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WORD MAPS, ISOTOPY AND ENTROPY

DAVID FRIED[1]

ABSTRACT. We find diffeomorphisms of low entropy in each isotopy class on $S^3 \times S^3$. These arise as word maps, a nonabelian analogue of toral automorphisms. Hyperbolic examples of equal entropy are also found. The group $\pi_0 \operatorname{Diff}(S^3 \times S^3)$ is computed.

The rule $x_{n+1} = x_n + x_{n-1}$ that generates the Fibonacci sequence can be applied to any initial pair of integers x_1, x_2: Fibonacci used 1, 1, Lucas used 1, 3, etc. One can use the corresponding multiplicative rule $x_{n+1} = x_n x_{n-1}$ for any group H and any initial pair of group elements. This defines a sort of discrete delay equation, or a "second order" transformation on H, that can be viewed as a "first-order" transformation σ on pairs of elements: $\sigma(x,y) = (y, yx)$, $\sigma: H \times H \hookleftarrow$. The Fibonacci and Lucas series are two orbits of σ for $H = \mathbf{Z}$.

This map σ is an instance of what we call a word map on two letters, since it assigns to the letters x, y a new pair of words in x, y and their inverses. Note σ is invertible and σ^{-1} is also a word map, $\sigma^{-1}(x,y) = (x^{-1}y, x)$. Such invertible word maps on two letters are determined by automorphisms of the free group $F_2 = F(x_1, x_2)$ on two generators. For example, σ corresponds to the automorphism sending x_1 to x_2 and x_2 to $x_2 x_1$. Similar remarks hold for word maps on n letters, $n \geq 1$.

The case $H = S^1$ is well known, since here an invertible word map on n letters is just an automorphism of the torus T^n of dimension n. These toral automorphisms are among the best-understood dynamical systems. Our interest is in the case H a compact connected Lie group and comparing its invertible word maps with toral automorphisms.

Suppose, for example, that H is the group $SO(d)$ of rotations of Euclidean space of dimension d. Then σ defines a transformation σ_d for each d. The case $d = 2$ gives the toral automorphism $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with topological entropy $h(\sigma_2) = \log \rho$, ρ the Golden Ratio $(1 + \sqrt{5})/2$. In §1, we will show that, surprisingly, $h(\sigma_3) = h(\sigma_2)$. In this sense, the Fibonacci transformation on $SO(3)$ is no more complicated than that for dimension 2. One has $h(\sigma_4) = 2h(\sigma_2)$, $h(\sigma_d) \geq [d/2]h(\sigma_2)$. The same holds true for any invertible word map. We do not know $h(\sigma_d)$ for $d \geq 5$.

Our main interest is in invertible word maps on $H = S^3$ on two letters. Besides computing their entropy we show in §2 that they furnish representatives for all isotopy classes of the 6-manifold $M = S^3 \times S^3$. This determines the group structure of $\pi = \pi_0 \operatorname{Diff} M$, the group of isotopy classes of diffeomorphisms of M. Previously one knew a certain three-step filtration of π without knowing what the extensions

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were: indeed, Kreck gave such a description for a large class of smooth manifolds [**Kr**].

This new information on π has a pleasant dynamical consequence. The natural homomorphism $\mathrm{Diff}\, M \to Gl_2\mathbf{Z}$ that assigns to diffeomorphism its action on $H_3(M;\mathbf{Z}) \cong \mathbf{Z}^2$ is onto, just like the homomorphism $\mathrm{Diff}\, T^2 \to Gl_2\mathbf{Z}$. But whereas the linear action of $Gl_2\mathbf{Z}$ on T^2 splits the latter map, there is no splitting of the former. In fact we show there is not even a splitting modulo isotopy; i.e., the extension $\pi \to Gl_2\mathbf{Z}$ does not split. This depends critically on the existence of exotic 7-spheres.

Given a diffeomorphism f of a compact manifold X, we let $s(f)$ be the spectral radius of the homology map $f_*: H_*(X;\mathbf{R}) \hookleftarrow$. Shub's entropy conjecture [**S**] is that $h(f) \geq \log s(f)$. On a simply connected manifold, $\log s(f)$ is the only topological constant available that might so bound entropy. This conjecture is not known for $X = M$, but our results show that each isotopy class on M has a representative f with $h(f) = \log s(f)$. So, modulo this difficult conjecture, such f's should be representatives of greatest "efficiency".[2]

We go further (in §3) and show that f can be chosen structurally stable. We do this by an isentropic deformation of a word map, i.e., one that does not change entropy. This f can even be chosen fitted, which is compatible with conjectures of Shub that the lowest entropy occurs for fitted diffeomorphisms, which are hyperbolic transformations built using handlebodies [**S**]. Some counterexamples for X not simply connected are known, e.g., for $X = T^4$; these also use word maps [**F2**]. It is conceivable that these give counterexamples on $M \times M = (S^3)^4$.

Thus, in passing from T^2 to M, the behavior of entropy in an isotopy class seems to persist. But we do not even know that word maps are points of lower semicontinuity for entropy on $\mathrm{Diff}\, M$, let alone that they are minima on their components. It would be interesting to show they are local minima.

In the course of our computations, we find a surprising deformation of the action of $PGl_2\mathbf{Z} = \Gamma$ on $T^2/\pm 1$ induced from the linear action of $Gl_2\mathbf{Z}$ on T^2. The deformation passes through analytic actions of Γ on S^2 that preserve area, and it ends with a finite group of orthogonal motions of S^2. We do not know whether the intermediate actions are ergodic.

We thank Dan Asimov and Bill Goldman for their interesting observations.

1. Word maps. Let F_n be the free group on generators $x_1, \ldots, x_n$, and let Φ_n be the automorphism group of F_n. Given any group H, there is a natural bijection of H^n with the collection $\mathrm{Hom}(F_n, H)$ of homomorphisms from F_n to H given by evaluating the homomorphism on the generators. This gives an action of Φ_n on H^n. If $\alpha \in \Phi_n$, then we denote the corresponding bijection of H^n by α_H. We call α_H a word map on n letters since it assigns to an n-tuple in H an n-tuple of words in the entries.

For example, suppose $n = 2$, $\alpha(x_1) = x_2$ and $\alpha(x_2) = x_2x_1$. Then $\alpha_H(h_1, h_2) = (h_2, h_2h_1)$ for $h_1, h_2 \in H$, so $\alpha_H = \sigma$ is the Fibonacci map discussed above.

When $H = \mathbf{R}$ or $\mathbf{C}$, the α_H are linear maps. Indeed, when H is abelian, Φ_n acts by group automorphisms and the action factors through $Gl_n\mathbf{Z}$, the automorphism group of the abelianization of F_n.

[2] Yomdin has recently proven the conjecture for $C^\infty f$.

For general H, H acts on H^n by conjugation:

$$h \cdot (h_1, \ldots, h_n) = (hh_1h^{-1}, \ldots, hh_nh^{-1}).$$

Let $C_n = C_n(H)$ be the orbit space. Then the actions of Φ_n and H on H^n commute, so Φ_n acts on C_n. In this action, inner automorphisms act trivially, and so there is an induced action of the outer automorphism group $\operatorname{Out} F_n$ ($= \Phi_n/F_n$ for $n \geq 2$) on C_n. We denote by $\overline{\alpha}_H$ the map on C_n induced by α_H.

Now we fix $n = 2$. We need some results due to Nielsen [**N**, **MKS**]. First, each $\alpha \in \Phi_2$ "preserves" the commutator of $x_1^{-1}x_2^{-1}x_1x_2 = c(x_1, x_2)$; i.e., $\alpha(c(x_1, x_2)) = gc(x_1, x_2)^{\varepsilon}g^{-1}$, for some $\varepsilon = \pm 1$, $g \in F_2$. This implies that the commutator map $c\colon H^2 \to H$, $c(x, y) = x^{-1}y^{-1}xy$, induces a map $\overline{c}\colon C_2 \to C_1$ that is equivariant for the actions of $\operatorname{Out} F_i$, $i = 1, 2$, under the homomorphism $\varepsilon\colon F_2 \to \operatorname{Out} F_1$. Second, the natural map $\operatorname{Out} F_2 \to Gl_2\mathbf{Z}$ is an isomorphism. So $\ker(\varepsilon) = Sl_2\mathbf{Z}$ acts on C_2, and $\overline{c}$ is a $Sl_2\mathbf{Z}$-invariant function. Note $\overline{c}$ is constant iff H is abelian.

Now we fix a compact connected Lie group H. Let T be a maximal torus and $W = N(T)/T$ the Weyl group, where $N(T)$ is the normalizer of T in H. For the obvious action of W on T, it is well known that the map $T/W \to C_1$ is a homeomorphism, where T/W is the orbit space of T under the finite group W. This identifies the image of $\overline{c}$ with the orbifold T/W of dimension $r = \dim T$.

We trivialize the tangent space of H^2 by left translation and fix an ad-invariant inner product on the Lie algebra $\mathcal{H}$ of H. This gives H, H^2 bi-invariant Riemannian metrics. Then the permutation $\alpha(x_1) = x_2$, $\alpha(x_2) = x_1$ determines an isometry α_H of H^2, and the shear $\beta(x_1) = x_1x_2$, $\beta(x_1) = x_2$ has differential $T\beta_H = \left(\begin{smallmatrix} \Delta & 0 \\ I & I \end{smallmatrix}\right)$ at (h_1, h_2), where $\Delta = \operatorname{ad} h_2 \in O(\mathcal{H})$. Taking $\gamma \in \Phi_2$ to be any positive word in α, β, we find

$$T\gamma_H = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where the operator norms of A_{ij} satisfy $\|A_{ij}\| \leq a_{ij}$, where the a_{ij} are the entries of the image $\gamma_{\mathbf{R}}$ of γ in $Gl_2\mathbf{Z} \subset Gl_2\mathbf{R}$. This implies that $\|T\gamma_H\| \leq \|\gamma_{\mathbf{R}}\|$ at all points of H^2.

Because the action of H on H^2 is isometric, there is an induced Riemannian metric on the dense open set $\mathcal{O}$ of points in C_2 with least isotropy. This gives a Riemannian metric on a regular level of $\overline{c}_2$ in $\mathcal{O}$ on which $\|T\overline{\gamma}_H\| \leq \|\gamma_{\mathbf{R}}\|$

But any α in $Sl_2\mathbf{Z}$ of infinite order is conjugate to $\pm\gamma$, γ of the above form. We compute in $PSl_2\mathbf{Z} = Sl_2\mathbf{Z}/\pm 1 \cong \langle a, b | a^2 = b^3 = 1\rangle$, where $a = \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $b = \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. Then $ab = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $ab^{-1} = \pm\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. One can conjugate any element of infinite order to $\prod_i (ab)^{c_i}(ab^{-1})^{d_i}$. Now write $ab^{-1} = \pm\beta_{\mathbf{Z}}$ and $ab = \pm\alpha_{\mathbf{Z}}\beta_{\mathbf{Z}}\alpha_{\mathbf{Z}}$.

It follows that for any α, $\|T\alpha_H^n\|$ is of the order $\|\alpha_{\mathbf{R}}^n\|$, for all $n > 0$, hence of the order r_α^n, where r_α is the spectral radius of $\alpha_{\mathbf{R}}$.

We specialize to the case $H = S^3$, the group of unit quaternions. In this case we can identify C_2 with a compact region P in $\mathbf{R}^3$ via the real part of the graph of the group law

$$\tau[x, y] = (\operatorname{Re} x, \operatorname{Re} y, \operatorname{Re} xy),$$

where $[x, y]$ denotes the conjugacy class in C_2 of the pair $(x, y) \in S^3 \times S^3$. Clearly τ is well defined because the real part of a quaternion is invariant under conjugation.

One can check that τ is 1-1 by conjugating (x, y) to normal form with $x \in \mathbf{C}$ and $y \in \mathbf{C} + \mathbf{R}j$.

We identify C_1 with $[0,1]$ by the map $N[x] = \|1-x\|^2/4$. Then $\bar{c}\colon C_2 \to C_1$ can be factored through the cubic polynomial

$$p(r,s,t) = 1 - (r^2+s^2+t^2) + 2rst;$$

i.e., $p\tau[x,y] = N\bar{c}[x,y]$. Again this is easily checked in normal form. One sees that p has critical points $(0,0,0)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $|\varepsilon_i| = 1$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 = 1$. It is clear that P lies in $[-1,1]^3 \cap p^{-1}[0,1]$, and one checks these are equal. This was noted independently by Dan Asimov, who observed that P is the region spanned by the three possible pairs of correlations of three independent random variables. He coined the term "tetrahedral pillow" for P. This captures the fact that P is obtained from a tetrahedron inscribed in $[-1,1]^3$ by rounding the edges so that only the four vertices are singularities of ∂P. Note that P meets each side of this cube in a diagonal segment.

We denote the level set $p^{-1}(v) \cap P$ by L_v. For $v = 0$, $L_0 = \partial P \cong S^1 \times S^1/\pm 1$ under τ, $S^1 \subset S^3$ any one parameter subgroup. For $0 < v < 1$, L_t is an analytic S^2. For $v = 1$, L_1 is the origin.

The action of $Gl_2\mathbf{Z}$ preserves these levels. The action on $\partial P = L_0$ is the usual action on $T^2/\pm 1$ (note the vertex singularities in ∂P correspond to the fixed points of the involution of T^2).

The action of $Gl_2\mathbf{Z}$ on P extends to a polynomial action on $\mathbf{R}^3$ that preserves volume. This can be checked on generators using the normal form: e.g., for the shear β,

$$\overline{\beta}_H(r,s,t) = (t, s, 2st - r).$$

It follows that each level L_v, $0 < v < 1$, inherits an analytic area form invariant by Γ.

Now we use S. Katok's theorem that the entropy of a diffeomorphism f of a compact manifold is bounded by $\log s(f^{\#})$, where $f^{\#}$ is the action of f on differential forms and s denotes spectral radius [**K**]. Using the invariant area, we see that the action $f_2^{\#}$ of $f = \overline{\alpha}_{S^3}|L_v$, $0 < v < 1$, on 2-forms has spectral radius zero. Because the same holds trivially for 0-forms, $s(f^{\#}) = s(f_1^{\#}) \leq \log r_\alpha$, by our estimates on $\|T\alpha_{S^3}^n\|$, $n > 0$. Altogether $h(f) \leq \log r_\alpha$.

For $v = 0$, $h(\overline{\alpha}_{S^3}|S_0) = h(\overline{\alpha}_{S^1}) = \log r_\alpha$. Thus by [**B**, Corollary 18], $h(\overline{\alpha}_{S^3}) = \sup_v h(\overline{\alpha}_{S^3}|L_v) = \log r_\alpha$. Since $\overline{\alpha}_{S^3}$ is the factor of α_{S^3} by the action of the compact group H and the entropy of α_{S^3} on each H-orbit is clearly zero, [**B**, Theorem 17] implies $h(\alpha_{S^3}) = h(\overline{\alpha}_{S^3}) = \log r_\alpha$.

It follows that for all $\alpha \in \Phi_2$ (even with $\varepsilon = -1$) $h(\alpha_{S^3}) = \log r_\alpha$, since both sides double when α is replaced by α^2. Note that Φ_1 acts trivially on C_1, so the action of $Gl_2\mathbf{Z}$ on C_2 leaves $\bar{c}$ invariant.

For $f = \alpha_{S^3}$, the action of f_* on $H^3(S^3 \times S^3; \mathbf{Z}) \cong \mathbf{Z}^2$ is just $\alpha_{\mathbf{Z}}$. Hence r_α is the spectral radius $s(f)$ of the action of f on real homology. We have

THEOREM 1. *For $\alpha \in \Phi_2$, $h(\alpha_{S^3}) = h(\overline{\alpha}_{S^3}) = h(\alpha_{S^1}) = \log r_\alpha$, where r_α is the spectral radius of $\alpha_{\mathbf{R}}$ or, equally, the spectral radius of the map induced by α_{S^3} (or α_{S^1}) on real homology.*

The induced action of $Gl_2\mathbf{Z}$ on $C_2(S^3)$ preserves the function $\|xy - yx\|^2$. The level sets of this function define a deformation of the standard action of $Gl_2\mathbf{Z}/\pm 1$

on $T^2/\pm 1$ to the linear action on S^2 defined by the orthogonal representation

$$\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\pm\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the last statement, one checks that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ corresponds to $(r, s, t) \to (s, r, 2rs - t)$, which has order two; hence the action of Φ_2 factors through $Gl_2\mathbf{Z}/\pm 1$. Then one blows up the origin, i.e. replaces it by the sphere of directions using spherical coordinates, to obtain the time $v = 1$ action for our deformation. In this deformation the curvature of the quotient metric goes from a delta measure at the four vertices to a constant function.

Note that the entropy calculations of Theorem 1 are valid for $H = SO(3) \cong S^3/\pm 1$ as well. For $H = (S^3)^k$, one has $h(\alpha_H) = h(\overline{\alpha}_H) = k \log r_\alpha$ and the same holds for H covered by $(S^3)^k$.

To try to generalize Theorem 1 further, one might consider various unitary embeddings of the compact connected Lie group and use the resulting functions $N(x, y) = \|xy - yx\|^2$ to lower the dimension. This fails even for $H = SU(3)$: since its irreducible complex representations are tensor products of the usual representation and its dual, all representations give essentially the same N even though $\dim T = 2$. One can pass to level sets of $\overline{c}$ instead. Then $S = \overline{c}^{-1}[1]$ is the image of $T \times T$, since commuting elements of H lie in some conjugate of T, and so on this level set one has entropy $(\dim T) \log r_\alpha = h(\alpha_H|S)$. It is not clear any longer that other level sets have entropy bounded by this: They may have larger dimension than S, for instance, hence more expansion. Namely, if H is semisimple (one easily reduces to this case), then generic level sets of $\overline{c}$ have dimension $\dim H - \dim T$, S has dimension $2 \dim T$ and $\dim H > 3 \dim T$ unless H is covered by $(S^3)^k$. We do not know the precise entropy in these cases: perhaps it is $\log r_\alpha \cdot \frac{1}{2}(\dim H - \dim T)$.

It would also be interesting to know whether the action of $PSl_2\mathbf{Z}$ on each sphere L_v, $0 < v < 1$, is ergodic.

W. Goldman informed us that the invariance of $p(r, s, t)$ under the given action of $Sl_2\mathbf{Z}$ was known to Fricke and Klein. Their interest lay in a certain noncompact level surface disjoint from P which serves as Teichmuller space for the punctured torus. They viewed (r, s, t) in terms of traces of elements in $Sl_2\mathbf{R}$. Our application is to S^3, the compact form of the Lie algebra of $Sl_2\mathbf{R}$, and some of the algebra carries over.

Note that for any connected compact Lie group H with maximal torus K and any $\alpha \in \Phi_n$, $n \geq 1$, the submanifold $K^n \subset H^n$ is invariant for α_H. This gives $h(\alpha_H) \geq h(\alpha_K) = k \log r_\alpha$, where $k = \dim K$. The real cohomology ring of H is a k-fold tensor product of cohomology rings of odd-dimensional spheres. The action of α_H on cohomology is the k-fold tensor product of the cohomology action α_{S^1} (aside from differences in grading), and so $s(\alpha_H) = r_\alpha^k$. Thus Shub's entropy conjecture holds for word maps. Taking $H = SO(d)$, one obtains the estimates mentioned in the introduction.

2. Isotopy classes on $S^3 \times S^3$. We show that every isotopy class of $M = S^3 \times S^3$ contains word maps. It suffices to show this for isotopy classes that fix $H_3(M;\mathbf{Z})$ (i.e., are homotopic to the identity) and word maps arising from inner automorphisms, $\alpha = i(g): a \to gag^{-1}$ $(a, g \in F_2)$. For the group K of such isotopy classes, Kreck gives an exact sequence [**Kr**, Theorem 2, Lemma 3b]

$$0 \to \theta_7 \to K \xrightarrow{\chi} H^3(M;\mathbf{Z}) \to 0,$$

where $\theta_7 \cong \mathbf{Z}_{28}$ is the group of smooth homotopy 7-spheres. The map χ is defined as follows. Choose an $S^3 \subset M$ with trivial normal bundle and a diffeomorphism f that fixes $H_3(M;\mathbf{Z})$ and leaves S^3 pointwise fixed. Then the normal part of the derivative of f on S^3 defines a map $S^3 \to Gl_3^+\mathbf{R}$. Passing to the class of this map in $\pi_3(Gl_3^+\mathbf{R}) = \pi_3(SO(3)) = \mathbf{Z}$, we obtain a pairing $K \otimes H_3(M;\mathbf{Z}) \to \mathbf{Z}$, and hence a homomorphism χ.

We first compute the map $\overline{\chi}: F_2 \to H^3(M;\mathbf{Z})$ that assigns to $g \in F_2$ the class $\chi(f(g))$, where $f(g) = i(g)_{S^3}$. Choose $S^3 = S^3 \times 1 \subset M$. Then $f(g)$ fixes S^3 pointwise. The normal bundle of S^3 is trivialized by left translation, so that the normal differential of $f(g)$ at $(x,1)$ corresponds to conjugation by x^a, where $a = a(g)$ is the sum of the exponents of x_1 in g, i.e., the x_1 component of the abelianization map $F_2 \to \mathbf{Z}^2$. For $a = 1$ this conjugation is the usual action of S^3 on quaternions with zero real part that gives the generator for $\pi_3(SO(3))$. Reasoning similarly for the other coordinate 3-sphere $1 \times S^3 \subset M$, we see $\overline{\chi}$ is onto with kernel the commutator subgroup $F_2' = [F_2, F_2]$.

It now suffices to show that the isotopy class $\xi = [f(g)] \in K$, $g = c(x_1, x_2)$, corresponds to a generator of θ_7. Note $i(g) = c(i(x_1), i(x_2))$, so that if d_j, $j = 1, 2$, is isotopic to $f(x_j)$ then $d = c(d_1, d_2)$ represents ξ. Choose a map $u: S^3 \to S^3$ homotopic to the identity so that $u^{-1}(1)$ is a neighborhood of 1 and set $d_1(x,y) = (x, u(x)yu(x)^{-1})$, $d_2(x,y) = (u(y)xu(y)^{-1}, y)$. Note that d_1 fixes a tubular neighborhood of the coordinate sphere $1 \times S^3$ and preserves a tubular neighborhood of the other coordinate sphere $S^3 \times 1$, and vice versa for d_2. Thus, d_1 and d_2 commute near $S^3 \vee S^3$, so d acts on $\mathbf{R}^6 = M - S^3 \vee S^3$ with compact support. Let ϕ be the corresponding diffeomorphism of $S^6 = \mathbf{R}^6 \cup \{\infty\}$. Using ϕ to glue two copies of the 7-disc along their boundaries gives the homotopy 7-sphere δ corresponding to ξ under Kreck's exact sequence.

Let $v \in \pi_3 SO(3)$ be the generator mentioned above, given by the conjugation action of unit quaternions. The commutator construction above was generalized by Milnor [**M**, pp. 17–18] to a bilinear pairing $\pi_3 SO(3) \otimes \pi_3 SO(3) \to \theta_7$, so (up to sign) $v \otimes v$ goes to δ. Milnor notes that δ generates θ_7, and this can be checked by showing that the Eells-Kuiper homomorphism $\mu: \theta_7 \to \mathbf{Q}/\mathbf{Z}$ takes the value $\pm 1/28$ on δ; cf. [**L**].

Let $\pi = \pi_0 \operatorname{Diff} M$. Let $W: \Phi_2 \to \pi$ be the homomorphism sending $\alpha \in \Phi_2$ to the isotopy class of the word map α_{S^3}. We have shown W is onto and $\operatorname{Ker} W \subset F_2'$ with $F_2'/\operatorname{Ker} W \cong \mathbf{Z}_{28}$. We now compute this kernel $S = \operatorname{Ker} W$.

Let $c_{ij} = c(x_1^i, x_2^j) \in F_2'$. Then using the bilinearity of Milnor's pairing, one sees that $W(c_{ij}) = ij\delta \in \theta_7 \subset \pi$. Now identify F_2 with $\pi_1 X$, X a figure eight, so that $F_2' = \pi_1 \tilde{X}$ where the grid $\tilde{X} = \mathbf{R} \times \mathbf{Z} \cup \mathbf{Z} \times \mathbf{R} \subset \mathbf{R}^2$ is the maximal abelian covering of X. Then $W: F_2' \to \theta_7$ induces $\overline{W}: H_1(\tilde{X}; \mathbf{Z}) \to \mathbf{Z}_{28}$. The class c_{ij}, $i, j \in \mathbf{Z} - \{0\}$, corresponds to a rectangular loop in $\tilde{X}$ based at $(0,0)$. These loops clearly generate $H_1\tilde{X}$. Thus $\overline{W}$ is the homomorphism which assigns to a loop γ its enclosed area $A(\gamma)$ reduced modulo 28. Note that $\ker A = [F_2, F_2']$ and that $F_2/[F_2, F_2']$ is the nilpotent group

$$H_{\mathbf{Z}} = \langle x, y, z | z = c(y,z), c(x,z) = 1 = c(y) \rangle.$$

Clearly $H_{\mathbf{Z}}$ is isomorphic to the integral Heisenberg group, i.e., the integral 3×3 matrices of form $I + N$ with N strictly upper triangular. We have shown

THEOREM 2. *There is an exact sequence*

$$1 \to S \to \Phi_2 \xrightarrow{W} \pi_0 \operatorname{Diff}(S^3 \times S^3) \to 1,$$

where $[F_2, F_2'] \subset S \subset F_2' \subset F_2 \subset \Phi_2$. *In the quotient* $F_2'/[F_2, F_2'] \cong \mathbf{Z}$, $S/[F_2, F_2']$ *corresponds to the subgroup* $28\mathbf{Z}$.

In particular, $K = H_{\mathbf{Z}}/z^{28}$ is nilpotent and χ is a central extension. These properties of K (and their analogues for other manifolds of high connectivity) are due to Bernhard Schmidt [**Sc**].

Let $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) \in Gl_2\mathbf{Z}$, $J^6 = I$. Then we show for the natural map $\Phi_2 \to Gl_2\mathbf{Z}$ induced by $F_2 \to \mathbf{Z}^2$:

LEMMA. *If* $B \in \Phi_2$ *covers* J, *then* $B^6 \notin S$.

The next corollary follows immediately from the lemma.

COROLLARY. *The natural map* $\pi \to Gl_2\mathbf{Z}$ *from isotopy classes to homotopy classes does not split.*

Let $A \in \Phi_2$ with $Ax = y$, $Ay = yx^{-1}$. Then A covers J and $A^3 = i(yx^{-1}) \circ \phi$, where $\phi x = x^{-1}$, $\phi y = y^{-1}$. By Nielsen's result any B covering J is of the form $B = i(g) \circ A$ for some $g \in F_2$. Then $B^3 = i(h) \circ \phi$ for $h = g \cdot Ag \cdot A^2 g \cdot yx^{-1}$.

Since $I + J + J^2$ has even entries, $h \equiv x^i y^j z^k \bmod S$ with i, j odd. Thus

$$B^6 = (B^3)^2 = i(h \cdot \phi(h)) \circ \phi^2 = i(x^i y^j z^k x^{-i} y^{-j} z^k) \equiv i(z^l) \mod S,$$

for l odd. Since z has even order $\bmod S$ (namely $28 = |\theta_7|$), $B^6 \notin S$, as desired. This proves the lemma and corollary.

Note the essential use of the nontriviality of the subgroup θ_7 of homotopy 7-spheres. Indeed for the quotient group π/θ_7, the natural map $\pi/\theta_7 \to Gl_2\mathbf{Z}$ does split, meaning that the usual action of $Gl_2\mathbf{Z}$ on $\mathbf{Z}^2$ lifts to an action by automorphisms of $H_{\mathbf{Z}}$. This can easily be seen by noting that the real Heisenberg group is the free two-step nilpotent Lie group generated by $\mathbf{R}^2$ and that $H_{\mathbf{Z}}$ is generated by $\mathbf{Z}^2 \subset \mathbf{R}^2$.

Also note that B^6 has infinite order in Φ_2 (since F_2 is torsionfree and centerless), so the word map B_{S^3} has infinite order for all B (the only trivial word map on S^3 comes from $1 \in \Phi_2$). We do not know if there is a periodic map of M inducing J on H_3M; if it is smooth, then we see its period must be divisible by 24.

3. Isentropic approximation. By the results of §1, we see that every isotopy class on M contains diffeomorphisms of entropy $\log s$, where s is the spectral radius of the action on real homology. Thus if Shub's entropy conjecture holds on M, these word maps minimize entropy in their isotopy class. But it is not even known whether the entropy function on Diff M is lower semicontinuous at a word map, aside from the trivial case of zero entropy.

We now show that a word map on two letters for S^3 or $SO(3)$ can be isotoped to an isentropic (equal entropy) fitted diffeomorphism. It suffices to work on $H = SO(3)$, because the isotopy can be lifted to M.

The function

$$(\operatorname{Re} x)^2 + (\operatorname{Re} y)^2 + (\operatorname{Re} xy)^2 - 2(\operatorname{Re} x)(\operatorname{Re} y)(\operatorname{Re} xy)$$

on M that (essentially) arose in §1 induces a function $g\colon H^2 \to [0,1]$ regular over $(0,1)$. The critical level $\tilde{L}_0 = g^{-1}0$ is a submanifold consisting of a single H-orbit corresponding to pairs of 180° rotations about perpendicular axes. The other critical level $\tilde{L}_1 = g^{-1}1$ corresponds to pairs of commuting rotations. $\tilde{L}_1 - (I,I)$ is a smooth submanifold. By assigning to a commuting pair $\neq (I,I)$ the common axis of rotation, one obtains a flat fibration of $\tilde{L}_1 - (I,I)$ over $\mathbf{R}P^2$, with fiber a punctured torus $T^2 - 0$ and monodromy ± 1. There is a vector field X on H^2 such that

(a) X is H-invariant.

(b) $Xg \leq 0$ with $Xg = 0$ precisely on $\tilde{L}_0 \cup \tilde{L}_1$.

(c) $X = 0$ on $\tilde{L}_0$ and on those points of $\tilde{L}_1$ outside an ε-neighborhood of (I,I).

(d) (I,I) is a source of X.

(e) $\tilde{L}_1 - (I,I)$ is a normally hyperbolic repeller, and $\tilde{L}_0$ is a normally hyperbolic attractor.

Let ϕ_t be the corresponding flow. For $t \gg r_\alpha > 1$, $\beta = \phi_t \circ \alpha_H$ has a chain recurrent set $R(\beta)$ consisting of $\tilde{L}_0$, (I,I) and a family $M_1 \subset \tilde{L}_1 - (I,I)$ of DA attractors indexed by $\mathbf{R}P^2$. Moreover, $\beta|\tilde{L}_0 = \alpha_H|L\tilde{L}0$ has finite order. Choose an equivariant Morse function on $\tilde{L}_0$ and extend its gradient vector field to a vector field Y_0 supported near $\tilde{L}_0$. Then choose a Morse function on $\mathbf{R}P^2$, lift its gradient vector field to M_1 using the flat connection and extend to a vector field Y_1 supported on a neighborhood of M_1. Let ψ_t be the flow generated by $Y_0 + Y_1$. Then for $\varepsilon > 0$ and small enough, $\gamma = \psi_\varepsilon \circ \beta$ is Axiom A-No Cycles, with $R(\gamma)$ consisting of the source (I,I), finitely many points in $\tilde{L}_0$ and finitely many DA attractors in M_1. Finally, double DA these DA attractors to get an Axiom A-No Cycles map δ with $R(\delta)$ zero dimensional. The Artin-Mazur zeta functions satisfy

$$\varsigma(\delta) = \varsigma(\alpha_{S^1})^k \cdot R,$$

where k is the number of critical points of the Morse function on $\mathbf{R}P^2$ and R is a finite product of terms $(1 - t^n)^{\pm 1}$, so $h(\delta) = h(\alpha_{S^1}) = \log r_\alpha$. It is then easy to find a fitted diffeomorphism Ω-conjugate to δ and isotopic to δ.

The case $r_\alpha = 1$ requires an isotopy to Morse-Smale. The same procedure clearly works: no DA attractors arise, and one has to use instead a gradient flow on T^2.

We have shown

THEOREM 3. *Every isotopy class of $S^3 \times S^3$ has a fitted representative f with entropy $\log s(f)$. The same holds for any isotopy class of $SO(3) \times SO(3)$ that contains a word map.*

See [**F1**] for results and a discussion of the corresponding question for word maps on S^1, i.e., toral automorphisms. Even for T^3 some isotopy classes are not known to have a fitted representative with entropy $\log s$. Specifically for $\det(x - A) = x^3 - x - 1$, $A \in Sl(3, \mathbf{Z})$, it is not known whether the linear map A has an isentropic fitting. On T^4 there are examples where no isentropic Axiom A representative exists in the homotopy class of a certain linear map [**F2**].

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DEPARTMENT OF MATHEMATICS, BOSTON UNIVERSITY, 111 CUMMINGTON STREET, BOSTON, MASSACHUSETTS 02215