Theorem 5.1, $U\mathcal{W}_1(gl_N)_{loc}$ is the quotient of the Lie algebra $U_N(\widehat{\mathcal{D}})_{loc}$ by the ideal generated by $s_n, n \in \mathbb{Z}$.

Corollary 5.1. *$\mathcal{W}_1(gl_N)$ is a simple VOA.*

Proof. Any ideal of $\mathcal{W}_1(gl_N)$ as VOA can be regarded as a $\widehat{\mathcal{D}}$ -module via ε_N . Since $\mathcal{W}_1(gl_N) \cong V_N$ as a $\widehat{\mathcal{D}}$ -module is irreducible, there are no nontrivial ideals of $\mathcal{W}_1(gl_N)$ as VOA. \square

Corollary 5.2. *Any representation of the VOA $\mathcal{W}_1(gl_N)$ can be canonically lifted to a representation of the Lie algebra $\widehat{\mathcal{D}}$ with central charge N .*

Let \mathfrak{h} be the Cartan subalgebra of gl_N (as in Sect. 4). For $\gamma \in \mathfrak{h}^*$ let $\gamma_i = \gamma(E_{ii})$, $i = 1, \dots, N$. Recall that each $\pi_\gamma(\gamma \in \mathfrak{h}^*)$ is a representation space of the VOA π_0 , hence of the VOA $\mathcal{W}_1(gl_N)$. Denote by $V_N(\gamma)$ the irreducible quotient of the $\mathcal{W}_1(gl_N)$ -submodule of $\widehat{\pi}_\gamma$ generated by the highest weight vector $|\gamma\rangle$.

Proposition 5.1. (a) *The modules $V_N(\gamma)$ are all up to isomorphism irreducible modules over the VOA $\mathcal{W}_1(gl_N)$.*

(b) *The lifting of a module $V_N(\gamma)$ to $\widehat{\mathcal{D}}$ is isomorphic to the primitive $\widehat{\mathcal{D}}$ -module with exponents $\gamma_1, \dots, \gamma_N$.*

Proof. (a) follows from the fact that irreducible $\mathcal{W}_1(gl_N)$ -modules are determined by the highest weights. In order to prove (b) note that $V_N(\gamma)$ is an irreducible highest weight \mathcal{D} -module with $c = N$ and one finds by a direct computation that

$$\Delta_\lambda(x) = \sum_{i=1}^N \frac{e^{\gamma_i x} - 1}{e^x - 1} . \quad \square$$

Thus, the primitive $\widehat{\mathcal{D}}$ -modules with central charge N produce all irreducible $\mathcal{W}_1(gl_N)$ -modules. In particular, from Theorem 2.1 and formula (2.5) one obtains the complete and specialized characters of all irreducible $\mathcal{W}_1(gl_N)$ -modules. The $\mathcal{W}_1(gl_N)$ -modules $V_N(\gamma)$ with integral γ were considered in [BMP], where they were used in the study of semi-infinite cohomology of \mathcal{W} -gravity models.

Remark 5.2. Fix $r \in \mathbb{C}$ and consider the associative algebra Cl_r on generators $\psi_j (j \in -r + \mathbb{Z})$ and $\psi_j^* (j \in r + \mathbb{Z})$ with defining relations (2.8), and let $\psi(z) = \sum_{j \in -r + \mathbb{Z}} \psi_j z^{-j-r-1}, \psi^*(z) = \sum_{j \in r + \mathbb{Z}} \psi_j^* z^{-j+r}$. Let \mathcal{F}_r denote the unique irreducible Cl -module such that

$$\psi_j |r\rangle = 0 \text{ if } j + 1 + r > 0, \quad \psi_j^* |r\rangle = 0 \text{ if } j - r > 0 .$$

Then formula (5.1) gives a primitive representation of $\widehat{\mathcal{D}}$ in $\mathcal{F}_r^{(0)}$, with central charge 1 and exponent r .

Remark 5.3. Summarizing, any positive energy $\widehat{\mathcal{D}}$ -module M gives rise to a module over the associated VOA M_c . If $c \notin \mathbb{Z}$, then $M_c = V_c$. If $c \in \mathbb{Z}_+$, then any primitive $\widehat{\mathcal{D}}$ -module gives rise to a module over the VOA V_c and all irreducible modules over V_c are thus obtained.

Remark 5.4. In [Zh], Zhu constructed an associative algebra $A(V)$ corresponding to an arbitrary VOA V and established a one to one correspondence between irreducible representations of V and irreducible representations of $A(V)$. In our case, one can show that the associative algebra $A(M_c)$, $c \in \mathbb{C}$, is isomorphic to the polynomial algebra in infinitely many variables w_0, w_1, \dots , which correspond to the generating fields $J^0(z), J^1(z), \dots$, and that $A(V_N)$ for $N \in \mathbb{Z}_+$ is isomorphic to the polynomial algebra $\mathbb{C}[w_0, w_1, \dots, w_{N-1}]$.

6. Towards Fusion Rules for $\widehat{\mathcal{D}}$

We can define spaces of conformal blocks for the Lie algebra $\widehat{\mathcal{D}}$ in the same fashion as for the Virasoro or affine algebras, using coinvariants, cf., e.g., [FFu].

For simplicity we will restrict ourselves to the genus 0 case. Consider the projective line $\mathbb{C}\mathbb{P}^1$ with a global coordinate t and n marked points: z_1, \dots, z_n . We assume that $z_i \neq \infty$ for all $i = 1, \dots, n$. Around the point z_i we have a local coordinate $t - z_i$. Denote by $\mathcal{D}(z_i)$ the Lie algebra of differential operators on the formal punctured disc around z_i . Elements of this Lie algebra are finite sums

$$\sum_{m \in \mathbb{Z}_+} f_m(t - z_i)(\partial_t)^m, \quad \text{where } f_m(t - z_i) \in \mathbb{C}((t - z_i)).$$

Now let $\widehat{\mathcal{D}}(z_1, \dots, z_n)$ be the central extension by \mathbb{C} of the Lie algebra $\oplus_{i=1}^n \mathcal{D}(z_i)$, such that its restriction to each of the summands $\mathcal{D}(z_i)$ coincides with the one defined by the 2-cocycle (1.2).

Denote by $\mathcal{D}_{z_1, \dots, z_n}$ the Lie algebra of regular differential operators on $\mathbb{C}\mathbb{P}^1 \setminus \{z_1, \dots, z_n\}$. We have a natural embedding of this Lie algebra into the Lie algebra $\oplus_{i=1}^n \mathcal{D}(z_i)$, obtained by expanding a differential operator around each of the points z_i . One can check that the restriction of the 2-cocycle to $\mathcal{D}_{z_1, \dots, z_n}$ is trivial, and therefore we obtain an embedding $\mathcal{D}_{z_1, \dots, z_n} \rightarrow \widehat{\mathcal{D}}(z_1, \dots, z_n)$.

Let M_1, \dots, M_n be highest weight $\widehat{\mathcal{D}}$ -modules with the same central charge. Then the tensor product $M_1 \otimes \dots \otimes M_n$ is a $\widehat{\mathcal{D}}(z_1, \dots, z_n)$ -module. We define the space of conformal blocks, corresponding to these modules, as the dual space to the space of coinvariants of $M_1 \otimes \dots \otimes M_n$ with respect to the Lie algebra $\mathcal{D}_{z_1, \dots, z_n}$. We denote this space by $H(M_1, \dots, M_n)$. In particular, the case $n = 3$ corresponds to the so-called fusion rules, which can also be defined via the intertwining operators introduced in [FHL], cf. [W].

A primitive $\widehat{\mathcal{D}}$ -module with positive integral central charge N is called *integral* if all its exponents r_1, \dots, r_N are integral; denote this module by $V(\mathbf{r})$ (recall that all these modules are unitary). Let $P = \mathbb{Z}^N$ and $P^+ = \{\mathbf{r} \in P \mid r_1 \geq \dots \geq r_N\}$. Putting the exponents in a decreasing order, we see that the integral primitive

$\widehat{\mathcal{D}}$ -modules are parametrized by $P^+ : \mathbf{r} \mapsto V(\mathbf{r})$. On the other hand, we may view P as the weight lattice of the group $GL_N(\mathbb{C})$. Then P^+ parametrizes the finite-dimensional rational irreducible representation of $GL_N(\mathbb{C}) : \mathbf{r} \mapsto F(\mathbf{r})$, where $F(\mathbf{r})$ denotes the finite-dimensional irreducible representation of $gl_N(\mathbb{C})$ with highest weight \mathbf{r} , (i.e., $\mathbf{r}(E_{ii}) = r_i$).

Conjecture 6.1. *The space $H(V(\mathbf{r}_1), \dots, V(\mathbf{r}_n))$ is isomorphic to the space of $gl_N(\mathbb{C})$ -invariants in the tensor product $F(\mathbf{r}_1) \otimes \dots \otimes F(\mathbf{r}_n)$. In particular, for $\mathbf{r}, \mathbf{s} \in P^+$, let*

$$F(\mathbf{r}) \otimes F(\mathbf{s}) = \bigoplus_{\mathbf{m} \in P^+} c_{\mathbf{r}\mathbf{s}}^{\mathbf{m}} F(\mathbf{m})$$

be the decomposition of the tensor product of $gl_N(\mathbb{C})$ -modules. Then the fusion rules of primitive integral $\widehat{\mathcal{D}}$ -modules with central charge N are given by the same formula:

$$V(\mathbf{r}) \cdot V(\mathbf{s}) = \bigoplus_{\mathbf{m} \in P^+} c_{\mathbf{r}\mathbf{s}}^{\mathbf{m}} V(\mathbf{m}).$$

Example 6.1 (a) Since $\mathcal{W}(gl_1)$ is isomorphic to the VOA $\pi_0(\mathbb{C})$, we have the fusion rules $V(r) \cdot V(s) = V(r + s)$.

(b) $\mathcal{W}(gl_2)$ with $c = 2$ is isomorphic to the tensor product of the irreducible vacuum module with $c = 1$ over the Virasoro algebra and $\pi_0(\mathbb{C})$. Denote by $[n]$ the irreducible module over the Virasoro algebra with central charge 1 and highest weight $n^2/4, n \in \mathbb{Z}_+$. Conjecture 6.1 states that the fusion rules for these modules are given by

$$[m] \cdot [n] = \sum_{k \in \mathcal{P}_{m,n}} [k],$$

where $\mathcal{P}_{m,n} = \{k \mid |m - n| \leq k \leq m + n, m + n + k \text{ is even}\}, m, n, k \in \mathbb{Z}_+$.

In order to provide some evidence for Conjecture 6.1, let

$$M = \bigoplus_{\gamma \in P} \pi_\gamma,$$

where π_γ is the irreducible $\widehat{\mathfrak{h}}$ -module defined in Sect. 4, \mathfrak{h} being the Cartan subalgebra of $\mathfrak{g} = gl_N(\mathbb{C})$. Then the classical vertex operator construction [FK] gives M a structure of a unitary $\widehat{\mathfrak{g}}$ -module of level 1. More explicitly, M decomposes into a direct sum of irreducible unitary $\widehat{\mathfrak{g}}$ -submodules of level 1 with highest weight vectors $|\gamma\rangle \in \pi_\gamma$, where $\gamma \in P$ are such that $\gamma_i - \gamma_{i+1} = \delta_{is}$ for some $1 \leq s \leq N$ (here we put $\gamma_{N+1} = \gamma_1$), the corresponding $\widehat{\mathfrak{g}}$ -submodule being $\bigoplus_{\alpha \in \gamma + Q} \pi_\alpha$, where $Q = \{\beta \in P \mid \sum_i \beta_i = 0\}$.

Viewed as a \mathfrak{g} -module, M decomposes into a direct sum of finite-dimensional irreducible modules, which can be integrated to $GL_N(\mathbb{C})$. On the other hand, each π_γ is a module over the VOA π_0 , hence over the VOA $\mathcal{W}_1(gl_N)$. Due to Theorem 5.1, we see that each π_γ has a canonical structure of a $\widehat{\mathcal{D}}$ -module with central charge N . Thus, M is a $\widehat{\mathcal{D}}$ -module (and a $\mathcal{W}_1(gl_N)$ -module) with central charge N . Moreover, it follows from the proof of Theorem 4.2 that the action of $\widehat{\mathcal{D}}$ and $GL_N(\mathbb{C})$ on M commute.

Theorem 6.1. *With respect to the commuting pair $(\widehat{\mathcal{D}}, GL_N(\mathbb{C}))$ the module M decomposes as follows:*

$$M = \bigoplus_{\mathbf{r} \in P^+} V(\mathbf{r}) \otimes F(\mathbf{r}), \tag{6.1}$$

the highest weight vector of $V(\mathbf{r}) \otimes F(\mathbf{r})$ being $|\mathbf{r}\rangle$.

Proof. Recall that by the specialized character formula (2.5) we have

$$\mathrm{tr}_{V(\mathbf{r})} q^{L_0(\frac{1}{2})} = q^{\frac{1}{2} \sum_i r_i(r_i+1)} \prod_{1 \leq i < j \leq N} (1 - q^{r_i - r_j + j - i}) / \varphi(q)^N. \tag{6.2}$$

On the other hand, denote by $\mathscr{U}(\mathbf{r})$ the direct sum of all $GL_N(\mathbb{C})$ -submodules of M isomorphic to $F(\mathbf{r})$. Since $\widehat{\mathscr{D}}$ commutes with $GL_N(\mathbb{C})$, this is a $\widehat{\mathscr{D}}$ -module. It is clear that $|\mathbf{r}\rangle \in V(\mathbf{r}) \subset \mathscr{U}(\mathbf{r})$ is the vector with minimal eigenvalue of $L_0(\frac{1}{2})$ on $\mathscr{U}(\mathbf{r})$. Comparing (6.2) with the character of $\mathscr{U}(\mathbf{r})$ computed in [K1] (see also [K2, Exercise 12.17]), we see that $\mathrm{tr}_{V(\mathbf{r})} q^{L_0(\frac{1}{2})} = \mathrm{tr}_{\mathscr{U}(\mathbf{r})} q^{L_0(\frac{1}{2})}$. It follows that $\mathscr{U}(\mathbf{r}) = V(\mathbf{r})$. \square

Due to (5.3) we have the following equivalent formulation of Theorem 6.1:

Theorem 6.2. *The representation of \widehat{gl} in $\mathscr{F}^{\otimes N}$ given by $\sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1} w^{-j} \mapsto \sum_{k=1}^N \psi^k(z) \psi^{*k}(w)$ and the representation of $gl_N(\mathbb{C})$ in $\mathscr{F}^{\otimes N}$ given by $E_{ij} \mapsto \int : \psi^i(z) \psi^{*j}(z) : dz$ ($i, j = 1, \dots, N$) commute. The decomposition of $\mathscr{F}^{\otimes N}$ with respect to the commuting pair $(\widehat{gl}, gl_N(\mathbb{C}))$ is as follows:*

$$\mathscr{F}^{\otimes N} = \bigoplus_{\mathbf{r} \in P^+} V \left(\sum_i \widehat{A}_{r_i} \right) \otimes F(\mathbf{r}). \tag{6.3}$$

(By restricting to $\widehat{\mathscr{D}}$ via the embedding ϕ_0 this decomposition coincides with the decomposition (6.1) with respect to $(\widehat{\mathscr{D}}, gl_N(\mathbb{C}))$.)

Remark 6.1. The decomposition (6.1) is easy for $N = 1$. For $N = 2$ an equivalent form of (6.1) was established in [K]. The decomposition (6.3) for general N was proved by another method in [F1, Theorem 1.6]; it also follows from [KP1, Proposition 1].

Remark 6.2. Another motivation of Conjecture 6.1 is the fact that fusion rules given by this conjecture for $c = N$ coincide with the limit of the fusion rules for the p^{th} unitary minimal model as p goes to infinity.

Note added. After this paper was finished, we saw on hep-th net the paper by H. Awata, M. Fukuma, Y. Matsuo, and S. Odake ‘‘Character and determinant formulae of quasifinite representation of the $\mathscr{W}_{1+\infty}$ algebra’’ (hep-th/9405093), where character formulas for a certain subclass of quasi-finite modules are given.

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