5. *wM-Spaces and Closed Maps*

By Tadashi ISHII
Utsunomiya University

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1970)

1. Introduction. In our previous paper [5], we introduced the notion of wM-spaces, which is a generalization of M-spaces (due to K. Morita [8]). A topological space X is called a wM-space if there exists a sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying the condition below:

$$(\mathrm{M}_2)\begin{cases}\text{If } \{K_n\} \text{ is a decreasing sequence of non-empty subsets of } X \text{ such} \\ \text{that } K_n \subset \mathrm{St}^2(x_0, \mathfrak{A}_n) \text{ for each } n \text{ and for some fixed point } x_0 \text{ of } X, \\ \text{then } \cap \bar{K}_n \neq \emptyset.\end{cases}$$

In the above definition, we may assume without loss of generality that $\{\mathfrak{A}_n\}$ is decreasing. Throughout this paper, we assume at least T_1 for every topological space unless otherwise specified.

The purpose of this paper is to show the following theorems:

(I) The image of a wM-space under a quasi-perfect map is also a wM-space (Theorem 2.1).[1]

(II) If $f: X \to Y$ is a closed continuous map of a wM-space X onto a space Y, then $Y = \bigcup_{n=0}^{\infty} Y_n$, where Y_n is discrete in Y for $n = 1, 2, \cdots$, and $f^{-1}(y)$ is countably compact for $y \in Y_0$ (Theorem 3.1).

(III) Let X be a regular space which has a sequence $\{\mathfrak{A}_n\}$ of point finite coverings of X satisfying the condition (*) below:

$$(*)\begin{cases}\text{If } \{K_n\} \text{ is a decreasing sequence of non-empty subsets of } X \text{ such} \\ \text{that } K_n \text{ is contained in some } U_n \in \mathfrak{A}_n \text{ for each } n, \text{ then } \cap \bar{K}_n \neq \emptyset.\end{cases}$$

If $f: X \to Y$ is a closed continuous map of X onto a regular space Y, then $Y = \bigcup_{n=0}^{\infty} Y_n$, where Y_n is discrete in Y for $n = 1, 2, \cdots$, and $f^{-1}(y)$ is countably compact for $y \in Y_0$ (Theorem 3.2).

(II) was proved by N. Lašnev [6] for metric speces and by V. V. Filippov [3] for paracompact p-spaces (due to A. Arhangel'skii [1]), and (III) was proved by A. Arhangel'skii [2] for point-paracompact G_δ-spaces.[2] It should be noted that, in a space X with a complete structure, any closed and countably compact subset of X is compact.

---

1) A quasi-perfect map $f: X \to Y$ is a closed continuous surjective map such that $f^{-1}(y)$ is countably compact for $y \in Y$.

2) Paracompact p-spaces are identical with paracompact M-spaces. Filippov [3] essentially proved (II) for M-spaces.

2. The images of *wM*-spaces under quasi-perfect maps.

Theorem 2.1. *Let $f: X \to Y$ be a quasi-perfect map of a wM-space X onto a space Y. Then Y is also a wM-space.*

Proof. Let $\{\mathfrak{A}_n\}$ be a decreasing sequence of open coverings of X satisfying (M_2). Let us put

$$V_n(y) = Y - f(X - \mathrm{St}(f^{-1}(y), \mathfrak{A}_n)),$$
$$\mathfrak{B}_n = \{V_n(y) \mid y \in Y\}$$

for each n and for each point y of Y. Then it is easy to verify that $V_n(y)$ are open subsets of Y such that $y \in V_n(y)$, $V_{n+1}(y) \subset V_n(y)$ and $f^{-1}(V_n(y)) \subset \mathrm{St}(f^{-1}(y), \mathfrak{A}_n)$. We now prove that the sequence $\{\mathfrak{B}_n\}$ of open coverings of Y satisfies (M_2). For this purpose, by [5, Theorem 2.1], it is sufficient to prove that, for any discrete sequence $\{y_n\}$ of points of Y, $\{\mathrm{St}(y_n, \mathfrak{B}_n) \mid n=1,2,\cdots\}$ is locally finite in Y. Suppose that this is not valid for some discrete sequence $\{y_n\}$ of points of Y. Then there exist a point y_0 of Y and a sequence $\{n(i) \mid i=1,2,\cdots\}$ of positive integers such that $V_i(y_0) \cap \mathrm{St}(y_{n(i)}, \mathfrak{B}_{n(i)}) \neq \emptyset$, $i=1,2,\cdots$, and $n(1) < \cdots < n(i) < \cdots$. Let $z_i \in V_i(y_0) \cap \mathrm{St}(y_{n(i)}, \mathfrak{B}_{n(i)})$. Then, from $z_i \in V_i(y_0)$, it follows that the sequence $\{z_i\}$ has an accumulation point in Y. Indeed, let $t_i \in f^{-1}(z_i)$. Then $t_i \in f^{-1}(V_i(y_0)) \subset \mathrm{St}(f^{-1}(y_0), \mathfrak{A}_i)$, and hence it is easily proved that the sequence $\{t_i\}$ has an accumulation point in X, because $f^{-1}(y_0)$ is countably compact. Therefore the sequence $\{z_i\}$ has an accumulation point in Y. On the other hand, from $z_i \in \mathrm{St}(y_{n(i)}, \mathfrak{B}_{n(i)})$, it follows that the sequence $\{z_i\}$ has no accumulation point in Y. Indeed, let u_i be the points of Y such that $y_{n(i)} \in V_{n(i)}(u_i)$ and $z_i \in V_{n(i)}(u_i)$. Then, since $f^{-1}(V_{n(i)}(u_i)) \subset \mathrm{St}(f^{-1}(u_i), \mathfrak{A}_{n(i)})$, the sets $f^{-1}(y_{n(i)})$ and $f^{-1}(z_{n(i)})$ are contained in $\mathrm{St}(f^{-1}(u_i), \mathfrak{A}_{n(i)})$. Hence we have $f^{-1}(u_i) \cap \mathrm{St}(f^{-1}(y_{n(i)}), \mathfrak{A}_{n(i)}) \neq \emptyset$. Let $s_i \in f^{-1}(u_i) \cap \mathrm{St}(f^{-1}(y_{n(i)}), \mathfrak{A}_{n(i)})$. Since $\{f^{-1}(y_n) \mid n=1,2,\cdots\}$ is a discrete collection of subsets of a *wM*-space X, $\{\mathrm{St}(f^{-1}(y_n), \mathfrak{A}_n) \mid n=1,2,\cdots\}$ is locally finite in X by [5, Theorem 2.1], and hence the sequence $\{s_i\}$ has no accumulation point in X. Accordingly, the sequence $\{u_i\}$ has no accumulation point in Y, because f is closed and $u_i = f(s_i)$. Therefore, by [5, Theorem 2.1], $\{\mathrm{St}(f^{-1}(u_i), \mathfrak{A}_{n(i)}) \mid i=1,2,\cdots\}$ is locally finite in X. This implies that $\{f^{-1}(z_i)\}$ is also locally finite in X, because $f^{-1}(z_i) \subset \mathrm{St}(f^{-1}(u_i), \mathfrak{A}_{n(i)})$. Hence the sequence $\{z_i\}$ has no accumulation point in Y, which is a contradiction. Thus we complete the proof.

As an application of Theorem 2.1, we can prove the following theorem.

Theorem 2.2. *Let Y be the image under a closed continuous map f of a completely regular wM-space X. Then the following statements are equivalent.*

(1) *Y is a wM-space.*

(2) *Y is a q-space (due to E. Michael* [7]).

(3) *The boundary $\mathfrak{B}f^{-1}(y)$ of the set $f^{-1}(y)$ is countably compact for every point y of Y.*

In our previous paper [4], we proved a similar result for normal M-spaces. Before proving Theorem 2.2, we mention a lemma.

Lemma 2.3. *If X is a completely regular wM-space which is pseudo-compact, then it is countably compact.*

Proof. Let X be a completely regular wM-space with a sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2), and suppose that X is pseudo-compact but not countably compact. Then there exists a discrete sequence $\{x_n\}$ of points of X, and hence $\{\mathrm{St}(x_n, \mathfrak{A}_n) \mid n=1, 2, \cdots\}$ is locally finite in X by [5, Theorem 2.1]. Since X is completely regular, there exists, for each n, a real-valued continuous function $h_n(x)$ on X such that $h_n(x_n)=n$ and $h_n(x)=0$ for $x \in X-\mathrm{St}(x_n, \mathfrak{A}_n)$. Let us put $h(x)=\Sigma h_n(x)$. Then $h(x)$ is an unbounded continuous function on X. This is a contradiction, because X is pseudo-compact. Thus we complete the proof.

Proof of Theorem 2.2. (1)→(2). This implication is trivial. (2)→(3). If Y is a q-space, then $\mathfrak{B}f^{-1}(y)$ is pseudo-compact for each point y of Y by a theorem of E. Michael [7, Theorem 2.1]. Since $\mathfrak{B}f^{-1}(y)$ is closed in X, it is a wM-space as a subspace of X. Hence $\mathfrak{B}f^{-1}(y)$ is countably compact by Lemma 2.3.

(3)→(1). By Theorem 2.1, this implication is proved along the same line as in the proof of [4, Theorem 4.1]. Hence we omitt the proof.

3. Closed maps and countably compact sets.

Theorem 3.1. *Let $f: X \to Y$ be a closed continuous map of a wM-space X onto a space Y. Then $Y=\bigcup_{n=0}^{\infty} Y_n$, where Y_n is discrete for $n=1, 2, \cdots$, and $f^{-1}(y)$ is countably compact for $y \in Y_0$.*

Proof. Let X be a wM-space with a decreasing sequence $\{\mathfrak{A}_n\}$ of open coverings of X satisfying (M_2). Let us put

$$H_n(f^{-1}(y))=f^{-1}(Y-f(X-\mathrm{St}(f^{-1}(y), \mathfrak{A}_n)))$$

for each n and for each point y of Y. Then, since f is closed, $H_n(f^{-1}(y))$ are open subsets of X such that $f^{-1}(y) \subset H_n(f^{-1}(y)) \subset \mathrm{St}(f^{-1}(y), \mathfrak{A}_n)$. For a given n, we denote by Y_n a subset of Y consisting of points y such that $f^{-1}(y)$ is contained in no $H_n(f^{-1}(y'))$ for $y' \neq y$. We shall prove that Y_n is discrete in Y for each n. For this purpose, it is sufficient to prove that $\{f^{-1}(y) \mid y \in Y_n\}$ is a discrete collection of subsets of X, because f is closed. Let x_0 be an arbitrary point of X, and put $y_0=f(x_0)$. If $y_0 \notin Y_n$, then a neighborhood $H_n(f^{-1}(y_0))$ of x_0 cannot intersect every member of $\{f^{-1}(y) \mid y \in Y_n\}$; otherwise, $H_n(f^{-1}(y_0))$ intersects some $f^{-1}(y)$ such that $y \in Y_n$, which implies $f^{-1}(y) \subset H_n(f^{-1}(y_0))$

by the definition of $H_n(f^{-1}(y_0))$, but this is impossible, because $y \in Y_n$. If $y_0 \in Y_n$, then a neighborhood $H_n(f^{-1}(y_0))$ of x_0 cannot intersect every member of $\{f^{-1}(y) | y \in Y_n, y \neq y_0\}$ by the same argument as above. Consequently, $\{f^{-1}(y) | y \in Y_n\}$ is a discrete collection, and hence Y_n is discrete in Y. It remains to prove that $f^{-1}(y)$ is countably compact for $y \in Y_0 = Y - \bigcup_{n=1}^{\infty} Y_n$. Suppose that $f^{-1}(y_0)$ is not countably compact for some point y_0 of Y_0; then there exists a discrete sequence $\{x_n\}$ of points of $f^{-1}(y_0)$. Since $y_0 \in Y_0$, there exists, for each n, a point z_n of Y such that $f^{-1}(y_0) \subset H_n(f^{-1}(z_n))$. Let $x_0 \in f^{-1}(y_0)$. Then $\mathrm{St}(x_0, \mathfrak{A}_n) \cap f^{-1}(z_n) \neq \emptyset$ for $n=1, 2, \cdots$, because $x_0 \in f^{-1}(y_0) \subset \mathrm{St}(f^{-1}(z_n), \mathfrak{A}_n)$. Let $u_n \in \mathrm{St}(x_0, \mathfrak{A}_n) \cap f^{-1}(z_n)$. Then the sequence $\{u_n\}$ has an accumulation point in X by (M_1), and hence the sequence $\{z_n\}$ has also an accumulation point in Y. On the other hand, since $x_n \in f^{-1}(y_0) \subset H_n(f^{-1}(z_n))$, we have $\mathrm{St}(x_n, \mathfrak{A}_n) \cap f^{-1}(z_n) \neq \emptyset$. Let $v_n \in \mathrm{St}(x_n, \mathfrak{A}_n) \cap f^{-1}(z_n)$. Then the sequence $\{v_n\}$ has no accumulation point in X, because $\{\mathrm{St}(x_n, \mathfrak{A}_n) | n=1, 2, \cdots\}$ is locally finite in X by [5, Theorem 2.1]. Hence the sequence $\{z_n\}$ has no accumulation point in Y by closedness of f, which is a contradiction. Thus we complete the proof.

Theorem 3.2. *Let X be a regular space with a sequence $\{\mathfrak{A}_n\}$ of point-finite open coverings of X satisfying $(*)$, and $f: X \to Y$ a closed continuous map of X onto a regular space Y. Then $Y = \bigcup_{n=0}^{\infty} Y_n$, where Y_n is discrete in Y for each n, and $f^{-1}(y)$ is countably compact for $y \in Y_0$.*

Before proving Theorem 3.2, we mention some definitions (due to J. Nagata [9]) and lemmas. A space X is a called a quasi-k-space if a subset F of X is closed if and only if $F \cap C$ is closed in X for every countably compact subset C of X. A sequence $\{U_n(x)\}$ of open neighborhoods of a point x of a space X is called a q-sequence of neighborhoods if $U_1(x) \supset \overline{U_2(x)} \supset U_2(x) \supset \overline{U_3(x)} \supset \cdots$ and if any sequence $\{x_n\}$ of points of X satisfying $x_n \in U_n(x)$ for each n has an accumulation point in X.

Lemma 3.3. *If X is a regular space such that each point x of X has a q-sequence $\{U_n(x)\}$ of neighborhoods, then X is a quasi-k-space.*

Proof. Suppose that there exists a subset F of X such that $F \cap C$ is closed in X for every countably compact subset C of X and that F is not closed. Let $x_0 \in \bar{F} - F$, and $C(x_0) = \bigcap U_n(x_0)$. Since $C(x_0)$ is countably compact, $F \cap C(x_0)$ is closed in X. Hence, if we put $V_n(x_0) = U_n(x_0) - F \cap C(x_0)$, $V_n(x_0)$ are open subsets of X containing x_0. Since X is a regular space, there exists a q-sequence $W_n(x_0)$ of neighborhoods of x_0 such that $W_n(x_0) \subset V_n(x_0)$ for each n. Let $x_n \in W_n(x_0) \cap F$, $n=1, 2, \cdots$, and $A = \{x_n | n=1, 2, \cdots\}$. Then $(\bar{A} - A) \cap (F \cap C(x_0))$

$=\emptyset$, because $\overline{W_n(x_0)}\cap(F\cap C(x_0))=\emptyset$ for $n\geqq 2$. This implies that $(\bar{A}-A)\cap F=\emptyset$, and hence we have $\bar{A}\cap F=A\cap F=A$. Since $\bar{A}$ is countably compact in $X, \bar{A}\cap F$ is closed in X, which shows that A is closed. Therefore A has no accumulation point in X, because $A\cap C(x_0)=\emptyset$. This is a contradiction. Thus we complete the proof.

Lemma 3.4. *Let X be a regular quasi-k-space, and $\mathfrak{A}$ a point-finite open covering of X. If $f:X\to Y$ is a closed continuous map of X onto a regular space Y, then*

$$N=\{y\in Y \mid \text{no finite } \mathfrak{A}'\subset\mathfrak{A} \text{ covers } f^{-1}(y)\}$$

is discrete in Y.

Since the lemma can be proved by the similar way as in the proof of [2, Lemma 1.2], we omitt the proof. But it should be noted that if $f:X\to Y$ is a closed continuous map of a regular quasi-k-space X onto a regular space Y, then Y is also a quasi-k-space by a theorem of J. Nagata [9, Theorem 1].

Proof of Theorem 3.2. For each point x of X, we select a sequence $\{U_n\}$ such that $x\in U_n$ and $U_n\in\mathfrak{A}_n$ for each n. Since X is a regular space and $\{\mathfrak{A}_n\}$ satisfies $(*)$, there exists a q-sequence $\{V_n(x)\}$ of neighborhoods of x such that $V_n(x)\subset U_n$. Hence, by Lemma 3.3, X is a quasi-k-space. Let us put

$$Y_n=\{y\in Y \mid \text{no finite } \mathfrak{A}'_n\subset\mathfrak{A}_n \text{ covers } f^{-1}(y)\}$$

for each n. Then, by Lemma 3.4, Y_n is discrete in Y for each n. It remains to prove that $f^{-1}(y)$ is countably compact for every $y\in Y_0 =Y-\bigcup_{n=1}^{\infty} Y_n$. Suppose that $f^{-1}(y_0)$ is not countably compact for a point y_0 of Y_0; then there exists a discrete sequence $\{x_n\}$ of points of $f^{-1}(y_0)$. Since $y_0\notin\bigcup_{n=1}^{\infty} Y_n$, $f^{-1}(y_0)$ is covered with finitely many members of $\mathfrak{A}_n$ for each n. Therefore we can select with no difficulty a subsequence $\{x'_n\}$ of $\{x_n\}$ such that $\{x'_k \mid k\geqq n\}$ is contained in some $U_n\in\mathfrak{A}_n$. Consequently the sequence $\{x'_n\}$ has an accumulation point in X by $(*)$, which is a contradiction. Thus we complete the proof.

References

[1] A. Arhangel'skii: On a class of spaces containing all metric and all locally comact spaces. Dokl. Akad. Nauk SSSR, **151**, 751–754 (1963); Soviet Math. Dokl., **4**, 1051–1055 (1963).

[2] ——: On closed mappings, bicompact spaces, and a problem of P. Aleksandrov. Pacific J. Math., **18**, 201–208 (1966).

[3] V. V. Filippov: On feathered paracompacta. Soviet Math. Dokl., **9**, 161–164 (1968).

[4] T. Ishii: On closed mappings and *M*-spaces. II. Proc. Japan Acad., **43**, 757–761 (1967).

[5] ——: On *wM*-spaces. I. Proc. Japan Acad., **46**, 5–10 (1970).

[6] N. Lašnev: Continuous decompositions and closed mappings of metric spaces. Dokl. Akad. Nauk SSSR, **165**, 756 (1965); Soviet Math. Dokl., **6**, 1504–1506 (1965).
[7] E. Michael: A note on closed maps and compact sets. Israel J. Math, **2**, 173–176 (1964).
[8] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365–382 (1964).
[9] J. Nagata: Quotient and bi-quotient spaces of *M*-spaces. Proc. Japan Acad., **45**, 25–29 (1969).