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# With or Without U? The Appropriate Test for a U-Shaped Relationship*

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# With or Without U?

## The appropriate test for a U shaped relationship.

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**Abstract**

Non-linear relationships are common in economic theory, and such relationships are also frequently tested empirically. We argue that the usual test of non-linear relationships is flawed, and derive the appropriate test for a U shaped relationship. Our test gives the exact necessary and sufficient conditions for the test of a U shape in both finite samples and for a large class of models.

**JEL: C12, C20**

**Keywords: U shape, hypothesis test, Kuznets curve, Fieller interval**

# 1 Introduction

Economics is full of humps and U's. It is often seen as an intriguing possibility that a relationship which is positive in some ranges, may turn negative in other ranges. Famous examples of non-monotone relationships are the "Laffer curve", the "Kuznets curve" (an inverted U between income and inequality, Kuznets (1955)), and the "Environmental Kuznets curve" (inverted U between income and pollution, Selden and Song (1994) and Grossman and Krueger (1995)). In growth theory, poverty traps are generated if the growth rate of per capita capital stock first increases and then decreases with income

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(Nelson 1956). In industrial organization it has been found that innovation is most intense at intermediate levels of competition (Aghion et al. 2005). In political science, a finding is that countries with an intermediate level of democracy are more prone to war compared to both dictatorships and democracies (Hegre et al. 2001). Finally, it has been argued that there is a hump shaped relationship between union bargaining centralization and wage growth (Calmfors and Driffill 1988), although few econometric studies of this exists mostly due to the scarcity of data.

For many of the examples above, the existence of genuinely U shaped (or hump shaped) relationships have been the subject of intense debate. The debates takes many forms and takes issue with conceptual, empirical, and theoretical problems. There is one important common empirical question, however, that to the best of our knowledge has not been adequately addressed anywhere in this diverse literature. Namely: *Given the estimates of a regression model, what is the test at level $\alpha$ of the presence of a U shape?*

In most empirical work trying to identify U shapes, the researcher include a non-linear (usually quadratic) term in an otherwise standard regression model. If this term is significant and, in addition, the estimated extremum point is within the data-range, it is common to conclude that there is a U shaped relationship. We argue in this paper that this criteria is too weak. The problem arises when the true relationship is convex but monotone. A quadratic approximation will then erroneously yield an extreme point and hence a U shape.

In order to test properly for the presence of a U shape, on some interval of values, we need to test whether the relationship is decreasing at low values within this interval and increasing at high values within the interval. As the distribution of the estimated slope at any point is readily available, a test for the sign of the slope at any given point is straightforward. A test for a U shape gets more involved, however, as the null hypothesis is that the relationship is increasing at the left hand side of the interval *and/or* is decreasing at the right hand side. For this composite null hypothesis, standard testing methodology is no longer suitable. Fortunately, a general framework for such tests has been developed by Sasabuschi (1980). We adopt his general framework to test for the presence of a U shaped relationship. The extension to an inverse U shape is of course trivial.

We have searched through all articles published in the American Economic Review since 2001 and found seven articles that uses regression techniques to identify a U or inverted-U shape in their data. Many of them conclude that they have found significant non-monotone relationships. In this paper we will present the appropriate test and then contrast it to the tests employed by the seven articles. In addition we provide a detailed example of the test in the case of a Kuznets curve.

## 2 A test of a U shape

In order to allow the regression to have a U shape, the standard approach has been to include a quadratic or an inverse term in a linear model. A more general formulation is

$$y_i = \alpha + \beta x_i + \gamma f(x_i) + \xi' z_i + \varepsilon_i \ , \ i = 1 \dots n \tag{1}$$

Here $x$ is the explanatory variable of main interest (e.g. income) while $y$ is the variable to be explained (e.g. inequality), $\varepsilon$ is an error term, and $z$ is a vector of control variables. The known function $f$ gives (1) a curvature and, depending on the parameters $\gamma$ and $\beta$, (1) may be a U or not. We assume that $f$ is chosen so that the relationship has at most one extreme point. In that case the relationship is either hump-shaped, U shaped, or monotone. In the following we focus on the case of testing for a U shape[1]

Given (1) and the assumption of only one extreme point, the requirement for a U shape is that the slope of the curve is negative at the start and positive at the end of a reasonably chosen interval of $x$-values $[x_l, x_h]$. The natural choice of interval in many contexts is the observed data range $[\min(x), \max(x)]$. If we want to make sure that the inverse U shape is not only a marginal phenomenon the interval could also be in the interior of the domain of $x$. To assure at most one extreme point on $[x_l, x_h]$, as assumed above, we require $f'$ to be monotone on this interval. A U shape is then implied by the conditions

$$\beta + \gamma f'(x_l) < 0 < \beta + \gamma f'(x_h) \tag{2}$$

If either of these inequalities are violated the curve is not U shaped but inversely U shaped or monotone.

In order to test whether the conditions in (2) are supported by the data, we need to test whether the combined null hypothesis

$$H_0 : \beta + \gamma f'(x_l) \geq 0 \text{ and/or } \beta + \gamma f'(x_h) \leq 0 \tag{3}$$

[1]The test for an inverse U is achieved by changing the sign on $y$.

can be rejected in favour of the combined alternative hypothesis

$$H_1 : \beta + \gamma f'(x_l) < 0 \text{ and } \beta + \gamma f'(x_h) > 0. \quad (4)$$

Due to the linearity of the specification (1) with respect to $\beta$ and $\gamma$ , the test of (3) vs. (4) is simply a test of linear restrictions on $\beta$ and $\gamma$. The difficulty is that the test involves a set of inequality constraints. Hence the set of $(\beta, \gamma)$ that satisfy $H_1$ is a sector in $\mathbb{R}^2$ contained between the two lines $\beta + \gamma f'(x_l) = 0$ and $\beta + \gamma f'(x_h) = 0$.

Assuming that $\varepsilon_i \sim NID(0, \sigma^2)$, Sasabuchi (1980) shows that a test of $H_0$ in (3) based on the likelihood ratio principle takes the form

Reject $H_0$ at the $\alpha$ level of confidence only if either
$H_0^L$ or $H_0^H$ or both can be rejected at the $\alpha$ level of confidence.

where $H_0^L$ or $H_0^H$ are the null hypotheses in the two standard one-sided tests

$$\begin{aligned} H_0^L &: \beta + \gamma f'(x_l) \geq 0 \text{ vs } H_1^L : \beta + \gamma f'(x_l) < 0 \\ H_0^H &: \beta + \gamma f'(x_h) \leq 0 \text{ vs } H_1^H : \beta + \gamma f'(x_h) > 0 \end{aligned}$$

The rejection area is

$$\begin{aligned} R_\alpha &= \left\{ (\beta, \gamma) : \frac{\beta + \gamma f'(x_l)}{\sqrt{F_l' \hat{\Sigma}^{-1} F_l}} < -t_\alpha \text{ and } \frac{\beta + \gamma f'(x_h)}{\sqrt{F_h' \hat{\Sigma}^{-1} F_h}} > t_\alpha \right\} \quad (5) \\ \text{where } F_i &= \begin{pmatrix} 1 \\ f'(x_i) \end{pmatrix}, \; i = l, h \end{aligned}$$

and where $\hat{\Sigma}$ is the estimated covariance matrix of the estimated $\beta$ and $\gamma$ and $t_\alpha$ the $\alpha$-level tail probability of the t distribution with the appropriate degrees of freedom. $R_\alpha$ is also a convex cone in parameter space.

The test above is also known as an intersection-union test (see e.g. Casella and Berger 2002, Ch 8.2.3); Sasabuchi's (1980) main contribution is to show that in our case, the likelihood ratio test takes the form of an intersection-union test.

The two most common specifications of (1) is the quadratic form

$$y_i = \alpha + \beta x_i + \gamma x_i^2 + \xi' z_i + \varepsilon_i \quad (6)$$

and the inverse form

$$y_i = \alpha + \beta x_i + \gamma x_i^{-1} + \xi' z_i + \varepsilon_i. \quad (7)$$

In the first case, the presence of a U implies $\beta + 2\gamma x_l < 0$ and $\beta + 2\gamma x_h > 0$. A U shape in the second implies $\beta - \gamma x_l^{-2} < 0$ and $\beta - \gamma x_h^{-2} > 0$. In both cases, the test is easily carried out as two ordinary t-tests. [2]

An equivalent test can be found by constructing a confidence interval for the minimum point and checking whether this confidence interval is contained within the interval $[x_l, x_h]$. Particularly, notice that the estimated extreme point of (1) is

$$\hat{x}^{\min} = f'^{-1}\left(-\frac{\hat{\beta}}{\hat{\gamma}}\right)$$

where $f'^{-1}$ is the inverse of the derivative of $f$. From the work of Fieller (1943), it is known how to construct an exact confidence interval for the ratio of two normally distributed estimates. In our case a $(1-2\alpha)$ confidence interval for $-\hat{\beta}/\hat{\gamma}$ is given by

$$\hat{\theta}_l, \hat{\theta}_h = \frac{s_{12}T_\alpha^2 - \hat{\beta}\hat{\gamma} \pm T_\alpha\sqrt{\left(s_{12}^2 - s_{22}s_{11}\right)T_\alpha^2 + \hat{\gamma}^2 s_{11} + \beta^2 s_{22} - 2s_{12}\hat{\beta}\hat{\gamma}}}{\left(\hat{\gamma}^2 - s_{22}T_\alpha^2\right)}, \tag{8}$$

and a $(1-2\alpha)$ confidence interval for $\hat{x}^{\min}$ is $[\widetilde{x}_l, \widetilde{x}_h] = f'^{-1}\left(\left[-\hat{\theta}_l, \hat{\theta}_h\right]\right)$. To perform the test of (3) vs. (4) at the $\alpha$ level of significance is then equivalent to see whether the $(1-2\alpha)$ confidence interval for $\hat{x}^{\min}$ is inside the data range, $[\widetilde{x}_l, \widetilde{x}_h] \subset [x_l, x_h]$. To find a confidence interval for $\hat{x}^{\min}$, one could also use the delta method, which would be only asymptotically correct. For finite samples, however, this may be severely biased. In addition, also when using the delta method the $(1-2\alpha)$ interval is the proper interval to use when testing for a U shape at $\alpha-$level.

The testing strategy is easily carried over to more involved estimations. First, it is readily extended to a more general structure[3] like $y_i = \alpha + \sum_{j=1}^{H}\beta_j f_j(x) + z_i'\gamma + \varepsilon_i$ where $f_j$ is a set of known functions. The appropriate test of the presence of a inverse U shaped

[2] A routine to perform the test in these two cases in the software package Stata is available from the authors

[3] The difficulty with this general specification is that although the relationship is decreasing at the left hand side of the relevant interval and increasing at the right hand side, it may not be an inverse U shaped relationship inside, but instead for instance a M shaped relationship. It is then necessary to perform a joint test of the whole shape of the relationship, which is much more complicated. See Doveh et al. (2002) for some developments in this direction.

relationship between $x$ and $y$ is now

$$\begin{aligned} & H_0 : \sum \beta_j f_j' (x_l) \leq 0 \text{ and/or } \sum \beta_j f_j' (x_h) \geq 0 \\ \text{vs.} & \\ & H_1 : \sum \beta_j f_j' (x_l) > 0 \text{ and } \sum \beta_j f_j' (x_h) < 0. \end{aligned} \tag{9}$$

The test is also applicable to studies of U shaped relationships in the full class of generalized linear models, encompassing most models of limited dependent variables. As the estimated parameters are asymptotically jointly normally distributed, the distribution of the test is asymptotically as explained above. In these cases it is only asymptotically valid, but as parameters estimates also only have known properties in large samples, this is not a limiting factor.

# 3 A comparison with applied work

We have read through a large number of applied econometrics works where U shapes are identified using (6). Most authors focus on the significance and sign of both $\hat{\beta}$ and $\hat{\gamma}$. How does the significance of these parameters relate to the test in (5)?

If the data range is the whole of $\mathbb{R}$ the significance of $\hat{\gamma}$ (with right sign) is necessary and sufficient for rejecting $H_0$. If the data range is any subset of $\mathbb{R}$, the significance of $\hat{\gamma}$ is still necessary but not sufficient.

If the data range is exactly $\mathbb{R}^+$ or $\mathbb{R}^-$ the significance (with right sign) individually of both $\hat{\beta}$ and $\hat{\gamma}$ is necessary and sufficient for rejecting $H_0$. If the data range is a subset of $\mathbb{R}^+$ (or of $\mathbb{R}^-$) the individual significance of both $\hat{\beta}$ and $\hat{\gamma}$ is necessary but not sufficient.

If the data range is neither $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{R}^-$, the necessary and sufficient conditions are only found using (5). Obviously, a simple first check is whether the estimated minimum point $\left(\hat{x}^{\min} = -\hat{\beta}/\left(2\hat{\gamma}\right)\right)$ itself is within the date range.

Most works use the criteria that if both $\hat{\beta}$ and $\hat{\gamma}$ are significant and if the implied extreme point is within the data range, they have found a U. This is a sensible criteria but it is neither sufficient nor necessary. It is insufficient as the estimated extreme point may be too close, given the uncertainty, to a end point of the data range. It is not generally necessary as $\hat{\beta}$ may be zero if the data range extends to both sides of $x = 0$.

A small number of works use the delta method to calculate the standard deviation of the extreme point. This method, though sound, is only reliable with a very large number of observations.

We have in particular looked at articles in the American Economic Review. There are seven articles since 2001 that uses regression techniques to identify a U shape. All use a formulation similar to (6).

Abdul and Mody (2006) uses several specifications, the closest they come to assessing the significance of the U shape is when they estimate (6) and find that both $\hat{\gamma}$ and $\hat{\beta}$ have the right sign and are individually significant. Based on this they conclude that there is an inverted U shape. They do not, however, calculate the estimated extreme point.

Aghion, Griffith, and Howitt (2006) uses their estimates to calculate predicted values $\hat{y}$. They find that the estimated minimum is inside the $x-$range, but they do not explicitly asses the significance of the result.

Carr, Markusen, and Maskus (2001) do not test their U shape against the alternatives of monotone relationships. Their specification does only include $\gamma x^2$ and not $\beta x$ while their $x$-range covers both negative and positive values. Hence, they rule out monotonically increasing or decreasing relationships between $x$ and $y$.

Imbs and Wacziarg (2003) first use nonparametric techniques to find the relationship between per capita income $x$ and sectoral concentration $y$ and find a U shape. They use this technique also to construct 95% intervals for the minimum point. They then turn to a parametric specification with $x$ and $x^2$ in the right hand side. Both coefficients have the right sign and are individually significant. The significance of the U shape is assessed by plotting $x$ and $\hat{y}$ versus the scatter plot of the data. The regression curve fits the data very well and based on the plot alone one can safely conclude that the U shape is significant at all conventional levels. But, the precise significance level is unclear and so is the precision of the estimated minimum point.

McKinnish (2004) and Kalemli-Ozcan, Sørensen, and Yosha (2003) find that both $\hat{\beta}$ and $\hat{\gamma}$ have the right sign and are individually significant. Based on this both conclude that there is an inverted U shape.

Sigman (2002) includes $x$ and $x^2$ and find that both $\hat{\beta}$ and $\hat{\gamma}$ have the right sign in all specifications. She also calculates the maximum point and find it to be within the data range. She also writes (p.1157): "The GDP coefficients always appear to follow an inverted U shaped pattern, but are jointly statistically significant at 5 percent only in columns (3) and (4)". Testing for joint significance of $\hat{\beta}$ and $\hat{\gamma}$ is, however, not particularly

Table 1: Estimates of the Kuznets curve

| Dependent variable: Gini index | | |
|---|---|---|
| log per capita GDP ($X_i$) | $\hat{\beta} =$ | 32.27 |
| | | $(11.63)^{***}$ |
| log per capita GDP squared ($X_i^2$) | $\hat{\gamma} =$ | $-1.88$ |
| | | $(0.65)^{***}$ |
| Slope at $X_l$ | $\hat{\beta} + 2\hat{\gamma}X_l =$ | 8.15 |
| | | $(3.65)^{**}$ |
| Slope at $X_h$ | $\hat{\beta} + 2\hat{\gamma}X_h =$ | $-4.33$ |
| | | $(2.34)^{*}$ |
| Sasabuchi test for inverse U shape | | 1.85 |
| | | [0.033] |
| Extremum point | $-\hat{\beta}/(2\hat{\gamma}) =$ | 8.60 |
| 90% confidence interval, Fieller method | | $[7.44, 9.58]$ |
| 90% confidence interval, Delta method | | $[7.73, 9.47]$ |

*Robust standard errors in parenthesis and p-values in square brackets.*

***, ** and * denotes significant at the 1%, 5%, and 10% level.*

relevant when looking for a U shape. As we have argued above, the individual significance of $\hat{\gamma}$ is always a necessary condition in the test of a U shape.

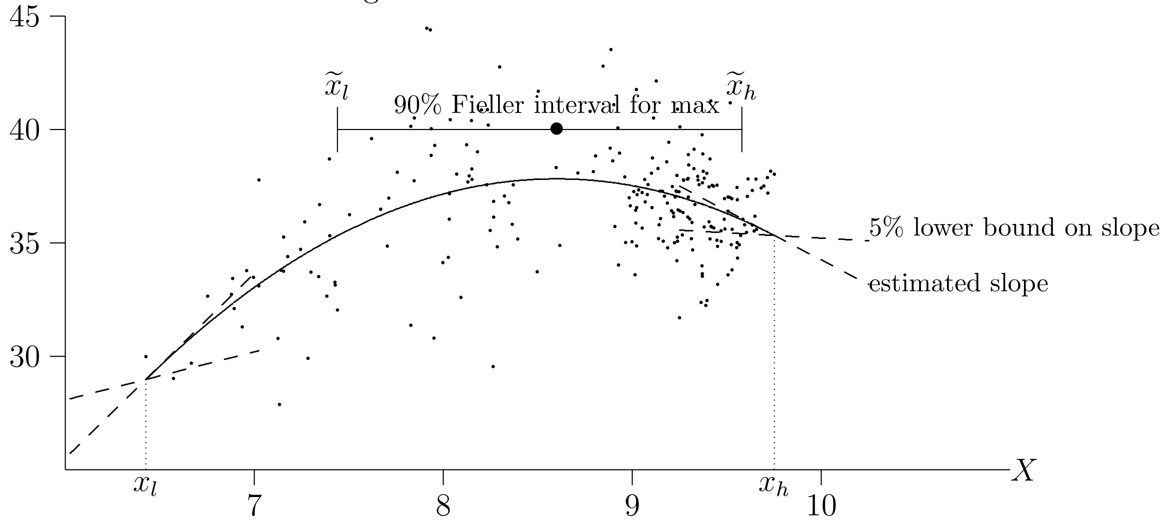
# 4 Illustration

As a concrete illustration of our methodology, we use a recent study by Chambers (2007). The paper contains a standard Kuznets regression between GDP and inequality. The data are an unbalanced panel of 29 countries giving a total of 232 observations. We have information on the level of inequality measured by the Gini coefficient, log of PPP adjusted GDP per capita, and a set of control variables (see Chambers (2007) for details).

Results from the regression analysis, test statistics from the Sasabuschi-test, and the derived Fieller interval are reported in Table 1. The conventional method would simply test the significance of $\hat{\beta}$ and $\hat{\gamma}$, which in this case yields a t-values of 2.77 and 2.89. With the appropriate Sasabuchi-test, however, we see that the test for a positive slope at $x_h$ yields a t-value of only 1.85 and hence a p-value of 0.032 with the required one-sided test.

Figure 1 illustrates the estimated relationship[4], the confidence interval for the maximum and the two lower bounds on the slopes in each endpoint. We see that the turning point of the relationship is quite close to $x_h$, and that the slope of the curve at $x_h$ is negative, but not very steep and only just significant at the 5% level. Hence there is a significant hump shaped relationship over the range of the data, but the significance of this relationship is weaker than what would be detected by traditional approaches.

Figure 1: Illustrative Kuznets curve




# 5 Conclusion

In this paper we have provided an appropriate test of a U shaped relationship in a regression model. In the applied econometrics literature a large number of articles tries to identify non-monotone relationships using regression analysis. Hardly any of these use adequate formal procedures when they test for the presence of a U shape. To the best of our knowledge none has used the simple test that we are suggesting. Most works, nevertheless, seems to be on fairly safe ground when they claim to have found a U shape. The reason is that the common practice is to check two necessary conditions, namely that the second derivative has the right sign and that the extremum point is within the data

[4]We have used average values for all the controls.

range. This criteria will be misleading, however, if the estimated extremum point is too close to the end point of the data range. Our test gives the exact necessary and sufficient conditions for the test of a U shape. In addition, the interval interpretation provides a confidence interval for the extremum point.

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