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W_V Paths on the Torus

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Abstract. We say that a cell complex has the W_V property provided any two vertices can be joined by a path that never returns to a facet once it leaves it. The boundaries of convex polytopes have been shown to have the W_V property for polytopes of dimensions at most 3. We extend the three-dimensional result to polyhedral maps on the torus by showing that all such maps have the W_V property. We also show that the W_V property for all polyhedral maps on manifolds of a given genus is equivalent to the property that for all maps on manifolds of that genus each two faces lie in a subcomplex that is a cell.

1. Introduction

One of the most important unsolved problems in the theory of convex polytopes is the Hirsch conjecture, that each two vertices of a d-polytope with n facets can be joined by a path of length at most $n-d$.

We say that a path in a cell complex is a W_V *path* if and only if its intersection with each face is connected (i.e., it never returns to a face once it leaves it). We say that a cell complex has the W_V *property* if and only if each two vertices can be joined by a W_V path. The truth of the Hirsch conjecture is implied by the W_V property for polytopes (see [3]). The W_V conjecture is known to be true for polytopes of dimensions at most 3 [3].

Larman [4] has found a topological 2-cell complex that does not have the W_V property. Mani and Walkup [5] have found a triangulated 3-sphere whose dual fails to have the W_V property. The author has proved the W_V property for projective plane polyhedral maps [2] and in this paper we prove it for toroidal polyhedral maps. For polyhedral maps in other surfaces it is not known whether the W_V property holds.

2. Definitions

If a graph G is embedded in a manifold M, then the closures of the connected components of $M-G$ are called the *faces* of G. If each face is a closed cell, each vertex is of valence at least three and each two faces meet on a vertex, an edge, or not at all, then G is a *polyhedral map* and we say that faces *meet properly*. If M is the torus, then we call G a *toroidal polyhedral map*, or TPM.

If P is a path in G with vertices x and y on P, then $P[x, y]$ denotes the portion of P joining x and y. If we wish to exclude an endpoint of $P[x, y]$ we use a parenthesis rather than a bracket. A path is *simple* provided it has no self-intersections. All paths in this paper are simple paths.

If P is a path in G whose intersection with a face F is not connected, then there exist vertices x and y of F such that $P[x, y] \cap F = \{x, y\}$. The path $P[x, y]$ is called a *revisit* of F by the path P. Let x and y be two vertices of a face F of a TPM in a torus T and let Γ_1 be a path along F from x to y. If $P[x, y]$ is a revisit such that $P[x, y] \cup \Gamma_1$ bounds a cell that is a subset of T, then we say that $P[x, y]$ is a *planar revisit*, otherwise we say that $P[x, y]$ is *nonplanar*.

By a *chain of faces* in a polyhedral map we mean a sequence of faces $F_1, F_2, \ldots, F_n$ such that $F_i \cap F_{i+1}$ is an edge for $1 \le i \le n-1$.

If P is a path with vertices x and y, then $d(x, y)$ denotes the number of edges of $P[x, y]$. If there is possible ambiguity we say "$d(x, y)$ *along* P." For any path $P[x, y]$, $d(x, y)$ is called its *length*.

3. The Main Theorem

We prove the existence of W_V paths joining any two vertices x and y in a TPM by proving that joining x and y is a path having only planar revisits. It then follows from the following lemma of the author that a W_V path joins x and y.

Lemma 1. *If x and y are two vertices of a polyhedral manifold M joined by a path having only planar revisits, then a W_V path joins x and y in M.*

The proof of Lemma 1 is found in [2].

Theorem 1. *Any two vertices of a TPM can be joined by a W_V path.*

Proof. Let x be a vertex of a TPM M. Let V be the set of all vertices of M which can be joined to x by a W_V path. Suppose there exist vertices of M not in V. Then there will exist a vertex y_0 not in V and a vertex y_1 in V such that y_0y_1 is an edge. Let P_1 be a W_V path from x to y_1 and let $P_0 = P_1 \cup y_0y_1$. We assume that among all choices of y_0, y_1, and P_1 we have chosen ones which minimize the length of P_0.

We now show that P_0 can be modified to produce a path from x to y_0 that is either a W_V path or a path with only planar revisits. Since $y_0 \notin V$, there is a face F revisited by P_0. Since P_1 has no revisits, $P_0 \cap F = \{y_0\} \cup P_2[x_1, x_2]$ where P_2 is a subpath of P_1. We assume that the order of the vertices on P_0 is x, x_1, x_2, y_1, y_0. Among all faces having nonplanar revisits by P_0 we assume that F is chosen

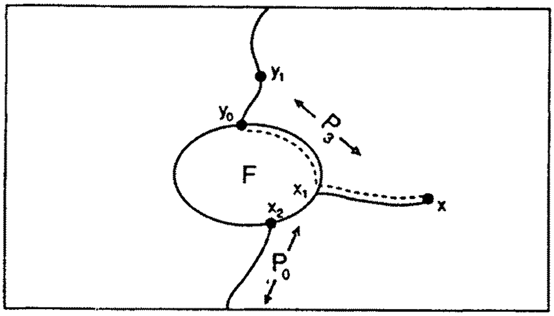


Fig. 1

such that $d(x, x_1)$ along P_0 is minimal. If there is no such face F, then by Lemma 1 we are done.

We replace $P_0[y_0, x_1]$ by a path along F as in Fig. 1, producing a path P_3 from x to y_0. If P_3 has a nonplanar revisit to a face F_1, then F_1 must meet $P_3(x_1, x]$, for otherwise F_1 and F would meet improperly.

The face F_1 will not meet y_0 because the minimality of $d(x, x_1)$ would be violated. Thus $F_1 \cap P_3 = P_4[x_3, x_4] \cup P_5[x_5, x_6]$ where $P_4 \subseteq P_3(x_1, x]$, $P_5 \subseteq P_3(y_0, x)$ and the order of the vertices on P_3 is $x, x_3, x_4, x_1, x_5, x_6, y_0$ (of course, some of these can be the same vertex, e.g., x_6 could equal x_5).

Topologically there is only one way $F_1 \cup P_3[x_4, x_5]$ can be embedded (see Fig. 2). We note that since P_1 is a W_V path, F_1 misses $P_0[y_0, x_2]$. We now assume that F_1 is chosen so that $d(x, x_3)$ along P_3 is minimized. We now replace $P_3[x_6, x_3]$ by a path along F_1, as in Fig. 2, producing a path P_6 from x to y_0. Let A be the cell bounded by $P_6[y_0, x_4] \cup P_0[x_4, y_0]$ (see Fig. 2(a)) or by $P_6[y_0, x_3] \cup P_1[x_3, y_0]$ (see Fig. 2(b)).

Suppose x is in A. For P_6 to have a nonplanar revisit, a face G would have to lie outside of A (for the revisit to be nonplanar) and meet y_0 and x_3 (with $x_3 = x_4$) or x_6 and x_3 (with $x_6 = x_5$). The first case is ruled out by the minimality

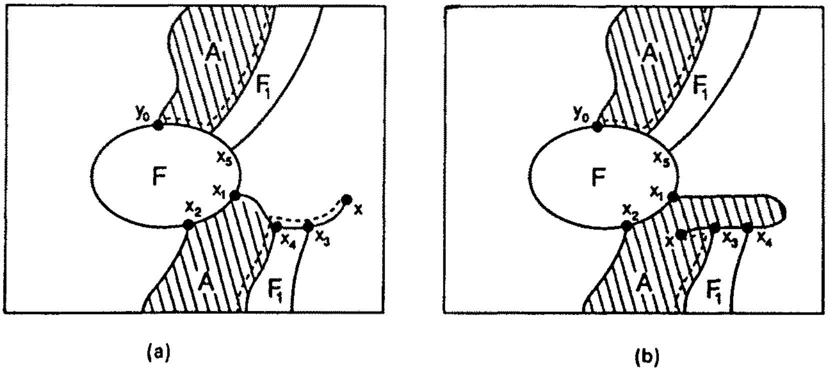
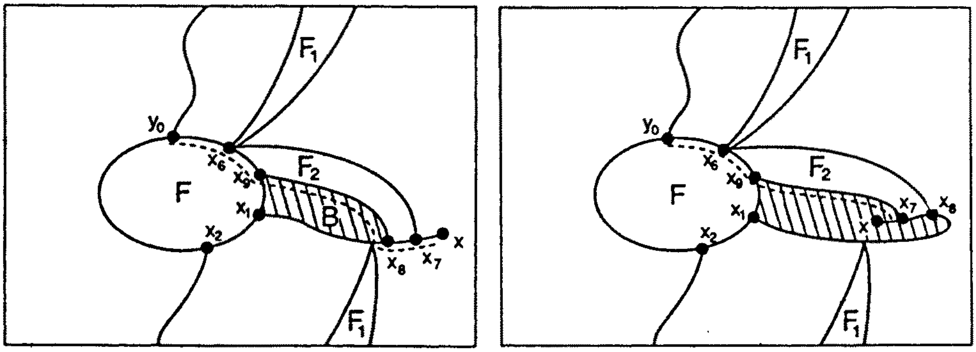


Fig. 2



(a) (b)

Fig. 3

condition on F (F minimizes $d(x, x_1)$). In the second case, since the revisit is nonplanar, F_1 meets G improperly. Thus all revisits to P_6 are planar when x is in A.

Suppose x is not in A. The only nonplanar revisits possible would be to faces lying outside A meeting vertices of $P_6[x, x_4]$ and vertices of $P_6[x_6, y_0)$. The only vertex of $P_6[x_6, y_0)$ accessible to such a face would be x_6 (in the case $x_6 = x_5$).

Let F_2 be such a revisited face. Then $F_2 \cap P_6 = \{x_6\} \cup P_7[x_7, x_8]$ where $P_7 \subseteq P_6[x, x_3)$ and the order of the vertices is x, x_7, x_8, x_3, x_4, etc. (Note F_2 misses $P_0[x_3, x_4]$ because F_2 and F_1 must meet properly.) We assume that F_2 is chosen such that $d(x_7, x)$ is minimal.

Let $F_2 \cap F = P_9[x_6, x_9]$. We construct a path P_8 from P_3 by replacing $P_3[x_6, x_7]$ by a path along F_2 as in Fig. 3. Let B be the cell bounded by $P_8[x_9, x_8] \cup P_0[x_1, x_8]$ together with the path along F from x_9 to x_1, missing y_0 (see Fig. 3(a)) or $P_8[x_9, x_7] \cup P_0[x_1, x_7]$ together with a path along F (see Fig. 3(b)). Note that F_2 misses $P_0(x_8, x_1)$ because P_1 is a W_V path, thus B is a cell. Any face in B can have only planar revisits by P_8. If a face F_3 lies outside B and has a nonplanar revisit by P_8, then it must meet $P_8[y_0, x_6]$ and $P_8[x_8, x]$.

If F_3 meets x_6 and $P_7[x_7, x_8]$ and if the revisit is nonplanar, then F_2 and F_3 meet improperly. The face F_1 prevents F_3 meeting $P_7[x_7, x_8]$ and vertices of $P_8(y_0, x_6)$. The minimality condition on F prevents F_3 from meeting y_0. If, however, F_3 meets x_6 and $P_8(x_7, x]$, then the minimality of $d(x_7, x)$ is violated. Thus all revisits of P_8 are planar.

In all cases we have obtained a path from x to y_0 with only planar revisits and by Lemma 1 we have a W_V path from x to y_0 contradicting the assumption that $y_0 \notin V$. Thus V is the entire set of vertices of M and we are done. □

4. A Necessary and Sufficient Condition for W_V Paths

It is interesting to consider the duals of W_V paths. We need one more lemma to do so.

Lemma 2. *The dual of a TPM is a TPM.*

This follows from a theorem by the author [1]

Corollary 1. *Any two faces of a TPM lie in a subcomplex that is a cell.*

Proof. Let F_1 and F_n be two faces of TPM T and let x_1 and x_n be the corresponding vertices in the dual T^* of T. Let $P = x_1x_2 \cdots x_n$ be a W_V path in T^* and let $F_1, F_2, \ldots, F_n$ be the corresponding chain of faces in T. We show by induction on k that $F_1 \cup F_2 \cup \cdots \cup F_k$ is a cell. Clearly, F_1 is a cell, we assume $F_1 \cup \cdots \cup F_{k-1}$ is a cell. It now suffices to show that $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ is an edge. Suppose F_k and $F_j, j < k$, have a vertex in common. Then x_k and x_j lie on a face F in T^*, and thus the path $x_jx_{j+1} \cdots x_k$ lies on that face. It follows that $F_j, F_{j+1}, \ldots, F_k$ meet at a vertex x. By the construction of the dual, $F_k \cap F_{k-1}$ is an edge e, and x is a vertex of e. Thus $F_k \cap (F_1 \cup \cdots \cup F_{n-1}) = e$ and $F_1 \cup \cdots \cup F_k$ is a cell. □

We note that the proof of Corollary 1 works for polyhedral maps on manifolds of any genus whenever all polyhedral manifolds of that genus have the W_V property.

Theorem 2. *Polyhedral maps on manifolds of a given genus have the W_V property if and only if each two faces of every polyhedral map on the manifolds of that genus lie in a subcomplex that is a cell.*

Proof. The above observation shows that the W_V property implies each two faces lie in a cell. Assume now that each two faces of every polyhedral map of genus g lie in a cell. Assume x_1 and x_n are two vertices of a polyhedral map on a manifold T of genus g. By Lemma 2, the dual T^* of T is a polyhedral map. Let F_1 and F_n be the faces of T^* corresponding to x_1 and x_n. Let C be a subcomplex of T^* that is a cell containing F_1 and F_n. Let $F_1, F_2, \ldots, F_n$ be a chain of faces in C such that $F_i \cap F_{i+1}$ is an edge for $i = 1, \ldots, n-1$. Corresponding to this chain is a path $P = x_1x_2 \cdots x_n$ in T.

We assume that the dual of T is constructed in the usual way with vertices of T^* in the faces of T and two vertices a and b joined whenever the corresponding faces meet on an edge e, with the edge ab crossing the edge e. Now P lies in the cell C. Any face F of T revisited by P will correspond to a vertex z of T^*.

Case I. Vertex z lies in the interior of C. In this case F lies in C. Since P lies in C, the revisit is planar.

Case II. Vertex z lies on the boundary of C. Since C is a cell the boundary of C will contain exactly two edges e_1 and e_2 meeting z. These two edges will separate the set of faces of T^* meeting z into two sets of faces, $H_1, H_2, \ldots, H_k$ lying in C and $H_{k+1}, \ldots, H_j$ lying outside C, with $H_1, H_2, \ldots, H_k, \ldots, H_j$ the cyclic ordering about z. The path P meets F at vertices that correspond to $H_1, \ldots, H_k$, thus there is a path in T along F lying in C joining any two connected components of $P \cap F$. Thus any such revisit is planar.

Since all revisits of P are planar, we are done by Lemma 1. □

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