Worst-Case Scenario Portfolio Optimization: A New Stochastic Control Approach

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Abstract

We consider the determination of portfolio processes yielding the highest worst-case bound for the expected utility from final wealth if the stock price may have uncertain (down) jumps. The optimal portfolios are derived as solutions of non-linear differential equations which itself are consequences of a Bellman principle for worst-case bounds. A particular application of our setting is to model crash scenarios where both the number and the height of the crash are uncertain but bounded. Also the situation of changing market coefficients after a possible crash is analyzed.

Keywords: Optimal portfolios, crash modelling, Bellman principle, equilibrium strategies, worst-case scenario, changing market coefficients

1 Introduction

A market crash is a synonym for a worst-case scenario of an investor trading at a security market. Therefore, to be prepared for such a situation is a desirable goal. One can of course do this by buying suitable put options, but being in such a well-insured situation is quite expensive. In contrast to this we will show below that it is possible to be indifferent between the occurrence or non-occurrence of a crash by following a suitable investment strategy in bond and stocks.

Modelling of a crash or – more general – of large stock price movements is an actively researched field in financial mathematics (see e.g. Aase [1], Merton [8], Eberlein and Keller [2], or Embrechts, Klüppelberg and Mikosch [3]). Most of the work done relies on modelling stock prices as Levy processes or other types of processes with heavy tailed distributions. As a contrast to that, we will take on the view of a semi-specialized stock price process. More precisely, we distinguish between so called "normal times" where the stock prices are assumed to follow Black Scholes type diffusions and "crash times" where all stock prices fall suddenly and simultaneously. Further, we are laying more stress on avoiding large losses in bad situations via trying to put the worst-case bound for the utility of terminal wealth as high as possible.

This approach is already looked at in a recent paper by Korn and Wilmott [6] where the authors determine optimal portfolios under the threat of a crash in the case of logarithmic utility for final wealth. There, the main aim is to show that still suitable investment in stocks can be more profitable than playing safe and investing all the funds in the riskless bond if a crash of the stock price can occur. The corresponding optimal strategy is found via the solution of a balance problem between obtaining good worst-case bounds in case of a crash on one hand and also a reasonable performance if no crash occurs at all.

Using the approach of Korn and Wilmott [6] but relaxing the assumption of the logarithmic utility function is the main aim of this paper. Our main findings are analogues to both the classical Bellman principle and the classical Hamilton-Jacobi-Bellman equation (for an introduction into this subject see Fleming and Soner [4]). We demonstrate their usefulness by solving the benchmark examples of log-utility and HARA-utility explicitly.

The paper is organized as follows: Section 2 describes the set up of the model and contains the main theoretical results if at most one crash can occur. In section 3 these results will be applied to both the log-utility and the HARA-utility functions. The main result will be generalized to a setting including changing market coefficients after a possible crash in section 4 in the case of log-utility.

2 Worst-case scenario portfolio optimization: The set up and main theoretical results

As in Korn and Wilmott [6] we start with the most basic setting of a security market consisting of a riskless bond and one risky security with prices given by

$$dP_0(t) = P_0(t)\, r\, dt\,, \qquad P_0(0) = 1\,, \tag{1}$$

$$dP_1(t) = P_1(t)\left[\mu\, dt + \sigma\, dW(t)\right]\,, \qquad P_1(0) = p_1\,, \tag{2}$$

with constant market coefficients $\mu > r$ and $\sigma \neq 0$ in "normal times". We further assume that until the time horizon T at most one crash can happen. At the "crash time" the stock price suddenly falls, i.e. we assume that the sudden relative fall of the stock price lies in the interval $[0, k^*]$ where the constant $0 < k^* < 1$ ("the worst possible crash") is given. We make no probabilistic assumption about the distribution of either the crash time or the crash height. As we assume that the investor can realize that the crash has happened we model its occurrence via a jump process $N(t)$ which is zero before the jump time and equals one from the jump time onwards. To model the fact that the investor is able to recognize that a jump of the stock price has happened we assume that the investor's decisions are adapted to the P-augmentation $\{\mathcal{F}_t\}$ of the filtration generated by both the Brownian motion $W(t)$ and the jump process $N(t)$.

Definition 2.1
Let $A(s, x)$ be the set of admissible portfolio processes $\pi(t)$ (i.e. the processes describing the fraction of wealth invested in the stock) corresponding to an initial capital of $x > 0$ at time s, i.e. $\{\mathcal{F}_t, s \leq t \leq T\}$-predictable processes such that

(i) the wealth equation in the usual crash-free setting

$$d\tilde{X}^\pi(t) = \tilde{X}^\pi(t)\left[(r + \pi(t)[\mu - r])\, dt + \pi(t)\sigma\, dW(t)\right], \tag{3}$$
$$\tilde{X}^\pi(s) = x \tag{4}$$

has a unique non-negative solution $\tilde{X}^\pi(t)$ and satisfies

$$\int_s^T \left[\pi(t)\tilde{X}^\pi(t)\right]^2 dt < \infty \quad P\text{-a.s.}. \tag{5}$$

(ii) the corresponding **wealth process $X^\pi(t)$ in the crash model**, *defined as*

$$X^\pi(t) = \begin{cases} \tilde{X}^\pi(t) & \text{for } s \leq t < \tau \\ [1 - \pi(\tau)k]\, \tilde{X}^\pi(t) & \text{for } t \geq \tau \geq s, \end{cases} \tag{6}$$

given the occurrence of a jump of height k at time τ, is strictly positive.

(iii) $\pi(t)$ has left-continuous paths with right limits.

We use $A(x)$ as an abbreviation for $A(0, x)$.

We can now state our worst-case problem. For details on its motivation see Korn and Wilmott [6].

Definition 2.2

1. *Let $U(x)$ be a utility function (i.e. a strictly concave, monotonously increasing and differentiable function). Then the problem to solve*

$$\sup_{\pi(\cdot)\in A(x)} \inf_{\substack{0\leq\tau\leq T,\\ 0\leq k\leq k^*}} \mathbb{E}\left[U\left(X^\pi(T)\right)\right], \tag{7}$$

where the final wealth $X^\pi(T)$ in the case of a crash of size k at time τ is given by

$$X^\pi(T) = [1 - \pi(\tau)\, k]\, \tilde{X}^\pi(T)\,, \tag{8}$$

with $\tilde{X}^\pi(t)$ as above, is called the **worst-case scenario portfolio problem** *with* **value function**

$$\nu_1(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ 0\le k\le k^*}} \mathbb{E}\left[U\left(X^\pi(T)\right)\right]. \tag{9}$$

2. *Let $\nu_0(t,x)$ be the* **value function** *for the usual optimization problem* **in the crash-free Black-Scholes setting**, *i.e.*

$$\nu_0(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \mathbb{E}\left[U\left(\tilde{X}^\pi(T)\right)\right], \tag{10}$$

(i.e. we obtain $\nu_0(t,x)$ by dropping the infimum in (9) above and maximizing over all usually admissible portfolio processes of the crash-free setting).

Remark 2.3
It is easy to see that under the assumption of $\mu > r$ each portfolio process that has a possibility to attain negative values cannot deliver the highest worst-case bound. As further the worst possible jump should not lead to a negative wealth process for an optimal portfolio process, we can thus without loss of generality restrict ourselves to portfolio processes satisfying

$$\frac{1}{k^*} \ge \pi(t) \ge 0 \quad \text{for all } t \in [0,T] \text{ a.s.} \tag{11}$$

which also implies that we only have to look at bounded portfolio processes.

We first state an obvious fact, the optimality of having all the money invested in the bond at the final time, if no crash has happened yet.

Proposition 2.4
If $U(x)$ is strictly increasing then an optimal portfolio process $\pi(t)$ for the worst-case problem has to satisfy

$$\pi(T,\omega) = 0 \qquad \textit{for all } \omega \textit{ where no crash happens in } [0,T]. \tag{12}$$

Remark 2.5
Although the assertion of Proposition 2.4 is trivial (see Korn and Wilmott [6] for a formal proof), it is very helpful to derive an analogue to the classical Bellman principle of dynamic programming:

Theorem 2.6 (Dynamic programming principle)
If $U(x)$ is strictly increasing in x, then we have

$$\nu_1(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{t\le\tau\le T} \mathbb{E}\left[\nu_0\left(\tau, \tilde{X}^\pi(\tau)\,[1-\pi(\tau)k^*]\right)\right]. \tag{13}$$

Proof: After the crash it is optimal to follow the optimal portfolio of the crash-free setting leading to an optimal expected utility of $\nu_0(\tau, z)$ if the wealth just *after* the crash at time τ equals z. As $\nu_0(\tau, \cdot)$ is strictly increasing in the second variable, a crash of maximum size k^* would be the worst thing to happen for an investor following a positive portfolio process at time τ. As by (11) we only have to consider non-negative portfolio processes and as by Proposition 2.4 we have

$$\mathbb{E}\left[\nu_0\left(T, \tilde{X}^\pi(T)\left[1-\pi(T)k^*\right]\right)\right] = \mathbb{E}\left[\nu_0\left(T, \tilde{X}^\pi(T)\right)\right] = \mathbb{E}\left[U\left(\tilde{X}^\pi(T)\right)\right],$$

the right hand side of equation (13) also includes the case where no crash happens at all. More precisely, the supremum is not changed as the formulation on the right hand side of (13) does not exclude candidates for the optimal portfolio process. Thus, it indeed coincides with the value function of the worst-case scenario portfolio problem. □

Theorem 2.7 (Dynamic programming equation)
Let the assumptions of Theorem 2.6 be satisfied, let $\nu_0(t, x)$ be strictly concave in x, and let there exist a continuously differentiable solution $\hat{\pi}(t)$ of

$$\left.\begin{array}{c} (\nu_0)_t(t,x) + (\nu_0)_x(t,x)\left[r + \hat{\pi}(t)(\mu - r)\right]x + \frac{1}{2}(\nu_0)_{xx}(t,x)\,\sigma^2\hat{\pi}(t)^2x^2 \\ -(\nu_0)_x(t,x)\,x\frac{\hat{\pi}'(t)}{1-\hat{\pi}(t)k^*}k^* = 0 \qquad \text{for } (t,x) \in [0,T[\times(0,\infty)\,, \end{array}\right\} \quad (14)$$

with boundary condition

$$\hat{\pi}(T) = 0\,. \quad (15)$$

With the notation of

$$\hat{\nu}(t,x) := \mathbb{E}^{t,x}\left[U\left(\tilde{X}^{\hat{\pi}}(T)\right)\right]$$

for the expected utility corresponding to the $\hat{\pi}(t)$ given the crash has not yet occurred at time t, we assume further that

$$f(x,y;t) := (\nu_0)_x(t,x)\left[y - \hat{\pi}(t)\right]\left[\mu - r\right]x + \frac{1}{2}(\nu_0)_{xx}(t,x)\,\sigma^2\left[y^2 - \hat{\pi}(t)^2\right]x^2 \quad (16)$$

is a concave function in (x, y) for all $t \in [0, T)$. Moreover, let the following implication be valid

$$\left.\begin{array}{c} \mathbb{E}^{0,x}\left[\hat{\nu}\left(t,\tilde{X}^\pi(t)\right)\right] \le \mathbb{E}^{0,x}\left[\hat{\nu}\left(t,\tilde{X}^{\hat{\pi}}(t)\right)\right] \quad \text{and} \quad \mathbb{E}^{0,x}\left[\pi(t)\right] \ge \hat{\pi}(t) \\ \text{for some } t \in [0,T),\ \pi(\cdot) \in A(x). \\ \Longrightarrow \mathbb{E}^{0,x}\left[\nu_0\left(t,\tilde{X}^\pi(t)\left[1-\pi(t)k^*\right]\right)\right] \le \mathbb{E}^{0,x}\left[\hat{\nu}\left(t,\tilde{X}^{\hat{\pi}}(t)\right)\right]. \end{array}\right\} \quad (17)$$

Then, $\hat{\pi}(t)$ is indeed the optimal portfolio process before the crash in our portfolio problem with at most one crash. The optimal portfolio process after the *crash has happened coincides with the optimal one in the crash-free setting. The corresponding value function before the crash is given by*

$$\begin{aligned} \nu_1(t,x) &= \nu_0\left(t, x\left[1-\hat{\pi}(t)k^*\right]\right) \\ &= \mathbb{E}^{t,x}\left[\nu_0\left(s, \tilde{X}^{\hat{\pi}}(s)\left[1-\hat{\pi}(s)k^*\right]\right)\right] \quad \textit{for } 0 \le t \le s \le T\,. \end{aligned} \tag{18}$$

Remark 2.8

1. Equation (14) can also be written as

$$\begin{aligned} \hat{\pi}'(t) &= \frac{1}{k^*}\frac{1-\hat{\pi}(t)k^*}{(\nu_0)_x(t,x)\,x}\Bigg[(\nu_0)_t(t,x) + (\nu_0)_x(t,x)\left[r+\hat{\pi}(t)(\mu-r)\right]x \\ &\quad + \frac{1}{2}(\nu_0)_{xx}(t,x)\,\sigma^2\hat{\pi}(t)^2x^2\Bigg] \qquad \text{for } (t,x)\in[0,T[\times(0,\infty)\,. \end{aligned}$$

   As ν_0 solves the usual HJB-equation for the portfolio problem in the crash-free setting, the term in the bracket non-negative zero. As its multiplier is positive, the optimal strategy $\hat{\pi}(t)$ is thus decreasing. Given the optimal strategy in the crash-free model

$$\pi^*(t,x) = -\frac{\mu-r}{\sigma^2}\frac{(\nu_0)_x(t,x)}{x(\nu_0)_{xx}(t,x)}\,, \tag{19}$$

   we can reduce the above equation to

$$\hat{\pi}'(t) = \frac{1}{k^*}\left[1-\hat{\pi}(t)k^*\right]\left[\frac{(\nu_0)_t(t,x)}{x(\nu_0)_x(t,x)} + r + \hat{\pi}(t)\left[\mu-r\right]\left[1-\frac{1}{2}\frac{\hat{\pi}(t)}{\pi^*(t,x)}\right]\right]. \tag{20}$$

   Furthermore, note $\hat{\pi}(t) \le \pi^*(t,x)$ for all $t\in[0,T]$. This is due to $\hat{\pi}(0) \le \pi^*(0,x)$ (as otherwise $\hat{\pi}(.)$ would not be optimal !) and $\hat{\pi}'(t) \le 0$ for all $t\in[0,T]$ as it has been shown above.

2. Equations (20) and (19) yield that equation (14) is only well-defined if $\frac{(\nu_0)_t(t,x)}{x(\nu_0)_x(t,x)}$ and $\frac{(\nu_0)_x(t,x)}{x(\nu_0)_{xx}(t,x)}$ are independent of x. By Proposition 3.11 in Menkens [7] for this it is sufficient that $\frac{(\nu_0)_x(t,x)}{x(\nu_0)_{xx}(t,x)}$ is independent of x. Then $\hat{\pi}$ is also independent of x as it has been tacitly assumed in equation (14).

Proof of Theorem 2.7:

i) As we have $\nu_0\,(t,x) \in C^{1,2}$ and $\hat{\pi}(t) \in C^1$, Itô's rule leads to

$$
\begin{aligned}
&\nu_0\left(s, \tilde{X}^{\hat{\pi}}(s)\left[1-\hat{\pi}(s)k^*\right]\right) \\
&= \nu_0\left(t, x\left[1-\hat{\pi}(t)k^*\right]\right) + \int_t^s (\nu_0)_t\left(u, \tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right]\right) du \\
&\quad + \int_t^s (\nu_0)_x\left(u, \tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right]\right)\left[r+\hat{\pi}(u)\left(\mu-r\right)\right]\tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right] du \\
&\quad - \int_t^s (\nu_0)_x\left(u, \tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right]\right)\tilde{X}^{\hat{\pi}}(u)\hat{\pi}'(u)k^*\, du \\
&\quad + \int_t^s \frac{1}{2}(\nu_0)_{xx}\left(u, \tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right]\right)\sigma^2\hat{\pi}(u)^2\tilde{X}^{\hat{\pi}}(u)^2\left[1-\hat{\pi}(u)k^*\right]^2 du \\
&\quad + \int_t^s (\nu_0)_x\left(u, \tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right]\right)\sigma\hat{\pi}(u)\tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right] dW(u) \\
&= \nu_0\left(t, x\left[1-\hat{\pi}(t)k^*\right]\right) \\
&\quad + \int_t^s (\nu_0)_x\left(u, \tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right]\right)\sigma\hat{\pi}(u)\tilde{X}^{\hat{\pi}}(u)\left[1-\hat{\pi}(u)k^*\right] dW(u)\,,
\end{aligned}
$$

where the last equality is due to the differential equation (14) for $\hat{\pi}(t)$. As $\hat{\pi}(t)$ is bounded and due to the properties of $\nu_0(t,x)$, we further obtain

$$
\nu_0\left(t, x\left[1-\hat{\pi}(t)k^*\right]\right) = \mathbb{E}^{t,x}\left[\nu_0\left(s, \tilde{X}^{\hat{\pi}}(s)\left[1-\hat{\pi}(s)k^*\right]\right)\right] \tag{21}
$$

which for the choice of $s = T$ implies

$$
\hat{\nu}\left(t,x\right) = \nu_0\left(t, x\left[1-\hat{\pi}(t)k^*\right]\right)\,.
$$

and that $\hat{\nu}\left(t, \tilde{X}^{\hat{\pi}}(t)\right)$ is a martingale.

ii) To prove optimality of $\hat{\pi}(t)$ and that $\hat{\nu}\left(t,x\right)$ coincides with the value function, we now consider $\hat{\nu}\left(t, \tilde{X}^{\pi}(t)\right)$ for an arbitrary admissible portfolio process

$\pi(t)$. With the help of Itô's formula we arrive at

$$\begin{aligned}
\hat{\nu}\left(t,\tilde{X}^{\pi}(t)\right) &= \nu_0\left(t,\tilde{X}^{\pi}(t)\left[1-\hat{\pi}(t)k^*\right]\right)\\
&= \nu_0\left(0,x\left[1-\hat{\pi}(0)k^*\right]\right)+\int_0^t \left(\nu_0\right)_t\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)du\\
&\quad+\int_0^t \left(\nu_0\right)_x\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)\Big[\left[r+\pi(u)\left(\mu-r\right)\right]Z\left(u,\pi,\hat{\pi}\right)\\
&\qquad -\tilde{X}^{\pi}(u)\hat{\pi}'(u)k^*\Big]du\\
&\quad+\int_0^t \frac{1}{2}\left(\nu_0\right)_{xx}\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)\sigma^2\pi(u)^2Z\left(u,\pi,\hat{\pi}\right)^2du\\
&\quad+\int_0^t \left(\nu_0\right)_x\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)\sigma\pi(u)Z\left(u,\pi,\hat{\pi}\right)dW(u),
\end{aligned}$$

where we have used the abbreviation

$$Z\left(t,\pi,\hat{\pi}\right):=\tilde{X}^{\pi}(t)\left[1-\hat{\pi}(t)k^*\right].$$

If we now use the differential equation (14) characterizing $\hat{\pi}(t)$ for the pairs

$$(t,x)=\left(u,\tilde{X}^{\pi}(u)\left[1-\hat{\pi}(u)k^*\right]\right)=\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)$$

in (14) to replace $-\tilde{X}^{\pi}(u)\hat{\pi}'(u)k^*$ in the equation above and simplify it afterwards, we obtain

$$\begin{aligned}
\hat{\nu}\left(t,\tilde{X}^{\pi}(t)\right) &= \nu_0\left(0,x\left[1-\hat{\pi}(0)k^*\right]\right)\\
&+ \int_0^t \left(\nu_0\right)_x\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)\left[\pi(u)-\hat{\pi}(u)\right]\left[\mu-r\right]Z\left(u,\pi,\hat{\pi}\right)du\\
&+ \int_0^t \frac{1}{2}\left(\nu_0\right)_{xx}\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)\sigma^2\left[\pi(u)^2-\hat{\pi}(u)^2\right]Z\left(u,\pi,\hat{\pi}\right)^2du\\
&+ \int_0^t \left(\nu_0\right)_x\left(u,Z\left(u,\pi,\hat{\pi}\right)\right)\sigma\pi(u)Z\left(u,\pi,\hat{\pi}\right)dW(u).
\end{aligned}$$

As we would like to prove optimality of $\hat{\pi}(t)$, we will in the following only consider portfolio processes $\pi(t)$ that might yield a higher worst-case bound than $\hat{\pi}(t)$. A necessary condition for $\pi(t)$ to yield a higher worst-case bound is of course

$$\pi(0) < \hat{\pi}(0)$$

as otherwise the worst-case bound corresponding to $\pi(t)$ can at most equal the one for $\hat{\pi}(t)$ (due to the martingale property of $\hat{\nu}\left(t, \tilde{X}^{\hat{\pi}}(t)\right)$).

iii) Assume now that there exists an admissible portfolio process $\pi(t)$ yielding a higher worst-case bound than $\hat{\pi}(t)$. As the inequality

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(s, \tilde{X}^{\pi}(s)\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(s, \tilde{X}^{\hat{\pi}}(s)\right)\right] \quad \text{for all } 0 \leq s \leq T, x > 0$$

would imply the non-existence of a higher worst-case bound for $\pi(t)$ (due to the martingale property of $\hat{\nu}\left(t, \tilde{X}^{\hat{\pi}}(t)\right)$), we can assume that we must have

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(s, \tilde{X}^{\pi}(s)\right)\right] > \mathbb{E}^{0,x}\left[\hat{\nu}\left(s, \tilde{X}^{\hat{\pi}}(s)\right)\right] \quad \text{for at least some } s > 0\,, \tag{22}$$

and in particular for $s = T$. This then leads to

$$\begin{aligned}\mathbb{E}\Big[& (\nu_0)_x \left(s, Z\left(s, \pi, \hat{\pi}\right)\right) \left[\pi(s) - \hat{\pi}(s)\right] \left[\mu - r\right] Z\left(s, \pi, \hat{\pi}\right) \\ & + \tfrac{1}{2} \left(\nu_0\right)_{xx} \left(s, Z\left(s, \pi, \hat{\pi}\right)\right) \sigma^2 \left[\pi(s)^2 - \hat{\pi}(s)^2\right] Z\left(s, \pi, \hat{\pi}\right)^2\Big] > 0\end{aligned} \tag{23}$$

for some $s > 0$. By assumption (16) and Jensen's inequality of the form $\mathbb{E}\left[f\left(X, Y\right)\right] \leq f\left(\mathbb{E}\left[X\right], \mathbb{E}\left[Y\right]\right)$ for concave functions applied to (23) with

$$X := Z\left(s, \pi, \hat{\pi}\right) = \tilde{X}^{\pi}(s)\left[1 - \hat{\pi}(s)k^*\right] \quad \text{and} \quad Y := \pi(s)\,,$$

we obtain

$$\begin{aligned} 0 < & \left(\nu_0\right)_x \left(s, \mathbb{E}\left[Z\left(s, \pi, \hat{\pi}\right)\right]\right) \left\{\mathbb{E}\left[\pi(s)\right] - \hat{\pi}(s)\right\} \left[\mu - r\right] \mathbb{E}\left[Z\left(s, \pi, \hat{\pi}\right)\right] \\ & + \tfrac{1}{2} \left(\nu_0\right)_{xx} \left(s, \mathbb{E}\left[Z\left(s, \pi, \hat{\pi}\right)\right]\right) \sigma^2 \left[\mathbb{E}\left[\pi(s)\right]^2 - \hat{\pi}(s)^2\right] \mathbb{E}\left[Z\left(s, \pi, \hat{\pi}\right)\right]^2 \end{aligned} \tag{24}$$

for some $s > 0$. Due to the HJB-equation for the portfolio problem of the crash-free setting and to equations (14) and (15) we must have

$$\hat{\pi}(s) \leq \pi^*(s) \quad \text{for all } s \in [0, T]\,. \tag{25}$$

This and the fact that $\pi^*(s)$ maximizes the right side of equation (24) (interpreted as a quadratic function in the variable $\mathbb{E}\left[\pi(s)\right]$) lead to either a contradiction in the case, when we have equality in (25) or to

$$\hat{\pi}(s) < \mathbb{E}\left[\pi(s)\right]\,. \tag{26}$$

iv) For an arbitrary admissible portfolio process $\pi(t)$ assumed to yield a higher worst-case bound than $\hat{\pi}(t)$ let

$$\bar{t} := \inf\left\{t > 0 \middle| \mathbb{E}\left[\pi(t)\right] \geq \hat{\pi}(t)\right\} . \tag{27}$$

Case 1: Assume first that we have $0 < \bar{t} < T$. We then obtain

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, \tilde{X}^{\pi}\left(\bar{t}\right)\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, \tilde{X}^{\hat{\pi}}\left(\bar{t}\right)\right)\right] ,$$

which together with assumption (17) implies

$$\begin{aligned} \mathbb{E}^{0,x}\left[\nu_0\left(\bar{t}, \tilde{X}^{\pi}\left(\bar{t}\right)\left[1 - \pi\left(\bar{t}\right) k^*\right]\right)\right] &\leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, \tilde{X}^{\hat{\pi}}\left(\bar{t}\right)\right)\right] \\ &= \mathbb{E}^{0,x}\left[\hat{\nu}\left(T, \tilde{X}^{\hat{\pi}}\left(T\right)\right)\right] \\ &= \mathbb{E}\left[U\left(\tilde{X}^{\hat{\pi}}\left(T\right)\right)\right] , \end{aligned}$$

if the infimum defining $\bar{t}$ is indeed attained. If it is not attained, then the above inequality together with (17) implies

$$\begin{aligned} \mathbb{E}^{0,x}\left[\nu_0\left(\check{t}, \tilde{X}^{\pi}\left(\check{t}\right)\left[1 - \pi\left(\check{t}\right) k^*\right]\right)\right] &\leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\check{t}, \tilde{X}^{\hat{\pi}}\left(\check{t}\right)\right)\right] \\ &= \mathbb{E}^{0,x}\left[\hat{\nu}\left(T, \tilde{X}^{\hat{\pi}}\left(T\right)\right)\right] \\ &= \mathbb{E}\left[U\left(\tilde{X}^{\hat{\pi}}\left(T\right)\right)\right] \end{aligned}$$

with $\check{t} = \bar{t} + \varepsilon$ for a suitable $\varepsilon > 0$. To see this, note that in case of

$$\mathbb{E}\left[\pi\left(\bar{t}\right)\right] \geq \hat{\pi}\left(\bar{t}\right) , \tag{28}$$

the relation is directly implied by assumption (17) for $\varepsilon = 0$. So let (28) be violated. As we have $0 < \bar{t} < T$ there is a $\delta > 0$ with

$$\delta < \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, \tilde{X}^{\hat{\pi}}\left(\bar{t}\right)\right)\right] - \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, \tilde{X}^{\pi}\left(\bar{t}\right)\right)\right] ,$$

which can be concluded by part iii)of this proof. But then continuity of $\tilde{X}^{\hat{\pi}}(t)$ and of $\tilde{X}^{\pi}(t)$ imply that there exists an $\varepsilon > 0$ with

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t} + \varepsilon, \tilde{X}^{\pi}\left(\bar{t} + \varepsilon\right)\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t} + \varepsilon, \tilde{X}^{\hat{\pi}}\left(\bar{t} + \varepsilon\right)\right)\right]$$

and the assertion then is a consequence of assumption (17).

Thus, both cases are contradicting the assumption that $\pi(t)$ yields a higher worst-case bound than $\hat{\pi}(t)$.

Case 2: In the case of $\bar{t} = T$ we would directly obtain

$$\begin{aligned}\mathbb{E}\left[U\left(\tilde{X}^{\pi}(T)\right)\right] &= \mathbb{E}^{0,x}\left[\nu_0\left(\bar{t}, \tilde{X}^{\pi}(\bar{t})\left[1-\pi(\bar{t})k^*\right]\right)\right] \\ &\leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, \tilde{X}^{\hat{\pi}}(\bar{t})\right)\right] \\ &= \mathbb{E}\left[U\left(\tilde{X}^{\hat{\pi}}(T)\right)\right],\end{aligned}$$

again a contradiction to the assumption that $\pi(t)$ yields a higher worst-case bound than $\hat{\pi}(t)$.

Case 3: In the case of $\bar{t} = 0$ we also obtain a contradiction to the assumption that $\pi(t)$ yields a higher worst-case bound than $\hat{\pi}(t)$. To see this note that the assumption of a higher worst-case bound for $\pi(t)$ can only be satisfied, if we have

$$\begin{aligned}\mathbb{E}^{0,x}\left[\nu_0\left(t, \tilde{X}^{\pi}(t)\left[1-\pi(t)k^*\right]\right)\right] &> \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, \tilde{X}^{\hat{\pi}}(t)\right)\right] \\ &= \mathbb{E}^{0,x}\left[\hat{\nu}\left(T, \tilde{X}^{\hat{\pi}}(T)\right)\right] \\ &= \mathbb{E}\left[U\left(\tilde{X}^{\hat{\pi}}(T)\right)\right],\end{aligned}$$

for all $0 < t \leq T$. On the other hand, for $t \downarrow 0$ the LCRL-property and the boundedness of $\hat{\pi}(t)$ and $\pi(t)$ together with the dominated convergence theorem imply

$$\begin{aligned}\nu_0\left(0, x\left[1-\hat{\pi}(0)k^*\right]\right) &= \lim_{t\downarrow 0}\mathbb{E}\left[\hat{\nu}\left(t, \tilde{X}^{\hat{\pi}}(t)\right)\right] \\ \mathbb{E}\left[\nu_0\left(0, x\left[1-\pi(0+)k^*\right]\right)\right] &= \lim_{t\downarrow 0}\mathbb{E}\left[\nu_0\left(t, \tilde{X}^{\pi}(t)\left[1-\pi(t)k^*\right]\right)\right].\end{aligned}$$

The concavity of ν_0 together with these limit relations lead to

$$\begin{aligned}\nu_0\left(0, x\left[1-\hat{\pi}(0)k^*\right]\right) &\leq \mathbb{E}\left[\nu_0\left(t, x\left[1-\pi(0+)k^*\right]\right)\right] \\ &\leq \nu_0\left(0, x\left[1-\mathbb{E}\left[\pi(0+)\right]k^*\right]\right).\end{aligned}$$

But by the definition of $\bar{t}$ and the assumed strict concavity this can only be true, if we have

$$\pi(0+) = \hat{\pi}(0) \quad \text{a.s.}$$

which then contradicts the assumption that $\pi(\cdot)$ yields a higher worst-case bound than $\hat{\pi}(\cdot)$.

Putting all three cases together, we have proved that there is no admissible portfolio process $\pi(t)$ yielding a higher worst-case bound than $\hat{\pi}(t)$.

In particular, we have also shown equation (18) by taking into account the relations proved in i) and the optimality of $\hat{\pi}(t)$. □

As the assumptions of Theorem 2.7 are hard to satisfy we will show below that we can weaken them if we only restrict to the class of deterministic portfolios.

Corollary 2.9
*Let $\hat{\pi}(\cdot)$ be the unique solution of (14). Moreover, assume that $\nu_0(t,x)$ is strictly increasing in x and strictly concave in x. Then $\hat{\pi}(\cdot)$ is the **best possible deterministic portfolio** (i.e. the one that solves the worst-case problem if we restrict to deterministic portfolios).*

Proof: For deterministic portfolio strategies inequality (23) reduces to

$$\begin{aligned} h(\pi) = \; & \mathbb{E}\left[(\nu_0)_x\left(s, Z\left(s,\pi,\hat{\pi}\right)\right) Z\left(s,\pi,\hat{\pi}\right)\right]\left[\pi(s)-\hat{\pi}(s)\right]\left[\mu - r\right] \\ & + \tfrac{1}{2}\mathbb{E}\left[(\nu_0)_{xx}\left(s, Z\left(s,\pi,\hat{\pi}\right)\right) Z\left(s,\pi,\hat{\pi}\right)^2\right]\sigma^2\left[\pi(s)^2-\hat{\pi}(s)^2\right]. \end{aligned}$$

Obviously, $h(\hat{\pi}) = 0$ and the function attains its maximum in

$$\pi^*\left(s, Z\left(s,\pi,\hat{\pi}\right)\right) = -\frac{\mathbb{E}\left[(\nu_0)_x\left(s, Z\left(s,\pi,\hat{\pi}\right)\right) Z\left(s,\pi,\hat{\pi}\right)\right]}{\frac{1}{2}\mathbb{E}\left[(\nu_0)_{xx}\left(s, Z\left(s,\pi,\hat{\pi}\right)\right) Z\left(s,\pi,\hat{\pi}\right)^2\right]}\frac{\mu - r}{\sigma^2}.$$

Furthermore, the function h is strictly increasing for $\pi < \pi^*$, strictly decreasing for $\pi > \pi^*$, and concave for all π. Since $\nu_0(t,x)$ is strictly increasing and strictly concave in x, $\pi^*(t,x)$ is strictly positive. This guarantees that $\hat{\pi} \le \pi^*$, as otherwise $\hat{\pi}$ cannot be a solution of (14) ($\hat{\pi}(T) = 0$ would yield a contradiction to $\hat{\pi} \ge \pi^* > 0$). Thus, $\hat{\pi} \le \pi^*$ implies $h(\pi) < 0$ for all $\pi < \hat{\pi}$. Observe now that condition (17) follows straightforward. Given

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(t, \tilde{X}^{\pi}(t)\right)\right] \le \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, \tilde{X}^{\hat{\pi}}(t)\right)\right] \quad \text{and} \quad \pi(t) \ge \hat{\pi}(t)$$

for some $t \in [0,T)$, $\pi(\cdot) \in A(x)$. This implies

$$\begin{aligned} \mathbb{E}^{0,x}\left[\nu_0\left(t, \tilde{X}^{\pi}(t)\left[1-\pi(t)k^*\right]\right)\right] & \le \mathbb{E}^{0,x}\left[\nu_0\left(t, \tilde{X}^{\pi}(t)\left[1-\hat{\pi}(t)k^*\right]\right)\right] \\ & = \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, \tilde{X}^{\pi}(t)\right)\right] \\ & \le \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, \tilde{X}^{\hat{\pi}}(t)\right)\right]. \end{aligned}$$

The assertion now follows as in the proof of Theorem 2.7, part iv). □

Remark 2.10
One could also determine the best constant portfolio process π^* for our worst-case problem. Due to space limitations this is left to the reader.

3 The log utility and the HARA utility case

i) The case of $U(x) = \ln(x)$ is already dealt with in Korn and Wilmott [6]. However, the treatment there uses special properties of the logarithmic function explicitly. Here, we will use Theorem 2.7. Note therefore that in this case we have (see Korn and Korn [5])

$$\nu_0(t,x) = \ln(x) + \left\{ r + \frac{1}{2}\left[\frac{\mu - r}{\sigma}\right]^2 \right\}[T-t] \ ,$$

which is strictly increasing and concave in x. This form allows a direct verification of assumption (16). Even more, it can also be shown that assumption (17) is satisfied, too. Hence, Theorem 2.7 is applicable and we obtain $\hat{\pi}(t)$ as the unique solution of the corresponding form of equation (14)

$$-\left[r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \right] + [r + \hat{\pi}(t)\,[\mu - r]] - \frac{1}{2}\sigma^2 \hat{\pi}(t)^2 = \frac{\hat{\pi}'(t)k^*}{1 - \hat{\pi}(t)k^*} \,.$$

with boundary condition and (15). Using $\pi^* = \frac{\mu - r}{\sigma^2}$ this can be written as

$$\hat{\pi}'(t) = -\frac{\sigma^2}{2k^*}\,[1 - \hat{\pi}(t)k^*]\,[\hat{\pi}(t) - \pi^*]^2 \ , \tag{29}$$

(compare to Equation (2.14) in Korn and Wilmott [6]). This equation together with the final condition $\hat{\pi}(T) = 0$ has a unique solution that can be computed explicitly up to some constant which has to be found numerically. For numerical examples we refer to Korn and Wilmott [6].

ii) The case of $U(x) = \frac{1}{\gamma}x^\gamma$ for $\gamma \leq 1$, $\gamma \neq 0$ is not covered by Korn and Wilmott [6]. Even worse, Theorem 2.7 cannot be used as the value function in the crash-free model

$$\nu_0(t,x) = \frac{1}{\gamma}x^\gamma \exp\left(\left[\gamma r + \frac{1}{2}\left[\frac{\mu - r}{\sigma}\right]^2 \frac{\gamma}{1-\gamma} \right] [T-t] \right) .$$

violates both condition (17)and condition (16).

However, Corollary 2.9 is still applicable and states that $\hat{\pi}(t)$ is at least the best deterministic strategy. In this case equation (14) reduces to

$$\hat{\pi}'(t) = -\frac{\sigma^2}{2k^*}\,[1-\gamma]\,[1 - \hat{\pi}(t)k^*]\,[\hat{\pi}(t) - \pi^*]^2 \,, \tag{30}$$

with $\pi^* = \frac{\mu - r}{\sigma^2}\frac{1}{1-\gamma}$. The unique solvability of this equation is ensured as in the log-utility case.

Remark 3.1
We can directly generalize the results of this section to the case of n possible crashes by an induction procedure. As this is straight forward we refer the interested reader to Menkens [7].

4 Changing Market Coefficients after a Possible Crash

So far in our model, a crash only has a temporary effect. However, in reality the occurrence of a crash can change the whole attitude of the market towards stock investment. We will take care for this by allowing for a change of market conditions change after a crash. Let therefore k (with $k \in [0, k^*]$) be the arbitrary size of a crash at time τ. The price dynamcis of the bond and the risky asset **after** the crash are then assumed to be given by

$$dP_{1,0}(t) = P_{1,0}(t)\, r_1\, dt\,, \qquad P_{1,0}(\tau) = P_{0,0}(\tau)\,, \tag{31}$$
$$dP_{1,1}(t) = P_{1,1}(t)\,[\mu_1\, dt + \sigma_1\, dW(t)]\,, \qquad P_{1,1}(\tau) = (1-k)\,P_{0,1}(\tau)\,, \tag{32}$$

with constant market coefficients r_1, μ_1 and $\sigma_1 \neq 0$ after the crash.

The initial market will be called market 0 while the market after a crash will be called market 1. We denote the corresponding market coefficients by r_0, μ_0, σ_0 and by r_1, μ_1, σ_1, respectively. For simplicity we concentrate on the case of the logarithmic utility function.

Definition 4.1
For $i = 0, 1$ the **optimal portfolio strategy in market *i***, *assuming that no crash will happen, is denoted by*

$$\pi_i^* := \frac{\mu_i - r_i}{\sigma_i^2}.$$

The **utility growth potential** *or* **earning potential** *of market i is defined as*

$$\Psi_i := r_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{\sigma_i}\right)^2 = r_i + \frac{\sigma_i^2}{2}\,(\pi_i^*)^2\ .$$

For deriving so-called **crash hedging strategy** that makes the investor indifferent between no crash occurring at all until the investment horizon and the worst possible crash to happen now, we have to compare the markets before and after the crash. As long as the worst possible crash is one of maximum height $k*$ we can use the same approach as in the setting of Section 2. This is in particular guaranteed if the utility growth potential in market 1 is at least as big as the riskless rate in market 0. We will consider this situation in the main theorem below. For other cases we refer the interested reader to Menkens [7].

Theorem 4.2
Let $0 \le {\pi_0}^* < \frac{1}{k^*}$. If $\Psi_1 \ge r_0$, then there exists a unique crash hedging strategy $\hat{\pi}$, which is given by the solution of the differential equation

$$\hat{\pi}'(t) = \left(\hat{\pi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\,(\hat{\pi}(t) - \pi_0^*)^2 + \Psi_1 - \Psi_0\right], \tag{33}$$
$$\hat{\pi}(T) = 0. \tag{34}$$

Moreover, this crash hedging strategy is bounded by $0 \leq \hat{\pi} < \frac{1}{k^*}$. *The optimal portfolio strategy* ***before*** *the crash for an investor who wants to solve her worst case scenario portfolio problem is given by*

$$\bar{\pi}(t) := \min\{\hat{\pi}(t), \pi_0^*\} \qquad \textit{for all } t \in [0, T]. \tag{35}$$

Proof: a)The form of the differential equation 33 for the crash hedging strategy $\hat{\pi}(t)$ can be derived from the balance equation

$$\hat{\nu}(t, x) = \nu_{0,1}(t, x[1 - \hat{\pi}(t)k^*]) \ .$$

as in the proof of Theorem 2.7 combined with the explicit calculations of the log-utility example of Section 3. Here, $\nu_{0,1}$ denotes the value function in the crash-free setting of market 1. The difference between the two markets is mirrored in the additional term $\Psi_1 - \Psi_0$ in the square bracket of equation 33. Unique existence of the solution to the equations 33 and 34 follows from an appropriate version of the standard Picard–Lindelöf Theorem (in fact, note that the right hand side of equation 33 is a polynomial in $\hat{\pi}$ and $\hat{\pi}(T) = 0$ then implies that $\hat{\pi}(.)$ is bounded on $[0, T]$).
b) As by the form of the differential equation 33 $\hat{\pi}(t)$ is decreasing with $\hat{\pi}(T) = 0$ under the assumption of $\Psi_1 \geq r_0$, only the following two cases can occur:

1. $\hat{\pi}(t) \leq \pi_0^*$ for all $t \in [0, T]$.
2. $\hat{\pi}(t) \geq \pi_0^*$ for all $t \in [0, S]$, $\hat{\pi}(t) \leq \pi_0^*$ for all $t \in [S, T]$ for a suitable $S \in [0, T)$

To prove optimality of the portfolio strategy $\bar{\pi}(t)$ in the first case we can either use an obvious modification of the corresponding proof in Korn and Wilmott [6] or of the proof of Theorem 2.7 combined with the explicit calculations of the log-utility example of Section 3.

To prove optimality in the second case note that from time S on the argument for the first case just given applies, too. Further, the second case can only occur if we have

$$\Psi_1 \geq \Psi_0.$$

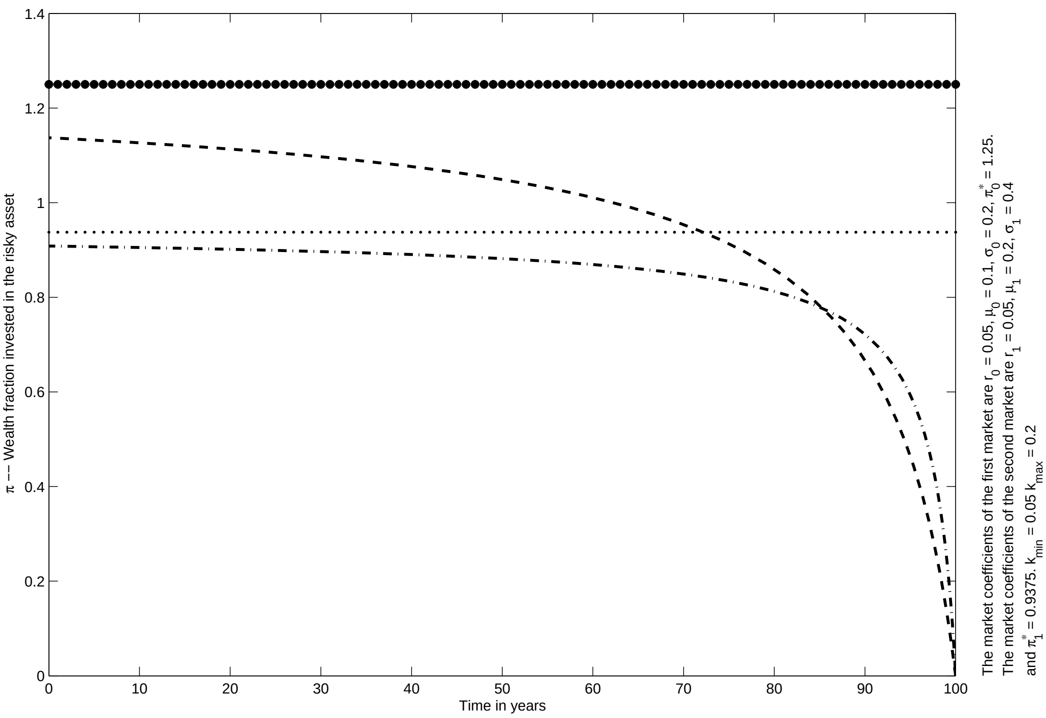
To see this, note that the crash hedging strategy $\hat{\pi}(t)$ is unique and for the strategy $\pi(t) = 0$ the worst case scenario is no crash. So, all strategies below $\hat{\pi}(t)$ must have a unique worst case scenario which is the no crash case. But then comparing the different expected utilities for the strategy π_0^* shows the above relation. Further, in the second case above, before time S, using π_0^* in stead of $\hat{\pi}(t)$ is better in both the no crash and the crash scenario. Even more, π_0^* outperforms all other portfolio strategies with respect to the expected log-utility in the no crash setting. As on the other side, we can only have $\hat{\pi}(t) \geq \pi_0^*$ on $[0, S]$ if the worst case on $[0, S]$ for an investor holding π_0^* is the no crash scenario, it is then clear that holding π_0^* on $[0, S]$ does indeed deliver the highest worst-case bound. □

4.1 Examples and Further Remarks

In order to compare the results in this paper with the results of Korn and Wilmott [6] let us name the optimal portfolio strategy of the market i given that the market conditions do not change after a crash $\hat{\phi}_i$. This is the situation of Korn and Wilmott [6].

Observe that it is possible that $\pi_0^* > \pi_1^*$, but $\hat{\phi}_0'(T) < \hat{\phi}_1'(T)$ and thus $\hat{\phi}_0(t) < \hat{\phi}_1(t)$ for $t \in [T-\epsilon, T]$ and for a suitable $\epsilon > 0$. However, if the time horizon T is long enough, it is valid that $\hat{\phi}_0(t) > \hat{\phi}_1(t)$ for some $t \in [0, \delta]$ with $\delta > 0$ being chosen suitable (see Figure 1).

Figure 1: Example $\pi_0^* > \pi_1^*$, but $\hat{\phi}_0'(T) < \hat{\phi}_1'(T)$

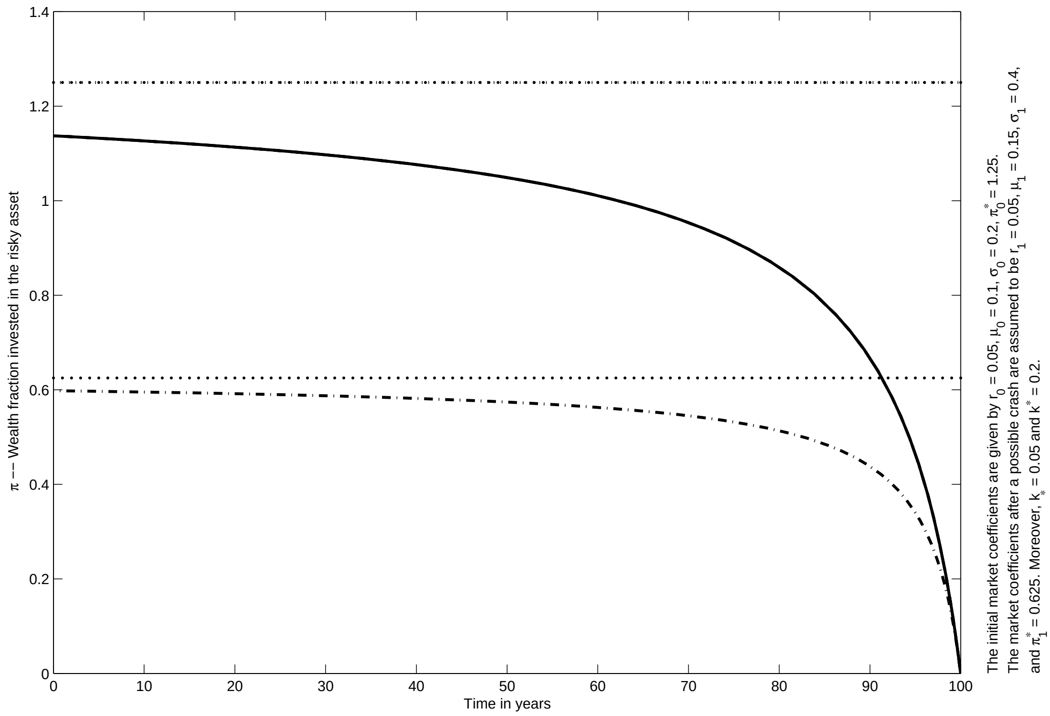


This graphic shows $\hat{\phi}_0$ (dash–dotted line), $\hat{\phi}_1$ (dashed line), π_0^* (upper dotted line), and π_1^* (lower dotted line).

1. $\mathbf{\Psi_1 = \Psi_0}$
   Be aware that this case includes the case of non–changing market coefficient (and it is not only this case). Moreover, this case is valid if the market conditions change in such a way that the utility growth potential does not change. So from an economic point of view the two markets are equivalent although the market coefficients are changing. However, as the log–optimal

portfolios in both markets differ, one obtains different crash hedging strategies compared to the case without changing market coefficients. Note that in this case(see Figure 2) $\hat{\pi} = \hat{\phi}_0 \neq \hat{\phi}_1$. The last inequality is due to the fact that in general $\pi_0^* \neq \pi_1^*$.

Figure 2: Example $\Psi_1 = \Psi_0$



This graphic shows $\hat{\pi} = \bar{\pi} = \hat{\phi}_0$ (dash–dotted line), π_0^* (upper dotted line), $\hat{\phi}_1$ (solid line), and π_1^* (lower dotted line).

2. $\mathbf{\Psi_1 > \Psi_0}$

There are several observations to make. First, note that the $\hat{\pi}$ in this case descents faster than $\hat{\phi}_0$. Thus, $\hat{\pi}(t) \geq \hat{\phi}_0(t)$ for all $t \in [0, T]$. This can also be verified in Figure 3. However, nothing comparable can be said about $\hat{\pi}$ and $\hat{\phi}_1$.

In this case it is possible that the crash hedging strategy will become greater than π_0^* given that the time horizon is large enough and $\pi_0^* < \frac{1}{k^*}$. To analyze this, define

$$t_0 := T + \frac{\ln(1 - \pi_0^* k^*)}{\Theta_2} + \frac{\pi_0^* - \frac{1}{k^*}}{\Delta_1 \cdot C} \arctan\left(\frac{\pi_0^*}{\Delta_1}\right) - \frac{1}{2\Theta_2} \ln\left(\frac{\Delta_1^2}{(\pi_0^*)^2 + \Delta_1^2}\right)$$

with

$$\Delta_1 := \sqrt{\frac{2}{\sigma_0^2}(\Psi_1 - \Psi_0)} \quad \text{and}$$
$$\Theta_2 := \frac{\sigma_0^2}{2}\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \Psi_1 - \Psi_0.$$

Hence, if $t_0 \in (0, T]$, then the optimal crash hedging strategy is

$$\bar{\pi}(t) := \begin{cases} \pi_0^*, & \text{for } t \leq t_0 \\ \hat{\pi}(t), & \text{for } t > t_0 \end{cases},$$

as it can be verified in Figure 3. Again, this has a clear economic reason. As the utility growth potential after a crash is bigger than before, the market situation after the crash is a better one. This results in the fact that one is not indifferent between occurrence and non-occurrence of a crash as long as there remains sufficient time to make use of the advantage of being in a better market after the crash. If this is satisfied (i.e. as long as we have $\pi_0^* = \bar{\pi}(t)$) the investor is indeed hoping for a crash.

3. $\boldsymbol{r_0 \leq \Psi_1 \leq \Psi_0}$
Note that $\hat{\pi}$ in this case descents slower than $\hat{\phi}_0$. This is, because the correction term $\Psi_1 - \Psi_0$ is negative. Thus, $\hat{\pi}(t) \leq \hat{\phi}_0(t)$ for all $t \in [0, T]$. This can also be verified in Figure 4. However, nothing comparable can be said about $\hat{\pi}$ and $\hat{\phi}_1$.

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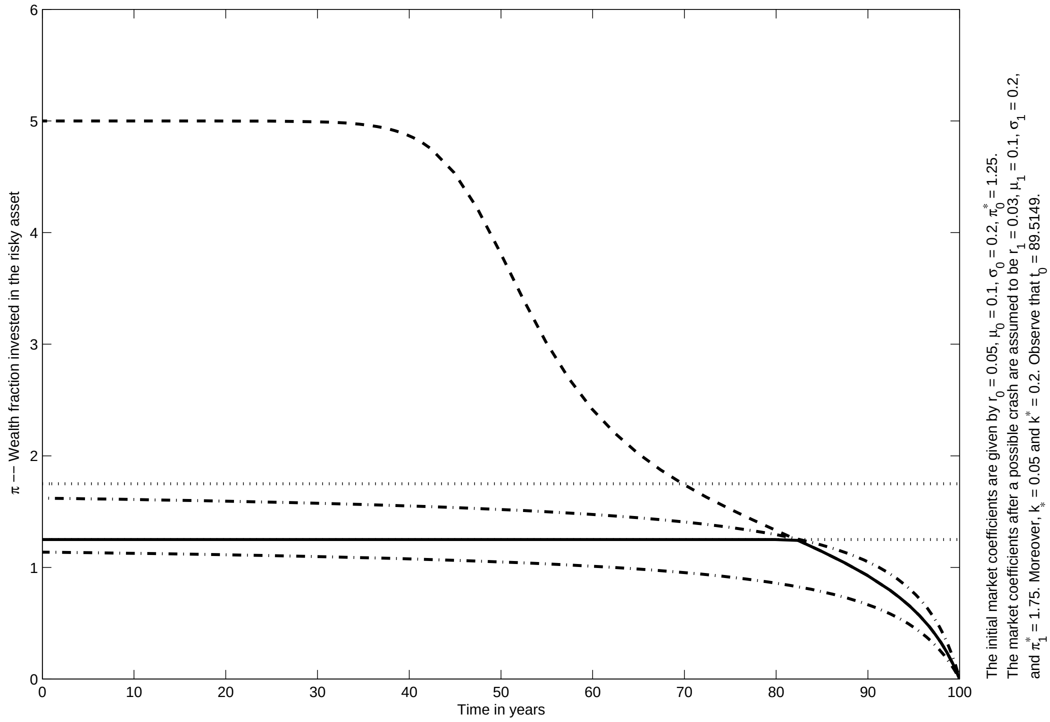
Figure 3: Example $\Psi_1 > \Psi_0$



This graphic shows $\hat{\pi}$ (dashed line), $\bar{\pi}$ (solid line), $\hat{\phi}_0$ (lower dash–dotted line), $\hat{\phi}_1$ (upper dash–dotted line), π_0^* (lower dotted line), and π_1^* (upper dotted line).

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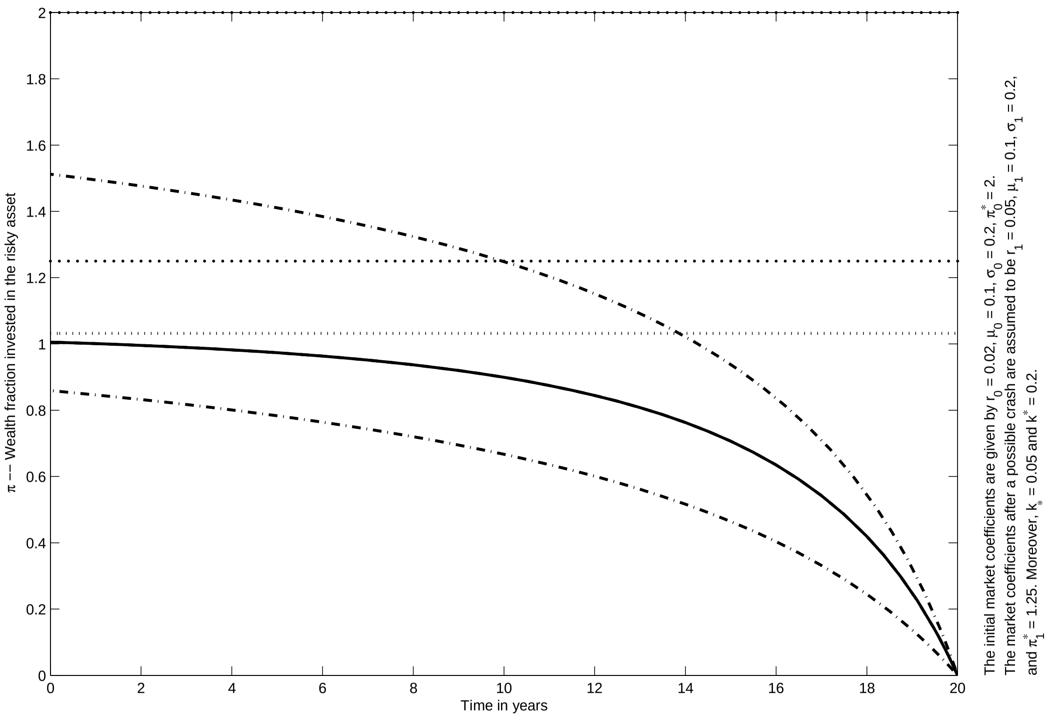
Figure 4: Example $r_0 \leq \Psi_1 \leq \Psi_0$



This graphic shows $\hat{\pi} = \bar{\pi}$ (solid line), $\hat{\phi}_0$ (upper dash–dotted line), $\hat{\phi}_1$ (lower dash–dotted line), $\pi_0^* = 2$ (upper dotted line), and π_1^* (lower dotted line).