$X = M$ THEOREM:
FERMIONIC FORMULAS AND RIGGED CONFIGURATIONS UNDER REVIEW

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ABSTRACT. We give a review of the current status of the $X = M$ conjecture. Here X stands for the one-dimensional configuration sum and M for the corresponding fermionic formula. There are three main versions of this conjecture: the unrestricted, the classically restricted and the level-restricted version. We discuss all three versions and illustrate the methods of proof with many examples for type $A_{n-1}^{(1)}$. In particular, the combinatorial approach via crystal bases and rigged configurations is discussed. Each section ends with a conglomeration of open problems.

CONTENTS

1. INTRODUCTION

The $X = M$ *Conjecture* asserts the equality between the generating function of highest weight tensor product crystal elements graded by the energy function and the fermionic formula [14, 15, 28]. This article concerns the $X = M$ *Theorem*, or more precisely, those cases in which the $X = M$ Conjecture has been proven. We describe the method of proof which uses the combinatorics of crystal bases and rigged configurations. We mostly focus on type $A_{n-1}^{(1)}$, but many of the constructions have analogues for other affine Kac–Moody algebras $\mathfrak{g}$. Instead of providing all details of the proofs, we illustrate the main concepts via examples. Each section ends with a conglomeration of open problems.

The *fermionic formula* is a q-analogue of the tensor product multiplicity $[\bigotimes_j W_{s_j}^{(r_j)}, V_\lambda]$, where $W_s^{(r)}$ is a $U_q(\mathfrak{g})$ Kirillov–Reshetikhin module indexed by a Dynkin node r and $s \in \mathbb{Z}_{>0}$, and V_λ is the irreducible highest weight $U_q(\bar{\mathfrak{g}})$-module with highest weight λ. Here $\bar{\mathfrak{g}}$ is the finite-dimensional classical algebra inside the affine Kac–Moody algebra $\mathfrak{g}$. Alternatively, since the procedure of taking the crystal limit does not change tensor product multiplicities, we can view the fermionic formula as a q-analogue of $[\bigotimes_j B^{r_j,s_j}, B(\lambda)]$, where $B^{r,s}$ is the Kirillov–Reshetikhin crystal and $B(\lambda)$ is the finite-dimensional highest weight crystal indexed by the dominant weight λ. Instead of labeling the fermionic formula by $B = \bigotimes_j B^{r_j,s_j}$ and $B(\lambda)$, we use the multiplicity array $L = (L_s^{(r)})$ and λ, where $L_s^{(r)}$ denotes the number of tensor factors $B^{r,s}$ in B. For type $A_{n-1}^{(1)}$ the fermionic formula is then given by

$$(1.1) \qquad \overline{M}(L, \lambda; q) = \sum_{\nu \in \overline{C}(L,\lambda)} q^{\mathrm{cc}(\nu)} \prod_{(a,i) \in \mathcal{H}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}.$$

Here $\overline{C}(L, \lambda)$ is the set of admissible (L, λ)-configurations, $\mathcal{H} = I \times \mathbb{Z}_{>0}$ with $I = \{1, 2, \ldots, n-1\}$, $m_i^{(a)}$ is the particle number and $p_i^{(a)}$ is the vacancy number. The precise definition of the various quantities is given in section 4.1. The q-binomial coefficient is defined as

$$\begin{bmatrix} p + m \\ m \end{bmatrix} = \frac{(q)_{p+m}}{(q)_p (q)_m}$$

for $p, m \in \mathbb{Z}_{\geq 0}$ and zero otherwise, where $(q)_m = (1 - q)(1 - q^2) \cdots (1 - q^m)$.

The q-binomial coefficient $\begin{bmatrix} p+m \\ m \end{bmatrix}$ is the generating function of partitions in a box of size $p \times m$. Using this interpretation, equation (1.1) can be rewritten in solely combinatorial terms as

$$\overline{M}(L, \lambda; q) = \sum_{(\nu,J) \in \overline{RC}(L,\lambda)} q^{\mathrm{cc}(\nu,J)},$$

where $\overline{RC}(L, \lambda)$ is the set of *rigged configurations* as defined in section 4.1. The one-dimensional configuration sum $\overline{X}(B, \lambda; q)$ is the generating function of highest weight paths $\overline{P}(B, \lambda)$ of weight λ weighted by the energy function D

$$\overline{X}(B, \lambda; q) = \sum_{b \in \overline{P}(B,\lambda)} q^{D(b)}.$$

The $X = M$ conjecture [14, 15] asserts that

$$(1.2) \qquad \overline{X}(B, \lambda; q) = \overline{M}(L, \lambda; q)$$

for all affine Kac–Moody algebras $\mathfrak{g}$.

The $X = M$ conjecture can be proved by establishing a statistics preserving bijection $\overline{\Phi}$: $\overline{P}(B, \lambda) \to \overline{RC}(L, \lambda)$ between the set of paths and the set of rigged configurations. More precisely, $\overline{\Phi}$ should have the property that $D(b) = \mathrm{cc}(\overline{\Phi}(b))$ for all $b \in \overline{P}(B, \lambda)$. For $B = \bigotimes_j B^{1,\mu_j}$ of type

$A_{n-1}^{(1)}$ such a bijection was given by Kerov, Kirillov and Reshetikhin [22, 23]. In fact, in this case the set of paths $\overline{\mathcal{P}}(B, \lambda)$ is in bijection with the set of semi-standard Young tableaux $\mathrm{SSYT}(\lambda, \mu)$ of shape λ and content $\mu = (\mu_1, \mu_2, \ldots)$, and the energy function corresponds to the cocharge of Lascoux and Schützenberger [27]. The bijection of Kerov, Kirillov and Reshetikhin [22, 23] is a bijection between semi-standard Young tableaux and rigged configurations and yields a fermionic formula for the Kostka–Foulkes polynomials. In [25], this bijection was generalized to $B = \bigotimes_j B^{r_j, s_j}$ of type $A_{n-1}^{(1)}$. In this case the set of paths $\overline{\mathcal{P}}(B, \lambda)$ is in bijection with Littlewood–Richardson tableaux and the bijection was in fact formulated as a bijection between Littlewood–Richardson tableaux and rigged configurations. For other types such bijections have also been given in special cases. In summary to date the following cases have been proven:

- $B = \bigotimes_j B^{r_j, s_j}$ of type $A_{n-1}^{(1)}$ [25];
- $B = \bigotimes_j B^{1, s_j}$ of all nonexceptional types [29, 38];
- $B = \bigotimes_j B^{r_j, 1}$ for type $D_n^{(1)}$ [34].

An important technique in studying fermionic formulas of nonsimply-laced types are virtual crystals and virtual rigged configuration [30, 31].

In this paper we provide a review of the bijective approach to the $X = M$ conjecture. We will mostly restrict our attention to type $A_{n-1}^{(1)}$ and set up the bijection between crystals and rigged configurations (rather than tableaux and rigged configurations).

The correspondence between the two combinatorial sets can be understood in terms of two approaches to solvable lattice models and their associated spin chain systems: the *Bethe Ansatz* [7] and the *corner transfer matrix method* [6].

In his 1931 paper [7], Bethe solved the Heisenberg spin chain based on the string hypothesis which asserts that the eigenvalues of the Hamiltonian form certain strings in the complex plane as the size of the system tends to infinity. The Bethe Ansatz has been applied to many further models proving completeness of the Bethe vectors. The eigenvalues and eigenvectors of the Hamiltonian are indexed by rigged configuration. However, numerical studies indicate that the string hypothesis is not always true [5]. The corner transfer matrix (CTM) method was introduced by Baxter and labels the eigenvectors by one-dimensional lattice paths. It turns out that these lattice paths have a natural interpretation in terms of Kashiwara's crystal base theory [18], namely as highest weight crystal elements in a tensor product of finite-dimensional crystals.

Even though neither the Bethe Ansatz nor the corner transfer matrix method are mathematically rigorous, they suggest that there should be a bijection between the two index sets, namely rigged configurations on the one hand and highest weight crystal elements on the other hand. This is schematically indicated in Figure 1. As explained above, the generating function of rigged configurations leads fermionic formulas. Fermionic formulas can be interpreted as explicit expressions for the partition function of the underlying physical models which reflect the particle structure. For more details regarding the physical background of fermionic formulas see [20, 21, 14].

The $X = M$ conjecture can be generalized in two different ways: to the *level-restricted* and the *unrestricted* case. Both of these cases will also be reviewed in this paper in the case of type $A_{n-1}^{(1)}$.

The set of paths $\overline{\mathcal{P}}(B, \lambda)$ is defined as the set of all $b \in B$ of weight λ that are highest weight with respect to the classical crystal operators. The Kirillov–Reshetikhin crystals are affine crystals and have the additional crystal operators e_0 and f_0, which can be used to define level-restricted paths. Hence it is natural to consider the generating functions of level-restricted paths, giving rise to a level-restricted version of X. The corresponding set of level-restricted rigged configurations was considered in [37]. The notion of level-restriction is also very important in the context of restricted-solid-on-solid (RSOS) models in statistical mechanics [6] and fusion models in conformal field theory [44]. The one-dimensional configuration sums of RSOS models are generating
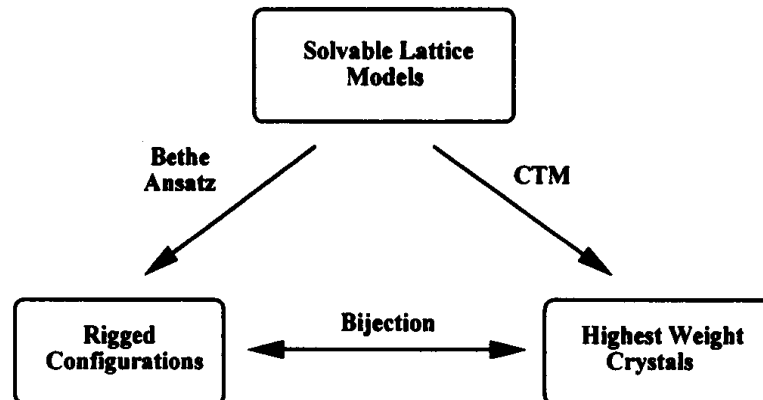
FIGURE 1. Bethe Ansatz versus corner transfer matrix method (CTM)

functions of level-restricted paths (see for example [3, 8, 16]). The structure constants of the fusion algebras of Wess–Zumino–Witten conformal field theories are exactly the level-restricted analogues of the tensor product multiplicities $X(B, \lambda; 1)$ or Littlewood–Richardson coefficients as shown by Kac [17, Exercise 13.35] and Walton [45, 46]. q-Analogues of these level-restricted Littlewood–Richardson coefficients in terms of ribbon tableaux were proposed in ref. [12].

Rigged configurations corresponding to highest weight crystal paths are only the tip of an iceberg. In [35] the definition of rigged configurations was extended to all crystal elements in types ADE by the explicit construction of a crystal structure on the set of *unrestricted* rigged configurations. The equivalence of the crystal structures on rigged configurations and crystal paths together with the correspondence for highest weight vectors yields the equality of generating functions in analogy to (1.2). Denote the unrestricted set of paths and rigged configurations by $\mathcal{P}(B, \lambda)$ and $RC(L, \lambda)$, respectively. The corresponding generating functions are unrestricted one-dimensional configuration sums or q-supernomial coefficients. A direct bijection $\Phi : \mathcal{P}(B, \lambda) \to RC(L, \lambda)$ for type $A_{n-1}^{(1)}$ along the lines of [25] is constructed in [9, 10].

The paper is organized as follows. In section 2 we present the Bethe Ansatz for the spin 1/2 XXX Heisenberg chain which first gave rise to rigged configurations. In section 3 we review the one-dimensional configuration sums and set the notation used in this article. The corresponding fermionic formulas for the classically restricted, unrestricted and level-restricted cases are subject of sections 4, 5 and 6, respectively. In particular for the $\overline{X} = \overline{M}$ case, we introduce rigged configurations and fermionic formulas in section 4.1, define certain splitting operations on crystals and rigged configurations in sections 4.2 and 4.3, which are necessary for the bijection $\overline{\Phi}$ between paths and rigged configurations of section 4.4. Section 4.5 features many of the properties of $\overline{\Phi}$. For the unrestricted version of the $X = M$ theorem, we define the crystal structure on rigged configurations in section 5.1. A characterization of unrestricted rigged configurations is given in section 5.2 which is used in section 5.3 to derive the fermionic formula. The affine crystal operators on rigged configurations are given in section 5.4. Section 6 deals with the level-restricted version of the $X^\ell = M^\ell$ theorem. Level-restricted rigged configurations are introduced in section 6.1 and the corresponding fermionic formula is derived in section 6.2. Each section ends with some open problems.

2. Bethe Ansatz and rigged configurations

In this section we discuss the algebraic Bethe Ansatz for the example of the spin 1/2 XXX Heisenberg chain and show how rigged configurations arise. Further details can be found in [11, 33].

The spin 1/2 XXX **Heisenberg chain** is a one-dimensional quantum spin chain on N sites with periodic boundary conditions. It is defined on the Hilbert space $\mathcal{H}_N = \bigotimes_{n=1}^{N} h_n$ where in this case $h_n = \mathbb{C}^2$ for all n. Associated to each site is a local spin variable $\vec{s} = \frac{1}{2}\vec{\sigma}$ where

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

are the Pauli matrices. The spin variable acting on the n-th site is given by

$$\vec{s}_n = I \otimes \cdots \otimes I \otimes \vec{s} \otimes I \otimes \cdots \otimes I$$

where I is the identity operator and $\vec{s}$ is in the n-th tensor factor. We impose periodic boundary conditions $\vec{s}_n = \vec{s}_{n+N}$.

The **Hamiltonian** of the spin 1/2 XXX model is

$$H_N = J \sum_{n=1}^{N} \left(\vec{s}_n \cdot \vec{s}_{n+1} - \frac{1}{4} \right).$$

Our goal is to determine the eigenvectors and eigenvalues of H_N in the antiferromagnetic regime $J > 0$ in the limit when $N \to \infty$.

The main tool is the **Lax operator** $L_{n,a}(\lambda)$, also called the local transition matrix. It acts on $h_n \otimes \mathbb{C}^2$ where $\mathbb{C}^2$ is an auxiliary space and is defined as

$$L_{n,a}(\lambda) = \lambda I_n \otimes I_a + i\vec{s}_n \otimes \vec{\sigma}_a.$$

Here I_n and I_a are unit operators acting on h_n and the auxiliary space $\mathbb{C}^2$, respectively; λ is a complex parameter, called the spectral parameter. Writing the action on the auxiliary space as a 2×2 matrix, we have

$$(2.1) \qquad L_n(\lambda) = \begin{pmatrix} \lambda + is_n^3 & is_n^- \\ is_n^+ & \lambda - is_n^3 \end{pmatrix}$$

where $s_n^{\pm} = s_n^1 \pm is_n^2$.

The crucial fact is that the Lax operator satisfies commutation relations in the auxiliary space $V = \mathbb{C}^2$. Altogether there are 16 relations which can be written compactly in tensor notation. Given two Lax operators $L_{n,a_1}(\lambda)$ and $L_{n,a_2}(\mu)$ defined in the same quantum space h_n, but different auxiliary spaces V_1 and V_2, the products $L_{n,a_1}(\lambda)L_{n,a_2}(\mu)$ and $L_{n,a_2}(\mu)L_{n,a_1}(\lambda)$ are defined on the triple tensor product $h_n \otimes V_1 \otimes V_2$. There exists an operator $R_{a_1,a_2}(\lambda - \mu)$ defined on $V_1 \otimes V_2$ such that

$$(2.2) \qquad R_{a_1,a_2}(\lambda - \mu)L_{n,a_1}(\lambda)L_{n,a_2}(\mu) = L_{n,a_2}(\mu)L_{n,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu).$$

Explicitly, the R-matrix $R_{a_1,a_2}(\lambda)$ is given by

$$R_{a_1,a_2}(\lambda) = \left(\lambda + \frac{i}{2} \right) I_{a_1} \otimes I_{a_2} + \frac{i}{2} \vec{\sigma}_{a_1} \otimes \vec{\sigma}_{a_2}.$$

Geometrically, the Lax operator $L_{n,a}(\lambda)$ can be interpreted as the transport between sites n and $n + 1$ of the quantum spin chain. Hence

$$T_{N,a}(\lambda) = L_{N,a}(\lambda) \cdots L_{1,a}(\lambda)$$

is the monodromy around the circle (recall that we assume periodic boundary conditions). In the auxiliary space write

$$T_N(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

with entries in the full Hilbert space $\mathcal{H}_N$. From (2.2) it is clear that the **monodromy matrix** satisfies the following commutation relation

(2.3) $$R_{a_1,a_2}(\lambda - \mu)T_{N,a_1}(\lambda)T_{N,a_2}(\mu) = T_{N,a_2}(\mu)T_{N,a_1}(\lambda)R_{a_1,a_2}(\lambda - \mu).$$

Let $\omega_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In the auxiliary space the Lax operator is triangular on ω_n

$$L_n(\lambda)\omega_n = \begin{pmatrix} \lambda + \frac{i}{2} & * \\ 0 & \lambda - \frac{i}{2} \end{pmatrix} \omega_n$$

where $*$ stands for an for us irrelevant quantity. This follows directly from (2.1). On the Hilbert space $\mathcal{H}_N$ we define $\Omega = \bigotimes_n \omega_n$ so that

$$T_N(\lambda)\Omega = \begin{pmatrix} \alpha^N(\lambda) & * \\ 0 & \delta^N(\lambda) \end{pmatrix} \Omega$$

where $\alpha(\lambda) = \lambda + \frac{i}{2}$ and $\delta(\lambda) = \lambda - \frac{i}{2}$. Equivalently this means that

$$\begin{aligned} C(\lambda)\Omega &= 0 \\ A(\lambda)\Omega &= \alpha^N(\lambda)\Omega \\ D(\lambda)\Omega &= \delta^N(\lambda)\Omega \end{aligned}$$

so that Ω is an eigenstate of $A(\lambda)$ and $D(\lambda)$ and hence also of $t_N(\lambda) = A(\lambda) + D(\lambda)$.

The claim is that the other eigenvectors of $t_N(\lambda)$ are of the form

$$\Phi(\lambda, \Lambda) = B(\lambda_1) \cdots B(\lambda_n)\Omega.$$

The lambdas $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ satisfy a set of algebraic relations, called the **Bethe equations**, which can be derived from (2.3)

(2.4) $$\left(\frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}} \right)^N = \prod_{\substack{\lambda' \in \Lambda \\ \lambda' \neq \lambda}} \frac{\lambda - \lambda' + i}{\lambda - \lambda' - i}$$

where $\lambda \in \Lambda = \{\lambda_1, \ldots, \lambda_n\}$.

Suggested by numerical analysis, it is assumed that in the limit $N \to \infty$ the λ's form strings. This hypothesis is called the **string hypothesis**. A string of length $\ell = 2M + 1$, where M is an integer or half-integer depending on the parity of ℓ, is a set of λ's of the form

$$\lambda_{jm}^M = \lambda_j^M + im$$

where $\lambda_j^M \in \mathbb{R}$ and $-M \leq m \leq M$ is integer or half-integer depending on M. The index j satisfies $1 \leq j \leq m_\ell$ where m_ℓ is the number of strings of length ℓ. A decomposition of $\{\lambda_1, \ldots, \lambda_n\}$ into strings is called a **configuration**. Each configuration is parametrized by $\{m_\ell\}$. It follows that

$$\sum_\ell \ell m_\ell = n.$$

Now take (2.4) and multiply over a string

$$\prod_{m=-M}^{M} \left(\frac{\lambda_j^M + i(m + \frac{1}{2})}{\lambda_j^M + i(m - \frac{1}{2})} \right)^N$$

(2.5)
$$= \prod_{m=-M}^{M} \prod_{\substack{M',j',m' \\ (M',j',m') \neq (M,j,m)}} \frac{\lambda_j^M - \lambda_{j'}^{M'} + i(m - m' + 1)}{\lambda_j^M - \lambda_{j'}^{M'} + i(m - m' - 1)}.$$

Many of the terms on the left and right cancel so that this equation can be rewritten as

(2.6)
$$e^{iNp_M(\lambda_j^M)} = \prod_{\substack{M',j' \\ (M',j') \neq (M,j)}} e^{iS_{MM'}(\lambda_j^M - \lambda_{j'}^{M'})},$$

in terms of the momentum and scattering matrix

$$e^{ip_M(\lambda)} = \frac{\lambda + i(M + \frac{1}{2})}{\lambda - i(M + \frac{1}{2})}$$

$$e^{iS_{MM'}(\lambda)} = \prod_{m=|M-M'|}^{M+M'} \frac{\lambda + im}{\lambda - im} \cdot \frac{\lambda + i(m + 1)}{\lambda - i(m + 1)}.$$

Taking the logarithm of (2.6) using the branch cut

$$\frac{1}{i} \ln \frac{\lambda + ia}{\lambda - ia} = \pi - 2 \arctan \frac{\lambda}{a}$$

we obtain

(2.7)
$$2N \arctan \frac{\lambda_j^M}{M + \frac{1}{2}} = 2\pi Q_j^M + \sum_{\substack{M',j' \\ (M',j') \neq (M,j)}} \Phi_{MM'}(\lambda_j^M - \lambda_{j'}^{M'}),$$

where

$$\Phi_{MM'}(\lambda) = 2 \sum_{m=|M-M'|}^{M+M'} \left(\arctan \frac{\lambda}{m} + \arctan \frac{\lambda}{m+1} \right).$$

The first term on the right is absent for $m = 0$. Here Q_j^M is an integer or half-integer depending on the configuration.

In addition to the string hypothesis, we assume that the Q_j^M classify the λ's uniquely: λ_j^M increases if Q_j^M increases and in a given string no Q_j^M coincide. As we will see shortly with this assumption one obtains the correct number of solutions to the Bethe equations (2.4).

Using $\arctan \pm\infty = \pm\frac{\pi}{2}$ we obtain from (2.7) putting $\lambda_j^M = \infty$

$$Q_\infty^M = \frac{N}{2} - \left(2M + \frac{1}{2} \right) (m_{2M+1} - 1) - \sum_{M' \neq M} (2 \min(M, M') + 1) m_{2M'+1}.$$

Since there are $2M + 1$ strings in a given string of length $2M + 1$, the maximal admissible $Q_{\max}^M$ is

$$Q_{\max}^M = Q_\infty^M - (2M + 1)$$

where we assume that if Q_j^M is bigger than $Q_{\max}^M$ then at least one root in the string is infinite and hence all are infinite which would imply $Q_j^M = Q_\infty^M$.

With the already mentioned assumption that each admissible set of quantum number Q_j^M corresponds uniquely to a solution of the Bethe equations we may now count the number of Bethe vectors. Since arctan is an odd function and by the assumption about the monotonicity we have

$$-Q_{\max}^M \le Q_1^M < \cdots < Q_{m_{2M}+1}^M \le Q_{\max}^M.$$

Hence defining p_ℓ as

$$p_\ell = N - 2\sum_{\ell'}\min(\ell,\ell')m_{\ell'}$$

so that

$$p_\ell + m_\ell = 2Q_{\max}^M + 1 \qquad \text{with } \ell = 2M + 1.$$

With this the number of Bethe vectors with configuration $\{m_\ell\}$ is given by

$$Z(N,n|\{m_\ell\}) = \prod_{\ell \ge 1}\binom{p_\ell + m_\ell}{m_\ell}$$

where $\binom{p+m}{m} = (p+m)!/p!m!$ is the binomial coefficient. The total number of Bethe vectors is

$$(2.8) \qquad Z(N,n) = \sum_{\substack{\{m_\ell\} \\ \sum_\ell \ell m_\ell = n}} \prod_{\ell \ge 1}\binom{p_\ell + m_\ell}{m_\ell}.$$

It should be emphasized that the derivation of (2.8) given here is not mathematically rigorous. Besides the various assumptions that were made we also did not worry about possible singularities of (2.5). However, (2.8) indeed yields the correct number of Bethe vectors.

To interpret (2.8) combinatorially let us view the set $\{m_\ell\}$ as a partition ν. A partition is a set of numbers $\nu = (\nu_1, \nu_2, \ldots)$ such that $\nu_i \ge \nu_{i+1}$ and only finitely many ν_i are nonzero. The partition has part i if $\nu_k = i$ for some k. The size of partition ν is $|\nu| := \nu_1 + \nu_2 + \cdots$. In the correspondence between $\{m_\ell\}$ and ν, m_ℓ specifies the number of parts of size ℓ in ν. For example, if $m_1 = 1$, $m_2 = 3$, $m_4 = 1$ and all other $m_\ell = 0$ then $\nu = (4, 2, 2, 2, 1)$.

It is well-known (see e.g. [1]) that $\binom{p+m}{m}$ is the number of partitions in a box of size $p \times m$, meaning, that the partition cannot have more than m parts and no part exceeds p. Let $\overline{RC}(N,n)$ be the set of all **rigged configurations** (ν, J) defined as follows. ν is a partition of size $|\nu| = n$ and J is a set of partition where J_ℓ is a partition in a box of size $p_\ell \times m_\ell$. Then (2.8) can be rewritten as

$$(2.9) \qquad Z(N,n) = \sum_{(\nu,J) \in \overline{RC}(N,n)} 1.$$

Example 2.1. Let $N = 5$ and $n = 2$. Then the following is the set of rigged configuration $\overline{RC}(5,2)$
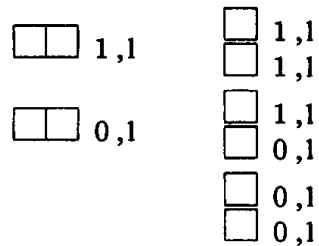
The underlying partition on the left is (2) and on the right (1,1). The partitions J attached to part length ℓ is specified by the first number next to each part. For example, the partition J for the top rigged configuration on the right is (1,1) whereas for the one in the middle and bottom is $J = (1)$ and $J_1 = \emptyset$, respectively. The numbers to the right of part ℓ is p_ℓ.

The rigged configurations introduced in this section correspond to the algebra A_1. In section 4.1, we introduce rigged configurations for the type A_{n-1} algebras and also define a statistics cc which turns (2.9) into a polynomial in q.

3. ONE-DIMENSIONAL CONFIGURATION SUMS AND CRYSTALS

One-dimensional configuration sums are generating functions of crystal elements. A detailed account on crystals can for example be found in [14, 15, 18, 28]. Here we review the main definitions to fix our notation. We restrict ourselves to crystals associated to $\mathfrak{g}$ of type $A^{(1)}_{n-1}$.

A crystal path is an element in the tensor product of crystals $B = B^{r_k,s_k} \otimes B^{r_{k-1},s_{k-1}} \otimes \cdots \otimes B^{r_1,s_1}$, where $B^{r,s}$ is the Kirillov–Reshetikhin crystal labeled by $r \in I = \{1, 2, \ldots, n-1\}$ and $s \in \mathbb{Z}_{>0}$. As a set the crystal $B^{r,s}$ of type $A^{(1)}_{n-1}$ is the set of all column-strict Young tableaux of shape (s^r) over the alphabet $\{1, 2, \ldots, n\}$. Kashiwara [18] introduced the notion of crystals and crystal graphs as a combinatorial means to study representations of quantum algebras. In particular, there are Kashiwara operators e_i, f_i defined on the elements in $B^{r,s}$ for $0 \le i < n$.

We first focus on e_i, f_i when $i \in I$. Let $b = b_k \otimes b_{k-1} \otimes \cdots \otimes b_1 \in B^{r_k,s_k} \otimes B^{r_{k-1},s_{k-1}} \otimes \cdots \otimes B^{r_1,s_1}$. Let $\text{row}(b) = \text{row}(b_k)\text{row}(b_{k-1}) \ldots \text{row}(b_1)$ be the concatenation of the row reading words of b. For a fixed i, consider the subword of $\text{row}(b)$ consisting of i's and $(i+1)$'s only. Successively bracket all pairs $i+1$ i. What is left is a subword of the form $i^a (i+1)^b$. Define

$$e_i(i^a(i+1)^b) = \begin{cases} i^{a+1}(i+1)^{b-1} & \text{if } b > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_i(i^a(i+1)^b) = \begin{cases} i^{a-1}(i+1)^{b+1} & \text{if } a > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.1. Let $b = \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 & 3 \\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2 & 3 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\\hline 3 \\\hline\end{array}$. Then $\text{row}(b) = 23122331$, $e_2(\text{row}(b)) = 23122231$ and $e_1(\text{row}(b)) = 23112331$, so that

$$e_1(b) = \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 & 3 \\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2 & 3 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\\hline 3 \\\hline\end{array}$$

$$e_2(b) = \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 & 3 \\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2 & 2 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\\hline 3 \\\hline\end{array}.$$

There are several sets of paths that will play an important role in the following. For a composition of nonnegative integers λ, the set of **unrestricted paths** is defined as

$$\mathcal{P}(B, \lambda) = \{b \in B \mid \text{wt}(b) = \lambda\}.$$

Here $\text{wt}(b) = (w_1, \ldots, w_n)$ is the weight of b where w_i counts the number of letters i in b. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the set of **classically restricted paths** is defined as

$$\overline{\mathcal{P}}(B, \lambda) = \{b \in B \mid \text{wt}(b) = \lambda, \quad e_i(b) = 0 \text{ for all } 1 \le i < n\}.$$

Example 3.2. For $B = B^{1,1} \otimes B^{2,2} \otimes B^{3,1}$ of type $A^{(1)}_3$ and $\lambda = (3, 3, 1, 1)$ the path

$$b = \begin{array}{|c|}\hline 2 \\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 & 4 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\\hline 2 \\\hline 3 \\\hline\end{array}$$

is in $\overline{\mathcal{P}}(B, \lambda)$.

There is a third set of level-restricted paths. The definition of these paths requires the affine Kashiwara crystal operators e_0 and f_0. The affine Dynkin diagram of type $A^{(1)}_{n-1}$ has a circular symmetry, which looks like a cycle with vertices labeled by $\mathbb{Z}/n\mathbb{Z}$ (see Figure 2). The affine
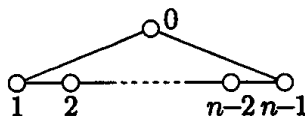
FIGURE 2. Dynkin diagram for $A_{n-1}^{(1)}$

crystal $B^{r,s}$ also has such a symmetry, where the map $i \mapsto i + 1$ (mod n) on the vertices of the Dynkin diagram corresponds to the **promotion** operator pr. Then the action of e_0 and f_0 is given by

$$(3.1) \qquad \begin{aligned} e_0 &= \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr}, \\ f_0 &= \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}. \end{aligned}$$

The promotion operator is a bijection pr : $B \to B$ such that the following diagram commutes for all $i \in \hat{I} = \{0, 1, \ldots, n-1\}$

$$(3.2) \qquad \begin{array}{ccc} B & \xrightarrow{\ \mathrm{pr}\ } & B \\ f_i \downarrow & & \downarrow f_{i+1} \\ B & \xrightarrow[\mathrm{pr}]{} & B \end{array}$$

and such that for every $b \in B$ the weight is rotated

$$(3.3) \qquad \langle h_{i+1}, \mathrm{wt}(pr(b)) \rangle = \langle h_i, \mathrm{wt}(b) \rangle.$$

Here subscripts are taken modulo n.

The promotion operator can be defined combinatorially using jeu de taquin [39]. Let $t \in B^{r,s}$ be a rectangular tableau of shape (s^r). Delete all letters n from t and use jeu de taquin to slide the boxes into the empty spaces until the shape of the new tableau is of skew shape $(s^r)/(\mu_n)$ where μ_n is the number of n in t. Add one to all letters and fill the empty spaces by 1s. The result is $\mathrm{pr}(t)$.

Example 3.3. Suppose $n = 5$ and let

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline 4 & 5 & 5 \\ \hline \end{array}.$$

Then removing the letters 5 and performing jeu de taquin, we obtain

$$\begin{array}{|c|c|c|} \hline \ & \ & 2 \\ \hline 1 & 3 & 3 \\ \hline 2 & 4 & 4 \\ \hline \end{array}.$$

Hence

$$\mathrm{pr}(t) = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 4 & 4 \\ \hline 3 & 5 & 5 \\ \hline \end{array}.$$

Example 3.4. Take t from the previous example. Then

$$f_0(t) = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}(t) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 4 & 4 & 5 \\ \hline \end{array}.$$

The set of **level-ℓ restricted paths** is now defined as

$$\mathcal{P}^\ell(B,\lambda) = \{b \in B \mid \mathrm{wt}(b) = \lambda, \quad e_i(b) = 0 \text{ for all } 1 \le i < n, \quad e_0^{\ell+1}(b) = 0\}.$$

There exists a crystal isomorphism $R : B^{r,s} \otimes B^{r',s'} \to B^{r',s'} \otimes B^{r,s}$, called the **combinatorial R-matrix**. Combinatorially it is given as follows. Let $b \in B^{r,s}$ and $b' \in B^{r',s'}$. The product $b \cdot b'$ of two tableaux is defined as the Schensted insertion of b' into b. Then $R(b \otimes b') = \tilde{b}' \otimes \tilde{b}$ is the unique pair of tableaux such that $b \cdot b' = \tilde{b}' \cdot \tilde{b}$.

The **local energy function** $H : B^{r,s} \otimes B^{r',s'} \to \mathbb{Z}$ is defined as follows. For $b \otimes b' \in B^{r,s} \otimes B^{r',s'}$, $H(b \otimes b')$ is the number of boxes of the shape of $b \cdot b'$ outside the shape obtained by concatenating (s^r) and $(s'^{r'})$.

Example 3.5. For

$$b \otimes b' = \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 & 4 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\\hline 3 \\\hline 4 \\\hline\end{array}$$

we have

$$b \cdot b' = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\\hline 2 & 2 & 4 \\\hline 4 \\\cline{1-1}\end{array} = \begin{array}{|c|}\hline 1 \\\hline 2 \\\hline 4 \\\hline\end{array} \cdot \begin{array}{|c|c|}\hline 1 & 3 \\\hline 2 & 4 \\\hline\end{array} = \tilde{b}' \cdot \tilde{b}.$$

so that

$$R(b \otimes b') = \tilde{b}' \otimes \tilde{b} = \begin{array}{|c|}\hline 1 \\\hline 2 \\\hline 4 \\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\\hline 2 & 4 \\\hline\end{array}.$$

Since the concatenation of ⬛ and ⬛ is ⬛, the local energy function $H(b \otimes b') = 0$.

Now let $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ be a k-fold tensor product of crystals. The **tail energy function** $D : B \to \mathbb{Z}$ is given by

$$D = \sum_{1 \le i < j \le k} H_{j-1} R_{j-2} \cdots R_{i+1} R_i,$$

where H_i (resp. R_i) is the local energy function (resp. combinatorial R-matrix) acting on the i-th and $(i+1)$-th tensor factors.

Definition 3.6. The **one-dimensional configuration sum** is the generating function of the corresponding set of paths graded by the tail energy function

$$X(B,\lambda;q) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{D(b)},$$

$$\overline{X}(B,\lambda;q) = \sum_{b \in \overline{\mathcal{P}}(B,\lambda)} q^{D(b)},$$

$$X^\ell(B,\lambda;q) = \sum_{b \in \mathcal{P}^\ell(B,\lambda)} q^{D(b)}.$$

The generating functions are called unrestricted, classically restricted and level-restricted one-dimensional configuration sums or generalized Kostka polynomials, respectively.

3.1. Open Problems.

- For types other than $A_{n-1}^{(1)}$, the existence of the Kirillov–Reshetikhin crystals $B^{r,s}$ has been conjectured in [14, 15]. The existence of $B^{r,s}$, their combinatorial structure and properties are not yet well-understood in general. For the nonsimply-laced cases, the theory of virtual crystals [30, 31] can be employed to obtain the combinatorial structure of these crystal in terms of the simply-laced cases.

- For types other than $A_{n-1}^{(1)}$, a combinatorial construction of R and D needs to be given.

4. $\overline{X} = \overline{M}$

In this section we consider the $\overline{X} = \overline{M}$ theorem for type $A_{n-1}^{(1)}$, which was proven in [25]. We begin by defining the fermionic formula $\overline{M}(L, \lambda; q)$ in section 4.1 and then describe the bijection $\overline{\Phi} : \overline{\mathcal{P}}(B, \lambda) \to RC(L, \lambda)$ and its properties in sections 4.4 and 4.5.

4.1. Fermionic formulas and rigged configurations.

As before let λ be a partition and $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$. Define the multiplicity array $L = (L_i^{(a)} \mid (a, i) \in \mathcal{H})$ where $L_i^{(a)}$ denotes the number of factors $B^{a,i}$ in B, $\mathcal{H} = I \times \mathbb{Z}_{>0}$ and $I = \{1, 2, \ldots, n - 1\}$. The sequence of partitions $\nu = \{\nu^{(a)} \mid a \in I\}$ is an (L, λ)-**configuration** if

$$(4.1) \qquad \sum_{(a,i) \in \mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{(a,i) \in \mathcal{H}} i L_i^{(a)} \Lambda_a - \lambda,$$

where $m_i^{(a)}$ is the number of parts of length i in partition $\nu^{(a)}$, $\Lambda_a = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_a$ are the fundamental weights and $\alpha_a = \epsilon_a - \epsilon_{a+1}$ are the simple roots of type A_{n-1}. Here ϵ_i is the i-th canonical unit vector of $\mathbb{Z}^n$. The constraint (4.1) is equivalent to the condition

$$(4.2) \qquad |\nu^{(k)}| = \sum_{j>k} \lambda_j - \sum_{a=1}^{L} s_a \max(r_a - k, 0)$$

on the size of $\nu^{(k)}$.

The **vacancy numbers** for the (L, λ)-configuration ν are defined as

$$p_i^{(a)} = \sum_{j \geq 1} \min(i, j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}} (\alpha_a | \alpha_b) \min(i, j) m_j^{(b)},$$

where $(\cdot \mid \cdot)$ is the normalized invariant form on the weight lattice P such that $(\alpha_a \mid \alpha_b)$ is the Cartan matrix. The (L, λ)-configuration ν is **admissible** if $p_i^{(a)} \geq 0$ for all $(i, a) \in \mathcal{H}$, and the set of admissible (L, λ)-configurations is denoted by $\overline{C}(L, \lambda)$. It was proven in [24, Lemma 10] that $p_i^{(a)} \geq 0$ for all existing parts i implies that $p_i^{(a)} \geq 0$ for all i.

Set

$$cc(\nu) = \frac{1}{2} \sum_{a,b \in I} \sum_{j,k \geq 1} (\alpha_a \mid \alpha_b) \min(j, k) m_j^{(a)} m_k^{(b)}.$$

With this notation we define the following fermionic formula. It was first conjectured in [24, 40] that it is an explicit expression for the generalized Kostka polynomials, stemming from the analogous expression of Kirillov and Reshetikhin [23] for the Kostka polynomial. This conjecture was proved in [25, Theorem 2.10].

Definition 4.1 (Fermionic formula). For a multiplicity array L and a partition λ such that $|\lambda| = \sum_{(a,i)\in\mathcal{H}} aiL_i^{(a)}$ define

$$(4.3) \qquad \overline{M}(L,\lambda;q) = \sum_{\nu\in\overline{C}(L,\lambda)} q^{\mathrm{cc}(\nu)} \prod_{(a,i)\in\mathcal{H}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}.$$

Expression (4.3) can be reformulated as the generating function over rigged configurations. To this end we need to define certain labelings of the rows of the partitions in a configuration. For this purpose one should view a partition as a multiset of positive integers. A rigged partition is by definition a finite multiset of pairs (i, x) where i is a positive integer and x is a nonnegative integer. The pairs (i, x) are referred to as strings; i is referred to as the length or size of the string and x as the label or quantum number of the string. A rigged partition is said to be a rigging of the partition ρ if the multiset, consisting of the sizes of the strings, is the partition ρ. So a rigging of ρ is a labeling of the parts of ρ by nonnegative integers, where one identifies labelings that differ only by permuting labels among equal sized parts of ρ.

A rigging J of the (L, λ)-configuration ν is a sequence of riggings of the partitions $\nu^{(a)}$ such that every label x of a part of $\nu^{(a)}$ of size i satisfies the inequalities

$$(4.4) \qquad 0 \le x \le p_i^{(a)}.$$

Alternatively, a rigging of a configuration ν may be viewed as a double-sequence of partitions $J = (J^{(a,i)} \mid (a,i) \in \mathcal{H})$ where $J^{(a,i)}$ is a partition that has at most $m_i^{(a)}$ parts each not exceeding $p_i^{(a)}$. The pair (ν, J) is called a **rigged configuration**. The set of riggings of admissible (L, λ)-configurations is denoted by $\overline{RC}(L, \lambda)$. Let $(\nu, J)^{(a)}$ be the a-th rigged partition of (ν, J). A string $(i, x) \in (\nu, J)^{(a)}$ is said to be **singular** if $x = p_i^{(a)}$, that is, its label takes on the maximum value.

Example 4.2. Let L be the multiplicity array of $B = (B^{1,1})^{\otimes 2} \otimes B^{1,4} \otimes B^{2,1} \otimes B^{2,3}$ and $\lambda = (6, 4, 3, 1)$. Then

$$(\nu, J) = \begin{array}{|c|c|c|}\hline & & \\\hline \end{array} \, 1\,,1 \quad \begin{array}{|c|c|c|}\hline & & \\\hline \end{array} \, 1\,,1 \quad \square \,\, 0\,,0 \in \overline{RC}(L,\lambda),$$

where the first number next to each part is the rigging and the second one is the vacancy number for the corresponding part.

The set of rigged configurations is endowed with a natural statistic cc defined by

$$(4.5) \qquad \mathrm{cc}(\nu, J) = \mathrm{cc}(\nu) + \sum_{(a,i)\in\mathcal{H}} |J^{(a,i)}|$$

for $(\nu, J) \in \overline{RC}(L, \lambda)$, where $|J^{(a,i)}|$ is the size of partition $J^{(a,i)}$. Since the q-binomial $\begin{bmatrix} p+m \\ m \end{bmatrix}$ is the generating function of partitions with at most m parts each not exceeding p, (4.3) can be rewritten as

$$(4.6) \qquad \overline{M}(L,\lambda;q) = \sum_{(\nu,J)\in\overline{RC}(L,\lambda)} q^{\mathrm{cc}(\nu,J)}.$$

The $\overline{X} = \overline{M}$ conjecture asserts that $\overline{M}(L, \lambda; q) = \overline{X}(B, \lambda; q)$ where L is the multiplicity array of B. For type A this was proven in [25] by showing that there is a bijection $\overline{\Phi} : \overline{P}(B, \lambda) \to \overline{RC}(L, \lambda)$ which preserves the statisitics.

Theorem 4.3. [25, Theorem 2.12] *For λ a partition, $B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}$ and L the corresponding multiplicity array such that $|\lambda| = \sum_j r_j s_j$ we have $\overline{M}(L, \lambda; q) = \overline{X}(B, \lambda; q)$.*

4.2. Operations on crystals. To define the bijection $\overline{\Phi}$ we first need to define certain maps on paths and rigged configurations. These maps correspond to the following operations on crystals:

(1) If $B = B^{1,1} \otimes B'$, let $\mathrm{lh}(B) = B'$. This operation is called **left-hat**.

(2) If $B = B^{r,s} \otimes B'$ with $s \geq 2$, let $\mathrm{ls}(B) = B^{r,1} \otimes B^{r,s-1} \otimes B'$. This operation is called **left-split**.

(3) If $B = B^{r,1} \otimes B'$ with $r \geq 2$, let $\mathrm{lb}(B) = B^{1,1} \otimes B^{r-1,1} \otimes B'$. This operation is called **box-split**.

In analogy we define $\mathrm{lh}(L)$ (resp. $\mathrm{ls}(L)$, $\mathrm{lb}(L)$) to be the multiplicity array of $\mathrm{lh}(B)$ (resp. $\mathrm{ls}(B)$, $\mathrm{lb}(B)$), if L is the multiplicity array of B. The corresponding maps on crystal elements are given by:

(1) Let $b = c \otimes b' \in B^{1,1} \otimes B'$. Then $\mathrm{lh}(b) = b'$.

(2) Let $b = c \otimes b' \in B^{r,s} \otimes B'$, where $c = c_1 c_2 \cdots c_s$ and c_i denotes the i-th column of c. Then $\mathrm{ls}(b) = c_1 \otimes c_2 \cdots c_s \otimes b'$.

(3) Let $b = \begin{array}{|c|}\hline b_1 \\\hline b_2 \\\hline \vdots \\\hline b_r \\\hline\end{array} \otimes b' \in B^{r,1} \otimes B'$, where $b_1 < \cdots < b_r$. Then $\mathrm{lb}(b) = \boxed{b_r} \otimes \begin{array}{|c|}\hline b_1 \\\hline \vdots \\\hline b_{r-1} \\\hline\end{array} \otimes b'$.

In the next subsection we define the corresponding maps on rigged configurations, and give the bijection in subsection 4.4.

4.3. Operations on rigged configurations. Suppose $L_1^{(1)} > 0$. The main algorithm on rigged configurations as defined in [23, 25] for admissible rigged configurations is called δ. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, let λ^- be the set of all nonnegative tuples $\mu = (\mu_1, \ldots, \mu_n)$ such that $\lambda - \mu = \epsilon_r$ for some $1 \leq r \leq n$. Define $\delta : \overline{RC}(L, \lambda) \to \bigcup_{\mu \in \lambda^-} \overline{RC}(\mathrm{lh}(L), \mu)$ by the following algorithm. Let $(\nu, J) \in \overline{RC}(L, \lambda)$. Set $\ell^{(0)} = 1$ and repeat the following process for $a = 1, 2, \ldots, n - 1$ or until stopped. Find the smallest index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $\mathrm{rk}(\nu, J) = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a + 1$. Set all undefined $\ell^{(a)}$ to ∞.

The new rigged configuration $(\tilde{\nu}, \tilde{J}) = \delta(\nu, J)$ is obtained by removing a box from the selected strings and making the new strings singular again. Explicitly

$$m_i^{(a)}(\tilde{\nu}) = m_i^{(a)}(\nu) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 0 & \text{otherwise.} \end{cases}$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $p_i^{(a)}(\nu)$ for $i = \ell^{(a)}$, adding a part of size $p_i^{(a)}(\tilde{\nu})$ for $i = \ell^{(a)} - 1$, and leaving it unchanged otherwise. Then $\delta(\nu, J) \in \overline{RC}(\mathrm{lh}(L), \mu)$ where $\mu = \lambda - \epsilon_{\mathrm{rk}(\nu, J)}$.

Example 4.4. Let (ν, J) be the rigged configuration of Example 4.2. Hence $\ell^{(1)} = 1$, $\ell^{(2)} = 3$ and $\ell^{(3)} = \infty$, so that $\mathrm{rk}(\nu, J) = 3$ and

$$\delta(\nu, J) = \begin{array}{|c|c|c|}\hline & & \\\hline\end{array} 1, 1 \quad \begin{array}{cc} \begin{array}{|c|c|}\hline & \\\hline\end{array} & 0, 0 \\ \begin{array}{|c|}\hline \\\hline\end{array} & 0, 0 \end{array} \quad \begin{array}{|c|}\hline \\\hline\end{array} 0, 0.$$

Also $\mathrm{cc}(\nu, J) = 8$.

Let $s \geq 2$. Suppose $B = B^{r,s} \otimes B'$ and L the corresponding multiplicity array. Note that $\overline{C}(L, \lambda) \subset \overline{C}(\mathrm{ls}(L), \lambda)$. Under this inclusion map, the vacancy number $p_i^{(a)}$ for ν increases by $\delta_{a,r} \chi(i < s)$. Hence there is a well-defined injective map $i : \overline{RC}(L, \lambda) \to \overline{RC}(\mathrm{ls}(L), \lambda)$ given by $i(\nu, J) = (\nu, J)$.

Suppose $r \geq 2$ and $B = B^{r,1} \otimes B'$ with multiplicity array L. Then there is an injection $j : \overline{RC}(L, \lambda) \to \overline{RC}(\mathrm{lb}(L), \lambda)$ defined by adding singular strings of length 1 to $(\nu, J)^{(a)}$ for $1 \leq a < r$. Moreover the vacancy numbers stay the same.

4.4. Bijection. The map $\overline{\Phi} : \overline{P}(B, \lambda) \to \overline{RC}(L, \lambda)$ is defined by various commutative diagrams. Note that it is possible to go from $B = B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}$ to the empty crystal via successive application of lh, ls and lb.

Definition 4.5. Define that map $\overline{\Phi} : \overline{P}(B, \lambda) \to \overline{RC}(L, \lambda)$ such that the empty path maps to the empty rigged configuration, and:

(1) Suppose $B = B^{1,1} \otimes B'$. Then the diagram

$$
\begin{array}{ccc}
\overline{P}(B, \lambda) & \xrightarrow{\ \phi\ } & \overline{RC}(L, \lambda) \\
{\scriptstyle \mathrm{lh}}\big\downarrow & & \big\downarrow {\scriptstyle \delta} \\
\displaystyle\bigcup_{\mu \in \lambda^-} \overline{P}(\mathrm{lh}(B), \mu) & \xrightarrow{\ \phi\ } & \displaystyle\bigcup_{\mu \in \lambda^-} \overline{RC}(\mathrm{lh}(L), \mu)
\end{array}
$$

commutes.

(2) Suppose $B = B^{r,s} \otimes B'$ with $s \geq 2$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\overline{P}(B, \lambda) & \xrightarrow{\ \phi\ } & \overline{RC}(L, \lambda) \\
{\scriptstyle \mathrm{ls}}\big\downarrow & & \big\downarrow {\scriptstyle i} \\
\overline{P}(\mathrm{ls}(B), \lambda) & \xrightarrow{\ \phi\ } & \overline{RC}(\mathrm{ls}(L), \lambda)
\end{array}
$$

(3) Suppose $B = B^{r,1} \otimes B'$ with $r \geq 2$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\overline{P}(B, \lambda) & \xrightarrow{\ \phi\ } & \overline{RC}(L, \lambda) \\
{\scriptstyle \mathrm{lb}}\big\downarrow & & \big\downarrow {\scriptstyle j} \\
\overline{P}(\mathrm{lb}(B), \lambda) & \xrightarrow{\ \phi\ } & \overline{RC}(\mathrm{lb}(L), \lambda)
\end{array}
$$

Theorem 4.6. [25] *The map* $\overline{\Phi} : \overline{P}(B, \lambda) \to \overline{RC}(L, \lambda)$ *is a bijection and preserves the statistics, that is,* $D(b) = \mathrm{cc}(\overline{\Phi}(b))$ *for all* $b \in \overline{P}(B, \lambda)$.

Note that Theorem 4.6 immediately implies Theorem 4.3.

Example 4.7. The path which corresponds to (ν, J) of Example 4.2 under $\overline{\Phi}$ is

$$
b = \boxed{3} \otimes \boxed{2} \otimes \boxed{1\ 1\ 3\ 4} \otimes \boxed{\begin{array}{c} 1 \\ 3 \end{array}} \otimes \boxed{\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}} \in \overline{P}(B, \lambda).
$$

We have $D(b) = \mathrm{cc}(\nu, J) = 8$. The steps of Definition 4.5 are summarized in Table 1.

4.5. Properties. As we have already seen in Section 4.4, the bijection $\overline{\Phi}$ preserves the statistics. In addition to this it satisfies a couple of other amazing properties, one of them being the evacuation theorem. The Dynkin diagram of type A_{n-1} has the symmetry τ which interchanges i and $n - i$. There is a corresponding map $*$ on crystals which satisfies

$$
\begin{aligned}
\mathrm{wt}(b^*) &= w_0 \mathrm{wt}(b) \\
e_i(b^*) &= f_{\tau(i)}(b)^* \\
f_i(b^*) &= e_{\tau(i)}(b)^*
\end{aligned}
$$

(4.7)

Step	(ν, J)	b
	⬚⬚ 1,1 ⬚ 1,1 ⬚⬚ 1,1 ⬚ 0,1 ⬚ 0,0	$\boxed{3} \otimes \boxed{2} \otimes \boxed{1\,1\,3\,4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	⬚⬚⬚ 1,1 ⬚⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{2} \otimes \boxed{1\,1\,3\,4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	⬚⬚ 1,1 ⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{1\,1\,3\,4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(2)	⬚⬚ 1,2 ⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{1} \otimes \boxed{1\,3\,4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	⬚⬚ 1,1 ⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{1\,3\,4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(2)	⬚⬚ 1,2 ⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{1} \otimes \boxed{3\,4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	⬚⬚ 1,1 ⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{3\,4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(2)	⬚⬚ 1,1 ⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{3} \otimes \boxed{4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	⬚ 1,1 ⬚ 0,0 / 0,0 ⬚ 0,0	$\boxed{4} \otimes \boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	$\emptyset$ ⬚ 0,0 $\emptyset$	$\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(3)	⬚ 1,1 ⬚ 0,0 $\emptyset$	$\boxed{3} \otimes \boxed{1} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	$\emptyset$ $\emptyset$ $\emptyset$	$\boxed{1} \otimes \boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$
(1)	$\emptyset$ $\emptyset$ $\emptyset$	$\boxed{\begin{smallmatrix}1\,1\,1\\2\,2\,2\end{smallmatrix}}$

TABLE 1. Explicit steps for Example 4.7

for all $i \in I$ where w_0 is the longest permutation of the symmetric group S_{n-1}. Explicitly an element $i \in B^{1,1}$ is mapped to $n + 1 - i$. For $b \in B^{r,s}$, b^* is the tableau obtained by replacing every entry c of b by c^* and then rotating by 180 degrees. The resulting tableau is sometimes called the antitableau of b. For $b = b_k \otimes b_{k-1} \otimes \cdots \otimes b_1 \in B^{r_k,s_k} \otimes B^{r_{k-1},s_{k-1}} \otimes \cdots \otimes B^{r_1,s_1}$ define $b^* = b_1^* \otimes b_2^* \otimes \cdots \otimes b_k^*$.

Example 4.8. For type $A_4^{(1)}$

$$\boxed{\begin{smallmatrix}1\,1\\2\,3\end{smallmatrix}}^* = \boxed{\begin{smallmatrix}3\,4\\5\,5\end{smallmatrix}}.$$

By (4.7) the map $*$ maps classical components to classical components. By weight considerations, these components have to be of the same classical highest weight. Let $\mathrm{ev}(b)$ be the highest weight vector in the same classical component as b^*.

Example 4.9. Let b be the path of Example 4.7. Then

$$b^* = \boxed{\begin{smallmatrix}3\,3\,3\\4\,4\,4\end{smallmatrix}} \otimes \boxed{\begin{smallmatrix}2\\4\end{smallmatrix}} \otimes \boxed{1\,2\,4\,4} \otimes \boxed{3} \otimes \boxed{2}$$

and

$$\text{ev}(b) = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\\hline 2 & 3 & 4 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 2 \\\hline 3 \\\hline\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 2 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\\hline\end{array}.$$

On rigged configurations define θ to be the complementation of quantum numbers. More precisely, if (i, x) is a string in $(\nu, J)^{(k)}$, replace this string by $(i, p_i^{(k)} - x)$. The **Evacuation Theorem** [25, Theorem 5.6] asserts that ev and θ correspond under the bijection $\overline{\Phi}$.

Example 4.10. For (ν, J) of Example 4.2 we have

$$\theta(\nu, J) = \begin{array}{c}\begin{array}{|c|c|c|}\hline & & \\\hline\end{array}\ 0,1 \\ \begin{array}{|c|}\hline \\\hline\end{array}\ 0,1\end{array} \quad \begin{array}{c}\begin{array}{|c|c|c|}\hline & & \\\hline\end{array}\ 0,1 \\ \begin{array}{|c|}\hline \\\hline\end{array}\ 1,1\end{array} \quad \begin{array}{|c|}\hline \\\hline\end{array}\ 0,0$$

and it is easy to check that $\theta(\overline{\Phi}(b)) = \overline{\Phi}(\text{ev}(b))$ with b as in Example 4.7.

The combinatorial R matrix on crystals is the identity on rigged configurations under the bijection $\overline{\Phi}$. See for example [25, Lemma 8.5] or [38, Theorem 8.6]. This shows in particular that the polynomial $\overline{X}(B, \lambda; q)$ does not depend on the order of the tensor factors in B.

Example 4.11. Take b from Example 4.7. Then R_1 is the combinatorial R-matrix applied to the first two tensor factors and

$$R_1(b) = \begin{array}{|c|}\hline 3 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 2 \\\hline\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\\hline 2 & 2 & 3 \\\hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\\hline 2 \\\hline\end{array}.$$

It can be checked that $\overline{\Phi}(b) = \overline{\Phi}(R_1(b)) = (\nu, J)$ is the rigged configuration of Example 4.2.

The bijection $\overline{\Phi}$ is also well-behaved with respect to **transpose duality**. Define

$$\text{tr}_\mathcal{P} : B = B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1} \to B^{s_k, r_k} \otimes B^{s_{k-1}, r_{k-1}} \otimes \cdots \otimes B^{s_1, r_1} =: B^t$$

as follows. For $b = b_k \otimes \cdots \otimes b_1 \in B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$ rotate each rectangular tableau b_i by 90° clockwise to obtain $\tilde{b}_i$. Suppose the letter a occurs in cell c of $\tilde{b}_i$. Then replace letter a in cell c by $\tilde{a}$ where $\tilde{a}$ is chosen such that the letter a in cell c is the $\tilde{a}$-th letter a in row(b) reading from right to left. Since heighest-weight crystal elements are mapped to heighest-weight elements this induces a map

$$\text{tr}_\mathcal{P} : \overline{\mathcal{P}}(B, \lambda) \to \overline{\mathcal{P}}(B^t, \lambda^t).$$

It should be noted that we are assuming here that n is big enough so that both B^{i, s_i} and B^{s_i, r_i} are $A_{n-1}^{(1)}$ crystals.

The analogous map on rigged configurations is

$$\text{tr}_{\text{RC}} : \overline{\text{RC}}(L, \lambda) \to \overline{\text{RC}}(L^t, \lambda^t),$$

where L^t is the multiplicity array of B^t. Let $(\nu, J) \in \overline{\text{RC}}(L, \lambda)$ and let ν have the associated matrix m with entries m_{ai} as in [24, (9.2)]

$$m_{ai} = \sum_{i \le j} m_j^{(a-1)} - m_j^{(a)}.$$

Note that $\sum_{i \le j} m_j^{(a)}$ is the size of the i-th column of the partition $\nu^{(a)}$. Here $m_j^{(0)}$ is defined to be zero. The configuration ν^t in $(\nu^t, J^t) = \text{tr}_{\text{RC}}(\nu, J)$ is defined by its associated matrix m^t given by

$$m_{ai}^t = -m_{ia} + \chi((i, a) \in \lambda) - \sum_{j=1}^k \chi((i, a) \in (s_j^{r_j})).$$

Here $(i, a) \in \lambda$ means that the cell (i, a) is in the Ferrers diagram of the partition λ with i specifying the row and a the column.

Recall that the riggings J can be viewed as a double sequence of partitions $J = (J^{(a,i)})$ where $J^{(a,i)}$ is a partition inside the rectangle of height $m_i^{(a)}$ and width $p_i^{(a)}$. The partition $J^{t(i,a)}$ corresponding to $(\nu^t, J^t) = \mathrm{tr_{RC}}(\nu, J)$ is defined as the transpose of the complementary partition to $J^{(a,i)}$ in the rectangle of height $m_i^{(a)}$ and width $p_i^{(a)}$.

The **Transpose Theorem** [25, Theorem 7.1] asserts that $\overline{\Phi}(\mathrm{tr}_P(b)) = \mathrm{tr_{RC}}(\overline{\Phi}(b))$ for all $b \in \overline{\mathcal{P}}(B, \lambda)$. This implies in particular the transpose symmetry [40, Theorem 7.1] [24, Conjecture 3]

$$\overline{X}(B, \lambda; q) = \widetilde{\overline{X}}(B^t, \lambda^t; q)$$

and similarly for $\overline{M}$, where $\widetilde{\overline{X}}(B, \lambda; q) = q^{n(B)}\overline{X}(B, \lambda; q^{-1})$ and

$$n(B) = \sum_{1 \le i < j \le k} \min(s_i, s_j)\min(r_i, r_j).$$

Example 4.12. As usual let b be the path of Example 4.7. Then

$$\mathrm{tr}_P(b) = \boxed{3} \otimes \boxed{4} \otimes \begin{array}{|c|}\hline 1 \\\hline 2 \\\hline 5 \\\hline 6 \\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 4 \\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 & 2 \\\hline 3 & 3 \\\hline\end{array}.$$

Similarly, let (ν, J) be the rigged configuration of Example 4.2. Then the matrix m and m^t are

$$m = \begin{pmatrix} 2 & -1 & -1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$m^t = \begin{pmatrix} -2 & -1 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 1 & \cdots \\ 1 & 0 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

so that

$$\mathrm{tr_{RC}}(\nu, J) = \begin{array}{l}\begin{array}{|c|c|}\hline & \\\hline\end{array}\\\begin{array}{|c|}\hline \\\hline\end{array}\end{array}\!\!\!{}^{1,1}_{0,1} \;\; \begin{array}{l}\begin{array}{|c|c|}\hline & \\\hline\end{array}\\\begin{array}{|c|}\hline \\\hline\end{array}\end{array}\!\!\!{}^{0,0}_{0,0} \;\; \begin{array}{l}\begin{array}{|c|c|}\hline & \\\hline\end{array}\\\begin{array}{|c|}\hline \\\hline\end{array}\end{array}\!\!\!{}^{0,1}_{0,1} \;\; \begin{array}{l}\begin{array}{|c|}\hline \\\hline\end{array}\\\begin{array}{|c|}\hline \\\hline\end{array}\end{array}\!\!\!{}^{0,0}_{0,0} \;\; \begin{array}{|c|}\hline \\\hline\end{array}{}^{0,0}.$$

It can be checked explicitly in this example that $\overline{\Phi}(\mathrm{tr}_P(b)) = \mathrm{tr_{RC}}(\overline{\Phi}(b))$.

Finally let us mention the **contragredient duality** which is of great importance for the notion of virtual crystals [30, 31]. On crystals define the map

$$\vee : B^{r,s} \to B^{n-r,s}$$

where each column $c = c_1 \ldots c_r$ of $b \in B^{r,s}$ is replaced by column $(n+1 - d_{n-r}) \ldots (n+1 - d_1)$ where $\{d_1 < d_2 < \cdots < d_{n-r}\}$ is the complement of $\{c_1 < c_2 < \ldots < c_r\}$ in $\{1, 2, \ldots, n\}$. Note that $e_i(b)^\vee = f_{n-i}(b^\vee)$.

Example 4.13. The contragredient dual of $b = \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 & 3 \\\hline\end{array}$ for $n = 4$ is $b^\vee = \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 & 4 \\\hline\end{array}$.

The map $\vee$ can be extended to a map on paths

$$\vee : \overline{\mathcal{P}}(B, \lambda) \to \overline{\mathcal{P}}(B^\vee, \lambda^\vee),$$

where $B^\vee = B^{n-r_k, s_k} \otimes \cdots \otimes B^{n-r_1, s_1}$, $\lambda^\vee = (N - \lambda_n, N - \lambda_{n-1}, \ldots, N - \lambda_1)$ and $N = s_1 + \cdots + s_k$, by mapping $b = b_k \otimes \cdots \otimes b_1$ to $b^\vee = b_k^\vee \otimes \cdots \otimes b_1^\vee$.
For given n, define

$$\text{rev} : \overline{\text{RC}}(L, \lambda) \to \overline{\text{RC}}(L^\vee, \lambda^\vee)$$

such that for $(\nu^\vee, J^\vee) = \text{rev}(\nu, J)$ we have $(\nu^\vee, J^\vee)^{(a)} = (\nu, J)^{(n-a)}$. Then we have [30, Theorem 5.7] that $\overline{\Phi}(b^\vee) = \text{rev}(\overline{\Phi}(b))$ for all $b \in \overline{\mathcal{P}}(B, \lambda)$. This implies the contragredient symmetry

$$\overline{X}(B^\vee, \lambda^\vee; q) = \overline{X}(B, \lambda; q)$$

and similarly for $\overline{M}$.

Example 4.14. Employing one last time b of Example 4.7 we obtain

$$b^\vee = \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline 3 & 3 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline \end{array}$$

and for (ν, J) of Example 4.2

$$\text{rev}(\nu, J) = \Box \ _{0,0} \quad \begin{array}{c} \square\square\square \\ \square \end{array} \ _{0,1}^{1,1} \quad \begin{array}{c} \square\square\square \\ \square \end{array} \ _{1,1}^{1,1}$$

which is also $\overline{\Phi}(b^\vee)$.

The bijection $\overline{\Phi}$ has further properties. For example it is well-behaved under certain embeddings. We refer the interested reader to the literature [25, 40, 24, 38].

4.6. Open Problems.

- For nonexceptional types, the bijection $\overline{\Phi}$ was given in [29, 38] for the cases $B = B^{1, s_k} \otimes \cdots \otimes B^{1, s_1}$ and for type $D_n^{(1)}$ in the case $B = B^{r_k, 1} \otimes \cdots \otimes B^{r_1, 1}$ [34]. For all other cases, it is still an outstanding problem to prove that $\overline{\Phi}$ exists. In particular, the analogues of the splitting maps need to be found.

- It would be very nice to have a more conceptual definition of the bijection $\overline{\Phi}$ rather than the recursive definition in terms of the splitting and hatting maps. A possible avenue would be to give a definition of $\overline{\Phi}$ in terms of the affine crystal structure on rigged configurations. In section 5 we provide such a crystal structure for $B^{r,s}$ of type $A_{n-1}^{(1)}$. To obtain $\overline{\Phi}$, one would need the affine crystal structure on tensor products $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$. Compare with section 5.5.

5. $X = M$

In this section we deal with the unrestricted version of the $X = M$ conjecture for type $A_{n-1}^{(1)}$. In particular it is our aim to find a fermionic formula for the unrestricted configuration sum $X(B, \lambda; q)$ of Definition 3.6. This has recently been achieved in [35] by extending the set of rigged configurations to the set of unrestricted rigged configurations by imposing a crystal structure in this set. A direct bijection between unrestricted paths and unrestricted rigged configurations along the lines of Definition 4.5 was given in [10]. Here we mostly follow [35] and derive the fermionic formula $M(B, \lambda; q)$ from the crystal structure on rigged configurations.

5.1. Crystal structure on rigged configurations. The set of unrestricted rigged configurations $RC(L)$ can be introduced by defining a crystal structure generated from highest weight vectors given by elements in $\overline{RC}(L) = \bigcup_\lambda \overline{RC}(L, \lambda)$ by the Kashiwara operators e_a, f_a.

Definition 5.1. Let L be a multiplicity array. Define the set of **unrestricted rigged configurations** $RC(L)$ as the set generated from the elements in $\overline{RC}(L)$ by the application of the operators f_a, e_a for $a \in I$ defined as follows:

(1) Define $e_a(\nu, J)$ by removing a box from a string of length k in $(\nu, J)^{(a)}$ leaving all colabels fixed and increasing the new label by one. Here k is the length of the string with the smallest negative rigging of smallest length. If no such string exists, $e_a(\nu, J)$ is undefined.

(2) Define $f_a(\nu, J)$ by adding a box to a string of length k in $(\nu, J)^{(a)}$ leaving all colabels fixed and decreasing the new label by one. Here k is the length of the string with the smallest nonpositive rigging of largest length. If no such string exists, add a new string of length one and label -1. If the result is not a valid unrestricted rigged configuration $f_a(\nu, J)$ is undefined.

Let $(\nu, J) \in RC(L)$. If f_a adds a box to a string of length k in $(\nu, J)^{(a)}$, then the vacancy numbers change according to

$$(5.1) \qquad p_i^{(b)} \mapsto p_i^{(b)} - (\alpha_a | \alpha_b) \chi(i > k),$$

where $\chi(S) = 1$ if the statement S is true and $\chi(S) = 0$ if S is false. Similarly, if e_a adds a box of length k to $(\nu, J)^{(a)}$, then the vacancy numbers change as

$$p_i^{(b)} \mapsto p_i^{(b)} + (\alpha_a | \alpha_b) \chi(i \geq k).$$

We may define a weight function $\mathrm{wt} : RC(L) \to P$ as

$$(5.2) \qquad \mathrm{wt}(\nu, J) = \sum_{(a,i) \in \mathcal{H}} i(L_i^{(a)} \Lambda_a - m_i^{(a)} \alpha_a)$$

for $(\nu, J) \in RC(L)$. It is clear from the definition that $\mathrm{wt}(f_a(\nu, J)) = \mathrm{wt}(\nu, J) - \alpha_a$. Define

$$RC(L, \lambda) = \{(\nu, J) \in RC(L) \mid \mathrm{wt}(\nu, J) = \lambda\}.$$

Example 5.2. Let $\mathfrak{g}$ be of type $A_2^{(1)}$. Let $\lambda = (3, 2, 3)$, $L_1^{(1)} = L_3^{(1)} = L_2^{(2)} = 1$ and all other $L_i^{(a)} = 0$. Then

is in $RC(L, \lambda)$, where the parts of the rigging $J^{(a,i)}$ are written next to the parts of length i in partition $\nu^{(a)}$. We have

Example 5.3. Let $\mathfrak{g}$ be of type $A_2^{(1)}$. Let $\lambda = (4, 5, 6)$, $L_1^{(1)} = 15$ and all other $L_i^{(a)} = 0$. Then

is in $\mathrm{RC}(L, \lambda)$. We have

$$e_1(\nu, J) = \boxed{} \quad \text{and} \quad e_2(\nu, J) = \boxed{} \ .$$

The following Theorem was proven in [35] for all simply-laced algebras.

Theorem 5.4. [35, Theorem 3.7] *The graph generated from* $(\overline{\nu, J}) \in \overline{\mathrm{RC}}(L, \lambda)$ *and the crystal operators* e_a, f_a *of Definition 5.1 is isomorphic to the crystal graph* $B(\lambda)$ *of highest weight* λ.

Example 5.5. Consider the crystal $B(\boxed{})$ of type A_2 in $B = (B^{1,1})^{\otimes 3}$. Here is the crystal graph in the usual labeling and the rigged configuration labeling:
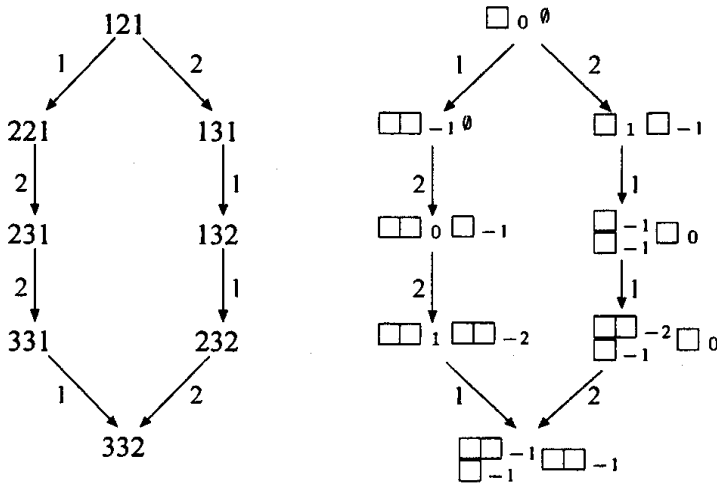
Theorem 5.6. [35, Theorem 3.9] *The cocharge* cc *as defined in* (4.5) *is constant on connected crystal components.*

Example 5.7. The cocharge of the connected component in Example 5.5 is 1.

Combining the various results yields a generalization of Theorem 4.6.

Theorem 5.8. [35, Theoren 3.10] *Let* λ *be a composition,* B *be as in Theorem 4.6 and* L *the corresponding multiplicity array. Then there is a bijection* $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L, \lambda)$ *which preserves the statistics, that is,* $D(b) = \mathrm{cc}(\Phi(b))$ *for all* $b \in \mathcal{P}(B, \lambda)$.

Proof. By Theorem 4.6 there is such a bijection for the maximal elements $b \in \overline{\mathcal{P}}(B)$. By Theorems 5.4 and 5.6 this extends to all of $\mathcal{P}(B, \lambda)$. $\square$

Extending the definition of (4.6) to

$$(5.3) \qquad M(L, \lambda; q) = \sum_{(\nu, J) \in \mathrm{RC}(L, \lambda)} q^{\mathrm{cc}(\nu, J)},$$

we obtain the corollary:

Corollary 5.9. [35, Corollary 3.10] *With all hypotheses of Theorem 5.8, we have* $X(B, \lambda; q) = M(L, \lambda; q)$.

Example 5.10. Let $n = 4$, $B = B^{2,2} \otimes B^{2,1}$ and $\lambda = (2,2,1,1)$. Then the multiplicity array is $L_1^{(2)} = 1, L_2^{(2)} = 1$ and $L_i^{(a)} = 0$ for all other (a,i). There are 7 possible unrestricted paths in $\mathcal{P}(B,\lambda)$. For each path $b \in \mathcal{P}(B,\lambda)$ the corresponding rigged configuration $(\nu, J) = \Phi(b)$ together with the tail energy and cocharge is summarized below.

$$b = \boxed{\begin{array}{cc}1&1\\2&2\end{array}} \otimes \boxed{\begin{array}{c}3\\4\end{array}} \quad (\nu, J) = \square\,0 \quad \boxed{\begin{array}{c}-1\\-1\end{array}} \quad \square\,0 \quad D(b) = 0 = \mathrm{cc}(\nu, J)$$

$$b = \boxed{\begin{array}{cc}1&1\\2&4\end{array}} \otimes \boxed{\begin{array}{c}2\\3\end{array}} \quad (\nu, J) = \square\,-1 \quad \boxed{\begin{array}{c}0\\0\end{array}} \quad \square\,0 \quad D(b) = 1 = \mathrm{cc}(\nu, J)$$

$$b = \boxed{\begin{array}{cc}1&2\\2&3\end{array}} \otimes \boxed{\begin{array}{c}1\\4\end{array}} \quad (\nu, J) = \square\,0 \quad \boxed{\begin{array}{c}0\\0\end{array}} \quad \square\,-1 \quad D(b) = 1 = \mathrm{cc}(\nu, J)$$

$$b = \boxed{\begin{array}{cc}1&2\\2&4\end{array}} \otimes \boxed{\begin{array}{c}1\\3\end{array}} \quad (\nu, J) = \square\,0 \quad \boxed{\begin{array}{c}0\\-1\end{array}} \quad \square\,0 \quad D(b) = 1 = \mathrm{cc}(\nu, J)$$

$$b = \boxed{\begin{array}{cc}1&3\\2&4\end{array}} \otimes \boxed{\begin{array}{c}1\\2\end{array}} \quad (\nu, J) = \square\,0 \quad \boxed{\begin{array}{c}0\\0\end{array}} \quad \square\,0 \quad D(b) = 2 = \mathrm{cc}(\nu, J)$$

$$b = \boxed{\begin{array}{cc}1&1\\2&3\end{array}} \otimes \boxed{\begin{array}{c}2\\4\end{array}} \quad (\nu, J) = \square\,-1 \quad \boxed{\;\;}\,0 \quad \square\,-1 \quad D(b) = 0 = \mathrm{cc}(\nu, J)$$

$$b = \boxed{\begin{array}{cc}1&2\\3&4\end{array}} \otimes \boxed{\begin{array}{c}1\\2\end{array}} \quad (\nu, J) = \square\,-1 \quad \boxed{\;\;}\,1 \quad \square\,-1 \quad D(b) = 1 = \mathrm{cc}(\nu, J)$$

The unrestricted Kostka polynomial in this case is $M(L, \lambda; q) = 2 + 4q + q^2 = X(B, \lambda; q)$.

5.2. Characterization of unrestricted rigged configurations. In this section we give an explicit description of the elements in $\mathrm{RC}(L, \lambda)$ for type $A_{n-1}^{(1)}$. Generally speaking, the elements are rigged configurations where the labels lie between the vacancy number and certain lower bounds defined explicitly. This characterization will be used in the next section to write down an explicit fermionic formula $M(L, \lambda; q)$ for the unrestricted configuration sum $X(B, \lambda; q)$.

Let $L = (L_i^{(a)} \mid (a,i) \in \mathcal{H})$ be a multiplicity array and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the n-tuple of nonnegative integers. The set of (L, λ)-configurations $C(L, \lambda)$ is the set of all sequences of partitions $\nu = (\nu^{(a)} \mid a \in I)$ such that (4.1) holds. As discussed in Section 4.1, in the usual setting a rigged configuration $(\nu, J) \in \overline{\mathrm{RC}}(L, \lambda)$ consists of a configuration $\nu \in \overline{C}(L, \lambda)$ together with a double sequence of partitions $J = \{J^{(a,i)} \mid (a,i) \in \mathcal{H}\}$ such that the partition $J^{(a,i)}$ is contained in a $m_i^{(a)} \times p_i^{(a)}$ rectangle. In particular this requires that $p_i^{(a)} \geq 0$. The unrestricted rigged configurations $(\nu, J) \in \mathrm{RC}(L, \lambda)$ can contain labels that are negative, that is, the lower bound on the parts in $J^{(a,i)}$ can be less than zero.

To define the lower bounds we need the following notation. Let $\mathcal{X} = (c_1, c_2, \ldots, c_{n-1})^t$, where $c_k = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_n$ is the length of the k-th column of $\mathcal{X}$, and let $\mathcal{A}(\lambda')$ be the set of tableaux of shape λ' such that the entries are strictly decreasing along columns, and the letters in column k are from the set $\{1, 2, \ldots, c_{k-1}\}$ with $c_0 = c_1$.

Example 5.11. For $n = 4$ and $\lambda = (0,1,1,1)$, the set $\mathcal{A}(\lambda')$ consists of the following tableaux

$$\boxed{\begin{array}{ccc}3&3&2\\2&2&\\1&&\end{array}} \quad \boxed{\begin{array}{ccc}3&3&2\\2&1&\\1&&\end{array}} \quad \boxed{\begin{array}{ccc}3&2&2\\2&1&\\1&&\end{array}} \quad \boxed{\begin{array}{ccc}3&3&1\\2&2&\\1&&\end{array}} \quad \boxed{\begin{array}{ccc}3&3&1\\2&1&\\1&&\end{array}} \quad \boxed{\begin{array}{ccc}3&2&1\\2&1&\\1&&\end{array}}.$$

Remark 5.12. Denote by $t_{j,k}$ the entry of $t \in \mathcal{A}(\lambda')$ in row j and column k. Note that $c_k - j + 1 \leq t_{j,k} \leq c_{k-1} - j + 1$ since the entries in column k are strictly decreasing and lie in the set $\{1, 2, \ldots, c_{k-1}\}$. This implies $t_{j,k} \leq c_{k-1} - j + 1 \leq t_{j,k-1}$, so that the rows of t are weakly decreasing.

Given $t \in \mathcal{A}(\lambda')$, we define the **lower bound** as

$$M_i^{(a)}(t) = -\sum_{j=1}^{c_a} \chi(i \geq t_{j,a}) + \sum_{j=1}^{c_{a+1}} \chi(i \geq t_{j,a+1}),$$

where recall that $\chi(S) = 1$ if the the statement S is true and $\chi(S) = 0$ otherwise.

Let $M, p, m \in \mathbb{Z}$ such that $m \geq 0$. A (M, p, m)-quasipartition μ is a tuple of integers $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ such that $M \leq \mu_m \leq \mu_{m-1} \leq \cdots \leq \mu_1 \leq p$. Each μ_i is called a part of μ. Note that for $M = 0$ this would be a partition with at most m parts each not exceeding p.

The following theorem shows that the set of unrestricted rigged configurations can be characterized via the lower bounds.

Theorem 5.13. [35, Theorem 4.6] *Let* $(\nu, J) \in \mathrm{RC}(L, \lambda)$. *Then* $\nu \in \mathrm{C}(L, \lambda)$ *and* $J^{(a,i)}$ *is a* $(M_i^{(a)}(t), p_i^{(a)}, m_i^{(a)})$-*quasipartition for some* $t \in \mathcal{A}(\lambda')$. *Conversely, every* (ν, J) *such that* $\nu \in \mathrm{C}(L, \lambda)$ *and* $J^{(a,i)}$ *is a* $(M_i^{(a)}(t), p_i^{(a)}, m_i^{(a)})$-*quasipartition for some* $t \in \mathcal{A}(\lambda')$ *is in* $\mathrm{RC}(L, \lambda)$.

Example 5.14. Let $n = 4$, $\lambda = (2, 2, 1, 1)$, $L_1^{(1)} = 6$ and all other $L_i^{(a)} = 0$. Then

$$(\nu, J) \quad = \quad \boxed{}\ -2 \quad \boxed{}\ 0 \quad \boxed{}\ -1$$

is an unrestricted rigged configuration in $\mathrm{RC}(L, \lambda)$, where we have written the parts of $J^{(a,i)}$ next to the parts of length i in partition $\nu^{(a)}$. To see that the riggings form quasipartitions, let us write the vacancy numbers $p_i^{(a)}$ next to the parts of length i in partition $\nu^{(a)}$:

$$\boxed{}\ 0 \quad \boxed{}\ 0 \quad \boxed{}\ -1.$$

This shows that the labels are indeed all weakly below the vacancy numbers. For

$$\begin{array}{|c|c|c|}
\hline
4 & 4 & 1 \\
\hline
3 & 3 \\
\cline{1-2}
2 \\
\cline{1-1}
1 \\
\cline{1-1}
\end{array} \in \mathcal{A}(\lambda')$$

we get the lower bounds

$$\boxed{}\ -2 \quad \boxed{}\ 0 \quad \boxed{}\ -1,$$

which are less or equal to the riggings in (ν, J).

For type A_1 we have $\lambda = (\lambda_1, \lambda_2)$ so that $\mathcal{A} = \{t\}$ contains just the single tableau

$$t = \begin{array}{|c|}
\hline
\lambda_2 \\
\hline
\lambda_2 - 1 \\
\hline
\vdots \\
\hline
1 \\
\hline
\end{array}.$$

In this case $M_i(t) = -\sum_{j=1}^{\lambda_2} \chi(i \geq t_{j,1}) = -i$. This agrees with the findings of [42].

As we will see in section 6 the characterization of unrestricted rigged configurations is similar to the characterization of level-restricted rigged configurations [37, Definition 5.5]. Whereas the unrestricted rigged configurations are characterized in terms of lower bounds, for level-restricted rigged configurations the vacancy number has to be modified according to tableaux in a certain set.

5.3. Fermionic formula. With the explicit characterization of the unrestricted rigged configurations of Section 5.2, it is possible to derive an explicit formula for the polynomials $M(L, \lambda)$ of (5.3).

Let $\mathcal{SA}(\lambda')$ be the set of all nonempty subsets of $\mathcal{A}(\lambda')$ and set

$$M_i^{(a)}(S) = \max\{M_i^{(a)}(t) \mid t \in S\} \quad \text{for } S \in \mathcal{SA}(\lambda').$$

By inclusion-exclusion the set of all allowed riggings for a given $\nu \in C(L, \lambda)$ is

$$\bigcup_{S \in \mathcal{SA}(\lambda')} (-1)^{|S|+1}\{J \mid J^{(a,i)} \text{ is a } (M_i^{(a)}(S), p_i^{(a)}, m_i^{(a)})\text{-quasipartition}\}.$$

The q-binomial coefficient $\begin{bmatrix} m+p \\ m \end{bmatrix}$, defined as

$$\begin{bmatrix} m + p \\ m \end{bmatrix} = \frac{(q)_{m+p}}{(q)_m (q)_p},$$

where $(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$, is the generating function of partitions with at most m parts each not exceeding p. Hence the polynomial $M(L, \lambda)$ may be rewritten as

$$(5.4) \quad M(L, \lambda; q) = \sum_{S \in \mathcal{SA}(\lambda')} (-1)^{|S|+1} \sum_{\nu \in C(L,\lambda)} q^{cc(\nu) + \sum_{(a,i) \in \mathcal{H}} m_i^{(a)} M_i^{(a)}(S)}$$

$$\times \prod_{(a,i) \in \mathcal{H}} \begin{bmatrix} m_i^{(a)} + p_i^{(a)} - M_i^{(a)}(S) \\ m_i^{(a)} \end{bmatrix}$$

called **fermionic formula**. By Corollary 5.9 this is also a formula for the unrestricted configuration sum $X(B, \lambda; q)$. This formula is different from the fermionic formulas of [13, 19] which exist in the special case when L is the multiplicity array of $B = B^{1,s_k} \otimes \cdots \otimes B^{1,s_1}$ or $B = B^{r_k,1} \otimes \cdots \otimes B^{r_1,1}$.

5.4. The Kashiwara operators e_0 and f_0. The Kirillov–Reshetikhin crystals $B^{r,s}$ are affine crystals and admit the Kashiwara operators e_0 and f_0. As we have seen in (3.1) they can be defined in terms of the **promotion operator** pr as

$$e_0 = \text{pr}^{-1} \circ e_1 \circ \text{pr} \quad \text{and} \quad f_0 = \text{pr}^{-1} \circ f_1 \circ \text{pr}.$$

We are now going to define the promotion operator on unrestricted rigged configurations.

Definition 5.15. Let $(\nu, J) \in \text{RC}(L, \lambda)$. Then $\text{pr}(\nu, J)$ is obtained as follows:

(1) Set $(\nu', J') = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_n^{\lambda_n}(\nu, J)$ where f_n acts on $(\nu, J)^{(n)} = \emptyset$.

(2) Apply the following algorithm ρ to (ν', J') λ_n times: Find the smallest singular string in $(\nu', J')^{(n)}$. Let the length be $\ell^{(n)}$. Repeatedly find the smallest singular string in $(\nu', J')^{(k)}$ of length $\ell^{(k)} \geq \ell^{(k+1)}$ for all $1 \leq k < n$. Shorten the selected strings by one and make them singular again.

Example 5.16. Let $B = B^{2,2}$, L the corresponding multiplicity array and $\lambda = (1, 0, 1, 2)$. Then

$$(\nu, J) = \square \, 0 \quad \begin{array}{c} \boxed{} \\ \boxed{} \end{array} \begin{array}{c} -1 \\ -1 \end{array} \quad \boxed{} \, -1 \quad \in \text{RC}(L, \lambda)$$

corresponds to the tableau $b = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 & 4 \\ \hline \end{array} \in \mathcal{P}(B, \lambda)$. After step (1) of Definition 5.15 we have

$$(\nu', J') = \boxed{} \, -1 \quad \begin{array}{c} \boxed{} \\ \boxed{} \end{array} \, 1 \, 0 \quad \begin{array}{c} \boxed{} \\ \boxed{} \end{array} \, -1 \, -1 \quad \boxed{} \, -1.$$

Then applying step (2) yields

$$\mathrm{pr}(\nu, J) = \quad \emptyset \quad \boxed{}_0 \quad \boxed{}_{-1}$$

which corresponds to the tableau $\mathrm{pr}(b) = \begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 & 4 \\\hline\end{array}$.

Lemma 5.17. [35, Lemma 4.10] *The map* pr *of Definition 5.15 is well-defined and satisfies* (3.2) *for* $1 \le a \le n - 2$ *and* (3.3) *for* $0 \le a \le n - 1$.

Lemma 7 of [39] states that for a single Kirillov–Reshetikhin crystal $B = B^{r,s}$ the promotion operator pr is uniquely determined by (3.2) for $1 \le a \le n - 2$ and (3.3) for $0 \le a \le n - 1$. Hence by Lemma 5.17 pr on $\mathrm{RC}(L)$ is indeed the correct promotion operator when L is the multiplicity array of $B = B^{r,s}$.

Theorem 5.18. [35, Theorem 4.11] *Let* L *be the multiplicity array of* $B = B^{r,s}$. *Then* $\mathrm{pr} :$ $\mathrm{RC}(L) \to \mathrm{RC}(L)$ *of Definition 5.15 is the promotion operator on rigged configurations.*

Conjecture 5.19. [35, Conjecture 4.12] *Theorem 5.18 is true for any* $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$.

Unfortunately, the characterization [39, Lemma 7] does not suffice to define pr uniquely on tensor products $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$.

5.5. Open Problems.

- In [10] a bijection $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L, \lambda)$ is defined via a direct algorithm. It is expected that Conjecture 5.19 can be proven by showing that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}(B) & \xrightarrow{\ \Phi\ } & \mathrm{RC}(L) \\
{\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}} \\
\mathcal{P}(B) & \xrightarrow[\ \Phi\ ]{} & \mathrm{RC}(L).
\end{array}
$$

Alternatively, an independent characterization of pr on tensor factors would give a new, more conceptual way of defining the bijection Φ between paths and (unrestricted) rigged configurations. A proof that the crystal operators f_a and e_a commute with Φ for $a = 1, 2, \ldots, n - 1$ is given in [10].

- Stembridge's local characterization of simply-laced crystals [41] was used in [35] to show that f_a and e_a of Definition 5.1 are in fact crystal operators. For nonsimply-laced types a local characterization of crystals is not known yet. It can be shown via virtual crystals what the crystal operators are in this case. See for example [30, 31, 36]

- Hatayama et al. [13] derived a different fermionic formula $M(L, \lambda; q)$ for the cases $B = B^{1, s_k} \otimes \cdots \otimes B^{1, s_1}$ and $B = B^{r_k, 1} \otimes \cdots \otimes B^{r_1, 1}$. In [32] this formula was interpreted in terms of "ribbon" rigged configurations. It would be very interesting to relate the two fermionic formulas, in particular the two different rigged configurations. As the fermionic formula of [13] is a special case of the Lascoux–Leclerc–Thibon (LLT) spin generating function [26], this would yield a proof of a conjecture by Kirillov and Shimozono [24, Conjecture 5] that the LLT spin generating function labeled by a partition whose k-quotient is a sequence of rectangles is the same as the unrestricted generalized Kostka polynomial $X(B, \lambda; q)$.

- The unrestricted rigged configurations for the A_1 case also appeared in a paper by Takagi [42] in the study of box-ball systems. A similar link should be given for the general A_{n-1} case.

- Bailey's lemma is a powerful tool to prove Rogers–Ramanujan-type identities. Andrews [2] showed that Bailey's lemma has an iterative structure which relies on a transformation property of the q-binomial coefficients. This iterative structure allows to derive infinite families of Rogers–Ramanujan identities from a single seed identity. Since the unrestricted configuration sums X yield a generalization of the q-binomial coefficients, it is expected that they also satisfy certain transformation properties which would give rise to a Bailey lemma. For type A_2 this has been achieved in [4]. The explicit formula M for the unrestricted configuration sum might trigger further progress on generalizations of the Bailey lemma to higher rank and other types.

- For type $D_n^{(1)}$, a simple characterization in terms of lower bounds for the parts of a configuration $\nu \in C(L)$ does not seem to exist. For example take $B = B^{2,1}$ of type $D_4^{(1)}$ so that $L_1^{(2)} = 1$ and all other $L_i^{(a)} = 0$. Then the unrestricted rigged configurations

$$\boxed{}\, 0 \quad \boxed{\begin{matrix} 0 \\ 0 \end{matrix}}\, \quad \boxed{}\, 0 \quad \boxed{}\, 0 \quad \text{and} \quad \boxed{}\, 0 \quad \boxed{\begin{matrix} 0 \\ -1 \end{matrix}}\, \quad \boxed{}\, 0 \quad \boxed{}\, 0\,,$$

which correspond to the crystal elements $\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}$ and $\boxed{\begin{matrix} 3 \\ 3 \end{matrix}}$ respectively, occur in $\mathrm{RC}(L)$, but

$$\boxed{}\, 0 \quad \boxed{\begin{matrix} -1 \\ -1 \end{matrix}}\, \quad \boxed{}\, 0 \quad \boxed{}\, 0$$

on the other hand does not appear. It remains to determine a closed form fermionic expression in this case.

6. $X^\ell = M^\ell$

The fermionic formula for the level-restricted $X^\ell = M^\ell$ theorem has a similar structure to the unrestricted fermionic formula. Instead of modifying the lower bounds for the rigged configurations, the upper bounds are adapted.

6.1. Level-restricted rigged configurations.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is restricted of level ℓ if $\lambda_1 - \lambda_n \le \ell$. Here λ has at most n parts, some of which may be zero. Fix a shape λ that is restricted of level ℓ and let L be a multiplicity array such that $L_i^{(a)} = 0$ if $i > \ell$. Call such a multiplicity array level-ℓ restricted. Define $\widetilde{\ell} = \ell - (\lambda_1 - \lambda_n)$, which is nonnegative by assumption.

Set $\lambda' = (\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)^t$ and denote the set of all column-strict tableaux of shape λ' over the alphabet $\{1, 2, \ldots, \lambda_1 - \lambda_n\}$ by $\mathrm{CST}(\lambda')$. Define a table of modified vacancy numbers depending on $\nu \in C(L, \lambda)$ and $t \in \mathrm{CST}(\lambda')$ by

$$(6.1) \qquad p_i^{(k)}(t) = p_i^{(k)} - \sum_{j=1}^{\lambda_k - \lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k}) + \sum_{j=1}^{\lambda_{k+1} - \lambda_n} \chi(i \ge \widetilde{\ell} + t_{j,k+1})$$

for all $i, k \ge 1$ and $t_{j,k}$ is the (j, k)-th entry of t. Finally let $x_i^{(k)}$ be the largest part of the partition $J^{(k,i)}$; if $J^{(k,i)}$ is empty set $x_i^{(k)} = 0$.

Definition 6.1. Say that $(\nu, J) \in \overline{\mathrm{RC}}(L, \lambda)$ is restricted of level ℓ provided that

(1) $\nu_1^{(k)} \le \ell$ for all k.
(2) There exists a tableau $t \in \mathrm{CST}(\lambda')$, such that for every $i, k \ge 1$,

$$x_i^{(k)} \le p_i^{(k)}(t).$$

Let $C^\ell(L, \lambda)$ be the set of all $\nu \in \overline{C}(L, \lambda)$ such that the first condition holds, and denote by $\mathrm{RC}^\ell(L, \lambda)$ the set of $(\nu, J) \in \overline{\mathrm{RC}}(L, \lambda)$ that are restricted of level ℓ.

Note in particular that the second condition requires that $p_i^{(k)}(t) \geq 0$ for all $i, k \geq 1$.

Example 6.2. Let us consider Definition 6.1 for two classes of shapes λ more closely:

(1) Vacuum case: Let $\lambda = (a^n)$ be rectangular with n rows. Then $\lambda' = \emptyset$ and $p_i^{(k)}(\emptyset) = p_i^{(k)}$ for all $i, k \geq 1$ so that the modified vacancy numbers are equal to the vacancy numbers.

(2) Two-corner case: Let $\lambda = (a^\alpha, b^\beta)$ with $\alpha + \beta = n$ and $a > b$. Then $\lambda' = (a^{a-b})$ and there is only one tableau t in $\mathrm{CST}(\lambda')$, namely the Yamanouchi tableau of shape λ'. Since $t_{j,k} = j$ for $1 \leq k \leq \alpha$ we find that

$$p_i^{(k)}(t) = p_i^{(k)} - \delta_{k,\alpha} \max\{i - \tilde{\ell}, 0\}$$

for $1 \leq i \leq \ell$ and $1 \leq k < n$.

We define the level-restricted rigged configuration generating function as

$$(6.2) \qquad M^\ell(L, \lambda; q) = \sum_{(\nu, J) \in \mathrm{RC}^\ell(L, \lambda)} q^{\mathrm{cc}(\nu, J)}.$$

The $X^\ell = M^\ell$ conjecture was proven in [37].

Theorem 6.3. [37, Theorem 5.7] *For a level-ℓ restricted partition λ and a level-ℓ restricted multiplicity array L we have $X^\ell(B, \lambda; q) = M^\ell(L, \lambda; q)$.*

Example 6.4. Consider $n = 3$, $\ell = 2$, $\lambda = (3, 2, 1)$, $L_1^{(1)} = 4$, $L_2^{(1)} = 1$ and all other $L_i^{(a)} = 0$. Then

$$(6.3) \qquad \boxed{}\ 0 \atop \boxed{}\ 0 \atop \boxed{}\ 0 \qquad \boxed{}\ 1 \qquad \text{and} \qquad \boxed{}\ 1 \atop 2 \qquad \boxed{}\ 0$$

are in $C^\ell(L, \lambda)$ where again the vacancy numbers are indicated to the left of each part. The set $\mathrm{CST}(\lambda')$ consists of the two elements

$$\begin{array}{|c|c|}\hline 1 & 1 \\\hline 2 \\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\\hline 2 \\\cline{1-1}\end{array}.$$

Since $\tilde{\ell} = 0$ the three rigged configurations

$$\boxed{}\ 0 \atop \boxed{}\ 0 \atop \boxed{}\ 0 \quad \boxed{}\ 0, \qquad \boxed{}\ 0 \atop 0 \quad \boxed{}\ 0 \qquad \text{and} \qquad \boxed{}\ 0 \atop 1 \quad \boxed{}\ 0$$

are restricted of level 2 with charges $2, 3, 4$, respectively. The riggings are given on the right of each part. Hence $M^\ell(L, \lambda; q) = q^2 + q^3 + q^4$.

In contrast to this, the rigged configuration generating function $\overline{M}(L, \lambda; q)$ is obtained by summing over both configurations in (6.3) with all possible riggings below the vacancy numbers. This amounts to $\overline{M}(L, \lambda; q) = q^2 + 2q^3 + 2q^4 + 2q^5 + q^6$.

6.2. Level-restricted fermionic formula.

Similarly to the unrestricted case of section 5.3, one can rewrite the expression of the level-restricted rigged configuration generating function of (6.2) in fermionic form. It was shown in [37, Lemma 6.1] that $p_i^{(k)}(t) = 0$ for all $i \geq \ell$.

Let $\mathrm{SCST}(\lambda')$ be the set of all nonempty subsets of $\mathrm{CST}(\lambda')$. Furthermore set $p_i^{(k)}(S) = \min\{p_i^{(k)}(t)|t \in S\}$ for $S \in \mathrm{SCST}(\lambda')$. Then by inclusion-exclusion the set of allowed rigging for a given configuration $\nu \in C^\ell(L, \lambda)$ is given by

$$\sum_{S \in \mathrm{SCST}(\lambda')} (-1)^{|S|+1} \{J | x_i^{(k)} \leq p_i^{(k)}(S)\}.$$

Since the q-binomial $\left[{m+p \atop m}\right]$ is the generating function of partitions with at most m parts each not exceeding p and since $p_\ell^{(k)}(S) = 0$ by [37, Lemma 6.1] the level-ℓ restricted fermionic formula has the following form.

Theorem 6.5. [37, Theorem 6.2]

$$M^\ell(L, \lambda; q) = \sum_{S \in \text{SCST}(\lambda')} (-1)^{|S|+1} \sum_{\nu \in C^\ell(L,\lambda)} q^{\text{cc}(\nu)} \prod_{i=1}^{\ell-1}\prod_{k=1}^{n-1} \left[{m_i^{(k)} + p_i^{(k)}(S) \atop m_i^{(k)}}\right].$$

6.3. Open Problems.

- In [37, Conjecture 8.3] it was conjectured that the bijection $\overline{\Phi}$ is also well-behaved with respect to fixing certain subtableaux in the set of Littlewood-Richardson tableaux. In the crystal language let $\rho \subset \lambda$ be a partition and $b_\rho = b_k \otimes \cdots \otimes b_1 \in B_\rho = B^{\rho_k^t,1} \otimes \cdots \otimes B^{\rho_1^t,1}$ where b_i is the column tableau of height ρ_i^t with row$(b_i) = \rho_i^t \ldots 21$. Denote the set of all paths in $\mathcal{P}^\ell(B \otimes B_\rho, \lambda)$ with fixed subpath b_ρ by $\mathcal{P}^\ell(B, \lambda, \rho)$. Set $\rho' = (\rho_1 - \rho_n, \ldots, \rho_{n-1} - \rho_n)^t$ and

$$M_i^{(k)}(t) = \sum_{j=1}^{\rho_k - \rho_n} \chi(i \le \rho_1 - \rho_n - t_{j,k}) - \sum_{j=1}^{\rho_{k+1} - \rho_n} \chi(i \le \rho_1 - \rho_n - t_{j,k+1})$$

for all $t \in \text{CST}(\rho')$. Then define $\text{RC}^\ell(L, \lambda, \rho)$ to be the set of all $(\nu, J) \in \text{RC}^\ell(L \cup L_\rho, \lambda)$ such that there exists a $t \in \text{CST}(\rho')$ such that $M_i^{(k)}(t) \le x$ for $(i, x) \in (\nu, J)^{(k)}$ and $M_i^{(k)}(t) \le p_i^{(k)}$ for all $i, k \ge 1$. Here L_ρ is the multiplicity array of B_ρ. Note that the second condition is obsolete if i occurs as a part in $\nu^{(k)}$ since by definition $M_i^{(k)}(t) \le x \le p_i^{(k)}$ for all $(i, x) \in (\nu, J)^{(k)}$. Conjecture 8.3 of [37] asserts that $\mathcal{P}^\ell(B, \lambda, \rho)$ and $\text{RC}^\ell(L, \lambda, \rho)$ correspond under $\overline{\Phi}$.

- It is still an open problem to provide a combinatorial formula for the fusion coefficients of the Verlinde algebra [43, 44]. The fermionic formulas of this section only provide such a formula for rectangular tensor factors.

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