Witten index, axial anomaly, and Krein's spectral shift function in supersymmetric quantum mechanics

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A new method is presented to study supersymmetric quantum mechanics. Using relative scattering techniques, basic relations are derived between Krein's spectral shift function, the Witten index, and the anomaly. The topological invariance of the spectral shift function is discussed. The power of this method is illustrated by treating various models and calculating explicitly the spectral shift function, the Witten index, and the anomaly. In particular, a complete treatment of the two-dimensional magnetic field problem is given, without assuming that the magnetic flux is quantized.

I. INTRODUCTION

Since the first observation of fractionally charged states in certain field theoretic soliton models,[1] various techniques to obtain a more detailed understanding of that phenomenon have been developed.[2] Furthermore, the possible phenomenological realization of these states in one-dimensional polymers such as polyacetylene strongly stimulated this development.[3–6]

Among the different existing approaches[2] the treatment of external field problems offers the simplest possibility to study fractional charge quantum numbers. In this context, one starts from a Dirac operator with some external potential with nontrivial asymptotics. For example, in one dimension this can be realized in the easiest way by considering the following operator, acting on two-component wave functions:

$$Q_m = \begin{pmatrix} m & A^* \\ A & -m \end{pmatrix}, \quad A = \frac{d}{dx} + \phi, \tag{1.1}$$

where $\phi(x)$ and $m(x)$ are space-dependent "mass" terms. Nontrivial (solitonlike) asymptotics is then expressed by $\lim_{x\to\pm\infty} \phi(x) = \phi_\pm$, in comparison with the trivial case $\lim_{x\to\pm\infty} \phi(x) = \phi_0$. Since, in a field theoretic context, the transition from one case to the other corresponds to the passage from one representation of the canonical anticommutation relations to an inequivalent one, the relative charge is usually defined through a regularization procedure. It turns out that under suitable conditions on the Dirac Hamiltonian, the charge is given by half of the associated η_m invariant.[2,7–12]

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The method described above (for $m = 0$) is closely connected with supersymmetry, a subject of current interest in different fields of physics.[13,14] Indeed, the Hamiltonian defined as

$$H = Q^2 = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix} \tag{1.2}$$

represents two Schrödinger operators, A^*A and AA^*, which are non-negative and which have the same spectrum, except perhaps for zero modes. The investigation of such supersymmetric quantum mechanical models is important. They serve as a laboratory to test and to understand supersymmetry breakdown in realistic field theories.[2,14–16] Furthermore, they provide a simple recipe for generating partner potentials, which can be used successfully in many physical problems. See Ref. 13 and references therein.

To study supersymmetric systems, Witten[16] introduced a quantity Δ, counting the difference in the number of bosonic and fermionic zero-energy modes of the Hamiltonian. This quantity, called the Witten index, has to be regularized if the threshold of the continuous spectrum of A^*A (AA^*) extends down to zero (see, e.g., Refs. 2 and 16–19). Here we will use the resolvent regularization, viz., Ref. 17,

$$\begin{aligned} \Delta &= \lim_{z\to 0} \Delta(z), \\ \Delta(z) &= -z \operatorname{Tr}[(A^*A - z)^{-1} - (AA^* - z)^{-1}]. \end{aligned} \tag{1.3}$$

When A is Fredholm (i.e., if and only if the infimum of the essential spectrum of A^*A is strictly positive), this index Δ equals the Fredholm index $i(A) \equiv [\dim \operatorname{Ker}(A) - \dim \operatorname{Ker}(A^*)]$. When A is not Fredholm, this equality is, in general, destroyed and Δ can become noninteger; in fact, it can be any arbitrary real number,[20] due to threshold effects.

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Fractionization of Δ has been seen explicitly in a number of examples.[2,8,20–25]

In this paper, we develop a new method to study supersymmetric quantum mechanics without assuming the Fredholm property for the operator A. This method, based on relative scattering techniques (Levinson theorem-type arguments, etc.), has the advantage of being simple and mathematically rigorous at the same time. In particular, we derive a relationship between Krein's spectral shift function[26–28] and the Witten index Δ. Furthermore, we show how the topological invariance of the (resolvent) regularized Witten index leads to the corresponding invariance of the spectral shift function itself. These new results offer a useful tool for explicit model calculations. To illustrate this, we discuss several examples in detail. A short account of this work has appeared in Ref. 20.

The rest of this paper is organized as follows: In Sec. II, we recall the basic properties of Krein's spectral shift function, $\xi(\lambda)$, λ the energy, and its connection with (modified) Fredholm determinants.[29–31] In Sec. III, we consider supersymmetric quantum mechanical systems. We prove that under certain conditions on the Hamiltonian, the Witten index Δ is given as (minus) the jump of the spectral shift function $\xi(\lambda)$ at $\lambda = 0$ and that the axial anomaly $\mathscr{A}$ (Refs. 17 and 32) is equal to the limit of $\xi(\lambda)$ as $\lambda \to \infty$. Furthermore, we use the topological invariance of the resolvent regularized Witten index under "sufficiently small" perturbations to derive the corresponding invariance of Krein's spectral shift function itself. Finally, we discuss the spectral asymmetry η_m associated with Q_m in terms of $\xi(\lambda)$. Section IV illustrates the power of our method in explicit calculations by treating a number of models. Using the connection between Fredholm determinants and Wronskians[33] or exploiting the topological invariance discussed in Sec. III, we calculate in a straightforward way Krein's spectral shift function, the Witten index, and the anomaly for various examples on the line and on the half-line. Furthermore, we analyze the supersymmetric system describing a particle in a two-dimensional magnetic field without assuming the magnetic flux to be quantized. In this case, our method is the first rigorous and nonperturbative one that shows that the spectral shift function is piecewise constant, and thus that both the anomaly $\mathscr{A}$ and minus the index Δ are equal to the flux. Also, the spectral asymmetry for the corresponding two-dimensional Q_m model is calculated.

We end this introduction with the remark that Secs. III and IV are completely self-contained, so that they may be read independently of Sec. II, which offers a full account of the more technical results needed in the paper.

II. FREDHOLM DETERMINANTS AND KREIN'S SPECTRAL SHIFT FUNCTION

In this section, we present a full account of those basic, more technical results on Krein's spectral shift function and its connection with Fredholm determinants that we need in the rest of the paper. We start by introducing the following hypotheses. For any result, only some of the hypotheses will be assumed.

Hypothesis (i): Let $\mathscr{H}$ be some (complex, separable) Hilbert space, let H_j, $j = 1,2$, be two self-adjoint operators in $\mathscr{H}$ such that $(H_1 - z_0)^{-1} - (H_2 - z_0)^{-1} \in \mathscr{B}_1(\mathscr{H})$ for some $z_0 \in \rho(H_1) \cap \rho(H_2)$.

[Here $\mathscr{B}_p(\mathscr{H})$, $p \in [1,\infty)$ denote the usual trace ideals[31] and $\rho(\cdot)$ denotes the resolvent set.]

Hypothesis (ii): In addition to Hypothesis (i), assume that H_j, $j = 1,2$, are bounded from below. Suppose that $H_1 = H_2 \dot{+} V_{12}$ (here $\dot{+}$ denotes the form sum), where V_{12} can be split into two parts, $V_{12} = v_{12} u_{12}$ such that $u_{12}(H_2 - z)^{-1} v_{12}$ is analytic with respect to $z \in \rho(H_2)$ in the $\mathscr{B}_1(\mathscr{H})$ topology and such that $u_{12}(H_2 - z_0)^{-1}$, $(H_2 - z_0)^{-1} v_{12} \in \mathscr{B}_2(\mathscr{H})$ for some $z_0 \in \rho(H_2)$.

Clearly, Hypothesis (ii) resembles the Rollnik trick of splitting a self-adjoint multiplication operator $V(x)$ into $V(x) = |V(x)|^{1/2} |V(x)|^{1/2} \operatorname{sgn} V(x)$.[34] Next, we introduce a "high-energy" assumption of the following type.

Hypothesis (iii): Assume Hypothesis (ii) and

$$\lim_{\substack{|z| \to \infty \\ \operatorname{Im} z \neq 0}} \det[1 + u_{12}(H_2 - z)^{-1} v_{12}] = 1 .$$

Finally, we introduce two assumptions which will allow generalizations in the sense that the Fredholm determinant used later on can be replaced by a modified one. This generalization is critical in higher-dimensional systems where Hypothesis (iii) is known to fail (cf., e.g., Refs. 35 and 36).

Hypothesis (iv): Suppose Hypothesis (ii) is satisfied except that $u_{12}(H_2 - z)^{-1} v_{12}$ is now assumed to be analytic with respect to $z \in \rho(H_2)$ in the $\mathscr{B}_2(\mathscr{H})$ norm.

Hypothesis (v): Assume Hypothesis (iv) and

$$\lim_{\substack{z \to \infty \\ \operatorname{Im} z \neq 0}} \det{}_2[1 + u_{12}(H_2 - z)^{-1} v_{12}] = 1 .$$

We first recall the following.

Lemma 2.1: Assume Hypothesis (i). Then there exists a real-valued measurable function ξ_{12} on $\mathbb{R}$ (Krein's spectral shift function[26–28]) unique a.e. up to a constant with

$$\text{(a)}\quad (1 + |\cdot|^2)^{-1} \xi_{12} \in L^1(\mathbb{R}) ; \tag{2.1}$$

$$\text{(b)}\quad \operatorname{Tr}[(H_1 - z)^{-1} - (H_2 - z)^{-1}] = -\int_{\mathbb{R}} d\lambda\, \xi_{12}(\lambda)(\lambda - z)^{-2}, \quad z \in \rho(H_1) \cap \rho(H_2) ; \tag{2.2}$$

(c) if $S_{12}(\lambda)$ denotes the on-shell scattering operator for the pair (H_1, H_2), then

$$\det S_{12}(\lambda) = e^{-2\pi i \xi_{12}(\lambda)} \quad \text{for a.e. } \lambda \in \sigma_{\mathrm{ac}}(H_j) \tag{2.3}$$

[$\sigma_{\mathrm{ac}}(\cdot)$ denotes the absolutely continuous spectrum].

For a proof, see, e.g., Refs. 37 and 38. For an appropriate class of $C^1(\mathbb{R})$ functions Φ with $\Phi(H_1) - \Phi(H_2) \in \mathscr{B}_1(\mathscr{H})$, one gets similarly

$$\operatorname{Tr}[\Phi(H_1) - \Phi(H_2)] = \int_{\mathbb{R}} d\lambda\, \xi_{12}(\lambda) \Phi'(\lambda) \tag{2.4}$$

(cf. Refs. 37–39). Finally, the invariance principle for wave operators can be used to relate ξ_{12} associated with (H_1, H_2) and ξ_{12}^{Φ} corresponding to $(\Phi(H_1), \Phi(H_2))$ by[37]

$$\xi_{12}(\lambda) = \xi_{12}^{\Phi}(\Phi(\lambda)) \operatorname{sgn}(\Phi'(\lambda)) . \tag{2.5}$$

If $H_j, j=1,2$, are bounded from below, we define $\xi_{12}(\lambda)=0$ to the left of the spectra of H_1 and H_2 in order to guarantee uniqueness for ξ_{12}. For connections between Levinson's theorem and ξ_{12}, see, e.g., Refs. 28, 40, and 41.

Example 2.2: Let H_1 denote the Friedrichs extension of $(-d^2/dx^2+\alpha/x^2)|_{C_0^\infty(\mathbb{R}\setminus\{0\})}$ in $L^2(\mathbb{R})$, $\alpha \geqslant -\frac{1}{4}$ and

$$H_2 = -\frac{d^2}{dx^2}\Big|_{H^{2,2}(\mathbb{R})} . \tag{2.6}$$

Then the on-shell scattering operator $S_{12}(\lambda)$ in $\mathbb{C}^2$ reads[42]

$$S_{12}(\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{-i\pi[(\alpha+1/4)^{1/2}+1/2]}, \quad \lambda>0 . \tag{2.7}$$

Thus

$$\xi_{12}(\lambda) = \begin{cases} 0, & \lambda<0, \\ (\alpha+\frac{1}{4})^{1/2}, & \lambda>0, \end{cases} \tag{2.8}$$

and, e.g.,

$$\operatorname{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}] = (\alpha+\tfrac{1}{4})^{1/2} z^{-1}, \quad z\in\mathbb{C}\setminus[0,\infty) . \tag{2.9}$$

By a Laplace transform, Eq. (2.9) is equivalent to a result of Ref. 43. If H_1 equals the Neumann instead of Friedrich's extension of $(-d^2/dx^2+\alpha/x^2)|_{C_0^\infty(\mathbb{R}\setminus\{0\})}$, $\alpha > -\frac{1}{4}$, one obtains[42]

$$\xi_{12}(\lambda) = \begin{cases} 0, & \lambda<0, \\ -(\alpha+\frac{1}{4})^{1/2}, & \lambda>0 . \end{cases}$$

Next, we recall[28,29] the following.

Lemma 2.3: (a) Let $U,G\subset\mathbb{C}$ be open, $A\in\mathscr{B}_p(\mathscr{H})$ for some $p\in[1,\infty)$, and $\sigma(A)\subset U\subset G$, where ∂U is compact and consists of a finite number of closed rectifiable Jordan curves (cf., e.g., Ref. 44) oriented in the positive sense. [Here $\sigma(\cdot)$ denotes the spectrum and ∂U denotes the boundary of the set U.] Let f: $G\to\mathbb{C}$ be analytic with $f(0)=0$. Then $f(A)\in\mathscr{B}_p(\mathscr{H})$.

(b) Let A: $[a,b]\to\mathscr{B}_1(\mathscr{H})$ be continuously differentiable in the $\mathscr{B}_1(\mathscr{H})$ norm. Let $\cup_{t\in[a,b]}\sigma(A(t))\subset G$, where $G\subset\mathbb{C}$ is open. Let f: $G\to\mathbb{C}$ be analytic with $f(0)=0$. Then

$$\frac{d}{dt}\operatorname{Tr}[f(A(t))] = \operatorname{Tr}\left[f'(A(t))\frac{dA(t)}{dt}\right], \quad t\in(a,b) . \tag{2.10}$$

(c) Let $G\subset\mathbb{C}$ be open, and A: $G\to\mathscr{B}_1(\mathscr{H})$ be analytic in the $\mathscr{B}_1(\mathscr{H})$ norm. Then $\det[1+A(z)]$ is analytic with respect to $z\in G$ and

$$\frac{d}{dz}\ln\det[1+A(z)] = \operatorname{Tr}\left\{[1+A(z)]^{-1}\frac{dA(z)}{dz}\right\}, \quad -1\notin\sigma(A(z)), \quad z\in G . \tag{2.11}$$

Lemma 2.3 immediately implies the following.

Lemma 2.4: Assume Hypothesis (ii). Then

$$\operatorname{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}] = -\frac{d}{dz}\ln\det[1+u_{12}(H_2-z)^{-1}v_{12}], \quad z\in\rho(H_1)\cap\rho(H_2) . \tag{2.12}$$

Proof: By Lemma 2.3, cyclicity of the trace, and the resolvent equation one gets

$$\begin{aligned}\frac{d}{dz}\ln\det[1+u_{12}(H_2-z)^{-1}v_{12}] &= \operatorname{Tr}\{[1+u_{12}(H_2-z)^{-1}v_{12}]^{-1}u_{12}(H_2-z)^{-2}v_{12}\} \\ &= \operatorname{Tr}\{(H_2-z)^{-1}v_{12}[1+u_{12}(H_2-z)^{-1}v_{12}]^{-1} \times u_{12}(H_2-z)^{-1}\} \\ &= -\operatorname{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}], \\ & z\in\rho(H_1)\cap\rho(H_2) .\end{aligned}$$ □

In order to connect Krein's spectral shift function with Fredholm determinants, we formulate the following.

Lemma 2.5: Assume Hypothesis (iii) and assume that $(1+|\cdot|)^{-1}\xi_{12}\in L^1(\mathbb{R})$. Then

$$\int_{\mathbb{R}} d\lambda\, \xi_{12}(\lambda)(\lambda-z)^{-1} = \ln\det[1+u_{12}(H_2-z)^{-1}v_{12}], \quad z\in\rho(H_1)\cap\rho(H_2) . \tag{2.13}$$

If, in addition, ξ_{12} is bounded and piecewise continuous on $\mathbb{R}$, then

$$[\xi_{12}(\lambda_+)+\xi_{12}(\lambda_-)]/2 = \frac{1}{2\pi i}\lim_{\epsilon\to 0_+}\ln\frac{\det[1+u_{12}(H_2-\lambda-i\epsilon)^{-1}v_{12}]}{\det[1+u_{12}(H_2-\lambda+i\epsilon)^{-1}v_{12}]}, \quad \lambda\in\mathbb{R} . \tag{2.14}$$

Proof: By Lemma 2.4, we have

$$\begin{aligned}-\frac{d}{dz}\ln\det[1+u_{12}(H_2-z)^{-1}v_{12}] &= \operatorname{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}] \\ &= -\frac{d}{dz}\int_{\mathbb{R}} d\lambda\,\xi_{12}(\lambda)(\lambda-z)^{-1}, \quad z\in\rho(H_1)\cap\rho(H_2) .\end{aligned}$$

Thus Eq. (2.13) holds up to a constant. By Hypothesis (iii), this constant equals zero. Equation (2.14) results from standard properties of the Poisson kernel (cf., e.g., Ref. 45). □

Without the piecewise continuity of ξ_{12}, Eq. (2.14) holds a.e. in $\lambda\in\mathbb{R}$. Hypothesis (iii) is, in general, valid for one-dimensional systems (cf. Sec. IV) but breaks down in higher dimensions. Thus we formulate the following.

Lemma 2.6: Let $G\subset\mathbb{C}$ be open, and A: $G\to\mathscr{B}_2(\mathscr{H})$ be analytic in $\mathscr{B}_2(\mathscr{H})$ topology. Then the modified Fredholm determinant $\det_2[1+A(z)]$ is analytic with respect to $z\in G$ and

$$\begin{aligned}\frac{d}{dz}\ln\det{}_2[1+A(z)] &= \operatorname{Tr}\left\{([1+A(z)]^{-1}-1)\frac{dA(z)}{dz}\right\} \\ &= -\operatorname{Tr}\left\{[1+A(z)]^{-1}A(z)\frac{dA(z)}{dz}\right\}, \\ & -1\notin\sigma(A(z)), \quad z\in G .\end{aligned} \tag{2.15}$$

Proof: Obviously Eq. (2.15) holds for $A(z)\in\mathscr{B}_1(\mathscr{H})$, $z\in G$. The general case follows by a limiting argument. □

Lemma 2.7: Assume Hypothesis (iv). Then

$$\begin{aligned}&\mathrm{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}\\&\quad+(H_2-z)^{-1}V_{12}(H_2-z)^{-1}]\\&\quad=-\frac{d}{dz}\ln\det{}_2[1+u_{12}(H_2-z)^{-1}v_{12}]\,,\\&\quad z\in\rho(H_1)\cap\rho(H_2)\,.\end{aligned}\tag{2.16}$$

Proof: By Lemma 2.6 one gets

$$\begin{aligned}&\frac{d}{dz}\ln\det{}_2[1+u_{12}(H_2-z)^{-1}v_{12}]\\&\quad=\mathrm{Tr}\{([1+u_{12}(H_2-z)^{-1}v_{12}]^{-1}-1)\\&\qquad\times u_{12}(H_2-z)^{-2}v_{12}\}\\&\quad=-\mathrm{Tr}\{(H_1-z)^{-1}-(H_2-z)^{-1}\\&\qquad+(H_2-z)^{-1}V_{12}(H_2-z)^{-1}\}\,,\\&\qquad z\in\rho(H_1)\cap\rho(H_2)\,.\end{aligned}$$

□

For related work, see also Ref. 46.

Next, we assume the existence of some η_{12}: $[\lambda_0,\infty)\to\mathbb{R}$ such that

$$\begin{aligned}&\mathrm{Tr}[(H_2-z)^{-1}V_{12}(H_2-z)^{-1}]\\&\quad=\int_{\lambda_0}^{\infty}d\lambda\,\eta_{12}(\lambda)(\lambda-z)^{-2}\,,\quad z\in\rho(H_2)\,,\end{aligned}\tag{2.17}$$

and we define

$$\tilde\xi_{12}(\lambda)=\begin{cases}\xi_{12}(\lambda)-\eta_{12}(\lambda)\,, & \lambda>\lambda_0\,,\\ \xi_{12}(\lambda)\,, & \lambda<\lambda_0\,.\end{cases}\tag{2.18}$$

Lemma 2.8: Assume Hypothesis (v) and assume that $(1+|\cdot|)^{-1}\tilde\xi_{12}\in L^1(\lambda_0,\infty)$. Then

$$\begin{aligned}&\int_{\mathbb{R}}d\lambda\,\tilde\xi_{12}(\lambda)(\lambda-z)^{-1}\\&\quad=\ln\det{}_2[1+u_{12}(H_2-z)^{-1}v_{12}]\,,\\&\quad z\in\rho(H_1)\cap\rho(H_2)\,.\end{aligned}\tag{2.19}$$

If, in adddition, $\tilde\xi_{12}$ is piecewise continuous and bounded on $\mathbb{R}$, then

$$\begin{aligned}&[\tilde\xi_{12}(\lambda_+)+\tilde\xi_{12}(\lambda_-)]/2\\&\quad=\frac{1}{2\pi i}\lim_{\epsilon\to0_+}\ln\frac{\det_2[1+u_{12}(H_2-\lambda-i\epsilon)^{-1}v_{12}]}{\det_2[1+u_{12}(H_2-\lambda+i\epsilon)^{-1}v_{12}]}\,.\end{aligned}\tag{2.20}$$

Proof: Similar to that of Lemma 2.5. □

Example 2.9: Let $|V_{12}|^{1+s}\in L^1(\mathbb{R}^2)$, $(1+|\cdot|^s)V_{12}\in L^1(\mathbb{R}^2)$ for some $s>0$, respectively, $V_{12}\in L^1(\mathbb{R}^3)\cap R$ (R the Rollnik class,[34] i.e.,

$$\int_{\mathbb{R}^6}d^3x\,d^3y|V(x)||V(y)||x-y|^{-2}<\infty)$$

and define in $L^2(\mathbb{R}^n)$: $H_1=-\Delta\dot+V_{12}$ and $H_2=-\Delta|_{H^{2,2}(\mathbb{R}^n)}$, $n=2,3$. Then

$$\begin{aligned}&\mathrm{Tr}[(H_2-z)^{-1}V_{12}(H_2-z)^{-1}]\\&\quad=-\frac{1}{4\pi}\int_{\mathbb{R}^n}d^nx\,V_{12}(x)\\&\qquad\times\begin{cases}z^{-1}\,, & n=2\,,\\ (2\sqrt{-z})^{-1}\,, & n=3\,,\end{cases}\quad z\in\mathbb{C}\setminus[0,\infty)\end{aligned}\tag{2.21}$$

and hence $\lambda_0=0$ and (cf., e.g., Refs. 35 and 36)

$$\eta_{12}(\lambda)=\begin{cases}0\,,\ \lambda<0\,,\\ -\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^n}d^nx\,V_{12}(x)\begin{cases}1\,, & n=2,\\ \sqrt{\lambda}/\pi\,, & n=3\,,\end{cases}\quad\lambda>0\,.\end{cases}\tag{2.22}$$

Finally, assume Hypothesis (i) and define, for some $M\in\mathbb{R}$,

$$\begin{aligned}&\Delta_M(z)=-(z-M)\,\mathrm{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}]\,,\\&\qquad z\in\rho(H_1)\cap\rho(H_2)\,.\end{aligned}\tag{2.23}$$

Furthermore, define

$$\Delta_M=\lim_{\substack{z\to M\\ |\mathrm{Re}\,z-M|\leqslant C_0|\mathrm{Im}\,z|}}\Delta_M(z)\tag{2.24}$$

and, if in addition H_j, $j=1,2$, are bounded from below,

$$\mathscr{A}=-\lim_{\substack{z\to\infty\\ |\mathrm{Re}\,z|\leqslant C_1|\mathrm{Im}\,z|}}\Delta_M(z)\tag{2.25}$$

(C_0,C_1 positive constants). Then one has the following.

Lemma 2.10: Assume Hypothesis (i).

(a) Let $M\in\mathbb{R}$ and suppose that ξ_{12} is bounded on $\mathbb{R}$ and piecewise continuous in $(M-2\delta,M+2\delta)$ for some $\delta>0$. Then

$$\Delta_M=\xi_{12}(M_-)-\xi_{12}(M_+)\,.\tag{2.26}$$

(b) If H_j, $j=1,2$, are bounded from below and if ξ_{12} is bounded and $\lim_{\lambda\to\infty}\xi_{12}(\lambda)=\xi_{12}(\infty)$ exists, then

$$\mathscr{A}=\xi_{12}(\infty)\,.\tag{2.27}$$

Proof: Choose $\epsilon>0$ sufficiently small,

$$\begin{aligned}\Delta_M(z)&=(z-M)\int_{M-\epsilon}^{M+\epsilon}d\lambda\,\xi_{12}(\lambda)(\lambda-z)^{-2}+O(z-M)\\&=\xi_{12}(M_-)-\xi_{12}(M_+)+(z-M)\int_{M}^{M+\epsilon}d\lambda\,[\xi_{12}(\lambda)-\xi_{12}(M_+)](\lambda-z)^{-2}\\&\quad+(z-M)\int_{M-\epsilon}^{M}d\lambda\,[\xi_{12}(\lambda)-\xi_{12}(M_-)](\lambda-z)^{-2}+O(z-M)\,.\end{aligned}$$

Now

$$\int_M^{M+\epsilon} d\lambda\,[\xi_{12}(\lambda)-\xi_{12}(M_+)](z-M)(\lambda-z)^{-2}$$
$$=\int_M^{M+\epsilon} d\lambda\,[\xi_{12}(\lambda)-\xi_{12}(M_+)]\{(\operatorname{Re} z-M)[(\lambda-\operatorname{Re} z)^2-(\operatorname{Im} z)^2]-2(\lambda-\operatorname{Re} z)(\operatorname{Im} z)^2\}$$
$$\times[(\lambda-\operatorname{Re} z)^2+(\operatorname{Im} z)^2]^{-2}+i\int_M^{M+\epsilon} d\lambda\,[\xi_{12}(\lambda)-\xi_{12}(M_+)]\{(\operatorname{Im} z)[(\lambda-\operatorname{Re} z)^2-(\operatorname{Im} z)^2]$$
$$+2(\lambda-\operatorname{Re} z)(\operatorname{Re} z-M)\operatorname{Im} z\}[(\lambda-\operatorname{Re} z)^2+(\operatorname{Im} z)^2]^{-2}. \tag{2.28}$$

For example, the real part in Eq. (2.28) yields

$$\int_{-\infty}^{\infty} d\mu\left\{\left[\mu^2\frac{|\operatorname{Re} z-M|}{|\operatorname{Im} z|}\right]-\left[\frac{|\operatorname{Re} z-M|}{|\operatorname{Im} z|}\right]-2\mu\right\}(\mu^2+1)^{-2}$$
$$\times[\xi_{12}((\mu|\operatorname{Im} z|+|\operatorname{Re} z-M|)\operatorname{sgn}(\operatorname{Re} z-M)+M)-\xi_{12}(M_+)]\chi_{I(M,z)}(\mu)\to 0$$

as $z\to M$ and $|\operatorname{Re} z-M|\leqslant C_0|\operatorname{Im} z|$,

$$I(M,z)=[-|\operatorname{Re} z-M|/|\operatorname{Im} z|,\quad [\epsilon\operatorname{sgn}(\operatorname{Re} z-M)-|\operatorname{Re} z-M|]/|\operatorname{Im} z|]$$

by dominated convergence. (Here χ_I denotes the characteristic function of the interval $I\subset\mathbb{R}$.) The same analysis applies for the imaginary part in Eq. (2.28), proving Eq. (2.26). Similarly one proves Eq. (2.27). □

III. SUPERSYMMETRY AND KREIN'S SPECTRAL SHIFT FUNCTION

In this section we consider general supersymmetric quantum mechanical systems and we establish a basic relationship betwen Krein's spectral shift function $\xi_{12}(\lambda)$ and the Witten index, and between $\xi_{12}(\lambda)$ and the axial anomaly. Furthermore, we discuss the topological invariance of the (regularized) Witten index and the spectral shift function. Finally, the spectral asymmetry for Q_m-type models [cf. Eq. (1.1)] is related to $\xi_{12}(\lambda)$.

Let A be a closed, densely defined operator in $\mathscr{H}$ and define the "bosonic," respectively, "fermionic" Hamiltonian H_1 and H_2, by

$$H_1=A^*A,\quad H_2=AA^*. \tag{3.1}$$

The corresponding supercharge Q and the supersymmetric Hamiltonian H in $\mathscr{H}\oplus\mathscr{H}$ are, respectively,

$$Q=\begin{pmatrix}0 & A^*\\ A & 0\end{pmatrix},\quad H=Q^2=\begin{pmatrix}H_1 & 0\\ 0 & H_2\end{pmatrix}. \tag{3.2}$$

Assuming Hypothesis (i) throughout this section, Witten's (resolvent) regularized index $\Delta(z)$ is defined by[17]

$$\Delta(z)=-z\operatorname{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}],$$
$$z\in\mathbb{C}\setminus[0,\infty), \tag{3.3}$$

and Witten's index Δ (Ref. 16) is given by (cf. Sec. II)

$$\Delta=\lim_{\substack{z\to 0\\ |\operatorname{Re} z|<C_0|\operatorname{Im} z|}}\Delta(z) \tag{3.4}$$

(for some $C_0>0$) whenever the limit exists. Instead of the regularization (3.3), one could as well consider a (heat kernel) regularization $\tilde{\Delta}(s)$ of the type

$$\tilde{\Delta}(s)=\operatorname{Tr}[e^{-sH_1}-e^{-sH_2}],\quad s\geqslant 0, \tag{3.5}$$

and define Witten's index by

$$\Delta=\lim_{s\to\infty}\tilde{\Delta}(s). \tag{3.6}$$

In order to avoid technicalities, we restrict ourselves to Callias's regularization (3.3).

As a first result, we try to relate Δ and the Fredholm index $i(A)$ of A: We recall an operator is Fredholm[47] iff A is a closed operator with a closed range such that dim Ker(A) and dim Ker(A^*) are finite. The Fredholm index $i(A)$ is then given by

$$i(A)=\dim\operatorname{Ker}(A)-\dim\operatorname{Ker}(A^*). \tag{3.7}$$

We remark that A is Fredholm iff A^* (or A^*A) is.[47] In addition

$$\dim\operatorname{Ker}(A)=\dim\operatorname{Ker}(A^*A) \tag{3.8}$$

implying that

$$i(A)=\dim\operatorname{Ker}(H_1)-\dim\operatorname{Ker}(H_2). \tag{3.9}$$

Thus $i(A)$ describes precisely the difference of bosonic and fermionic zero-energy states (counting multiplicities).

We emphasize that we shall also use definition (3.7) for $i(A)$ in case A is not Fredholm. Of course, in this case $i(A)$ might lose some of the typical properties of an index.

We state the following.

Theorem 3.1: Assume Hypothesis (i) and suppose A is Fredholm. Then

$$\Delta=i(A). \tag{3.10}$$

Proof: We only sketch the major step. The fact that H_j, $j=1,2$, are Fredholm guarantees an expansion of the type

$$-z[(H_1-z)^{-1}-(H_2-z)^{-1}]$$
$$=P_1-P_2-z\sum_{n=0}^{\infty}z^n[T_1^{n+1}-T_2^{n+1}] \tag{3.11}$$

valid in the $\mathscr{B}_1(\mathscr{H})$ norm. Here P_j denotes the projection onto the eigenvalue zero of H_j, $j=1,2$, and T_j is the reduced resolvent, viz., Ref. 47,

$$T_j=n-\lim_{z\to 0}(H_j-z)^{-1}[1-P_j],\quad j=1,2. \tag{3.12}$$

Taking the trace in Eq. (3.11) and observing that

$$\mathrm{Tr}[P_1-P_2]=i(A) \tag{3.13}$$

completes the proof. □

What happens if A is not a Fredholm operator? Before trying to answer this question, let us consider an equivalent definition of the Fredholm property of A. Since $A^*A\geqslant 0$ and A is Fredholm iff A^*A is, we get the criterion that A is Fredholm iff $\inf\sigma_{\rm ess}(A^*A)>0$ [$\sigma_{\rm ess}(\cdot)$ denotes the essential spectrum]. The examples of the next section show that, in general, equality (3.10) is violated if A is not Fredholm. In fact, Δ may take on half-interger values in the first four examples of Sec. IV, whereas in the fifth example it can even take on arbitrary real values (see also Ref. 20).

To study also these non-Fredholm cases we now introduce Krein's spectral shift function ξ_{12} associated with (H_1,H_2) as discussed in Sec. II. We always assume Hypothesis (vi). Assume that ξ_{12} (or $\tilde{\xi}_{12}$) is bounded and piecewise continuous on $\mathbb{R}$ and $\xi_{12}(\lambda)=0$ for $\lambda<0$.

As can be seen from Lemma 2.5 (Lemma 2.8), this essentially requires continuity of the trace-norm (Hilbert–Schmidt norm) limits $u_{12}(H_2-\lambda\mp i0)^{-1}v_{12}$ with respect to $\lambda\in\mathbb{R}$. This can be checked explicitly in concrete examples (cf., e.g., Sec. IV).

Let us denote the threshold of H_j by

$$\sum=\inf\sigma_{\rm ess}(H_1)=(\inf\sigma_{\rm ess}(H_2)). \tag{3.14}$$

We observe that H_1 and H_2 are essentially isospectral[49] (cf. also Ref. 50), i.e.,

$$\sigma(H_1)\setminus\{0\}=\sigma(H_2)\setminus\{0\}$$

and

$$H_1f=Ef,\quad E\neq 0$$

implies $H_2(Af)=E(Af)$, $f\in\mathscr{D}(H_1)$,

$$H_2g=E'g,\quad E'\neq 0$$

implies $H_1(A^*g)=E'(A^*g)$, $g\in\mathscr{D}(H_2)$, (3.15)

with multiplicities preserved. Under the additional assumption that

$$\sum=\inf\sigma_{ac}(H_1)\quad[=\inf\sigma_{ac}(H_2)] \tag{3.16}$$

and that, e.g., $u_{12}(H_2-\lambda-i\epsilon)^{-1}v_{12}$, $\lambda\geqslant\Sigma$, has $\mathscr{B}_2(\mathscr{H})$-valued limits as $\epsilon\to 0_+$ and that the exceptional set

$$\delta=\{\lambda\geqslant\Sigma\,|\,\exists f\in\mathscr{H},\ f\neq 0 \text{ with } u_{12}(H_2-\lambda-i0)^{-1}v_{12}f=-f\} \tag{3.17}$$

is discrete (cf., e.g., Refs. 31 and 51), we get

$$\xi_{12}(\lambda)=\begin{cases}0, & \lambda<0,\\ \xi_{12}(0_+), & 0<\lambda<\Sigma,\\ -(2\pi i)^{-1}\ln\det S_{12}(\lambda), & \lambda>\Sigma.\end{cases} \tag{3.18}$$

The simple structure in Eq. (3.18) follows from the fact that the effects of all nonzero bound states of H_1 and H_2 cancel since they occur with the same multiplicity in both H_1 and H_2.[49] Under suitable conditions on V_{12},[37,51] the on-shell S matrix $S_{12}(\lambda)$ is continuous in trace norm in $\lambda>\Sigma$ [with $\det S_{12}(\lambda)\neq 0$], implying continuity of ξ_{12} for $\lambda>\Sigma$. [If $\Sigma=0$, then the second line of the rhs of Eq. (3.18) should be omitted.]

If we define the axial anomaly $\mathscr{A}$ by (cf. Refs. 17 and 32)

$$\mathscr{A}=-\lim_{\substack{z\to\infty\\ |\mathrm{Re}\,z|<C_1|\mathrm{Im}\,z|}}\Delta(z) \tag{3.19}$$

(for some $C_1>0$) we obtain from Lemma 2.10 the following.

Theorem 3.2: Assume Hypotheses (i) and (vi). Then

$$\Delta=-\xi_{12}(0_+). \tag{3.20}$$

If, in addition, $\lim_{\lambda\to\infty}\xi_{12}(\lambda)\equiv\xi_{12}(\infty)$ exists, then

$$\mathscr{A}=\xi_{12}(\infty). \tag{3.21}$$

If $\Sigma>0$, then $-\xi_{12}(0_+)$ describes precisely the difference of zero-energy bound states of H_1 and H_2 (counting multiplicity) since $\xi_{12}(\lambda)=0$ for $\lambda<0$. Thus $-\xi_{12}(0_+)=i(A)$ in agreement with Theorem 3.1. If $\Sigma=0$, then $\xi_{12}(0_+)$ might be fractional due to threshold resonances or bound states of H_1 or H_2 or due to relative long-range interactions as shown in Sec. IV.

We also recall that by Lemma 2.5, ξ_{12} can be recovered from the Fredholm determinants by

$$[\xi_{12}(\lambda_+)+\xi_{12}(\lambda_-)]/2=\frac{1}{2\pi i}\lim_{\epsilon\to 0_+}\ln\frac{\det[1+u_{12}(H_2-\lambda-i\epsilon)^{-1}v_{12}]}{\det[1+u_{12}(H_2-\lambda+i\epsilon)^{-1}v_{12}]} \tag{3.22}$$

assuming Hypotheses (iii) and (vi) and $(1+|\cdot|)^{-1}\xi_{12}\in L^1(\mathbb{R})$. Under the same assumptions, $\Delta(z)$ is given by [cf. Eq. (2.13)]

$$\begin{aligned}\Delta(z)&=-z\,\mathrm{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}]\\ &=z\frac{d}{dz}\int_{\mathbb{R}}d\lambda\,\xi_{12}(\lambda)(\lambda-z)^{-1}\\ &=z\frac{d}{dz}\ln\det[1+u_{12}(H_2-z)^{-1}v_{12}],\quad z\in\mathbb{C}\setminus[0,\infty).\end{aligned} \tag{3.23}$$

We omit the corresponding generalizations based on Hypothesis (v) in terms of modified Fredholm determinants. If an expansion of the type

$$\det[1+u_{12}(H_2-z)^{-1}v_{12}]=z^\alpha[1+O(z)]\quad\text{as } z\to 0 \tag{3.24}$$

holds, then obviously

$$\Delta=\alpha. \tag{3.25}$$

In the same way, a high-energy expansion determines the anomaly $\mathscr{A}$.

Next, we turn to an important invariance property of $\Delta(z)$ under sufficiently small perturbations of A. Let B be another closed operator in $\mathscr{H}$ infinitesimally bounded with respect to A, and introduce on $\mathscr{D}(A)$,

$$A_\beta=A+\beta B,\quad \beta\in\mathbb{R}. \tag{3.26}$$

The quantities $H_{1,\beta}$, $H_{2,\beta}$, $u_{12,\beta}$, $v_{12,\beta}$, $\xi_{12,\beta}$, and $\Delta(\beta,z)$ then result after replacing A by A_β. We have[48] the following.

Theorem 3.3: Fix $z_0\in\mathbb{C}\setminus[0,\infty)$ and assume that

(i) $(H_{1,\beta}-z_0)^{-1}-(H_{2,\beta}-z_0)^{-1}\in\mathscr{B}_1(\mathscr{H})$ for all $\beta\in\mathbb{R}$;

(ii) $B^*B(H_1-z_0)^{-1}$, $BB^*(H_2-z_0)^{-1}\in\mathscr{B}_\infty(\mathscr{H})$,

$$[A^*B+B^*A](H_1-z_0)^{-1},$$
$$[AB^*+BA^*](H_2-z_0)^{-1}\in\mathscr{B}_\infty(\mathscr{H});$$

(iii) $(H_1-z_0)^{-1}B^*B(H_1-z_0)^{-1}$, $(H_2-z_0)^{-1}BB^*(H_2-z_0)^{-1}\in\mathscr{B}_1(\mathscr{H})$, $(H_1-z_0)^{-1}[A^*B+B^*A](H_1-z_0)^{-1}$, $(H_2-z_0)^{-1}[AB^*+BA^*](H_2-z_0)^{-1}\in\mathscr{B}_1(\mathscr{H})$;

(iv) $(H_1-z_0)^{-M}B^*(H_2-z_0)^{-M}\in\mathscr{B}_1(\mathscr{H})$ for some $M\in\mathbb{N}$.

[Here $\mathscr{B}_1(\mathscr{H})$ and $\mathscr{B}_\infty(\mathscr{H})$ denote trace class and compact operators in $\mathscr{H}$, respectively.] Then

$$\Delta(\beta,z)=\Delta(z),\quad z\in\mathbb{C}\setminus[0,\infty),\quad \beta\in\mathbb{R}, \tag{3.27}$$

i.e., the regularized Witten index is invariant against small perturbations B of the above type.

Since a more general result (where A acts between different Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$) has been proven in Ref. 48, we only formally indicate the proof: By conditions (i)–(iii) one proves that the function

$$F(\beta,z)=\operatorname{Tr}[(H_{1,\beta}-z)^{-1}-(H_{2,\beta}-z)^{-1}],\quad z\in\mathbb{C}\setminus[0,\infty), \tag{3.28}$$

is differentiable with respect to β with derivative

$$\frac{\partial}{\partial\beta}F(\beta,z)=-\operatorname{Tr}\{(H_{1,\beta}-z)^{-1}[A_\beta^*B+B^*A_\beta](H_{1,\beta}-z)^{-1}-(H_{2,\beta}-z)^{-1}[A_\beta B^*+BA_\beta^*](H_{2,\beta}-z)^{-1}\}. \tag{3.29}$$

Using the commutation formulas[49]

$$(A_\beta^*A_\beta-z)^{-1}A_\beta^*\subseteq A_\beta^*(A_\beta A_\beta^*-z)^{-1},$$
$$(A_\beta A_\beta^*-z)^{-1}A_\beta\subseteq A_\beta(A_\beta^*A_\beta-z)^{-1},\quad z\in\mathbb{C}\setminus[0,\infty), \tag{3.30}$$

and cyclicity of the trace, the two terms on the rhs of Eq. (3.29) cancel. Thus

$$\frac{\partial}{\partial\beta}F(\beta,z)=0,\quad \beta\in\mathbb{R},\quad z\in\mathbb{C}\setminus[0,\infty), \tag{3.31}$$

implying the desired result $F(\beta,z)=F(0,z)$. Conditions (iii) and (iv) enter in a rigorous derivation of Eq. (3.31).[48]

The result (3.27) yields the topological invariance of the regularized index $\Delta(z)$ in the concrete examples of Sec. IV (cf. also Ref. 52). Moreover, it proves the topological invariance of Δ and $\mathscr{A}$ whenever the limits $z\to0$ and $z\to\infty$ of $\Delta(z)$ exist. In the case where A is Fredholm, the invariance of the Fredholm index $i(A)$ (and thus of Δ by Theorem 3.1), i.e.,

$$i(A+\beta B)=i(A),\quad \beta\in\mathbb{R}, \tag{3.32}$$

under relatively compact perturbations B with respect to A is a standard result.[47] Equation (3.27) works without assuming A to be Fredholm, but needs much stronger assumptions on the "smallness" of B than just relative compactness.

Another application of Eq. (3.27) concerns the invariance of Krein's spectral shift function. In fact, we get the following.

Theorem 3.4: Assume Hypothesis (vi) with A replaced by A_β and $(1+|\cdot|)^{-1}[\xi_{12,\beta}-\xi_{12}]\in L^1(\mathbb{R})$ for all $\beta\in\mathbb{R}$. If conditions (ii)–(iv) of Theorem 3.3 hold, then

$$[\xi_{12,\beta}(\lambda_+)-\xi_{12}(\lambda_+)]+[\xi_{12,\beta}(\lambda_-)-\xi_{12}(\lambda_-)]=0, \tag{3.33}$$

for all $\beta,\lambda\in\mathbb{R}$. In particular if $\xi_{12,\beta}$, $\beta\in\mathbb{R}$, and ξ_{12} are continuous at a point $\lambda\in\mathbb{R}$ then

$$\xi_{12,\beta}(\lambda)=\xi_{12}(\lambda),\quad \beta\in\mathbb{R}. \tag{3.34}$$

Proof: Equations (2.2) and (3.27) together with the Lebesgue dominated convergence theorem imply

$$0=\int_{\mathbb{R}}d\lambda\,[\xi_{12,\beta}(\lambda)-\xi_{12}(\lambda)](\lambda-z)^{-2}=\frac{d}{dz}\int_{\mathbb{R}}d\lambda\,[\xi_{12,\beta}(\lambda)-\xi_{12}(\lambda)](\lambda-z)^{-1} \tag{3.35}$$

and hence

$$\int_{\mathbb{R}}d\lambda\,[\xi_{12,\beta}(\lambda)-\xi_{12}(\lambda)](\lambda-z)^{-1}=0$$

by taking $|z|\to\infty$, $\operatorname{Im}z\neq0$. Thus Eq. (3.33) results from standard properties of the Poisson kernel (cf., e.g., Ref. 45). □

In the first four examples of the next section, $\xi_{12,\beta}(\lambda)$ coincides with a multiple of the relative phase shift between H_1 and H_2 and the Fredholm determinants in Eq. (3.22) are expressed in terms of Wronski determinants. In these cases the topological invariance property of $\Delta(z)$ and $\xi_{12}(\lambda)$ can be established by simple and explicit calculations.

Finally, we note that the following family of operators in $\mathscr{H}\oplus\mathscr{H}$:

$$Q_m=\begin{pmatrix} m & A^* \\ A & -m\end{pmatrix},$$
$$H_m=Q_m^2=\begin{pmatrix} H_1+m^2 & 0 \\ 0 & H_2+m^2\end{pmatrix},\quad m\in\mathbb{R}\setminus\{0\}, \tag{3.36}$$

can be treated analogously. In order to illustrate a simple application of the above results, we briefly discuss the invariance of the spectral asymmetry η_m (Refs. 7 and 9) under "small" perturbations. Under suitable conditions on H_m [cf., e.g., Eq. (3.17)], the (regularized) and spectral asymmetry can be defined by

$$\eta_m=\lim_{t\to0_+}\eta_m(t), \tag{3.37}$$
$$\eta_m(t)=\operatorname{Tr}[Q_mH_m^{-1/2}e^{-tH_m}],\quad m\in\mathbb{R}\setminus\{0\}. \tag{3.38}$$

(This definition resembles the ones available in the literature, e.g., in Refs. 2, 8, 12, 53, and 54.) Since

$$\operatorname{Tr}[Q_m(H_m+z^2)^{-1}e^{-tH_m}]=m\operatorname{Tr}[(H_1+m^2+z^2)^{-1}e^{-t(H_1+m^2)}-(H_2+m^2+z^2)^{-1}e^{-t(H_2+m^2)}],\quad t>0, \tag{3.39}$$

we can rewrite Eq. (3.38) in the form

$$\eta_m(t)=m\operatorname{Tr}[(H_1+m^2)^{-1/2}e^{-t(H_1+m^2)}-(H_2+m^2)^{-1/2}e^{-t(H_2+m^2)}] \tag{3.40}$$

and, using Eq. (2.4),

$$\eta_m(t) = m\int_0^\infty d\lambda\,\xi_{12}(\lambda)\,\frac{d}{d\lambda}\,[(\lambda+m^2)^{-1/2}e^{-t(\lambda+m^2)}]\,. \tag{3.41}$$

This implies

$$\eta_m = -\frac{m}{2}\int_0^\infty d\lambda\,\xi_{12}(\lambda)(\lambda+m^2)^{-3/2}\,. \tag{3.42}$$

Obviously, Eqs. (3.41) and (3.42) imply the invariance of η_m with respect to the substitution $A\to A_\beta = A+\beta B$ as a consequence of Theorem 3.4.

IV. SPECIFIC MODELS

We present a series of examples of explicit model calculations which illustrate the practical use of the abstract results of the foregoing section.

Example 4.1: Let $\mathscr{H} = L^2(\mathbb{R})$ and

$$A = \left(\frac{d}{dx}+\phi\right)\Big|_{H^{2,1}(\mathbb{R})}\,, \tag{4.1}$$

where ϕ fulfills the following requirements:

$$\begin{gathered} \phi,\phi'\in L^\infty(\mathbb{R}) \text{ are real valued} \\ \lim_{x\to\pm\infty}\phi(x) = \phi_\pm\in\mathbb{R}\,,\quad \phi_-^2\leqslant\phi_+^2\,, \\ \int_{\mathbb{R}} dx(1+|x|^2)|\phi'(x)|<\infty\,, \\ \pm\int_0^{\pm\infty} dx(1+|x|^2)|\phi(x)-\phi_\pm|<\infty\,. \end{gathered} \tag{4.2}$$

In this case, H_1 and H_2 explicitly read

$$H_j = \left(-\frac{d^2}{dx^2}+\phi^2+(-1)^j\phi'\right)\Big|_{H^{2,2}(\mathbb{R})}\,,\quad j=1,2\,. \tag{4.3}$$

Then

$$\Delta(z) = [\phi_+(\phi_+^2-z)^{-1/2}-\phi_-(\phi_-^2-z)^{-1/2}]/2\,,\quad z\in\mathbb{C}\setminus[0,\infty)\,, \tag{4.4}$$

and hence

$$\Delta = [\mathrm{sgn}\,(\phi_+)-\mathrm{sgn}(\phi_-)]/2\,,\quad \mathscr{A}=0\,, \tag{4.5}$$

$$\begin{aligned} \xi_{12}(\lambda) &= \pi^{-1}\{\theta(\lambda-\phi_+^2)\arctan[(\lambda-\phi_+^2)^{1/2}/\phi_+] \\ &\quad -\theta(\lambda-\phi_-^2)\arctan[(\lambda-\phi_-^2)^{1/2}/\phi_-]\} \\ &\quad +\theta(\lambda)[\mathrm{sgn}\,(\phi_-)-\mathrm{sgn}(\phi_+)]/2\,, \\ &\quad \phi_-\neq0\,,\quad \phi_+\neq0\,, \\ \xi_{12}(\lambda) &= \pi^{-1}\theta(\lambda-\phi_+^2)\arctan[(\lambda-\phi_+^2)^{1/2}/\phi_+] \\ &\quad -\theta(\lambda)[\mathrm{sgn}(\phi_+)]/2\,, \\ &\quad \phi_-=0\,,\quad \phi_+\neq0\,;\quad \lambda\in\mathbb{R}\,. \end{aligned} \tag{4.6}$$

[Here $\theta(x)=1$ for $x\geqslant0$ and $\theta(x)=0$ for $x<0$ and $\mathrm{sgn}(x)=\pm1$ for $x\gtrless0$ and $\mathrm{sgn}(0)=0$.] Equations (4.4)–(4.6) clearly demonstrate the topological invariance of these quantities as discussed in Sec. III since they only depend on the asymptotic values $\phi_\pm$ of $\phi(x)$ and not on its local properties. In fact, replace $\phi(x)$ by $\phi(x)+\beta\psi(x)$, $\beta\in\mathbb{R}$, where

$$\begin{gathered} \psi,\psi'\in L^\infty(\mathbb{R}) \text{ are real valued,} \\ \psi(x),\psi'(x) = O(|x|^{-3-\epsilon}) \quad \text{for some } \epsilon>0 \text{ as } |x|\to\infty\,. \end{gathered} \tag{4.7}$$

Then the perturbation B [cf. Eq. (3.26)] given by multiplication with ψ leaves the regularized index invariant since the hypotheses of Theorem 3.3 are satisfied.

Concerning zero-energy properties of H_j, $j=1,2$, see Table I.

These zero-energy results easily follow from the fact that the equations

$$Af=0\,,\quad A^*g=0 \tag{4.8}$$

have the solutions

$$\begin{aligned} f(x) &= f(0)\exp\left(-\int_0^x dt\phi(t)\right) \\ &= O(e^{-\phi_\pm x}) \quad \text{as } x\to\pm\infty\,, \end{aligned} \tag{4.9}$$

$$g(x) = g(0)\exp\left(\int_0^x dt\phi(t)\right) = O(e^{\phi_\pm x}) \quad \text{as } x\to\pm\infty\,.$$

In order to derive Eq. (4.4), we introduce Jost solutions $f_{j\pm}(z,x)$ associated with $H_j, j=1,2,$

TABLE I. Zero-energy properties of H_1 and H_2 in example 4.1.

	Zero-energy resonance of H_1	of H_2	Zero-energy bound state $\sigma_p(H_1)\cap\{0\}$	$\sigma_p(H_2)\cap\{0\}$	Δ	$i(A)$
$\phi_-<0<\phi_+$	no	no	$\{0\}$	ϕ	1	1
$\phi_+<0<\phi_-$	no	no	ϕ	$\{0\}$	-1	-1
$\phi_+,\phi_->0$ or $\phi_+,\phi_-<0$	no	no	ϕ	ϕ	0	0
$\phi_-=0, \phi_+\neq0$	yes	no	ϕ	ϕ	$\frac{1}{2}\mathrm{sgn}(\phi_+)$	0
$\phi_-=\phi_+=0$	yes	yes	ϕ	ϕ	0	0

$$f_{j\pm}(z,x)=e^{\pm ik_\pm x}-\int_x^{\pm\infty}dx'\,k_\pm^{-1}\sin[k_\pm(x-x')]\times[\phi^2(x')-\phi_\pm^2+(-1)^j\phi'(x')]f_{j\pm}(z,x'),\quad z\in\mathbb{C},\quad j=1,2,\tag{4.10}$$

where

$$k_\pm(z)=(z-\phi_\pm^2)^{1/2},\quad \operatorname{Im}k_\pm\geqslant 0.\tag{4.11}$$

The corresponding Fredholm integral equation reads

$$f_{1\pm}(z,x)=[T_{12}(z)]^{-1}f_{2\pm}(z,x)-\int_{\mathbb{R}}dx'\,g_2(z,x,x')[-2\phi'(x')]f_{1\pm}(z,x'),\quad z\in\mathbb{C}\setminus\sigma_p(H_2),\quad z\neq\phi_-^2,\tag{4.12}$$

where

$$g_2(z,x,x')=-[W(f_{2-}(z),f_{2+}(z))]^{-1}\times\begin{cases}f_{2+}(z,x)f_{2-}(z,x'), & x\geqslant x',\\ f_{2-}(z,x)f_{2+}(z,x'), & x\leqslant x',\end{cases}\quad z\in\mathbb{C}\setminus\sigma_p(H_2),\quad z\neq\phi_-^2,\tag{4.13}$$

$$g_2(z)=(H_2-z)^{-1},\quad z\in\rho(H_2),$$

and $T_{12}(z)$ denotes

$$T_{12}(z)=W(f_{2-}(z),f_{2+}(z))/W(f_{1-}(z),f_{1+}(z)),\quad z\in\mathbb{C}\setminus\sigma_p(H_2),\quad z\neq\phi_-^2.\tag{4.14}$$

Here

$$W(F,G)_x=F(x)G'(x)-F'(x)G(x)\tag{4.15}$$

denotes the Wronskian of F and G. (For more details on one-dimensional systems with nontrivial spatial asymptotics, cf. Ref. 23.) As can be seen, e.g., from Eq. (4.12), the relative interaction V_{12} reads

$$V_{12}(x)=-2\phi'(x).\tag{4.16}$$

Our first main step to derive Eq. (4.4) now consists of the observation that

$$\frac{W(f_{1-}(z),f_{1+}(z))}{W(f_{2-}(z),f_{2+}(z))}=\det[1-2|\phi'|^{1/2}\operatorname{sgn}(\phi')g_2(z)|\phi'|^{1/2}],\quad z\in\rho(H_2),\quad z\neq\phi_-^2,\tag{4.17}$$

such that (cf. Lemma 2.4)

$$\operatorname{Tr}[(H_1-z)^{-1}-(H_2-z)^{-1}]=-\frac{d}{dz}\ln\frac{W(f_{1-}(z),f_{1+}(z))}{W(f_{2-}(z),f_{2+}(z))},\quad z\in C\setminus[0,\infty).\tag{4.18}$$

Equality (4.17) can be proved along the lines of Ref. 33 using Eqs. (4.10) and (4.12) (cf. Ref. 23).

Next, we note that Eq. (3.15) also holds for distributional-type (e.g., Jost) solutions of H_1 and H_2. In fact, assume that $f_1(z,x)$, $z\neq0$, is normalized according to Eq. (4.10), i.e.,

$$f_{1\pm}(z,x)=e^{\pm ik_\pm x}+o(1)\quad\text{as } x\to\pm\infty,$$

then $(Af_{1\pm})(z,x)$ asymptotically fulfills

$$(Af_{1\pm})(z,x)=(\pm ik_\pm+\phi_\pm)e^{\pm ik_\pm x}+o(1)\quad\text{as } x\to\pm\infty.$$

Thus

$$\begin{cases}f_{1\pm}(z,x)\\ f_{2\pm}(z,x)=(\pm ik_\pm+\phi_\pm)^{-1}(Af_{1\pm})(z,x),\quad z\neq0,\end{cases}\tag{4.19}$$

are correctly normalized Jost solutions for H_1 and H_2. Equation (4.17) thus becomes

$$\det[1-2|\phi'|^{1/2}\operatorname{sgn}(\phi')g_2(z)|\phi'|^{1/2}]=(-ik_-+\phi_-)(ik_++\phi_+)W(f_{1-}(z),f_{1+}(z))\times[W((Af_{1-})(z),(Af_{1+})(z))]^{-1},\quad z\in\mathbb{C}\setminus[0,\infty).\tag{4.20}$$

Finally, a straightforward computation yields

$$W((Af)(z),(Ag)(z))=zW(f(z),g(z)),\quad z\in\mathbb{C},\tag{4.21}$$

where f,g are distributional solutions of

$$(A^*A\psi(z))(x)=-\psi''(z,x)+[\phi^2(x)-\phi'(x)]\psi(z,x)=z\psi(z,x),\quad z\in\mathbb{C}.\tag{4.22}$$

Consequently, Eq. (4.20) becomes

$$\det[1-2|\phi'|^{1/2}\operatorname{sgn}(\phi')g_2(z)|\phi'|^{1/2}]=(-ik_-+\phi_-)(ik_++\phi_+)/z,\quad z\in\mathbb{C}\setminus[0,\infty)\tag{4.23}$$

and Eq. (4.4) follows from Eqs. (3.23) and (4.23).

The result (4.4) was first derived by Callias,[17] and since then by numerous authors.[2,10,11,18,21,22,25,55] While our derivation is close to that in Ref. 22, it seems to be the shortest one since the trick based on Eq. (4.21) explicitly exploits supersymmetry and avoids the use of an additional comparison Hamiltonian in the approach of Ref. 22.

Next, we discuss an example on the half-line $(0,\infty)$.

Example 4.2: Let $\mathscr{H}=L^2(0,\infty)$ and

$$A=\left(\frac{d}{dr}+\tilde\phi(r)\right)\Big|_{H_0^{2,1}(0,\infty)},\tag{4.24}$$

where $\tilde\phi$ fulfills the following requirements:

$$\begin{aligned}&\tilde\phi,\tilde\phi'\in L^\infty(0,\infty)\ \text{are real valued},\\ &\lim_{r\to\infty}\tilde\phi(r)=\tilde\phi_+\in\mathbb{R},\quad \lim_{r\to0+}\tilde\phi(r)=\tilde\phi_0\in\mathbb{R},\\ &\int_0^\infty dr\,r(1+r)|\tilde\phi'(r)|<\infty,\\ &\int_0^\infty dr\,r(1+r)|\tilde\phi(r)-\tilde\phi_+|<\infty.\end{aligned}\tag{4.25}$$

In this case, H_1 and H_2 read

$$H_1=\left(-\frac{d^2}{dr^2}+\tilde\phi^2-\tilde\phi'\right)_F,\tag{4.26}$$

where F denotes the Friedrichs extension of the corresponding operator restricted to $C_0^\infty(0,\infty)$ and

$$H_2 = -\frac{d^2}{dr^2} + \tilde{\phi}^2 + \tilde{\phi}',$$

$$\mathscr{D}(H_2) = \{g \in L^2(0,\infty) | g, g' \in AC_{\text{loc}}(0,\infty);\quad g'(0_+) - \tilde{\phi}_0 g(0_+) = 0;\quad g'' \in L^2(0,\infty)\}, \tag{4.27}$$

With $AC_{\text{loc}}(a,b)$ the set of locally absolutely continuous functions on (a,b). Then

$$\Delta(z) = (z/2)(\tilde{\phi}_+^2 - z)^{-1/2}[\tilde{\phi}_+ + (\tilde{\phi}_+^2 - z)^{1/2}]^{-1},\quad z \in \mathbb{C} \setminus [0,\infty), \tag{4.28}$$

and hence

$$\Delta = \begin{cases} -[1 - \operatorname{sgn}(\tilde{\phi}_+)]/2, & \tilde{\phi}_+ \neq 0, \\ -\frac{1}{2}, & \tilde{\phi}_+ = 0, \end{cases} \quad \mathscr{A} = \tfrac{1}{2}, \tag{4.29}$$

$$\xi_{12}(\lambda) = \begin{cases} \pi^{-1}\theta(\lambda - \tilde{\phi}_+^2) \arctan[(\lambda - \tilde{\phi}_+^2)^{1/2}/\tilde{\phi}_+] + \theta(\lambda)\theta(-\tilde{\phi}_+), & \tilde{\phi}_+ \neq 0, \\ \theta(\lambda)/2, \quad \tilde{\phi}_+ = 0; \quad \lambda \in \mathbb{R}. \end{cases} \tag{4.30}$$

Again, Eqs. (4.28)–(4.30) exhibit the topological invariance of all these quantities since only $\tilde{\phi}_+$ enters. [The arguments in connection with Eq. (4.7) can easily be extended to the present situation.] Concerning zero-energy properties, see Table II.

In order to derive Eq. (4.28), we introduce the Jost solutions

$$\begin{aligned} f_{j\pm}(z,r) &= e^{\pm ik_+ r} - \int_r^\infty dr'\, k_+^{-1} \sin[k_+(r - r')] \\ &\quad \times [\tilde{\phi}^2(r') - \tilde{\phi}_+^2 + (-1)^j \tilde{\phi}'(r')] f_{j\pm}(z,r'), \\ &\quad z \in \mathbb{C},\quad j = 1,2, \end{aligned} \tag{4.31}$$

where

$$k_+(z) = (z - \tilde{\phi}_+^2)^{1/2},\quad \operatorname{Im} k_+ \geqslant 0, \tag{4.32}$$

and the regular solutions

$$\begin{aligned} \psi_1(z,r) &= k_+^{-1} \sin k_+ r + \int_0^r dr'\, k_+^{-1} \sin[k_+(r - r')] \\ &\quad \times [\tilde{\phi}^2(r') - \tilde{\phi}_+^2 - \tilde{\phi}'(r')]\, \psi_1(z,r'), \\ \psi_2(z,r) &= \cos k_+ r + \tilde{\phi}_0 k_+^{-1} \sin k_+ r \\ &\quad + \int_0^r dr'\, k_+^{-1} \sin[k_+(r - r')] \\ &\quad \times [\tilde{\phi}^2(r') - \tilde{\phi}_+^2 + \tilde{\phi}'(r')]\, \psi_2(z,r'),\quad z \in \mathbb{C}. \end{aligned} \tag{4.33}$$

Using again Eq. (3.15), we assume that $f_{1\pm}(z,r)$, $z \neq 0$ is normalized according to Eq. (4.31), i.e.,

$$f_{1\pm}(z,r) = e^{\pm ik_+ r} + o(1) \quad \text{as } r \to \infty.$$

Then $(Af_{1\pm})(z,r)$ fulfills

$$(Af_{1\pm})(z,r) = (\pm ik_+ + \tilde{\phi}_+) e^{\pm ik_+ r} + o(1) \quad \text{as } r \to \infty$$

such that the Jost functions

$$\begin{cases} f_{1\pm}(z,r), \\ f_{2\pm}(z,r) = (\pm ik_+ + \tilde{\phi}_+)^{-1}(Af_{1\pm})(z,r),\quad z \neq 0, \end{cases} \tag{4.34}$$

are correctly normalized. Similarly, we assume that $\psi_1(z,r)$, $z \neq 0$ fulfills

$$\psi_1(z,r) = r + o(r) \quad \text{as } r \to 0_+.$$

Then

$$(A\psi_1)(z,r) = 1 + \tilde{\phi}_0 r + o(r) \quad \text{as } r \to 0_+$$

and thus

$$\begin{cases} \psi_1(z,r), \\ \psi_2(z,r) = (A\psi_1)(z,r),\quad z \neq 0, \end{cases} \tag{4.35}$$

are correctly normalized regular solutions of H_1 and H_2. The rest is now identical to the treatment of example 4.1. First of all, one derives, as in Eq. (4.18) (cf., e.g., Ref. 30)

$$\operatorname{Tr}[(H_1 - z)^{-1} - (H_2 - z)^{-1}] = \frac{d}{dz} \ln \frac{W(\psi_2(z), f_{2+}(z))}{W(\psi_1(z), f_{1+}(z))},\quad z \in \mathbb{C} \setminus [0,\infty). \tag{4.36}$$

Then one calculates, as in Eq. (4.21), that

$$W((A\psi_1)(z), (Af_{1+})(z)) = zW(\psi_1(z), f_{1+}(z)),\quad z \in \mathbb{C}. \tag{4.37}$$

We now consider a generalization of this example which allows us to discuss n-dimensional spherically symmetric systems (cf., e.g., Refs. 2 and 13).

Example 4.3: Let $\mathscr{H} = L^2(0,\infty)$ and

$$A = \overline{\left(\frac{d}{dr} + \phi\right)\Big|_{C_0^\infty(0,\infty)}}, \tag{4.38}$$

where ϕ fulfills the following requirements:

$$\begin{aligned} &\phi(r) = \phi_0 r^{-1} + \tilde{\phi}(r),\quad \phi_0 \leqslant -\tfrac{1}{2},\quad r > 0, \\ &\tilde{\phi}, \tilde{\phi}' \in L^\infty(0,\infty) \text{ are real valued}, \\ &\lim_{r \to \infty} \tilde{\phi}(r) = \tilde{\phi}_+ \in \mathbb{R}, \\ &\int_0^\infty dr\, W_{\phi_0}(r)(|\tilde{\phi}'(r)| + r^{-1}|\tilde{\phi}(r) - \tilde{\phi}_+|) < \infty, \\ &\int_0^\infty dr\, W_{\phi_0}(r)|\tilde{\phi}(r) - \tilde{\phi}_+| < \infty, \end{aligned} \tag{4.39}$$

and the weight function W_{ϕ_0} is defined by

TABLE II. Zero-energy properties of H_1 and H_2 in example 4.2.

	Zero-energy resonance of H_1	Zero-energy resonance of H_2	Zero-energy bound state $\sigma_p(H_1) \cap \{0\}$	Zero-energy bound state $\sigma_p(H_2) \cap \{0\}$	Δ	$i(A)$
$\tilde{\phi}_+ > 0$	no	no	ϕ	ϕ	0	0
$\tilde{\phi}_+ < 0$	no	no	ϕ	$\{0\}$	-1	-1
$\tilde{\phi}_+ = 0$	no	yes	ϕ	ϕ	$-\frac{1}{2}$	0

$$W_{\phi_0}(r) = \begin{cases} r(1+r) & \text{if } \phi_0 < -\frac{1}{2}, \\ r(1+|\ln r|^2), \quad 0<r\leqslant\frac{1}{2} & \text{if } \phi_0 = -\frac{1}{2}, \\ r(1+r), \quad r\geqslant\frac{1}{2}. \end{cases} \tag{4.40}$$

Now H_1 and H_2 are given by

$$H_j = \left(-\frac{d^2}{dr^2} + \phi^2 + (-1)^j\phi'\right)_F, \quad j=1,2. \tag{4.41}$$

Explicitly, we have

$$\phi^2(r) \mp \phi'(r) = (\phi_0^2 \pm \phi_0)r^{-2} + 2\phi_0\tilde{\phi}_+ r^{-1} + \tilde{\phi}_+^2 + \tilde{\phi}^2(r) - \tilde{\phi}_+^2 \mp \tilde{\phi}'(r) + 2\phi_0[\tilde{\phi}(r) - \tilde{\phi}_+]r^{-1}, \quad r>0. \tag{4.42}$$

Then

$$\Delta(z) = (z/2)(\tilde{\phi}_+^2 - z)^{-1/2}[\tilde{\phi}_+ - (\tilde{\phi}_+^2 - z)^{1/2}]^{-1}, \quad z\in\mathbb{C}\setminus[0,\infty) \tag{4.43}$$

and hence

$$\Delta = \begin{cases} [1+\operatorname{sgn}(\tilde{\phi}_+)]/2, & \tilde{\phi}_+\neq 0, \\ \frac{1}{2}, & \tilde{\phi}_+ = 0, \end{cases} \quad \mathscr{A} = -\tfrac{1}{2}, \tag{4.44}$$

$$\xi_{12}(\lambda) = \begin{cases} \pi^{-1}\theta(\lambda - \tilde{\phi}_+^2)\arctan[(\lambda-\tilde{\phi}_+^2)^{1/2}/\tilde{\phi}_+] - \theta(\lambda)\theta(\tilde{\phi}_+), & \tilde{\phi}_+\neq 0, \\ -\theta(\lambda)/2, \quad \tilde{\phi}_+ = 0; \quad \lambda\in\mathbb{R}. \end{cases} \tag{4.45}$$

The topological invariance in Eqs. (4.43)–(4.45) is obvious. (See Table III.) If $\tilde{\phi}_+ = 0$, the result $\Delta = \frac{1}{2}$ is not due to a zero-energy (threshold) resonance, but due to the long-range nature of the relative interaction $V_{12}(r) = 2\phi_2 r^{-2} + o(r^{-2})$ as $r\to\infty$. Since Eq. (4.43) is independent of ϕ_0, this result holds in any dimension $\geqslant 2$ and for any value of the angular momentum.

In order to derive Eq. (4.43), one could follow the strategy of example 4.2 step by step since formula (4.36) remains valid in the present case for suitably normalized Jost and regular solutions (although we are dealing with a long-range problem!). To shorten the presentation, we will use instead a different approach based on the topological invariance property of $\Delta(z)$ and $\xi_{12}(\lambda)$ (this approach obviously also works in example 4.2). Indeed, because of Theorem 3.3, it suffices to choose $\tilde{\phi}(r) = \tilde{\phi}_+$, $r\geqslant 0$ in example 4.3. Then

$$H_j = \left(-\frac{d^2}{dr^2} + [\phi_0^2 - (-1)^j\phi_0]r^{-2} + 2\phi_0\tilde{\phi}_+ r^{-1} + \tilde{\phi}_+^2\right)_F, \quad j=1,2 \tag{4.46}$$

[cf. Eq. (4.42)] and hence[56]

$$S_j(\lambda) = \frac{\Gamma(2^{-1} + 2^{-1}(-1)^j - \phi_0 + i(\phi_0\tilde{\phi}_+/k_+))}{\Gamma(2^{-1} + 2^{-1}(-1)^j - \phi_0 - i(\phi_0\tilde{\phi}_+/k_+))} \times e^{i\pi[2^{-1} - (-1)^j 2^{-1} + \phi_0]}, \quad \lambda > \tilde{\phi}_+^2, \quad j=1,2 \tag{4.47}$$

[$k_+(\lambda)$ defined in Eq. (4.32)] implying

$$S_{12}(\lambda) = S_1(\lambda)S_2(\lambda)^{-1} = (\tilde{\phi}_+ - ik_+)/(\tilde{\phi}_+ + ik_+), \quad \lambda > \tilde{\phi}_+^2. \tag{4.48}$$

Equation (4.48) proves Eq. (4.45). Now Eq. (4.43) follows by explicit integration (Ref. 57, p. 556) in Eq. (3.23).

The result (4.43), in the special case $\tilde{\phi}(r)\equiv 0$, has been discussed in Ref. 21 by different methods.

Next, we briefly discuss nonlocal interactions.

Example 4.4: Let $\mathscr{H} = L^2(0,\infty)$ and

$$A = \frac{d}{dr}\bigg|_{H_0^{2,1}(0,\infty)} + B, \tag{4.49}$$

where

$$B, A^*B, AB^* \in \mathscr{B}_1(L^2(0,\infty)). \tag{4.50}$$

In this case the assumptions of Theorem 3.3 are trivially fulfilled, and hence Eqs. (4.28)–(4.30), in the special case $\phi(r)\equiv 0$, hold. In particular

$$\Delta(z) = \Delta = -\tfrac{1}{2}, \quad z\in\mathbb{C}\setminus[0,\infty), \quad \mathscr{A} = \tfrac{1}{2}. \tag{4.51}$$

In order to illustrate the possible complexity of zero-energy properties of H_1 and H_2 in spite of the simplicity of Eq. (4.51), it suffices to treat the following rank 2 example:

$$B = \alpha(f,\cdot)f + \beta(g,\cdot)g, \quad \alpha,\beta\in\mathbb{R},$$
$$f,g\in C_0^1(0,\infty), \quad f\geqslant 0, g\geqslant 0, \quad f\neq g. \tag{4.52}$$

By straightforward calculations, one obtains the information contained in Table IV. Here the following case distinction has been used:

TABLE III. Zero-energy properties of H_1 and H_2 in example 4.3.

	Zero-energy resonance of H_1	Zero-energy resonance of H_2	Zero-energy bound state $\sigma_p(H_1)\cap\{0\}$	Zero-energy bound state $\sigma_p(H_2)\cap\{0\}$	Δ	$i(A)$
$\tilde{\phi}_+>0$	no	no	$\{0\}$	ϕ	1	1
$\tilde{\phi}_+<0$	no	no	ϕ	ϕ	0	0
$\tilde{\phi}_+=0$	no	no	ϕ	ϕ	$\frac{1}{2}$	0

TABLE IV. Zero-energy properties of H_1 and H_2 in example 4.4.

	Zero-energy resonance of H_1	Zero-energy resonance of H_2	Zero-energy bound state $\sigma_p(H_1)\cap\{0\}$	Zero-energy bound state $\sigma_p(H_2)\cap\{0\}$	Δ	$i(A)$
Case I	no	yes	ϕ	ϕ	$-\frac{1}{2}$	0
Case II	yes	no	ϕ	$\{0\}$	$-\frac{1}{2}$	-1
Case III	no	yes	$\{0\}$	$\{0\}$	$-\frac{1}{2}$	0

case I, $\Psi(\alpha,\beta)\neq 0$;
case II, $\Psi(\alpha,\beta)=0$,
$\alpha\neq 2G(\infty)\{F(\infty)[(f,G)-(g,F)]\}^{-1}$;
case III, $\Psi(\alpha,\beta)=0$,
$\alpha=2G(\infty)\{F(\infty)(f,G)-(g,F)]\}^{-1}$;

where

$$F(x)=\int_0^x dx'\,f(x')\,,\quad G(x)=\int_0^x dx'\,g(x')\,,$$
$$\Psi(\alpha,\beta)=[1+\alpha(f,F)][1+\beta(g,G)]-\alpha\beta(f,G)(g,F)\,. \tag{4.53}$$

Finally, we consider in detail the following two-dimensional magnetic field problem.

Example 4.5: Let $\mathscr{H}=L^2(\mathbb{R}^2)$ and

$$A=\overline{[(-i\partial_1-a_1)+i(i\partial_2+a_2)]|_{C_0^\infty(\mathbb{R}^2)}}\,, \tag{4.54}$$

where

$$a=(\partial_2\phi,-\partial_1\phi)\,,\quad \partial_j\equiv\frac{\partial}{\partial x_j}\,,\quad j=1,2\,, \tag{4.55}$$

and ϕ fulfills the following requirements:

$$\begin{aligned}&\phi\in C^2(\mathbb{R}^2)\ \text{is real valued}\,,\\ &\phi(x)=-F\ln|x|+C+O(|x|)^{-\epsilon}\,,\\ &(\nabla\phi)(x)=-F|x|^{-2}x+O(|x|^{-1-\epsilon})\,,\\ &\qquad C,F\in\mathbb{R}\,,\quad \epsilon>0\ \text{as}\ |x|\to\infty\,,\\ &(\Delta\phi)^{1+\delta},\ (1+|\cdot|^\delta)(\Delta\phi)\in L^1(\mathbb{R}^2)\quad\text{for some }\delta>0\,.\end{aligned} \tag{4.56}$$

Then

$$H_j=[(-i\nabla-a)^2-(-1)^j b]|_{H^{2,2}(\mathbb{R}^2)}\,,\quad j=1,2\,, \tag{4.57}$$

where

$$b(x)=(\partial_1a_2-\partial_2a_1)(x)=-(\Delta\phi)(x)\,. \tag{4.58}$$

Introducing the magnetic flux F by

$$F=(2\pi)^{-1}\int_{\mathbb{R}^2}d^2x\,b(x) \tag{4.59}$$

we obtain

$$\Delta(z)=\Delta=-F\,,\quad z\in\mathbb{C}\setminus[0,\infty)\,,\quad \mathscr{A}=F\,, \tag{4.60}$$
$$\xi_{12}(\lambda)=F\theta(\lambda)\,,\quad \lambda\in\mathbb{R}\,. \tag{4.61}$$

Moreover, we have

$$\begin{aligned}&i(A)\,\mathrm{sgn}(F)\\ &\quad=\theta(-F)\dim\mathrm{Ker}(A)-\theta(F)\dim\mathrm{Ker}(A^*)\\ &\quad=\begin{cases}-N\ \text{if}\ |F|=N+\epsilon\,, & 0<\epsilon<1\,,\\ -(N-1)\ \text{if}\ |F|=N\,, & N\in\mathbb{N}\,.\end{cases}\end{aligned} \tag{4.62}$$

Since Eq. (4.62) has been derived in Ref. 58 (cf. also Refs. 8, 24, and 59–62), we concentrate on Eqs. (4.60) and (4.61). For this purpose we first study a special example (treated in Ref. 63). Let

$$\phi(R,r)=\begin{cases}-(Fr^2/2R^2)\,, & r\leqslant R\,,\\ -(F/2)[1+\ln(r^2/R^2)]\,, & r\geqslant R\,,\quad R>0\,,\end{cases} \tag{4.63}$$

and denote the corresponding Hamiltonian in (4.57) by $H_j(R)$, $j=1,2$. Next, define U_ϵ, $\epsilon\geqslant 0$, to be the unitary group of dilations in $L^2(\mathbb{R}^2)$, viz.,

$$(U_\epsilon g)(x)=\epsilon^{-1}g(x/\epsilon)\,,\quad \epsilon>0\,,\quad g\in L^2(\mathbb{R}^2)\,. \tag{4.64}$$

Then a simple calculation yields

$$U_\epsilon H_j(R)U_\epsilon^{-1}=\epsilon^2H_j(\epsilon R)\,,\quad \epsilon,R>0\,,\quad j=1,2\,. \tag{4.65}$$

If we denote by $S_{12}(R)$, the scattering operator in $L^2(\mathbb{R}^2)$ associated with the pair $(H_1(R),H_2(R))$, then $S_{12}(R)$ is decomposable with respect to the spectral representation of $H_2(R)P_{\rm ac}(H_2(R))$ [$P_{\rm ac}(\cdot)$ is the projection onto the absolutely continuous spectral subspace]. Let $S_{12}(\lambda,R)$ in $L^2(S^1)$ denote the fibers of $S_{12}(R)$, then Eq. (4.65) implies

$$\begin{aligned}&S_{12}(\lambda,R)=S_{12}(\epsilon^2\lambda,R/\epsilon)\,,\\ &\xi_{12}(\lambda,R)=\xi_{12}(\epsilon^2\lambda,R/\epsilon)\,,\quad \lambda>0\,.\end{aligned} \tag{4.66}$$

Applying now Theorem 3.4, we infer that $\xi_{12}(\lambda)$ cannot depend on $R>0$ as long as F is kept fixed in Eq. (4.63). Thus Eq. (4.63) implies $\xi_{12}(\lambda)=\xi_{12}(\epsilon^2\lambda)$, $\lambda>0$, which in turn implies that ξ_{12} is energy independent.

We will give two methods of computing this constant value of ξ_{12}, the first using heat kernels, the second, resolvents.

Method 1: By Eq. (2.4)

$$\begin{aligned}\mathrm{Tr}(e^{-tH_1}-e^{-tH_2})&=-t\int_0^\infty e^{-t\lambda}\xi_{12}(\lambda)d\lambda\\ &=-\xi_{12}\,.\end{aligned} \tag{4.67}$$

Let $H_0=-\Delta_{H^{2,2}(\mathbb{R}^2)}$. We will prove that

$$\lim_{t\downarrow 0}[\mathrm{Tr}(e^{-tH_1}-e^{-tH_0})]=-\tfrac{1}{2}F\,. \tag{4.68}$$

This, with the analogous calculation for H_2, yields

$$\xi_{12}=F\,. \tag{4.69}$$

To prove (4.68), we expand e^{-tH_1} perturbatively (Du Hamel expansion) and obtain

$$\mathrm{Tr}(e^{-tH_1}-e^{-tH_0})=\alpha+\beta\,, \tag{4.70}$$
$$\begin{aligned}&\alpha=-t\,\mathrm{Tr}(e^{-tH_0}b)\,,\\ &\beta=\int_0^t s\,\mathrm{Tr}(e^{-sH_0}be^{-(t-s)H_1}b)ds\,.\end{aligned} \tag{4.71}$$

Since $(e^{-tH_0})(x,x)=(4\pi t)^{-1}$, we have

$$\alpha=-t(4\pi t)^{-1}\int_{\mathbb{R}^2}b(x)d^2x=-\tfrac{1}{2}F \tag{4.72}$$

so we need only show that

$$\lim_{t\downarrow 0}\beta=0\,. \tag{4.73}$$

By the Schwarz inequality

$$\mathrm{Tr}(e^{-sH_0}be^{-(t-s)H_1}b)\leqslant\gamma^{1/2}\delta^{1/2}\,, \tag{4.74}$$

$$\begin{aligned}\gamma&=\mathrm{Tr}(e^{-2sH_0}b^2)=(8\pi s)^{-1}\int_{\mathbb{R}^2}b^2\,d^2x\,,\\ \delta&=\mathrm{Tr}(e^{-2(t-s)H_1}b^2)\leqslant e^{2(t-s)\|b\|_\infty}\,\mathrm{Tr}(e^{-2(t-s)H_0}b^2)\\ &=e^{2(t-s)\|b\|_\infty}(8\pi(t-s))^{-1}\int_{\mathbb{R}^2}b^2\,d^2x\,,\end{aligned} \tag{4.75}$$

where we have used the diamagnetic inequalities (see Ref. 59 and references therein). Thus

$$\beta \leqslant \left(\int_{\mathbb{R}^2} b^2 d^2x\right) e^{+2t\|b\|_\infty} (8\pi)^{-1} \int_0^t s^{1/2}(t-s)^{-1/2} ds \tag{4.76}$$

goes to zero as $t \downarrow 0$.

Method 2: This is essentially the Laplace transform of method 1. Since $\Delta(z) = \Delta$ is independent of z, we can calculate it in the $z \to \infty$ limit. To do this, we infer from the proof of Lemma 2.7 that

$$\begin{aligned}\Delta(z) &= z \operatorname{Tr}[(H_2-z)^{-1}V_{12}(H_2-z)^{-1}] \\ &\quad - z \operatorname{Tr}\{[1+u_{12}(H_2-z)^{-1}v_{12}]^{-1}u_{12}(H_2-z)^{-1} \\ &\quad \times v_{12}u_{12}(H_2-z)^{-2}v_{12}\}, \quad z \in \mathbb{C}\setminus[0,\infty).\end{aligned} \tag{4.77}$$

Next, we employ the resolvent equation giving

$$\begin{aligned}(H_2-z)^{-1}V_{12}(H_2-z)^{-1} &= (H_0-z)^{-1}V_{12}(H_0-z)^{-1} \\ &\quad - (H_2-z)^{-1}V_2(H_0-z)^{-1}V_{12}(H_0-z)^{-1} \\ &\quad - (H_0-z)^{-1}V_{12}(H_0-z)^{-1}V_2(H_2-z)^{-1} \\ &\quad + (H_2-z)^{-1}V_2(H_0-z)^{-1} \\ &\quad \times V_{12}(H_0-z)^{-1}V_2(H_2-z)^{-1}, \quad z \in \mathbb{C}\setminus[0,\infty),\end{aligned} \tag{4.78}$$

where

$$H_0 = -\Delta|_{H^{2,2}(\mathbb{R}^2)}, \quad V_{12}(x) = 2b(x),$$
$$V_2 = 2ia\nabla + i(\nabla a) + a^2 - b. \tag{4.79}$$

Then estimates of the type[35]

$$\|w(H_0-z)^{-1}\|_2^2 \leqslant C\|w\|_2^2|z|^{-1},$$
$$\operatorname{Im} z^{1/2} > 0, \quad w \in L^2(\mathbb{R}^2), \tag{4.80}$$

and, e.g.,

$$\begin{aligned}&\|(H_2-z)^{-1}V_2(H_0-z)^{-1}V_{12}(H_0-z)^{-1}\|_1 \\ &\quad \leqslant \|(H_2-z)^{-1/2}\| \, \|(H_2-z)^{-1/2}V_2\| \\ &\quad \times \|(H_0-z)^{-1}u_{12}\|_2 \, \|v_{12}(H_0-z)^{-1}\|_2 \\ &\quad \leqslant C|z|^{-1}|\operatorname{Im} z|^{-1/2}, \quad |\operatorname{Re} z| \leqslant C_1|\operatorname{Im} z|\end{aligned} \tag{4.81}$$

imply [cf. Eq. (2.21)] that

$$\begin{aligned}&\lim_{\substack{|z|\to\infty \\ |\operatorname{Re} z| \leqslant C_1|\operatorname{Im} z|}} z \operatorname{Tr}[(H_2-z)^{-1}V_{12}(H_2-z)^{-1}] \\ &\quad = \lim_{\substack{|z|\to\infty \\ |\operatorname{Re} z| \leqslant C_1|\operatorname{Im} z|}} z \operatorname{Tr}[(H_0-z)^{-1}V_{12}(H_0-z)^{-1}] \\ &\quad = -(2\pi)^{-1}\int_{\mathbb{R}^2} d^2x\, b(x) = -F.\end{aligned} \tag{4.82}$$

Similarly, we get

$$\begin{aligned}&\|[1+u_{12}(H_2-z)^{-1}v_{12}]^{-1}u_1(H_2-z)^{-1} \\ &\quad \times v_{12}u_{12}(H_{12}-z)^{-2}v_{12}\|_1 \\ &\quad \leqslant C\|u_{12}(H_2-z)^{-1}v_{12}\| \, \|(H_0-z)(H_2-z)^{-1}\|^2 \\ &\quad \times \|u_{12}(H_0-z)^{-1}\|_2 \, \|(H_0-z)^{-1}v_{12}\|_2 \\ &\quad \leqslant C'|z|^{-1}\|u_{12}(H_2-z)^{-1}v_{12}\| = o(|z|^{-1}) \\ &\quad \text{as } |z|\to\infty, \quad |\operatorname{Re} z| \leqslant C_1|\operatorname{Im} z|.\end{aligned} \tag{4.83}$$

Inequality (4.83) follows from the fact that

$$\|u_{12}(H_0-z)^{-1}v_{12}\|_2 \xrightarrow[|\operatorname{Re} z| \leqslant C_1|\operatorname{Im} z|]{|z|\to\infty} 0, \tag{4.84}$$

which in turn is a consequence of the Hankel function estimate

$$|H_0^{(1)}(\sqrt{z}|x-y|)|^2$$
$$\leqslant d_1 + d_2(\ln|x-y|)^2, \quad \operatorname{Im}\sqrt{z} \geqslant \mu > 0, \tag{4.85}$$

and dominated convergence. Relation (4.80) then shows

$$\|u_{12}(H_2-z)^{-1}v_{12}\|_2 \xrightarrow[|\operatorname{Re} z| \leqslant C_1|\operatorname{Im} z|]{|z|\to\infty} 0, \tag{4.86}$$

where we have again used the resolvent equation and Eq. (4.84). Thus we have shown that $\Delta(\infty) = -\mathscr{A} = -F$, which completes the derivation of Eq. (4.60).

The result of Aharonov–Casher[58] implies that $\dim \operatorname{Ker}(H_1) - \dim \operatorname{Ker}(H_2)$ differs from Δ by at most 1. It would be nice to know why this is true.

We remark that the result (4.60) has been obtained in Ref. 24 by using certain approximations in a path integral approach. The above treatment seems to be the first rigorous and nonperturbative one.

To complete this discussion, we still mention that the (regularized) spectral asymmetry, $\eta_m(t)$, associated with this magnetic field example (4.5) after replacing H_j by $H_j + m^2$ [Q by Q_m, cf. Eq. (3.36)] can be calculated using the result (4.61) and Eq. (3.41). One easily gets

$$\eta_m(t) = \operatorname{sgn}(m) F e^{-tm^2}, \quad m \in \mathbb{R}\setminus\{0\}, \quad t > 0, \tag{4.87}$$

containing in the limit $t \to 0_+$ the known result for η_m (cf., e.g., Ref. 2).

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