Wrapping the Mozartkugel

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Abstract

We study wrappings of the unit sphere by a piece of paper (or, perhaps more accurately, a piece of foil). Such wrappings differ from standard origami because they require infinitely many infinitesimally small "folds" in order to transform the flat sheet into a positive-curvature sphere. Our goal is to find shapes that have small area even when expanded to shapes that tile the plane. We characterize the smallest square that wraps the unit sphere, show that a 0.1% smaller equilateral triangle suffices, and find a 20% smaller shape that still tiles the plane.

Keywords: chocolate, marzipan, praline, nougat

1 Introduction

The Mozartkugel ("Mozart sphere") [9, 8] is a famous fine Austrian confectionery: a sphere with marzipan in its core, encased in nougat or praline cream, and coated with dark chocolate. It was invented in 1890 by Paul Fürst in Salzburg (where Wolfgang Amadeus Mozart was born), six years after he founded his confectionery company, Fürst. Fürst (the company) still to this day makes Mozartkugeln by hand, about 1.4 million per year, under the name "Original Salzburger Mozartkugel" [6]. At the 1905 Paris Exhibition, Paul Fürst received a gold medal for the Mozartkugel.

Many other companies now make similar Mozartkugeln, but Mirabell is the market leader with their "Echte (Genuine) Salzburger Mozartkugeln" [7]. Over 1.5 billion have been made, about 90 million per year, originally by hand but now by industrial methods, and Mirabell claims their product to be the only Mozartkugel that is perfectly spherical. They are also the only Mozartkugel to be taken into outer space, by the first Austrian astronaut Franz Viehböck as a gift to the Russian cosmonauts on the MIR space station. Despite industrial techniques, each Mozartkugel still takes about 2.5 hours to make.

Although most of a Mozartkugel is edible, each sphere is individually wrapped in a square of aluminum foil. To minimize the amount of this wasted, inedible material, it is natural to study the smallest piece of foil that can wrap a unit sphere. Because the pieces will be cut from a large sheet of foil, we would also like the unfolded shape to tile the plane.

We formalize this practical problem in the next section; the main difficulty is to allow a continuum of infinitesimal folds to curve the paper, a feature not normally modeled by mathematical origami. We then study wrappings by squares and equilateral triangles, and show that the latter leads to a small (0.1%) savings, which may prove significant on the many millions of Mozartkugel consumed each year. Even better, if we allow wrapping by arbitrary shapes that tile the plane, we show how to achieve a 20% savings. In addition to direct savings in material costs for Mozartkugel manufacturers, the reduced material usage also indirectly cuts down on CO_2 emissions, and therefore partially solves the global-warming problem and consequently the little-reported but equally important chocolate-melting problem.

2 Wrapping Problem

In standard mathematical origami [4, 5], a *piece of paper* is a two-dimensional manifold (usually flat), and a *folding* is an isometric mapping of this piece of paper into Euclidean 3-space. Here *isometric* means that distances are preserved, as measured by shortest paths on the piece of paper before and after mapping via the folding.

But there is no isometric folding of a square into a sphere: isometric folding preserves curvature. Therefore we define a new, less restrictive type of folding that allows changing curvature but still prevents stretching of the material. Namely, a *wrapping* is a continuous contractive mapping of a piece of paper into Euclidean 3-space. Here *contractive* means that every distance either decreases or stays the same, as measured by shortest paths on the piece of paper before and after mapping via the folding. This definition effectively assumes that the length contraction can be achieved by continuous infinitesimal pleating.

We can model one family of wrappings by expressing which distances are preserved isometrically. An optimal wrapping should be isometric along some path, for otherwise we could uniformly scale the entire wrapping and make a larger object. We call a path *stretched* if the wrapping is isometric along it. A *stretched wrapping* has the property that every point is covered by some stretched path. Such a wrapping can be specified by a set of stretched paths whose

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union covers the entire piece of paper. Although not all such specifications are valid—we need to check that all other paths are contractive—the specification does uniquely determine a wrapping. We specify all of our wrappings in this way, under the belief that stretched wrappings are generally the most efficient.

A special case of stretched wrapping is when the stretched paths consist of the shortest paths from one point x to every other point y. In this case, we are rolling geodesics in the piece of paper onto geodesics of the target surface. This situation corresponds to continuous unfoldings of smooth polyhedra as considered by Benbernou, Cahn, and O'Rourke [1]. Although perhaps the most natural kind of wrapping, this special case is too restrictive for our purposes, as it essentially forces the sphere to be wrapped by a disk of radius π, for those geodesics to reach around to the pole opposite x. We will show how to wrap with far less paper than this disk of area π^3.

Note that, if we start with an arbitrarily long and narrow rectangle, we can wrap the sphere using paper area arbitrarily close to the surface area 4π of the sphere [3]. This wrapping is not very practical, however; in particular, it makes it difficult to make a nondistorted logo on the surface of the sphere.

The only other known optimal wrapping result (where no contraction is necessary) is wrapping a unit cube with a square [2].

3 Petal Wrapping

Our wrappings are based on the following *k-petal wrapping*. On the sphere we first construct k stretched paths $p_1, p_2, \ldots, p_k$ from the south pole to the north pole, dividing the 2π angle around each pole into k equal parts of $2\pi/k$. To each path p_i we assign an "orange peel" with apex angles $2\pi/k$, centered on the path p_i and bounded by the Voronoi diagram of p_{i-1}, p_i, p_{i+1}. These orange peels partition the surface of the sphere into k equal pieces.

Then we construct a continuum of stretched paths to cover each orange peel. Specifically, for every point q along each path p_i, we construct two stretched paths emanating from q, proceeding along geodesics perpendicular to p_i in both directions, and stopping at the boundary of p_i's orange peel.

These stretched paths cover every point of the sphere (covering boundary points twice). It remains to find a suitable piece of paper that wraps according to these stretched paths. The main challenge is to unfold the half of an orange peel left of a path p_i. Then we can easily glue the two halves together along the (straight) unfolded path p_i, and finally join the resulting petals at the unfolded south pole.

To unroll half of a petal, we parameterize as shown in Figure 1. Here $B = \pi/k$ is the half-petal angle; c is a given amount that we traverse along the center path p_i starting at the south-pole endpoint; $A = \pi/2$ specifies that we turn perpendicular from that point; and b is the distance that we travel in that direction. Our goal is to determine b in terms of c.

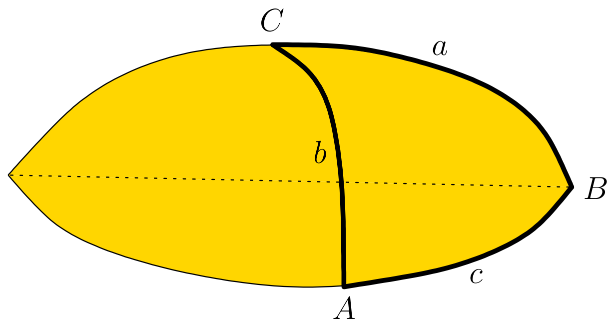


Figure 1: Half of a petal, labeled in preparation for spherical trigonometry.

By the spherical law of cosines,

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Now $\cos A = \cos(\pi/2) = 0$ and $\sin A = \sin(\pi/2) = 1$, so this equation simplifies to $\cos C = \sin B \cos c$. Hence, $\sin C = \sqrt{1 - \sin^2 B \cos^2 c}$. By the spherical law of sines,

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Substituting $\sin C = \sqrt{1 - \sin^2 B \cos^2 c}$, we obtain

$$\frac{\sin B}{\sin b} = \frac{\sqrt{1 - \sin^2 B \cos^2 c}}{\sin c},$$

i.e.,

$$\sin b = \frac{\sin B \sin c}{\sqrt{1 - \sin^2 B \cos^2 c}}.$$

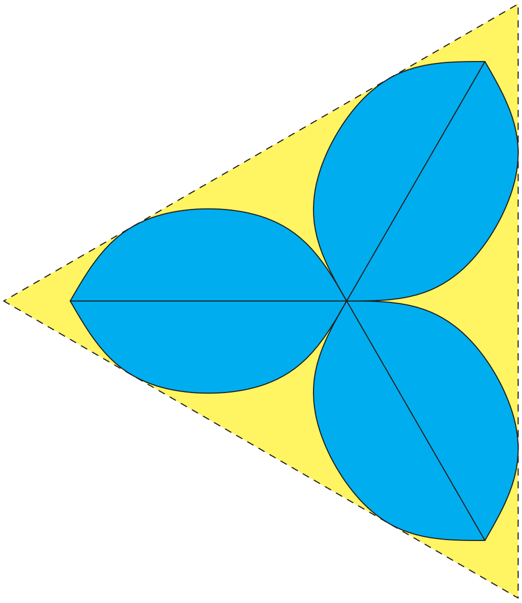
Taking arccos of both sides, we determine the value of b in terms of the parameter c and the known quantity $B = \pi/k$.

Figure 2 shows two examples of the resulting petal unfolding, with $k = 3$ and $k = 4$.

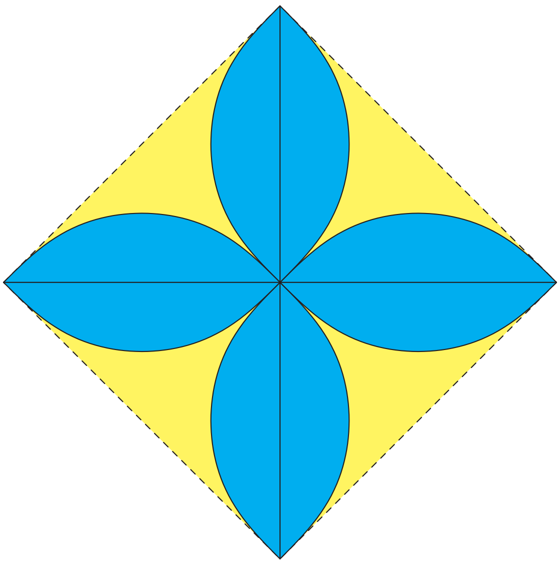
4 Square Wrapping

The angle at the tip of the petals can be computed by taking the derivative $\partial b/\partial c$ at $c = 0$. For $k = 4$, this derivative is 1 which implies a half angle of $\pi/4$. Because the petals are convex, the convex hull of the petal unfolding for $k = 4$ is exactly the square of diagonal 2π. No smaller square could wrap the unit sphere because the length of the path connecting the center of the square to the point mapped to the antipodal point must have length at least π. This square has area $2\pi^2$.

Note that the same area is attainable by a rectangle of dimensions $2\pi \times \pi$: draw one path p around the



(a) $k = 3$



(b) $k = 4$

Figure 2: Petal unfoldings.

equator of the sphere and cover the sphere by a continuum of stretched paths perpendicular to p emanating from every point of p until the north and the south pole of the sphere. The same rectangle is also exactly a 2-petal unfolding. Interestingly, the area of this rectangle wrapping is also 2π. The Echte Salzburger Mozartkugel is wrapped by Mirabell using the same rectangle (expanded a bit to ensure overlap) but with a slightly different folding.

5 Triangle Wrapping

For $k = 3$, the angle at the tip of the petals can be computed similarly to obtain $2\pi/3$, which is natural as the three petals meet at the north pole, their angles summing to 2π. However, the convex hull of the 3-petal unfolding is not a triangle. We compute its smallest enclosing equilateral triangle. The supporting lines of the triangle will be each tangent to two of the petals. The tangent point on the petal can be computed by finding the point (c, b) on its boundary that maximizes the direction $(-\cos(\pi/3), \sin(\pi/3))$. Plugging this into the previous equations, we obtain

$$c = \arccos\left(\frac{\sqrt{57}}{6} - \frac{1}{2}\right) \approx 0.710086.$$

This implies that the supporting line is at a distance

$$\frac{\pi}{2} - \frac{1}{2}\arccos\left(\frac{\sqrt{57}}{6} - \frac{1}{2}\right) + \frac{\sqrt{3}}{2}\arcsin\left(\frac{\sqrt{\sqrt{57}-5}}{\sqrt{\sqrt{57}-3}}\right) \approx 0.620190\pi$$

from the center. The area of the inscribing equilateral triangle is therefore $3h^2\tan(\pi/6) \approx 1.998626\,\pi^2$, about 0.1% less than the $2\pi^2$ area of the smallest wrapping square.

6 Tiling

Instead of expanding the petal unfoldings to tilable regular polygons, we can pack the petal unfoldings directly and expand them just to fill the extra space. Figure 3 shows an even better tiling resulting from the 3-petal unfolding. A quick computation shows that only about $1.6\,\pi^2$ area of paper is required for each wrapping, a substantial improvement.

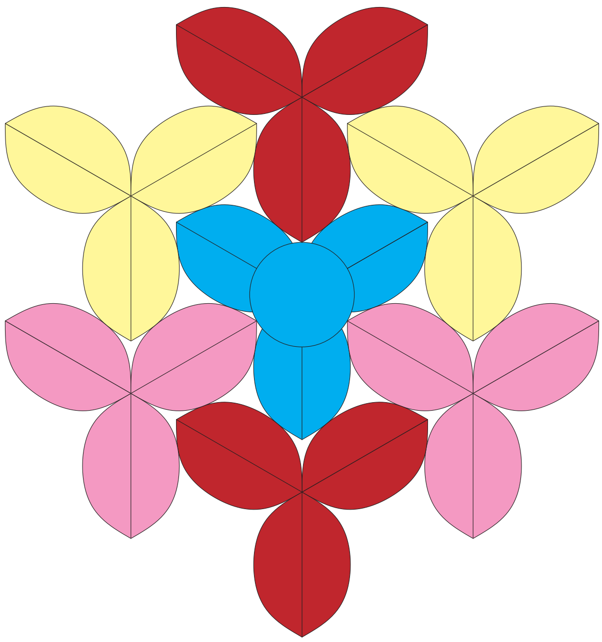


Figure 3: Packing the 3-petal unfolding.

7 Conclusion

This paper initiates a new research direction in the area of *computational confectionery*. We leave as open problems the study of wrapping other geometric confectioneries, or further improving our wrappings of the Mozartkugel. In particular, what is the optimal convex shape that can wrap a unit sphere? What is the optimal shape that also tiles the plane? What about smooth surfaces other than the sphere?

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