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X-INNER AUTOMORPHISMS OF FILTERED ALGEBRAS

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ABSTRACT. For the enveloping algebra of a finite-dimensional Lie algebra, and for the ring of differential polynomials over a commutative domain, we compute the group of those automorphisms which become inner when extended to the quotient division rings. Both of these results depend on a more general result about the automorphisms of a filtered algebra.

Let A be a prime ring. An X-*inner* automorphism σ of A is one which becomes inner when extended to the Martindale quotient ring $A_{\mathfrak{F}}$ of A; when A is a prime Goldie ring, this condition is equivalent to σ becoming inner on the classical quotient ring of A [5]. Thus the set of all X-inner automorphisms is a normal subgroup of $\mathrm{Aut}(A)$ which contains all the inner automorphisms; this set has proved useful in studying group actions on rings and crossed products. Recently the X-inner automorphisms have been computed for certain group rings [7] and for coproducts of domains [4].

In this note we consider a filtered algebra A such that the associated graded ring $\bar{A}$ is a commutative domain. We show that any X-inner automorphism of A preserves the filtration of A and induces the trivial automorphism on $\bar{A}$. We then give two applications of this result: we determine the X-inner automorphisms of $U(\mathfrak{g})$, the enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$, and also of the differential polynomial ring $A = R[x; d]$, where R is a commutative domain with nontrivial derivation d.

We require some definitions. For a prime ring A, the Martindale quotient ring of A is the left quotient ring of A with respect to the filter $\mathfrak{F}$ of all nonzero two-sided ideals; that is $A_{\mathfrak{F}} = \varinjlim_{I \in \mathfrak{F}} \mathrm{Hom}_A({}_A I, A)$. $A_{\mathfrak{F}}$ is also a prime ring, containing A, and for any $0 \neq x \in A$, there exists $I \in \mathfrak{F}$ so that $0 \neq Ix \subseteq A$ (for details see [3]). Moreover, for any $\sigma \in \mathrm{Aut}(A)$, σ has a unique extension to $A_{\mathfrak{F}}$.

Now assume that σ is X-inner. That is, for some $a \in A_{\mathfrak{F}}$, $r^\sigma = a^{-1}ra$, all $r \in A$. Let $I \in \mathfrak{F}$ be such that $0 \neq Ia \subseteq A$, and choose $b, c \in A$ with $0 \neq ba = c$. Then for any $r \in R$, $c(rb)^\sigma = ba(rb)^\sigma = b(rb)a = brc$. That is,

$$(*) \qquad brc = cr^\sigma b^\sigma, \quad \text{all } r \in A.$$

Conversely, it is not difficult to show that if for some $\sigma \in \mathrm{Aut}(A)$, there exist $0 \neq b, c \in A$ such that $brc = cr^\sigma b^\sigma$, all $r \in A$, then σ is X-inner [6].

Now let A be a filtered ring, say $A = \bigcup_{n>0} A_n$, where $1 \in A_0$. For any $a \in A$, let $f(a)$ denote the filtration of a; that is, $f(a) = n$ if $a \in A_n$ but $a \notin A_{n-1}$. The

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associated graded ring is $\bar{A} = \Sigma_{n \geq 0} \oplus A_n/A_{n-1}$. We note that $\bar{A}$ is a domain if and only if $f(ab) = f(a) + f(b)$, all $a, b \in A$. When $\bar{A}$ is a domain, so is A, and thus $A_{\mathcal{F}}$ is a prime ring.

Proposition 1. *Let A be a filtered algebra, as above, such that $\bar{A}$ is a commutative domain. Let σ be an X-inner automorphism of A. Then*:

(1) *σ preserves the filtration of A (that is, $A_n^\sigma = A_n$, all n),*

(2) *the induced action $\bar{\sigma}$ of σ on $\bar{A}$ is trivial.*

Proof. Since σ is X-inner, we may choose $b, c \in A$ such that

$$brc = cr^\sigma b^\sigma, \quad \text{for all } r \in A. \qquad (*)$$

Since $\bar{A}$ is a domain, $f(b) + f(r) + f(c) = f(c) + f(r^\sigma) + f(b^\sigma)$ for all $r \in A$; thus $f(b) + f(r) = f(r^\sigma) + f(b^\sigma)$. Letting $r = 1 = r^\sigma$, $f(r) = f(r^\sigma) = 0$, and so $f(b) = f(b^\sigma)$. It follows that $f(r) = f(r^\sigma)$, all $r \in A$. That is, σ preserves the filtration.

Now let $k = f(b)$, $l = f(c)$, and let $\bar{b} = b + A_{k-1}$, $\overline{b^\sigma} = b^\sigma + A_{k-1}$ be the "leading terms" of b and b^σ, respectively. We claim that $\bar{b} = \overline{b^\sigma}$. For, using $r = 1$ in $(*)$, $bc = cb^\sigma$. Since $\bar{A}$ is commutative, $cb^\sigma - b^\sigma c \in A_{k+l-1}$, and thus $(b - b^\sigma)c = cb^\sigma - b^\sigma c \in A_{k+l-1}$. Since $f(c) = l$, it follows that $f(b - b^\sigma) \leq k - 1$. Thus $b - b^\sigma \in A_{k-1}$, so $\bar{b} = \overline{b^\sigma}$.

Finally, consider any $x \in A_n$; $bxc = cx^\sigma b^\sigma$ from $(*)$, and $cx^\sigma b^\sigma - b^\sigma x^\sigma c \in A_{k+l+n-1}$ since $\bar{A}$ is commutative. Thus $(bx - b^\sigma x^\sigma)c \in A_{k+l+n-1}$; since $f(c) = l$, we obtain $f(bx - b^\sigma x^\sigma) \leq k + n - 1$, and so $bx - b^\sigma x^\sigma \in A_{k+n-1}$. Using $b - b^\sigma \in A_{k-1}$ from above, $(b - b^\sigma)x^\sigma \in A_{k+n-1}$, and so $b(x - x^\sigma) = bx - b^\sigma x^\sigma + b^\sigma x^\sigma - bx^\sigma \in A_{k+n-1}$. As before, as $f(b) = k$; this forces $x - x^\sigma \in A_{n-1}$. Thus $\bar{x} = \overline{x^\sigma}$, for any $x \in A_n$. We have proved that the induced automorphism $\bar{\sigma}$ (given by $\bar{x}^\sigma = \overline{x^\sigma}$) is trivial on $\bar{A}$. □

Our first application of Proposition 1 is to enveloping algebras. Thus, following [**1**, §4.3], let $\mathfrak{g}$ denote a finite-dimensional Lie algebra over a field k, with universal enveloping algebra $U = U(\mathfrak{g})$. Let $K = K(\mathfrak{g})$ denote the division ring of quotients of U. For P a completely prime ideal of U, $A = U/P$ also has a division ring of quotients, which we will denote by $Q(A)$.

Let ε be the adjoint representation of $\mathfrak{g}$ in A; that is if $a \in A$ and $i(x)$ is the image of $x \in \mathfrak{g}$ in A, then $\varepsilon(x)a = [i(x), a]$. For $\lambda \in \mathfrak{g}^*$, let $A_\lambda = \{a \in A \mid \varepsilon(x)a = \lambda(x)a$, all $x \in \mathfrak{g}\}$. We have $A_\lambda A_\mu \subset A_{\lambda+\mu}$, and the sum of the A_λ is direct. The subalgebra $S(A) = \Sigma_{\lambda \in \mathfrak{g}^*} \oplus A_\lambda$ is called the *semicentre* of A. More generally, as in 4.9.7 of [**1**], we may also define $Q(A)_\lambda$.

Give A the induced filtration from U, and thus $i(\mathfrak{g})$ generates A. When the associated graded ring $\bar{A}$ of $A = U/P$ is a domain, we are now able to completely determine those automorphisms of A which become inner on $Q(A)$; they are precisely those automorphisms which are given by conjugation by an element of some $Q(A)_\lambda$.

Theorem 1. *Let $A = U(\mathfrak{g})/P$, for $\mathfrak{g}$ a finite-dimensional Lie algebra, and assume that $\bar{A}$ is a domain. Let $0 \neq a \in Q(A)$. Then $a^{-1}Aa = A \Leftrightarrow$ there exists $\lambda \in \mathfrak{g}^*$ such that $a \in Q(A)_\lambda$.*

PROOF. First assume that $a \in Q(A)_\lambda$. Then $[i(x), a] = \lambda(x)a$, all $x \in \mathfrak{g}$, and so $i(x)a = ai(x) + \lambda(x)a = a(i(x) + \lambda(x))$. Thus $i(\mathfrak{g})a \subseteq aA$, and it follows that $Aa \subseteq aA$. Similarly $aA \subseteq aA$, and so $aA = Aa$. It follows that $a^{-1}Aa = A$.

Conversely, if $a^{-1}Aa = A$, then $\sigma \in \operatorname{Aut}(A)$ given by $r^\sigma = a^{-1}ra$ is an X-inner automorphism of A. By Proposition 1, $\bar{\sigma}$ is trivial on the associated graded ring A. In particular, for any $i(x) \in i(\mathfrak{g}) \subseteq A_1$, $i(x)^\sigma - i(x) \in A_0 = k \cdot 1$. That is, $i(x)^\sigma = i(x) + \lambda(x)$, some $\lambda = \lambda(x) \in k$, for each $x \in g$. Clearly $\lambda \in \mathfrak{g}^*$, and since $i(x)^\sigma = a^{-1}i(x)a = i(x) + \lambda(x)$, $[i(x), a] = \lambda(x)a$, all $x \in \mathfrak{g}$. Thus $a \in Q(A)_\lambda$. □

COROLLARY 1. *The subgroup of all X-inner automorphisms of $A = U(\mathfrak{g})/P$, for $\bar{A}$ a domain, is isomorphic to the additive subgroup of $\mathfrak{g}^*$ consisting of those λ with* $Q(A)_\lambda \neq 0$.

COROLLARY 2. *Consider $A = U(\mathfrak{g})/P$ as above, with $\bar{A}$ a domain, and let σ be any automorphism of A. Then*

(1) $(Q(A)_\lambda)^\sigma = Q(A)_\mu$, *for some* $\mu \in \mathfrak{g}^*$.

(2) σ *stabilizes* $S(A)$, *the semicentre of* A.

PROOF. Choose any $0 \neq a \in Q(A)_\lambda$. By Theorem 1, $a^{-1}Aa = A$ and thus $(a^\sigma)^{-1}A^\sigma a^\sigma = A^\sigma$, or $(a^\sigma)^{-1}Aa^\sigma = A$. Thus, again by Theorem 1, there exists $\mu \in \mathfrak{g}^*$ such that $a^\sigma \in Q(A)_\mu$. It follows that $Q(A)_\lambda^\sigma = Q(A)_\mu$, since σ preserves the center C of $Q(A)_\lambda$ and $Q(A)_\lambda = Ca$.

Clearly σ preserves $S(A)$, since $A_\lambda = Q(A)_\lambda \cap A$. □

We now turn to derivations of $U(\mathfrak{g})$. The next corollary was pointed out to us by Martha Smith, and we wish to thank her for allowing us to include it.

COROLLARY 3 (M. SMITH). *Let d be a derivation of $\mathfrak{g}$, a finite-dimensional Lie algebra over a field k of characteristic* 0, *and extend d to $U(\mathfrak{g})$. Then for any $\lambda \in \mathfrak{g}^*$ such that $U_\lambda \neq 0$,*

(1) $d(U_\lambda) \subseteq U_\lambda$,

(2) $\lambda(d(\mathfrak{g})) = 0$.

PROOF. We first note that by passing to the algebraic closure of k, we may assume that k is algebraically closed. Since d is a locally finite derivation on U, and the semicentre $S = S(U)$ is stable under $\operatorname{Aut}(U)$ by Corollary 2, we may apply a result of J. Krempa [2] to conclude that S is also d-stable.

Let $0 \neq a \in U_\lambda$, some $\lambda \in \mathfrak{g}^*$. Then $d(a) \in S$, and so $d(a) = \sum_\mu b_\mu$, where $b_\mu \in U_\mu$. Applying d to the equation $[x, a] = \lambda(x)a$, any $x \in \mathfrak{g}$, we see that $[x, d(a)] = -\lambda(d(x))a + \lambda(x)d(a)$. Now

$$\sum_\mu \mu(x)b_\mu = \sum_\mu [x, b_\mu] = [x, d(a)] = -\lambda(d(x))a + \lambda(x)\sum_\mu b_\mu,$$

and so

$$\sum_\mu (\mu(x) - \lambda(x))b_\mu = -\lambda(d(x))a. \qquad (**)$$

For each $u \neq \lambda$, this gives $(\mu(x) - \lambda(x))b_\mu = 0$, all $x \in \mathfrak{g}$. Since $\lambda(x_0) \neq \mu(x_0)$ for some $x_0 \in \mathfrak{g}$, it follows that $b_\mu = 0$, all $\mu \neq \lambda$. That is, $d(a) = b_\lambda \in U_\lambda$, proving

(1). Again using (**), it follows that $0 = -\lambda(d(x))a$. Thus $\lambda(d(x)) = 0$, all $x \in \mathfrak{g}$, proving (2). □

We also give an application to crossed products. For any k-algebra A, any subgroup $G \subseteq \operatorname{Aut}_k(A)$, and any factor set $t\colon G \times G \to k$, we may form the crossed product algebra $A *_t G$.

COROLLARY 4. *Let g be a finite-dimensional Lie algebra over a field k of characteristic* 0, *and let* $A = U(g)/p$ *as above, such that* $\bar{A}$ *is a domain. Let G be any subgroup of* $\operatorname{Aut}_k(A)$. *Then any crossed product* $A *_t G$ *is a prime ring.*

PROOF. Let $G_{\text{inn}} = \{g \in G \mid g \text{ is } X\text{-inner}\}$. By Corollary 1, G_{inn} is isomorphic to a subgroup of the dual space g^*, which is torsion-free when k has characteristic 0. Thus the crossed product is prime by [6, Theorem 2.8]. □

Our second application of Proposition 1 is to certain differential polynomial rings. Let R be a commutative domain with 1 with a nontrivial derivation d, and let $A = R[x; d]$, the differential polynomial ring, in which $xr = rx + d(r)$, for all $r \in R$. A is a filtered algebra, using $A_n = \{\text{all polynomials of degree} \leq n\}$, and the associated graded ring $\bar{A} \cong R[x]$, the ordinary polynomial ring over R. We let $F = Q(R)$, the quotient field of R, and let $D = Q(A)$, the division ring of quotients of A. We are now able to determine all X-inner automorphisms of A.

THEOREM 2. *Let* $A = R[x; d]$, *where d is a nontrivial derivation of the commutative domain R, and let* $\sigma \in \operatorname{Aut}(A)$. *Then σ becomes inner on* $D = Q(A) \Leftrightarrow \sigma$ *is conjugation by some* $q \in F$ *such that* $q^{-1}d(q) \in R$.

PROOF. First assume that σ is conjugation by some such q. Clearly $R^\sigma = R$, and $x^\sigma = q^{-1}xq = q^{-1}(qx + d(q)) = x + q^{-1}d(q) \in A$. Thus σ is an X-inner automorphism of A.

Conversely, assume that σ becomes inner on D; say that σ is induced by $b(x)a(x)^{-1}$, where $a(x) = a_m x^m + \cdots + a_1 x + a_0$ and $b(x) = b_n x^n + \cdots + b_0$, $a_i, b_i \in R$. We will show that σ is also induced by $q = b_n a_m^{-1} \in F$ (the fact that $q^{-1}d(q) \in R$ follows from the fact that $\sigma \in \operatorname{Aut}(A)$).

Now by Proposition 1, we know that σ preserves the filtration on A and that $\bar{\sigma}$ is trivial on $\bar{A}$. In particular, $r^\sigma = r$, all $r \in R$, and $x^\sigma = x + \alpha$, for some $\alpha \in R$.

To simplify the argument, note that $b(x) = b_1(x)b_n$, so $b(x)a(x)^{-1} = b_1(x)(b_n^{-1}a(x))^{-1}$; that is, we may assume $b(x)$ is monic. Moreover, since for any $f(x) \in A$, $f(x)^\sigma = a(x)b(x)^{-1}f(x)b(x)a(x)^{-1}$, replacing f by $b(x)f$, it follows that $a(x)f(x)b(x) = b(x)^\sigma f(x)^\sigma a(x)$, all $f(x) \in A$. Again since $\bar{\sigma}$ is trivial on $\bar{A}$, $b(x)^\sigma - b(x)$ has degree $< n$; that is, $b(x)^\sigma$ is also monic of degree n, say $b(x)^\sigma = c(x) = x^n + \cdots + c_0$. Thus

$$a(x)f(x)b(x) = c(x)f(x)^\sigma a(x), \quad \text{all } f \in A. \tag{+}$$

We first evaluate (+) for $f = r = r^\sigma$, $r \in R$:

$$\begin{aligned}(a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0)r(x^n + b_{n-1}x^{n-1} + \cdots + b_0)\\ = (x^n + \cdots + c_0)r(a_m x^m + \cdots + a_0),\end{aligned}$$

so

$$a_m r x^{m+n} + a_m m d(r) x^{m+n-1} + a_{m-1} r x^{m+n-1} + a_m r b_{n-1} x^{m+n-1} + \cdots$$
$$= r a_m x^{m+n} + n d(r a_m) x^{n+m-1} + c_{n-1} r a_m x^{m+n-1} + r a_{m-1} x^{n+m-1} + \cdots.$$

Equating coefficients of x^{n+m-1}, we see

$$m a_m d(r) + a_{m-1} r + a_m r b_{n-1} = n d(r) a_m + n r d(a_m) + c_{n-1} r a_m + r a_{m-1}.$$

Cancelling and letting $r = 1$,

$$n d(a_m) = a_m(b_{n-1} - c_{n-1}). \tag{++}$$

We also evaluate (+) for $f(x) = x$, $x^\sigma = x + \alpha$:

$$(a_m x^m + \cdots + a_0) x (x^n + \cdots + b_0)$$
$$= (x^n + \cdots + c_0)(x + \alpha)(a_m x^m + \cdots + a_0),$$
$$a_m x^{m+n+1} + a_m b_{n-1} x^{m+n} + a_{m-1} x^{m+n} + \cdots$$
$$= a_m x^{n+m+1} + (n+1) d(a_m) x^{n+m}$$
$$+ c_{n-1} a_m x^{n+m} + a_{m-1} x^{n+m} + \alpha a_m x^{n+m} + \cdots.$$

Equating coefficients of x^{n+m}, we see

$$a_m b_{n-1} + a_{m-1} = (n+1) d(a_m) + c_{n-1} a_m + a_{m-1} + \alpha a_m$$

and thus

$$a_m(b_{n-1} - c_{n-1}) = (n+1) d(a_m) + \alpha a_m.$$

Substituting (++),

$$n d(a_m) = (n+1) d(a_m) + \alpha a_m.$$

Thus $\alpha a_m = -d(a_m)$, and so $\alpha = -a_m^{-1} d(a_m) = a_m d(a_m^{-1})$. Using $q = a_m^{-1}$, we have proved that

$$x^\sigma = x + \alpha = x + q^{-1} d(q) = q^{-1} x q.$$

Since σ is determined by its action on x, the theorem is proved. □

We illustrate Theorems 1 and 2 with several examples.

EXAMPLE 1. Let $\mathfrak{g}$ be the 3-dimensional completely solvable Lie algebra over k with basis $\{x, y, z\}$ such that $[x, y] = y$, $[x, z] = z$, and $[y, z] = 0$. Since $[\mathfrak{g}, \mathfrak{g}] = \langle y, z \rangle$, and any $\lambda \in \mathfrak{g}^*$ such that $K_\lambda \neq 0$ must annihilate $[\mathfrak{g}, \mathfrak{g}]$, $\lambda(y) = 0 = \lambda(z)$. Thus λ is determined by $\lambda(x) = \alpha \in k$. We claim that $\alpha \in \mathbf{Z}$, an integer. For, if $0 \neq a \in K_\lambda$, $[y, a] = 0 = [z, a]$ and $[x, a] = \alpha a$. Using $U(\mathfrak{g}) \cong k[y, z][x; d]$, where $d(y) = y$, $d(z) = z$, we may assume $a = pq^{-1}$, where p and q are both polynomials in y and z. Since for a monomial m in y and z of degree l, $d(m) = lm$, $d(a) = \alpha a$ implies $d(p)q - pd(q) = \alpha pq$; it follows that $\alpha \in \mathbf{Z}$. When $\alpha = 1$, so $\lambda(x) = 1$, both $y, z \in U_\lambda$, and induce the X-inner automorphism $\sigma \in \operatorname{Aut}(U)$ given by $x^\sigma = x + 1$, $y^\sigma = y$, $z^\sigma = z$. Thus the group of X-inners of $U(\mathfrak{g})$ is $\langle \alpha \rangle$, the infinite cyclic group generated by σ.

EXAMPLE 2. Consider the Weyl algebra $A_1 = k[x, y]$, $xy - yx = 1$, where k has characteristic 0. Now $A_1 = k[y][x; d]$ where $d(y) = 1$, the usual derivative. Then A_1 has no nontrivial X-inner automorphisms. For by Theorem 2, if σ is X-inner, it is induced by some $p(y)/q(y) \in k(y)$ such that $(q(y)/p(y))d(p(y)/q(y)) \in R = k[y]$. Thus, $(q'p - qp')/pq$ must be a polynomial; this is impossible, as

$$\text{degree}(pq) > \text{degree}(q'p - qp').$$

However, enlarging A_1 slightly to $B = k[y]_{(y)}[x; d]$, the localization of A_1 at the powers of y, the group of X-inners of B is $\langle \sigma \rangle$, the infinite cyclic group generated by σ which is given by conjugation by y. For, by Theorem 2, any X-inner automorphism τ is induced by $py^k/q \in k(y)$, where $p, q \in k[y]$ are relatively prime, $y \nmid p, y \nmid q$, and $k \in \mathbf{Z}$. Since conjugation by y^k is easily seen to be X-inner (it preserves $k[y]_{(y)}$) and the X-inner automorphisms form a subgroup, $\tau_1 = \tau\sigma^{-k}$ is X-inner and is induced by p/q. As before, $(q/p)d(p/q) = (q'p - qp')/pq \in k[y]_{(y)}$, and this is impossible as $y \nmid pq$, unless $q' = p' = 0$, so $p/q = \alpha \in k$. Thus, τ is induced by y^k.

Finally, if we consider $C = k(y)[x; d]$, then conjugation by any nonzero element of $F = k(y)$ is an X-inner automorphism of C, and $p, q \in F$ induce the same automorphism $\Leftrightarrow p = \alpha q$, $\alpha \in k$. Thus the group of X-inners is isomorphic to $k(y)^\circ / k^\circ$, where K° denotes the multiplicative group.

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