WRINKLING IN FINITE PLANE-STRESS THEORY*

By

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Abstract. A general and complete formulation is given for the wrinkling phenomenon in the context of finite plane-stress theory. The planar portion of the true three-dimensional displacement field, called the pseudo-displacement field, is used as a basis for the necessary kinematic analysis. It is assumed that the principal directions associated with the *pseudo*-deformation field are the same as those associated with the true stress field. The true stress field is governed by equilibrium and the assumption that one of the principal stresses vanishes, and hence is statically determinate. The difference between the *pseudo*-strain and the *true* strain calculated from the true stress is a new tensor, called the wrinkle-strain tensor, and serves as a measure of the wrinkliness of the surface.

1. Introduction. Wrinkling is a phenomenon that is commonly associated with the deformation of a thin membrane surface. It is commonly assumed that the direction of a wrinkle is a principal direction. The principal stress resultant along a wrinkle is assumed to be tensile, while the transverse principal stress resultant is assumed to be zero. This assumption renders the equilibrium conditions statically determinate, and the resulting analysis is the so-called tension field theory. This theory has been studied extensively in the context of linear plane-stress theory by many authors [1-8].[1] The formulations used in these references are not quite the same but are all built upon the same basic assumption. The work reported in [7] was obtained without the knowledge of the earlier references and hence is not published. It should be mentioned, however, that the use of a *wrinkle strain* to measure the wrinkliness of a deformation was first explicitly introduced in [7].[2] In any case, the formulation needed for a linear analysis is complete.

When the deformation is finite, the system of equilibrium conditions is still statically determinate. This leads to the false impression that the necessary nonlinear analysis is just as straightforward as the linear analysis. But how does one get back to the displacement field from the stresses? This question cannot be answered without a detailed study of the kinematics based on physically reasonable assumptions. Formulations applicable to axially symmetric problems can be found in [9, 10]. The purpose of this paper is to give a complete

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[1] Anisotropic and nonlinear elastic properties were considered in [8].

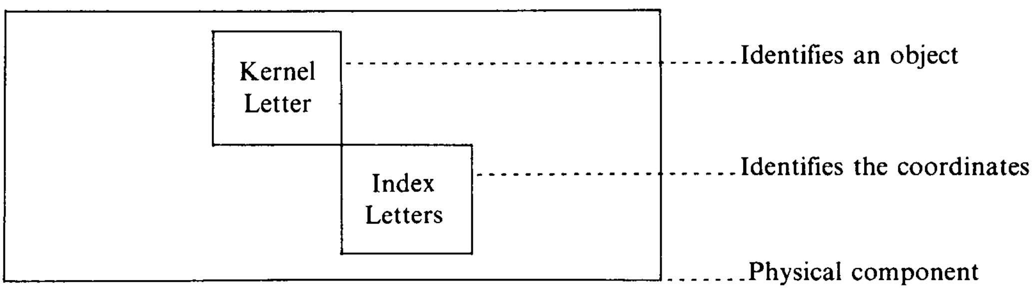
[2] To be sure, the variable Poisson's ratio and the maximum slope of the wrinkles introduced in [3] and [4], respectively, are also measures of the wrinkliness.

and general formulation of the wrinkling phenomenon in the context of finite plane-stress theory.

We shall begin by naming the planar portion of the true three-dimensional displacement field the *pseudo-displacement field.* The main assumption will be that the principal directions associated with the pseudo-displacement field are the same as those associated with the true stress field. The difference between the strains calculated from the pseudo-displacement field and strains calculated from the true stress field is a measure of the wrinkliness of the surface.

Some mathematical preliminaries, mainly having to do with Schouten's kernal-index notation, are reviewed in Sec. 2. This notation is convenient for our purpose, because we have to deal with several sets of different curvilinear coordinates. Finite plane-stress theory with wrinkles is introduced in Sec. 3. The presentation of Sec. 3 is somewhat fragmentary because of the mixed use of several coordinates. Sec. 4 is an attempt to summarize the equations obtained in Sec. 3. The class of rotationally symmetric problems is solved exactly in Sec. 5, and some results are presented in Sec. 6 with a set of initial data as parameters.

2. Mathematical preliminaries via Schouten's kernel-index notation. The objectives of this paper require the solution of the physical components of various field variables in different curvilinear coordinates. Schouten's kernel-index notation [11, 12] appears to be most suitable for our purposes. The notation uses a kernel letter to identify an object, and index letters to identify the reference bases. Moreover, the word "component" is always meant to be the physical component. This is summarized as follows:



Before proceeding, we set forth in Table 1, once and for all, the kernel letters and index letters to be assigned to various coordinates.

The base vectors associated with a set of coordinate axes are denoted by

$$\mathbf{e}_{\text{Index}} \qquad \text{Index} = (I, i, A, a) \tag{2.1}$$

and the associated local Cartesian unit vectors are

$$\mathbf{i}_{\text{Index}} = \mathbf{e}_{\text{Index}}/|\mathbf{e}_{\text{Index}}|. \tag{2.2}$$

It follows from our notation (Table 1) that

$$\mathbf{i}_{\text{Index}} = \mathbf{e}_{\text{Index}} \quad \text{for} \quad \text{Index} = I \text{ or } i. \tag{2.3}$$

Direction cosines are defined accordingly:

$$Q_{(\text{1st Index})(\text{2nd Index})} = \mathbf{i}_{(\text{1st Index})} \cdot \mathbf{i}_{(\text{2nd Index})} \tag{2.4}$$

where (1st Index) and (2nd Index) are associated with different bases. A characteristic of

TABLE 1. Coordinate systems (all subscripts range over the integers (1, 2)).

Coordinates	States	Undeformed State	Deformed State
Rectangular Cartesian	Indexed	Z_I I is a typical member of (I, J, K, L, M, N)	z_i i is a typical member of (i, j, k, l, m, n)
	Explicit	$Z_I = X$ $I = 1$ for $Z_I = Y$ $I = 2$	$z_i = x$ $i = 1$ for $z_i = y$ $i = 2$
Orthogonal Curvilinear	Indexed	X_A A is a typical member of (A, B, C, D, E, F)	x_a a is typical member of (a, b, c, d, e, f)
	Explicit	$X_A = U$ $A = 1$ for $X_A = V$ $A = 2$	$x_a = u$ $a = 1$ for $x_a = v$ $a = 2$
	Principal	$X_A = P$ $A = 1$ for $X_A = Q$ $A = 2$	$x_a = p$ $a = 1$ for $x_a = q$ $a = 2$

Schouten's notation is that

$$Q_{(\text{1st Index})(\text{2nd Index})} = Q_{(\text{2nd Index})(\text{1st Index})}. \tag{2.5}$$

This, however, is merely an identity but not a symmetry property. In terms of the direction cosines, physical components of a tensor **T** transform like Cartesian components, e.g.,

$$T_{IJ} = Q_{IA} Q_{JB} T_{AB}, \qquad T_{iJ} = Q_{ia} Q_{JB} T_{aB}, \qquad \text{etc.} \qquad (2.6)^3$$

Let the transformation between the Cartesian coordinates Z_I and the orthogonal curvilinear coordinates X_A be denoted by

$$Z_I = Z_I(X_A), \qquad X_A = X_A(Z_I). \tag{2.7}$$

Then, since X_A are orthogonal,

$$Z_{I,A} Z_{I,B} = 0 \quad \text{for} \quad A \neq B. \qquad (2.8)^4$$

The square of a line element dL is

$$(dL)^2 = dZ_I\, dZ_I = \delta X_A \delta X_A \tag{2.9}$$

where

$$\delta X_A \equiv H_A^{-1}\, dX_A \quad (\text{no sum}) \qquad (2.10)^5$$

[3] Summation over repeated subscripts is taken for granted.

[4] Subscripts preceded by a comma indicate partial differentiation.

[5] No sum is performed in similar situations.

and

$$H_A^{-1} = (Z_{I,A} Z_{I,A})^{1/2} = (X_{A,I} X_{A,I})^{-1/2}. \tag{2.11}$$

It follows from the above that

$$X_{A,I} = H_A^2 Z_{I,A}, \qquad Z_{I,A} = H_A^{-2} X_{A,I}. \tag{2.12}$$

We note in passing that δX_A is nothing but a convenient notation.

Let $\mathbf{P}$ be the position vector of a point; then

$$\mathbf{e}_A = \mathbf{P}_{,A}, \qquad |\mathbf{e}_A| = (Z_{I,A} Z_{I,A})^{1/2} = H_A^{-1}, \qquad \mathbf{i}_A = H_A \mathbf{e}_A. \tag{2.13}$$

We introduce directional differentiation defined by

$$\partial_A(\ ) = H_A(\ )_{,A}. \tag{2.14}$$

Then

$$\mathbf{i}_A = H_A \mathbf{e}_A = H_A \mathbf{P}_{,A} = \partial_A \mathbf{P}, \tag{2.15}$$

$$Q_{AI} = Q_{IA} = \mathbf{i}_A \cdot \mathbf{i}_I = \partial_A \mathbf{P} \cdot \mathbf{i}_I = \partial_A Z_I = H_A Z_{I,A} = H_A^{-1} X_{A,I}. \tag{2.16}$$

Some of the most useful identities are:

$$\begin{aligned} &\mathbf{i}_A = Q_{AI} \mathbf{i}_I, \qquad \mathbf{i}_I = Q_{IA} \mathbf{i}_A, \qquad \partial_A = Q_{AI} \partial_I, \\ &\partial_I = Q_{IA} \partial_A, \qquad \delta X_A = Q_{AI}\, dZ_I, \qquad dZ_I = Q_{IA} \delta X_A. \end{aligned} \tag{2.17}$$

Absolute differentiations in orthogonal curvilinear coordinates are denoted by a semicolon, and are defined by

$$T_{A;B} = Q_{AI} Q_{BJ} T_{I,J}, \qquad T_{AB;C} = Q_{AI} Q_{BJ} Q_{CK} T_{IJ,K}, \quad \text{etc.} \tag{2.18}$$

It follows from the above and (2.17) that

$$T_{A;B} = \partial_B T_A + W_{CAB} T_C, \qquad T_{AB;C} = \partial_C T_{AB} + W_{EAC} T_{EB} + W_{EBC} T_{AE}, \quad \text{etc.} \tag{2.19}$$

where

$$W_{CAB} = Q_{AI} \partial_B Q_{IC} \tag{2.20}$$

are the wryness coefficients. For orthogonal curvilinear coordinates, the only non-zero coefficients are

$$W_{BAB} = -W_{ABB} = \partial_A \ln H_B \qquad (A \neq B). \tag{2.21}$$

This completes the relations needed for the $\mathbf{Z} - \mathbf{X}$ transformation. Relations pertinent to the $\mathbf{z} - \mathbf{x}$ transformation may be obtained from the above by simply replacing all the subscripts by their lower-case counterparts.

3. Finite plane-stress theory with wrinkles. Let M be the domain of the (Z_1, Z_2)-plane characterizing the shape of a membrane surface in its undeformed configuration. We assume that the membrane is deformed to a wrinkly surface so that the position of a point (Z_1, Z_2) after deformation is (z_1, z_2, z_3). The deformation may be represented by a transformation

$$z_i = z_i(Z_I), \qquad z_3 = z(Z_I) \quad \text{for all} \quad \mathbf{Z} \in M, \tag{3.1}$$

where, as throughout this paper, all subscripts range over the integers (1, 2). The transformation (3.1) maps M onto a wrinkly surface m^* characterized by the fact that z_3 is not identically zero. Physical evidence seems to indicate that the distribution of wrinkles is a random process. Moreover, creases and folds caused by wrinkling and characterized by $z_3 = z(Z_I)$ are difficult to define analytically. To bypass these difficulties, we use the surface m defined by

$$m\colon z_i = z_i(Z_I), \qquad z_3 \equiv 0, \tag{3.2}$$

to give a reference description of the surface m^*. The difference between m and m^* will be described by a new strain-like kinematic variable to be defined in the development to follow. The surface m will be called the *pseudo-deformed surface* to emphasize the fact that it is not the true deformed surface. All quantities to be defined on the surface m will be prefixed by "pseudo-" to give the same implication. It is clear from (3.1) and (3.2) that the surface m is nothing but the projection of m^* on the plane. We do, however, make one assumption that the projection is one-to-one so that the mapping between (Z_1, Z_2) and (z_1, z_2) is one-to-one.

Let F_{iI} be the components of the *pseudo-deformation-gradient tensor* **F** associated with (3.2), whence

$$F_{iI} = z_{i,I}. \tag{3.3}$$

By polar decomposition, **F** has the representation

$$\begin{aligned}[F_{iI}] = [z_{i,I}] = \tfrac{1}{2}(\Lambda_1 + \Lambda_2)\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\\ + \tfrac{1}{2}(\Lambda_1 - \Lambda_2)\begin{bmatrix}\cos(2\alpha+\beta) & \sin(2\alpha+\beta)\\ \sin(2\alpha+\beta) & -\cos(2\alpha+\beta)\end{bmatrix},\end{aligned} \tag{3.4}$$

where Λ_1 and Λ_2 are the *pseudo-principal stretch ratios* associated with **F** and

$$\Lambda_1\Lambda_2 = J = \det[F_{iI}], \qquad \Lambda_1^2 + \Lambda_2^2 = I = F_{iI}F_{iI}. \tag{3.5, 3.6}$$

The angles α and β, depicted in Fig. 1, define the orientations of the *pseudo-principal coordinates* $(X_1 = P,\ X_2 = Q)$ and $(x_1 = p,\ x_2 = q)$.[6] We may now proceed to state our *Assumption I*: The projection on the plane of the set of principal coordinates associated with the true deformation experienced by the deformed surface m^* is just the set of pseudo-principal coordinates defined by F_{iI}.

Let Λ_1^* and Λ_2^* be the true principal stretch ratios experienced by the deformed surface m^*; then it is convenient to *construct* a deformation-gradient-like tensor $\mathbf{F}^*$ by the expression

$$[F_{iI}^*] = [\text{replace } (\Lambda_1, \Lambda_2) \text{ by } (\Lambda_1^*, \Lambda_2^*) \text{ in (3.4)}]. \tag{3.7}$$[7]

We emphasize that the term "deformation-gradient" here is nothing more than a convenient name because $\mathbf{F}^*$ is not the gradient of a deformation. Nevertheless, (3.7) implies that

$$\Lambda_1^*\Lambda_2^* = J^* = \det[F_{iI}^*], \qquad \Lambda_1^{*2} + \Lambda_2^{*2} = I^* = F_{iI}^*F_{iI}^*, \tag{3.8, 3.9}$$

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[6] Cf. Table 1 for notation.

[7] Unless otherwise stated, an asterisk is used to identify a true physical quantity defined on the true deformed surface m^*.

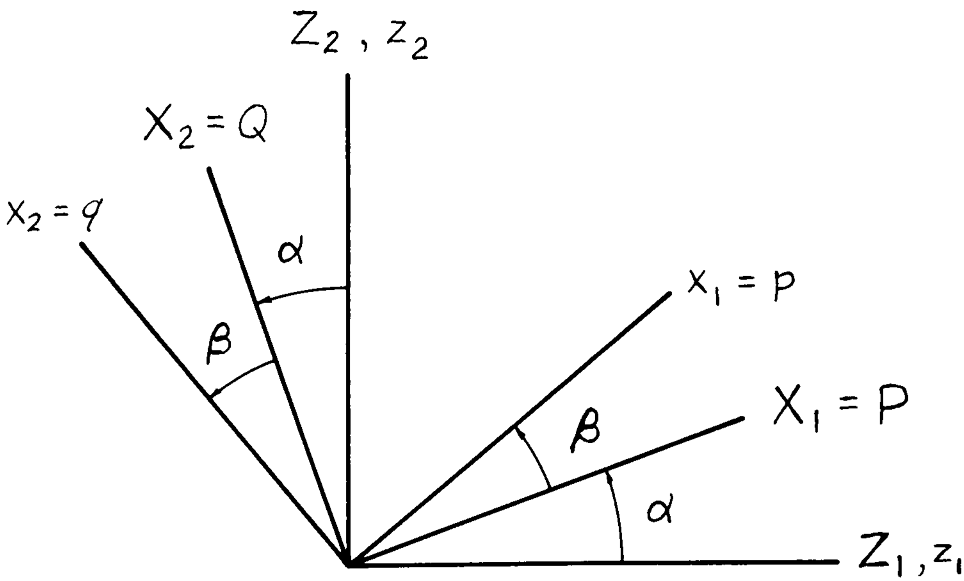

FIG. 1. Principal coordinates.

which, in turn, may be used to define the true strain energy density function

$$U = U(I^*, J^*). \tag{3.10}$$

Before proceeding, it is again convenient to introduce a new tensor **W** defined by

$$\mathbf{W} = \mathbf{F} - \mathbf{F}^*. \tag{3.11}$$

It follows from (3.4) and (3.7) that the principal quantities W_1 and W_2 associated with **W** are just

$$W_1 = \Lambda_1 - \Lambda_1^*, \qquad W_2 = \Lambda_2 - \Lambda_2^*. \tag{3.12}$$

Since (Λ_1, Λ_2) are the projections of $(\Lambda_1^*, \Lambda_2^*)$ on the plane, (W_1, W_2) must be either zero (no wrinkle) or negative. Thus, we assume without the loss of generality that the 1-direction is always the taut direction so that

$$W_1 = 0, \qquad \Lambda_1 = \Lambda_1^*, \tag{3.13}$$

$$W_2 = \Lambda_2 - \Lambda_2^* \leq 0. \tag{3.14}$$

Eq. (3.12) implies that

$$W_2 = \Lambda_2 - \Lambda_2^* = \frac{\text{Projection of deformed length} - \text{Deformed length}}{\text{Undeformed length}},$$

which suggests the name *wrinkle-strain tensor* for **W** defined by (3.11).

We proceed to state our *Assumption II*: The true principal Piola stresses experienced by the deformed surface m^* are

$$P_1 = \frac{\partial U}{\partial \Lambda_1^*} = 2\,\frac{\partial U}{\partial I^*}\,\Lambda_1^* + \frac{\partial U}{\partial J^*}\,\Lambda_2^*, \tag{3.15}$$

$$P_2 = \frac{\partial U}{\partial \Lambda_2^*} = 2\,\frac{\partial U}{\partial I^*}\,\Lambda_2^* + \frac{\partial U}{\partial J^*}\,\Lambda_1^*, \tag{3.16}$$

where

$$P_2 = 0 \qquad \text{if} \quad W_2 < 0. \tag{3.17}$$

In view of the fact that the principal Piola stresses are defined with respect to the undeformed surface M, and our Assumption I that the true principal directions are the same as the pseudo-principal directions, there is no need to differentiate pseudo-Piola stresses from Piola stresses as far as our notation and terminology are concerned. This leads to the one exception that an asterisk is not used to identify a true physical quantity, and the (first) Piola stress tensor is just

$$[P_{iI}] = [\text{replace } (\Lambda_1, \Lambda_2) \text{ by } (P_1, P_2) \text{ in } (3.4)], \tag{3.18}$$

and

$$[P_{iI}] = P_1 \begin{bmatrix} \cos(\alpha+\beta)\cos\alpha & \cos(\alpha+\beta)\sin\alpha \\ \sin(\alpha+\beta)\cos\alpha & \sin(\alpha+\beta)\sin\alpha \end{bmatrix} \quad \text{if} \quad P_2 = 0 \qquad (W_2 < 0). \tag{3.19}$$

Isotropy is, of course, presumed in deriving (3.18).

The true Cauchy stress tensor $\mathbf{T}^*$ (defined on m^*) and the *pseudo-Cauchy stress tensor* $\mathbf{T}$ (defined on m) may be expressed in terms of $\mathbf{P}$, $\mathbf{F}^*$ and $\mathbf{F}$. They are

$$\mathbf{T}^* = \frac{1}{J^*}\,\mathbf{P}\mathbf{F}^{*T}, \qquad (3.20)^{8}$$

$$\mathbf{T} = \frac{1}{J}\,\mathbf{P}\mathbf{F}^{T}. \tag{3.21}$$

In particular,

$$T_{ij} = \frac{1}{J}\,P_{iI}F_{jI}, \tag{3.22}$$

$$P_{iI} = JT_{ij}f_{Ij}, \tag{3.23}$$

where

$$f_{Ii} = Z_{I,i}. \tag{3.24}$$

The tensors $\mathbf{F}$ and $\mathbf{f}$ satisfy the relations

$$f_{Ii} = \frac{1}{J}\,e_{IL}e_{il}F_{lL}, \tag{3.25}$$

$$F_{iI} = Je_{il}e_{IL}f_{Ll}, \tag{3.26}$$

[8] The superscript T indicates transposition.

where e_{il}, e_{IL} are the two-dimensional alternators. We recall that $\mathbf{F}^*$ is *defined* by (3.7) and, as a result, no interpretation of the form (3.3) and (3.24) can be assigned to its components as well as the components of its inverse. Let (T_1^*, T_2^*) and (T_1, T_2) be, respectively, the principal Cauchy stresses and *principal pseudo-Cauchy stresses* associated with $\mathbf{T}^*$ and $\mathbf{T}$. The three sets of principal stresses are related by (3.20) and (3.21), and the explicit relations are

$$P_1 = \Lambda_2^* T_1^* = \Lambda_2 T_1, \tag{3.27}$$

$$P_2 = \Lambda_1^* T_2^* = \Lambda_1 T_2, \tag{3.28}$$

$$P_2 = T_2^* = T_2 = 0 \qquad \text{if} \quad W_2 < 0. \tag{3.29}$$

We note in passing that it is possible to make Λ_2 vanish by "wrinkling" an arc element to zero while keeping Λ_2^* finite. This is why T_1^* is always finite even though the associated T_1 may be infinite, a situation that appears very often in an actual solution.

In the absence of body forces, the Piola stresses satisfy the equations of equilibrium

$$P_{iI,I} = 0 \quad \text{on} \quad M. \tag{3.30}$$

Using (3.23–3.26), we find that the pseudo-Cauchy stresses satisfy the equations of equilibrium

$$T_{ij,j} = 0 \quad \text{on} \quad m. \tag{3.31}$$

As a further consequence of the fact that no relations of the form (3.24) can be assigned to F_{iI}^*, the equations

$$T_{ij,j}^* = 0 \quad \textit{are not valid conditions.} \tag{3.32}$$

In linear theory, however, there is no difference between (3.31) and (3.32). This completes the general formulation in Cartesian coordinates. It suffices to mention that the theory is a straightforward extension of the conventional membrane theory. The presence of a wrinkly region is indicated by the additional field equation $P_2 = 0$, (3.29), and the wrinkliness of the membrane is measured by the additional kinematic variable W_2.

To apply these equations to boundary-value problems, it is very often more convenient to introduce suitable curvilinear coordinates. Several curvilinear versions of these equations are derived in Appendix A.

4. A summary of the equations. The exposition presented in Section 3 is somewhat fragmentary in that too many sets of related coordinates, including the unknown principal coordinates, are involved. As a result, it is difficult to tell just which are the independent unknowns and what are their governing equations. It is therefore desirable to provide a summary to clarify the situation. We shall do this by employing the explicit notations (X, Y), (x, y), $(P\ Q)$, (p, q), etc., identified in Table 1. Certain equations will be repeated for the sole purpose of putting everything in one place. Also, for the sole purpose of summarizing and enumerating unknowns and equations, we have found it convenient to interpret the deformed principal coordinates (p, q) as the independent variables. Thus, quantities involved in this section are all to be considered as functions of p and q.

We begin with Eq. (A21) of Appendix A, the off-diagonal terms of which indicate that

$$P = P(p), \qquad Q = Q(q). \tag{4.1}$$

It follows from (A5), (A6) and (4.1) that

$$H_p = (x_{,p}^2 + y_{,p}^2)^{-1/2}, \qquad H_q = (x_{,q}^2 + y_{,q}^2)^{-1/2}, \tag{4.2}$$

$$H_P = \frac{dP}{dp}(X_{,p}^2 + Y_{,p}^2)^{-1/2}, \qquad H_Q = \frac{dQ}{dq}(X_{,q}^2 + Y_{,q}^2)^{-1/2}. \tag{4.3}$$

Using these relations and the diagonal terms of (A21), we obtain

$$\Lambda_1 = \frac{H_P}{H_p}\frac{dp}{dP} = [(x_{,p}^2 + y_{,p}^2)/(X_{,p}^2 + Y_{,p}^2)]^{1/2}, \tag{4.4}$$

$$\Lambda_2 = \frac{H_Q}{H_q}\frac{dq}{dQ} = [(x_{,q}^2 + y_{,q}^2)/(X_{,q}^2 + Y_{,q}^2)]^{1/2}. \tag{4.5}$$

The kinematic wrinkling conditions, (3.13) and (3.14), are

$$W_2 = \Lambda_2 - \Lambda_2^*, \tag{4.6}$$

$$\Lambda_1 = \Lambda_1^*, \tag{4.7}$$

where Λ_1^* and Λ_2^* are the true principal stretch ratios. They are determined from the stresses in equilibrium via the use of constitutive relations.

The assumption (3.29), together with (3.16), yields

$$2\frac{\partial U^*}{\partial I^*}\Lambda_2^* + \frac{\partial U^*}{\partial J^*}\Lambda_1^* = 0 \quad \text{or} \quad \Lambda_2^* = \Lambda^*(\Lambda_1^*), \tag{4.8}$$

where $\Lambda^*(\Lambda_1^*)$ is a root of the original equation, and may or may not have a simple explicit form. The non-zero principal Piola stress P_1 may be obtained from (3.15). It is

$$P_1 = P_1(\Lambda_1^*) = \left(2\frac{\partial U}{\partial I^*}\Lambda_1^* + \frac{\partial U}{\partial J^*}\Lambda_2^*\right)_{\Lambda_{2^*} = \Lambda^*(\Lambda_{1^*})} \tag{4.9}$$

The two equations of equilibrium (A27), (A28) now become

$$\frac{\partial}{\partial P}\left(\frac{P_1}{H_Q}\right) = 0 \quad \text{or} \quad \frac{\partial}{\partial p}[P_1(X_{,q}^2 + Y_{,q}^2)^{1/2}] = 0, \tag{4.10}$$

$$\frac{\partial H_p}{\partial q} = 0 \quad \text{or} \quad \frac{\partial}{\partial q}(x_{,p}^2 + y_{,p}^2)^{-1/2} = 0, \tag{4.11}$$

where (4.1) has been used in deriving (4.10). Finally, the two sets of principal coordinates must be orthogonal. The relations are just

$$x_{,p}x_{,q} + y_{,p}y_{,q} = 0, \tag{4.12}$$

$$X_{,p}X_{,q} + Y_{,p}Y_{,q} = 0. \tag{4.13}$$

The ten equations (4.4)–(4.13) completely determine the ten unknowns, (x, y), (X, Y), (Λ_1, Λ_2), $(\Lambda_1^*, \Lambda_2^*)$, W_2 and P_1 as functions of p and q.

It can be easily shown by using (4.11) that the p-(principal) coordinate curves are straight lines (see, e.g., [1–7]). Let En be the envelope of the family of straight lines; then the q-(principal) coordinate curves are just the involutes of En. It is therefore sometimes more convenient to define p and q through the use of an arbitrary envelope curve En. This will be illustrated in Sec. 5.

We conclude this section by mentioning that for a Mooney material

$$U = (I^* + J^{*-2}) + k(J^{*2} + I^*J^{*-2}), \tag{4.14}$$

and Eqs. (4.8) and (4.9) become

$$\Lambda_2^* = \Lambda^*(\Lambda_1^*) = \Lambda_1^{*-1/2}, \tag{4.15}$$

$$P_1 = P_1(\Lambda_1^*, k) = 2\Lambda_1^*\left(1 + \frac{k}{\Lambda_1^*}\right)\left(1 - \frac{1}{\Lambda_1^{*3}}\right). \tag{4.16}$$

In (4.14), k is defined by

$$k = C_2/C_1, \tag{4.17}$$

the ratio of the two Mooney constants, and U is nondimensionalized by C_1H with H being the constant undeformed thickness of the membrane.

5. Rotationally symmetric problems—exact solution. In terms of the polar coordinates (R, Θ) and (r, θ), the class of rotationally symmetric problems is defined by the relations

$$\begin{cases} r = r(R) \\ \theta = \Theta + \phi(R) \end{cases} \quad \text{or} \quad \begin{cases} R = R(r) \\ \Theta = \theta - \Phi(r) \end{cases}. \tag{5.1}$$

We shall study this class of problems in detail, and show that the system of equations may be reduced to quadratures. The required deductions parallel to those used in [10] for axially symmetric problems. Indeed, symmetric solutions are just special cases of (5.1).[9]

Since the p-coordinate curves are straight lines (Fig. 2c), the transformation between (r, θ) and (p, q) are given by (B10) and (B11) of Appendix B. We shall use (5.1), (B10) and (B11) to simplify the set of equations obtained in Sec. 4 to suit this particular class of problems. The explicit forms of the equations outlined in Sec. 4, however, are not always the most convenient ones to use for a given situation. Thus, when a citation is made of an equation from Sec. 4, we do not necessarily mean the explicit form of that equation, but rather one of its many equivalent variations.

We begin by substituting (5.1) into (B10) and (B11) to obtain p and q as functions of R and Θ. The results are

$$p = \Theta + p_0(R), \qquad q = \Theta + q_0(R), \tag{5.2}$$

where

$$p_0 = \left[\Phi(r) - \cos^{-1}\frac{r_e}{r} + \frac{1}{r_e}(r^2 - r_e^2)^{1/2}\right]_{r=r(R)}, \tag{5.3}$$

$$q_0 = \left[\Phi(r) - \cos^{-1}\frac{r_e}{r}\right]_{r=r(R)}. \tag{5.4}$$

[9] In linear elasticity, the term "rotationally symmetric solution" is exclusively reserved for the situation $r \equiv R$, $\theta = \Theta + \phi(R)$.

It follows that

$$\frac{\partial p}{\partial R} = \frac{dp_0}{dR} = \left\{ \frac{d\Phi}{dr} + \frac{1}{r}\left[\left(\frac{r}{r_e}\right)^2 - 1\right]^{1/2} \right\} \frac{dr}{dR}, \tag{5.5}$$

$$\frac{\partial q}{\partial R} = \frac{dq_0}{dR} = \left\{ \frac{d\Phi}{dr} - \frac{1}{r}\left[\left(\frac{r}{r_e}\right)^2 - 1\right]^{1/2} \right\} \frac{dr}{dR}. \tag{5.6}$$

Eq. (B12) now becomes

$$H_p = \frac{1}{r_e}, \qquad H_q = \frac{1}{r_e(p_0 - q_0)}. \tag{5.7}$$

For the undeformed principal coordinates, we use (4.3), (5.1) and (5.2) to obtain

$$H_P \frac{dp}{dP} = \left[\left(\frac{\partial p}{\partial R}\right)^2 + \frac{1}{R^2}\right]^{1/2} = \left[\left(\frac{dp_0}{dR}\right)^2 + \frac{1}{R^2}\right]^{1/2}, \tag{5.8}$$

$$H_Q \frac{dq}{dQ} = \left[\left(\frac{\partial p}{\partial R}\right)^2 + \frac{1}{R^2}\right]^{1/2} = \left[\left(\frac{dq_0}{dR}\right)^2 + \frac{1}{R^2}\right]^{1/2}. \tag{5.9}$$

The conditions (4.11) and (4.12) are identically satisfied by the choice of (p, q) defined by (B10) and (B11). The condition (4.13), after applying (4.1), (5.1) and (5.2), now becomes

$$\frac{\partial p}{\partial R}\frac{\partial q}{\partial R} + \frac{1}{R^2} = \frac{dp_0}{dR}\frac{dq_0}{dR} + \frac{1}{R^2} = 0. \tag{5.10}$$

A convenient expression for Λ_1 may be obtained from (4.4), (5.7) and (5.8). It is

$$\Lambda_1 = r_e H_P \frac{dp}{dP} = r_e\left[\left(\frac{dp_0}{dR}\right)^2 + \frac{1}{R^2}\right]^{1/2}. \tag{5.11}$$

In view of (4.7), the principal Piola stress P_1 defined by (4.16) is

$$P_1 = P_1(\Lambda_1) \tag{5.12}$$

where the functional form of $P_1(\Lambda_1)$ depends on the choice of the strain energy density function, and may or may not be an explicit expression. Integrating the first form of (4.10) yields

$$P_1(\Lambda_1) = \left(K \frac{dq}{dQ}\right) H_Q \tag{5.13}$$

where the quantity in front of H_Q is taken as the arbitrary function of integration, and hence K is an arbitrary function of Q. In view of (5.9) and the fact that P_1 can only be a function of R for rotationally symmetric problems, K can at most be a function of R. This, however, is impossible because R is a function of both P and Q. It follows that K is a constant.

We proceed to convert the governing equations to a system of uncoupled ordinary differential equations with Λ_1 as the independent variable. Applying (5.8), (5.10) and (5.11) to (5.9), we get

$$H_Q \frac{dq}{dQ} = \Lambda_1 \bigg/ \left(r_e R \frac{dp_0}{dR}\right). \tag{5.14}$$

Substituting (5.14) into (5.13) yields

$$r_e R \frac{dp_0}{dR} = K \frac{\Lambda_1}{P_1(\Lambda_1)}, \tag{5.15}$$

which, in turn, may be substituted into (5.11) to obtain

$$R^2 = \left[\frac{K}{P_1(\Lambda_1)}\right]^2 + \left(\frac{r_e}{\Lambda_1}\right)^2. \tag{5.16}$$

This is the relation that will be used to convert functions of R to functions of Λ_1. In particular, applying (5.16) and (5.15) we obtain

$$\begin{aligned}\frac{dp_0}{d\Lambda_1} &= \frac{dp_0}{dR}\frac{dR}{d\Lambda_1} \\ &= \frac{K}{r_e}\left\{-\frac{1}{P_1(\Lambda_1)} + \frac{1}{4}\frac{d}{d\Lambda_1}\left[\frac{\Lambda_1}{P_1(\Lambda_1)}\right]\right\} + \frac{1}{4}\frac{d}{d\Lambda_1}\left[\tan^{-1}\frac{r_e P_1(\Lambda_1)}{K\Lambda_1}\right].\end{aligned} \tag{5.17}$$

Similar substitutions may be applied to (5.10) to yield

$$\frac{dq_0}{d\Lambda_1} = \frac{r_e P_1(\Lambda_1)}{K\Lambda_1} + \frac{d}{d\Lambda_1}\left[\tan^{-1}\frac{r_e P_1(\Lambda_1)}{K\Lambda_1}\right] \tag{5.18}$$

These are the two equations that must be integrated. Depending on the form of $P_1(\Lambda_1)$, they may even be explicitly integrated, as in the case for a Mooney material which will be presented at the end of this section.

Eqs. (5.3) and (5.4) may be solved for r and Φ in terms of p_0 and q_0. They are

$$r = r_e[1 + (p_0 - q_0)^2]^{1/2}, \tag{5.19}$$

$$\Phi = \frac{\pi}{2} + q_0 - \tan^{-1}\frac{1}{p_0 - q_0}, \tag{5.20}$$

which, in view of (5.17) and (5.18), are again functions of Λ_1. Finally, Λ_2 is determined from (4.5), (5.7) and (5.12). It is

$$\Lambda_2 = \frac{r_e}{K}(p_0 - q_0)P_1(\Lambda_1). \tag{5.21}$$

The solution to the two equations (5.17) and (5.18) involves four arbitrary constants: r_e, K and two more from integration. All other variables are expressed in terms of p_0, q_0 and the "independent variable" Λ_1 by algebraic relations. We have thus completed the general solution.

We now give an examination of the physical meaning of the constant K. The pseudo Cauchy traction vector acting on an arc element $rd\theta$ in the pseudo-deformed surface m is in the direction of the p-coordinate (Fig. 2c). The magnitude is

$$\frac{P_1}{\Lambda_2}\cos\psi(r)\, rd\theta = K\, d\theta \tag{5.22}$$

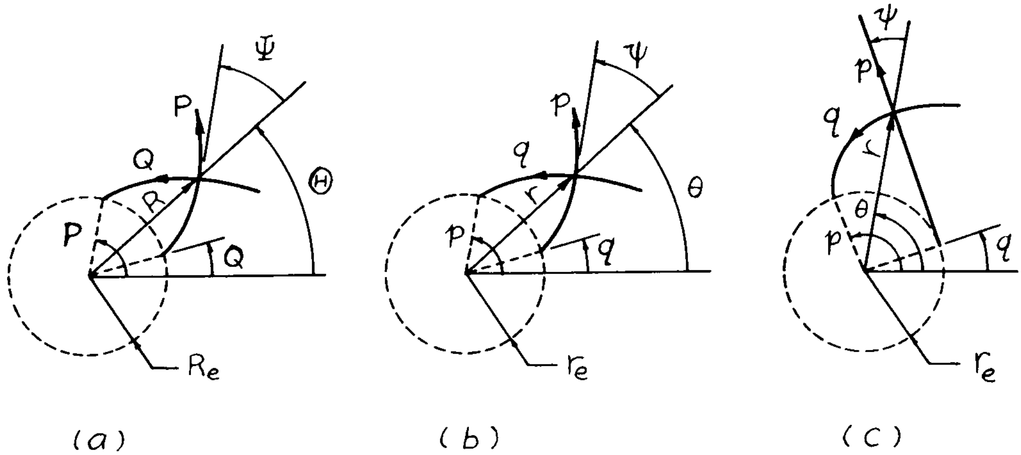

FIG. 2. Principal coordinates for rotationally symmetric problems.

where Fig. 2c, (5.2) and (5.21) have been used in the derivation. Since the p-coordinate curves are tangent to the circle $r = r_e$, the resultant moment of the traction on a circle is

$$C = \int_0^{2\pi} r_e K \, d\theta = 2\pi r_e K. \tag{5.23}$$

For axially symmetric problems, $r_e = 0$ and K/r is simply the pseudo Cauchy stress resultant in the radial direction (cf. [10]).

The principal coordinates (P, Q) in the undeformed configuration may be characterized by a single function $\Psi(R)$ and a constant R_e ((B1) and (B2) of Appendix B). Differentiating the first of (5.2) and using (B3), we get

$$\frac{dp}{dP} = \frac{dp_0}{dR}\frac{\partial R}{\partial P} + \frac{\partial \Theta}{\partial P} = R \sin \Psi \left[\frac{dp_0}{dR} \cos \Psi + \frac{1}{R} \sin \Psi \right]. \tag{5.24}$$

Eqs. (5.11), (5.24), (B4) and (5.15) now yield

$$\tan \Psi = \frac{r_e P_1(\Lambda_1)}{K\Lambda_1}. \tag{5.25}$$

The P- and Q-curves may then be determined from (B1) and (B2) by completing the integration. However, there is no need for such a direct integration. Using (5.25) and other relations, we find from (5.24) that

$$dp/dP = 1. \tag{5.26}$$

Similar calculations applied to the second of (5.2) yield

$$dq/dQ = 1. \tag{5.27}$$

The two reference radii r_e and R_e may be conveniently adjusted to satisfy the relations

$$r_e = r(R_e) \quad \text{or} \quad R_e = R(r_e). \tag{5.28}$$

Then, in view of (5.26) and (5.27), the undeformed principal coordinate curves are:

$$P = P_c:\ \Theta = P_c + [\phi(R_e) - p_0(R)], \tag{5.29}$$

$$Q = Q_c:\ \Theta = Q_c + [\phi(R_e) - q_0(R)]. \tag{5.30}$$

The images of these curves in the deformed configuration are, respectively,

$$p = P_c + \phi(R_e):\ \theta = [P_c + \phi(R_e)] + \cos^{-1}\frac{r_e}{r} - \frac{1}{r_e}(r^2 - r_e^2)^{1/2}, \tag{5.31}$$

$$q = Q_c + \phi(R_e):\ \theta = [Q_c + \phi(R_e)] + \cos^{-1}(r_e/r). \tag{5.32}$$

Finally, for a Mooney membrane surface with the P_1 function defined by (4.16), both (5.17) and (5.18) may be integrated. The results are:

$$p_0 = C_0 F_1(\Lambda_1, k) + \Psi(\Lambda_1, k, C_0) + c_p, \tag{5.33}$$

$$q_0 = \frac{1}{C_0} F_2(\Lambda_1, k) + \Psi(\Lambda_1, k, C_0) + c_q, \tag{5.34}$$

where

$$\begin{aligned} F_1(\Lambda_1, k) = \frac{1}{2}\Bigg[& \frac{k(1+k)}{\sqrt{3}(1+k^3)} \tan^{-1}\frac{2\Lambda_1 + 1}{\sqrt{3}} - \frac{k^3}{1+k^3}\ln(\Lambda_1 + k) \\ & - \frac{1}{3(1+k)}\ln(\Lambda_1 - 1) - \frac{2 + k - k^2}{6(1+k^3)}\ln(\Lambda_1^2 + \Lambda_1 + 1) + \frac{2\Lambda_1}{P_1(\Lambda_1, k)}\Bigg], \end{aligned} \tag{5.35}$$

$$F_2(\Lambda_1, k) = 2\left[\ln \Lambda_1 - \frac{k}{\Lambda_1} + \frac{1}{3\Lambda_1^3} + \frac{k}{4\Lambda_1^4}\right], \tag{5.36}$$

$$\Psi(\Lambda_1, k, C_0) = \tan^{-1}\frac{P_1(\Lambda_1, k)}{C_0\Lambda_1}, \tag{5.37}$$

$$P_1(\Lambda_1, k) = 2\Lambda_1\left(1 + \frac{k}{\Lambda_1}\right)\left(1 - \frac{1}{\Lambda_1^3}\right), \tag{5.38}$$

$$C_0 = K/r_e = C/2\pi r_e^2, \tag{5.39}$$

and c_p, c_q are two integration constants. The four arbitrary constants involved in the complete solution are r_e, K, c_p and c_q. If we set $r_e = 0$ and treating p_0 as r, (5.33) reduces to the result obtained in [10] for axially symmetric problems.

6. Rotationally symmetric initial-value problems. A wrinkly region is in general coupled with a taut region. A complete solution would then require the solutions to both regions. While we have solved the equations exactly for the wrinkly region, no such explicit solution is expected for the taut region where the full finite plane-stress equations must be applied. To bring out some of the physical features of the solution obtained in the previous section, we decided to consider the initial-value problem in detail.

Let the undeformed membrane surface M be defined by

$$M:\ R \geq R_e. \tag{6.1}$$

We assume that after deformation the pseudo-deformed surface m is

$$m\colon r \geq r_e. \tag{6.2}$$

For convenience, we scale all the length quantities by the radius r_e and define

$$r_0 = r/r_e, \qquad R_0 = R/r_e. \tag{6.3}$$

It follows that

$$M\colon R_0 \geq R_e/r_e, \qquad m\colon r_0 \geq 1. \tag{6.4}$$

To complete the initial data, we let

$$\Lambda_1 = \Lambda, \qquad \theta - \Theta = \phi_0 = \Phi_0 \quad \text{at} \quad R_0 = R_e/r_e. \tag{6.5}$$

The quantity C_0 defined by (5.39) is treated as a parameter.

We shall restrict ourselves to the explicit solution obtained for a Mooney material. The quantities p_0 and q_0 defined by (5.33) and (5.34) now become

$$p_0 - \phi_0 = C_0[F_1(\Lambda_1, k) - F_1(\Lambda, k)] + \Psi(\Lambda_1, k, C_0) - \Psi(\Lambda, k, C_0), \tag{6.7}$$

$$q_0 - \phi_0 = \frac{1}{C_0}[F_2(\Lambda_1, k) - F_2(\Lambda, k)] + \Psi(\Lambda_1, k, C_0) - \Psi(\Lambda, k, C_0), \tag{6.8}$$

where ϕ_0 is the initial value of both p_0 and q_0 by (5.3) and (5.4). Eqs. (5.16), (5.19), (5.21), (5.20), (4.7), (4.15) and (4.6) become

$$r_0 = \{1 + [(p_0 - \phi_0) - (q_0 - \phi_0)]^2\}^{1/2}, \tag{6.9}$$

$$R_0 = \left\{\left[\frac{C_0}{P_1(\Lambda_1, k)}\right]^2 + \left(\frac{1}{\Lambda_1}\right)^2\right\}^{1/2}, \tag{6.10}$$

$$\Lambda_2 = \frac{1}{C_0} P_1(\Lambda_1, k)[(p_0 - \phi_0) - (q_0 - \phi_0)], \tag{6.11}$$

$$\Phi - \Phi_0 = \phi - \phi_0 = \frac{\pi}{2} + (q_0 - \phi_0) - \tan^{-1}\frac{1}{(p_0 - \phi_0) - (q_0 - \phi_0)}, \tag{6.12}$$

$$\Lambda_1^* = \Lambda_1, \qquad \Lambda_2^* = \Lambda_1^{-1/2}, \tag{6.13}$$

$$W_2 = \Lambda_2 - \Lambda_2^*. \tag{6.14}$$

The above quantities are computed as functions of Λ_1 with Λ and C_0 as parameters. The data $\phi_0 = \Phi_0$ is not needed in the computation, but only appears in the final interpretation as a rigid body rotation.

Two sets of results are presented in Tables 2 and 3—Table 2: the initial data are $k = 0.1$, $\Lambda = 1.5$ and $C_0 = 4.0$; the deformed principal coordinates are given in Fig. 3; Table 3: the initial data are $k = 0.1$, $\Lambda = 3.0$ and $C_0 = 11.0$; the deformed and undeformed principal coordinates are given in Fig. 4. For both cases the images of the wrinkles in the undeformed state are almost straight. This fact can be observed from the relation

$$\tan \Psi = [(\Lambda_1 R_0)^2 - 1]^{-1/2} \tag{6.15}$$

derived from (5.16) and (5.25). The P-curve would be straight if Λ_1 were constant (cf. (B.9)). The variation of Λ_1 is rather small for the two cases calculated.

TABLE 2. Initial-value problem ($k = 0.1, \Lambda = 1.5, C_0 = 4.0$).

R_0	r_0	$\Phi - \Phi_0$	$\Lambda_1 = \Lambda_1^*$	Λ_2^*	W_2
1.897	1.000	0.000	1.500	0.817	−0.817
2.000	1.013	0.142	1.463	0.827	−0.740
2.502	1.346	0.638	1.338	0.865	−0.487
3.002	1.868	0.857	1.265	0.889	−0.344
3.497	2.427	0.958	1.218	0.906	−0.256
3.993	2.994	1.012	1.185	0.919	−0.196
4.504	3.576	1.044	1.160	0.929	−0.152
4.983	4.117	1.064	1.142	0.936	−0.122
5.486	4.681	1.077	1.127	0.942	−0.097
5.983	5.233	1.087	1.115	0.947	−0.079
6.435	5.786	1.093	1.105	0.951	−0.064
6.960	6.308	1.098	1.097	0.955	−0.052
7.447	6.837	1.102	1.090	0.958	−0.043
7.928	7.359	1.105	1.084	0.961	−0.035
8.484	7.958	1.108	1.078	0.963	−0.027
9.017	8.530	1.110	1.073	0.966	−0.021
9.499	9.046	1.111	1.069	0.967	−0.016
10.040	9.624	1.112	1.065	0.969	−0.012
10.490	10.110	1.113	1.062	0.971	−0.008
10.990	10.630	1.114	1.059	0.972	−0.005
11.550	11.220	1.115	1.056	0.973	−0.002
12.000	11.640	1.115	1.054	0.941	0.000

TABLE 3. Initial-value problem ($k = 0.1, \Lambda = 3.0, C_0 = 11.0$)

R_0	r_0	$\Phi - \Phi_0$	$\Lambda_1 = \Lambda_1^*$	Λ_2^*	W_2
1.872	1.000	0.000	3.000	0.577	−0.577
1.900	1.003	0.080	2.960	0.581	−0.537
1.950	1.027	0.221	2.890	0.588	−0.467
2.000	1.069	0.349	2.824	0.595	−0.403
2.050	1.127	0.462	2.782	0.602	−0.345
2.100	1.197	0.561	2.703	0.608	−0.290
2.150	1.277	0.646	2.647	0.615	−0.240
2.200	1.363	0.718	2.594	0.621	−0.193
2.250	1.454	0.780	2.544	0.627	−0.150
2.300	1.550	0.833	2.496	0.633	−0.110
2.350	1.649	0.879	2.450	0.639	−0.072
2.400	1.748	0.917	2.407	0.645	−0.038
2.450	1.848	0.951	2.365	0.650	−0.006
2.459	1.866	0.957	2.359	0.651	0.000

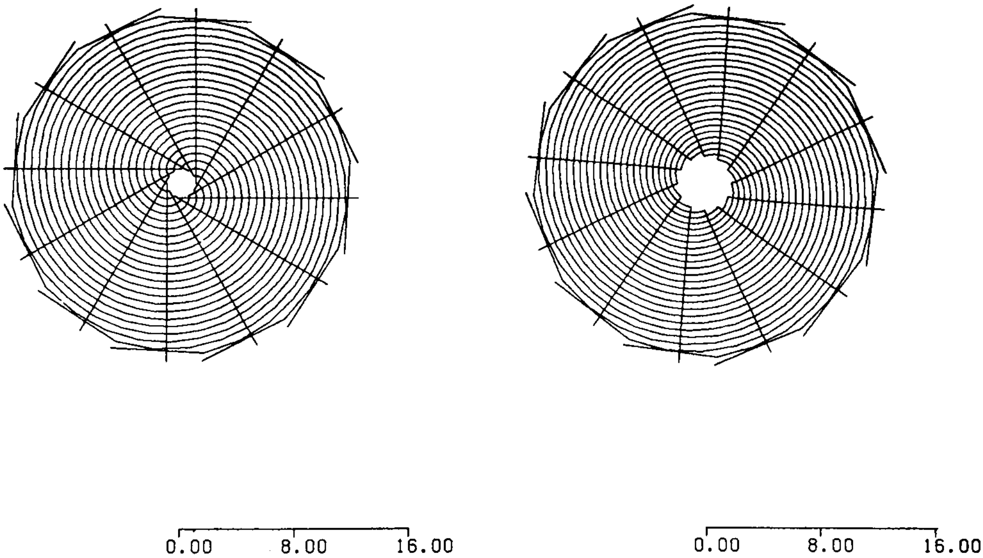

FIG. 3. Principal coordinates for the data given in Table 2. Deformed configuration is given on the left.

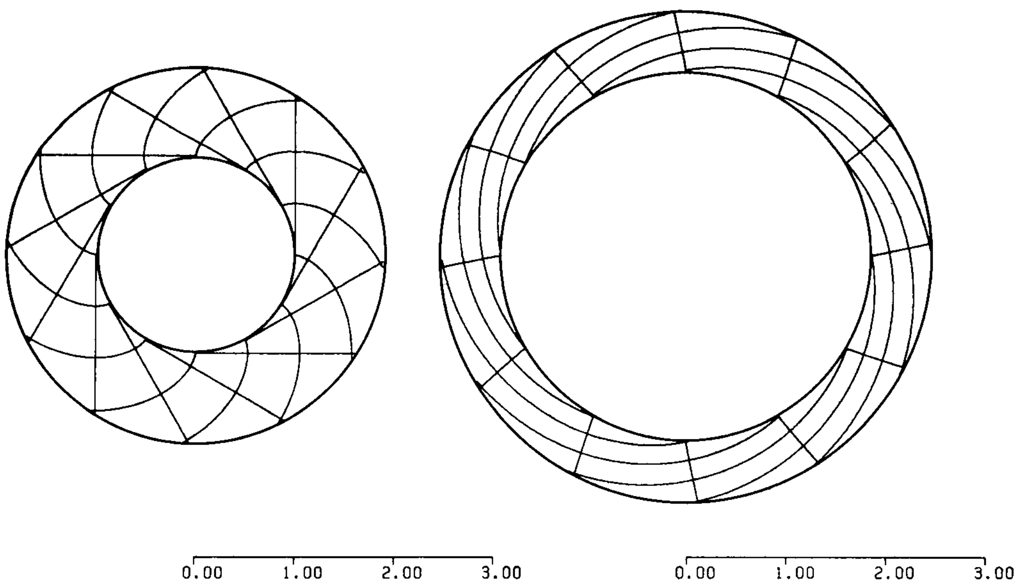

FIG. 4. Principal coordinates for the data given in Table 3. Deformed configuration is given on the left.

References

[1] H. Wagner, *Flat sheet metal girders with a very thin metal web*, Z. Flugtechnik u. Motorluftschiffahrt **20**, 200–314 (1929)
[2] E. Reissner, *On tension field theory*, in *Proc. 5th Int. Congr. Appl. Mech.*, 88–92 (1938)
[3] M. Stein, and J. M. Hedgepeth, *Analysis of partly wrinkled membranes*, NASA TN-D-813 1961
[4] E. H. Mansfield, *Tension field theory*, in *Proc. 12th Int. Congr. Appl. Mech.*, 305–320 (1969)
[5] E. H. Mansfield, *Load transfer via a wrinkled membrane*, Proc. Roy. Soc. Lond. (A) **316**, 269–289 (1970)
[6] G. P. Cherepanov, *On the buckling under tension of a membrane containing holes*, PMM **27**, 275–286 (1963)
[7] C. H. Wu, *Plane linear wrinkle elasticity without body force*, Report, Dept. Mat. Engrg, University of Ill. at Chicago Circle, 1974
[8] E. H. Mansfield, *Analysis of wrinkled membranes with anisotropic and nonlinear elastic properties*, Proc. Roy. Soc. Lond. (A) **353**, 475–498 (1977)
[9] C. H. Wu, *The wrinkled axisymmetric air bags made of inextensible membrane*, J. Appl. Mech. **41**, 963–968 (1974)
[10] C. H. Wu, *Nonlinear wrinkling of nonlinear membranes of revolution*, J. Appl. Mech. **45**, 533–538 (1978)
[11] J. A. Schouten, *Tensor analysis for physicists*, Oxford, Clarendon Press, 1953
[12] W. Jaunzemis, *Continuum mechanics*, Macmillan, New York, 1967

Appendix A: Equations in orthogonal curvilinear coordinates (cf. Table 1 for notation). Let $[X_A] = [U, V]$ and $[x_a] = [u, v]$ be, respectively, the curvilinear coordinates in the undeformed and deformed configurations. They are related to the Cartesian coordinates $[Z_I] = [X, Y]$ and $[z_i] = [x, y]$ by the relations

$$Z_I = Z_I(X_A), \qquad X_A = X_A(Z_I), \tag{A1}$$

$$z_i = z_i(x_a), \qquad x_a = x_a(z_i). \tag{A2}$$

These relations satisfy the orthogonality conditions

$$Z_{I,A} Z_{I,B} = 0 \quad \text{for} \quad A \neq B, \tag{A3}$$

$$z_{i,a} z_{i,b} = 0 \quad \text{for} \quad a \neq b. \tag{A4}$$

It follows from (2.11) that

$$H_A = (Z_{I,A} Z_{I,A})^{-1/2}, \qquad H_a = (z_{i,a} z_{i,a})^{-1/2}. \tag{A5, A6}$$

With respect to the orthogonal curvilinear coordinates, the components F_{aA} of the pseudo-deformation-gradient tensor **F** may be obtained from the Cartesian components F_{iI} via the transformation

$$F_{aA} = Q_{ai} Q_{AI} F_{iI}. \tag{A7}$$

Using (3.3), (2.16) and the Q_{ai}-counterpart of (2.16), we obtain from (A7)

$$[F_{aA}] = \left[\frac{H_A}{H_a}\frac{\partial x_a}{\partial X_A}\right] \quad \text{(no sum)},$$

$$= \begin{bmatrix} F_{uU} & F_{uV} \\ F_{vU} & F_{vV} \end{bmatrix} = \begin{bmatrix} \dfrac{H_U}{H_u}\dfrac{\partial u}{\partial U} & \dfrac{H_V}{H_u}\dfrac{\partial u}{\partial V} \\ \dfrac{H_U}{H_v}\dfrac{\partial v}{\partial U} & \dfrac{H_V}{H_v}\dfrac{\partial v}{\partial V} \end{bmatrix} \tag{A8}$$

The pseudo-Cauchy stress equations of equilibrium, (3.31), now become

$$T_{ab;\,b} = \partial_b T_{ab} + W_{eab} T_{eb} + W_{ebb} T_{ae} = 0. \tag{A9}$$

Using (2.20) and (2.21), we obtain

$$H_u T_{uu,u} + H_v T_{uv,v} - 2\frac{H_v}{H_u}\frac{\partial H_u}{\partial v} T_{uv} + \frac{H_u}{H_v}\frac{\partial H_v}{\partial u}(T_{vv} - T_{uu}) = 0, \tag{A10}$$

$$H_v T_{vv,v} + H_u T_{uv,u} - 2\frac{H_u}{H_v}\frac{\partial H_v}{\partial u} T_{uv} + \frac{H_v}{H_u}\frac{\partial H_u}{\partial v}(T_{uu} - T_{vv}) = 0. \tag{A11}$$

To derive the Piola stress equations of equilibrium, (3.30), we begin with the transformation

$$P_{aA;\,A} = Q_{ai} Q_{AI} Q_{AJ} P_{iI,\,J}. \tag{A12}$$

By repeatedly applying the chain rule of differentiation and (2.17), we finally arrive at

$$P_{aA;\,A} = \partial_A P_{aA} + W_{BAA} P_{aB} + W_{bac} F_{cA} P_{bA}. \tag{A13}$$

The two equations of equilibrium now become

$$\begin{aligned} &H_U P_{uU,U} + H_V P_{uV,V} - \frac{H_U}{H_V}\frac{\partial H_V}{\partial U} P_{uU} - \frac{H_V}{H_U}\frac{\partial H_U}{\partial V} P_{uV} \\ &\quad - \frac{H_v}{H_u}\frac{\partial H_u}{\partial v}[F_{uU} P_{vU} + F_{uV} P_{vV}] + \frac{H_u}{H_v}\frac{\partial H_v}{\partial u}[F_{vU} P_{vU} + F_{vV} P_{vV}] = 0, \end{aligned} \tag{A14}$$

$$\begin{aligned} &H_V P_{vV,V} + H_U P_{vU,U} - \frac{H_V}{H_U}\frac{\partial H_U}{\partial V} P_{vV} - \frac{H_U}{H_V}\frac{\partial H_V}{\partial U} P_{vU} \\ &\quad - \frac{H_u}{H_v}\frac{\partial H_v}{\partial u}[F_{vV} P_{uV} + F_{vU} P_{uU}] + \frac{H_v}{H_u}\frac{\partial H_u}{\partial v}[F_{uV} P_{uV} + F_{uU} P_{uU}] = 0. \end{aligned} \tag{A15}$$

We conclude this appendix by specializing these equations to suit two special sets of coordinates.

Polar coordinates ($[X_A] = [R, \Theta]$, $[x_a] = [r, \theta]$).

$$[F_{aA}] = \begin{bmatrix} F_{rR} & F_{r\Theta} \\ F_{\theta R} & F_{\theta\Theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} \\ \dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} \end{bmatrix}, \tag{A16}$$

$$T_{rr,r} + \frac{1}{r} T_{r\theta,\theta} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0, \tag{A17}$$

$$\frac{1}{r} T_{\theta\theta,\theta} + T_{r\theta,r} + \frac{2}{r} T_{r\theta} = 0, \tag{A18}$$

$$P_{rR,R} + \frac{1}{R} P_{r\Theta,\Theta} + \frac{1}{R} P_{rR} - \frac{1}{r}[F_{\theta R} P_{\theta R} + F_{\theta\Theta} P_{\theta\Theta}] = 0, \tag{A19}$$

$$\frac{1}{R} P_{\theta\Theta,\Theta} + P_{\theta R,R} + \frac{1}{R} P_{\theta R} + \frac{1}{r}[F_{\theta\Theta} P_{r\Theta} + F_{\theta R} P_{rR}] = 0. \tag{A20}$$

Principal coordinates ($[X_A] = [P, Q], [x_a] = [p, q]$).

$$[F_{aA}] = \begin{bmatrix} F_{pP} & F_{pQ} \\ F_{qP} & F_{qQ} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} = \begin{bmatrix} \dfrac{H_P}{H_p}\dfrac{\partial p}{\partial P} & \dfrac{H_Q}{H_p}\dfrac{\partial p}{\partial Q} \\ \dfrac{H_P}{H_q}\dfrac{\partial p}{\partial P} & \dfrac{H_Q}{H_q}\dfrac{\partial q}{\partial Q} \end{bmatrix}, \tag{A21}$$

$$[T_{ab}] = \begin{bmatrix} T_{pp} & T_{pq} \\ T_{qp} & T_{qq} \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \begin{bmatrix} \dfrac{P_1}{\Lambda_2} & 0 \\ 0 & \dfrac{P_2}{\Lambda_1} \end{bmatrix}, \tag{A22}$$

$$[T^*_{ab}] = \begin{bmatrix} T^*_{pp} & T^*_{pq} \\ T^*_{qp} & T^*_{qq} \end{bmatrix} = \begin{bmatrix} T^*_1 & 0 \\ 0 & T^*_2 \end{bmatrix} = \begin{bmatrix} \dfrac{P_1}{\Lambda^*_2} & 0 \\ 0 & \dfrac{P_2}{\Lambda^*_1} \end{bmatrix}, \tag{A23}$$

$$[P_{aA}] = \begin{bmatrix} P_{pP} & P_{pQ} \\ P_{qP} & P_{qQ} \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial U}{\partial \Lambda^*_1} & 0 \\ 0 & \dfrac{\partial U}{\partial \Lambda^*_2} \end{bmatrix}, \tag{A24}$$

$$H_p \frac{\partial T_1}{\partial p} + \frac{H_p}{H_q}\frac{\partial H_q}{\partial p}(T_2 - T_1) = 0, \tag{A25}$$

$$H_q \frac{\partial T_2}{\partial q} + \frac{H_q}{H_p}\frac{\partial H_p}{\partial q}(T_1 - T_2) = 0, \tag{A26}$$

$$H_p \frac{\partial P_1}{\partial P} - \frac{H_P}{H_Q}\frac{\partial H_Q}{\partial P} P_1 + \frac{H_p}{H_q}\frac{\partial H_q}{\partial p}\frac{H_Q}{H_q}\frac{\partial q}{\partial Q} P_2 = 0, \tag{A27}$$

$$H_Q \frac{\partial P_2}{\partial Q} - \frac{H_Q}{H_P}\frac{\partial H_P}{\partial Q} P_2 + \frac{H_q}{H_p}\frac{\partial H_p}{\partial q}\frac{H_P}{H_p}\frac{\partial p}{\partial P} P_1 = 0. \tag{A28}$$

Appendix B: Principal coordinates for rotationally symmetric problems. In terms of the polar coordinates (R, Θ), the most general representation for the principal coordinates (P, Q) associated with a rotationally symmetric problem may be written as

$$P = \Theta + \int_{R_e}^{R} \frac{\cot \Psi(\rho)}{\rho}\, d\rho, \tag{B1}$$

$$Q = \Theta - \int_{R_e}^{R} \frac{\tan \Psi(\rho)}{\rho}\, d\rho, \tag{B2}$$

where $\Psi(R)$ is the angle between P and R (Fig. 2a) and R_e is a constant reference radius. It follows that

$$\frac{\partial R}{\partial P} = R \sin \Psi \cos \Psi, \qquad \frac{\partial R}{\partial Q} = -R \sin \Psi \cos \Psi,$$

$$\frac{\partial \Theta}{\partial P} = \sin^2 \Psi, \qquad \frac{\partial \Theta}{\partial Q} = \cos^2 \Psi. \tag{B3}$$

$$H_P = \frac{1}{R \sin \Psi}, \qquad H_Q = \frac{1}{R \cos \Psi}. \tag{B4}$$

Similar relations may be obtained for the deformed coordinates (r, θ) and (p, q), viz.

$$p = \theta + \int_{r_e}^{r} \frac{\cot \psi(\rho)}{\rho}\, d\rho, \tag{B5}$$

$$q = \theta - \int_{r_e}^{r} \frac{\tan \psi(\rho)}{\rho}\, d\rho, \tag{B6}$$

$$\frac{\partial r}{\partial p} = r \sin \psi \cos \psi, \qquad \frac{\partial r}{\partial q} = -R \sin \psi \cos \psi,$$

$$\frac{\partial \theta}{\partial p} = \sin^2 \psi, \qquad \frac{\partial \theta}{\partial q} = \cos^2 \psi, \tag{B7}$$

$$H_p = \frac{1}{r \sin \psi}, \qquad H_q = \frac{1}{r \cos \psi}, \tag{B8}$$

where $\psi(r)$ is the angle between p and r (Fig. 2b), and r_e is a constant reference radius. If the p-coordinate curves (q = constant curves) are straight lines, then (Fig. 2c)

$$\tan \psi(r) = \frac{r_e}{(r^2 - r_e^2)^{1/2}}, \tag{B9}$$

$$p = \theta + \frac{1}{r_e}(r^2 - r_e^2)^{1/2} - \cos^{-1}\frac{r_e}{r}, \tag{B10}$$

$$q = \theta - \cos^{-1}\frac{r_e}{r}. \tag{B11}$$

Moreover,

$$H_p = \frac{1}{r_e}, \qquad H_q = (r^2 - r_e^2)^{1/2} = \frac{1}{r_e(p - q)}. \tag{B12}$$