The deviations are evidently too large ($o - u$ is $+2{\cdot}4$, $-1{\cdot}4$, $-2{\cdot}2$, $-1{\cdot}0$, $+2{\cdot}2$) to be due to the use of round numbers; the sum of the squares is also

$$220{\cdot}8 \text{ instead of } 3 \pm \sqrt{6},$$

consequently, no doubt, an over-adjustment.

The special adjustment of the second degree, $\Delta^4 = 0$, $\nabla\Delta^3 = 0$, and $\Delta^2 = 2$, gives for u_i and its differences:

$$\begin{array}{ccccc} 11{\cdot}6 & 19{\cdot}4 & 29{\cdot}2 & 41{\cdot}0 & 54{\cdot}8 \\ & 7{\cdot}8 & 9{\cdot}8 & 11{\cdot}8 & 13{\cdot}8 \end{array}$$

The deviations $o - u =$ $\cdot\,0{\cdot}4$, $-0{\cdot}4$, $-0{\cdot}2$, $0{\cdot}0$, $+0{\cdot}2$

nowhere reach $\frac{1}{2}$, and may consequently be due to the use of round numbers; the sum of the squares

$$4{\cdot}8 \text{ instead of } 3 \pm \sqrt{6}$$

also agrees very well. Indeed, a constant subtraction of 0·04 from u_i would lead to $(3{\cdot}4)^2$, $(4{\cdot}4)^2$, $(5{\cdot}4)^2$, $(6{\cdot}4)^2$, and $(7{\cdot}4)^2$, from which the example is taken.

Example 3. Between 4 points on a straight line the 6 distances

$$\begin{array}{ccc} o_{12}, & o_{13}, & o_{14} \\ & o_{23}, & o_{24} \\ & & o_{34}\,. \end{array}$$

are measured with equal exactness without bonds. By adjustment we find for instance

$$u_{12} = \tfrac{1}{2} o_{12} + \tfrac{1}{4}(o_{13} - o_{23}) + \tfrac{1}{4}(o_{14} - o_{24});$$

we notice that every scale $= \frac{1}{2}$. It is recommended actually to work the example by a millimeter scale, which is displaced after the measurement of each distance in order to avoid bonds.

XII. ADJUSTMENT BY ELEMENTS.

§ 51. Though every problem in adjustment may be solved in both ways, by correlates as well as by elements, the difficulty in so doing is often very different. The most frequent cases, where the number of equations of condition is large, are best suited for adjustment by elements, and this is therefore employed far oftener than adjustment by correlates.

The adjustment by elements requires the theory in such a form that *each observation is represented by one equation* which expresses the mean value $\lambda_1(o)$ explicitely as linear functions of unknown values, the "*elements*", x, y, ... z:

$$\left.\begin{array}{l}\lambda_1(o_1) = p_1 x + q_1 y + \ldots + r_1 z = u_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_1(o_n) = p_n x + q_n y + \ldots + r_n z = u_n\end{array}\right\} \qquad (85)$$

where the $p, q, \ldots r$ are theoretically given. All observations are supposed to be unbound.

The problem is then first to determine the adjusted values of these elements $x, y, \ldots z$, after which each of these equations (85), which we call "*equations for the observations*", gives the adjusted value u of the observation.

Constantly assuming that $\lambda_2(o)$ is known for each observation, we can from the system (85) deduce the following *normal equations:*

$$\left.\begin{array}{l}\left[\frac{p\lambda_1(o)}{\lambda_2(o)}\right] = \left[\frac{pp}{\lambda_2(o)}\right]x + \left[\frac{pq}{\lambda_2(o)}\right]y + \ldots + \left[\frac{pr}{\lambda_2(o)}\right]z = \left[\frac{po}{\lambda_2(o)}\right] \\ \left[\frac{q\lambda_1(o)}{\lambda_2(o)}\right] = \left[\frac{qp}{\lambda_2(o)}\right]x + \left[\frac{qq}{\lambda_2(o)}\right]y + \ldots + \left[\frac{qr}{\lambda_2(o)}\right]z = \left[\frac{qo}{\lambda_2(o)}\right] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \left[\frac{r\lambda_1(o)}{\lambda_2(o)}\right] = \left[\frac{rp}{\lambda_2(o)}\right]x + \left[\frac{rq}{\lambda_2(o)}\right]y + \ldots + \left[\frac{rr}{\lambda_2(o)}\right]z = \left[\frac{ro}{\lambda_2(o)}\right]\end{array}\right\} \qquad (86)$$

the rule of formation being apparent from the left hand terms. Of these normal equations we can prove, first that they, m in number, are suited for the determination of the m elements, so far as these, on the whole, can be determined by the equations (85), and then that the functions of the observations, which form their left hand terms are free of all the theoretical conditions of the problem, so that, as indicated by the last sign of equality in the normal equations, they can and must be determined by the directly observed values $o_1 \ldots o_n$.

For if we assume, as to the first proposition, that any of the normal equations can be deduced from the others, so that all the elements cannot be determined by these equations, then there must be m coefficients $h, k, \ldots l$, so that

$$\begin{array}{l}h\left[\frac{pp}{\lambda}\right] + k\left[\frac{qp}{\lambda}\right] + \ldots + l\left[\frac{rp}{\lambda}\right] = 0 \\ h\left[\frac{pq}{\lambda}\right] + k\left[\frac{qq}{\lambda}\right] + \ldots + l\left[\frac{rq}{\lambda}\right] = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h\left[\frac{pr}{\lambda}\right] + k\left[\frac{qr}{\lambda}\right] + \ldots + l\left[\frac{rr}{\lambda}\right] = 0\end{array}$$

(λ everywhere used for $\lambda_2(o)$); but if we multiply these again respectively by $h, k, \dots l$ and add, we get

$$\left[\frac{(hp+kq+\dots+lr)^2}{\lambda_2(o)}\right] = 0,$$

that is

$$hp_i + kq_i + \dots + lr_i = 0,$$

so that not only the normal equations, but the very equations for the observations can, consequently, all be written with $m-1$ or a smaller number of elements.

But further, the system of functions represented by the normal equations is free of every one of the conditions of the theory. The latter we can get by eliminating the elements $x, y, \dots z$ from the equations of the observations (85). But elimination of an element, say for instance x, leads to the functions $p_i\lambda_1(o_k) - p_k\lambda_1(o_i)$, and among the linear functions of these must be found the functions from which not only x but all the other elements are eliminated, and consequently the conditional equations of the theory. But it is easily seen that the functions

$$p_i\lambda_1(o_k) - p_k\lambda_1(o_i) \quad \text{and} \quad \left[\frac{p\lambda_1(o)}{\lambda_2(o)}\right]$$

are mutually free. The latter is the left hand side of the normal equation which is particularly aimed at the element x; it is formed by multiplying the equations (85) by the coefficient of x in each, and has the sum of the squares $\left[\frac{pp}{\lambda}\right]$ as the coefficient of this element; it has thus been proved to be free of all the conditions of the theory, and must therefore in the adjustment be computed by the directly observed values, for which reason we have been able in the equations (86) to rewrite the function as $\left[\frac{po}{\lambda_2(o)}\right]$. In the same way we prove that all the other normal equations are free of the theory, each through the elimination from (85) of its particularly prominent element. While, in the adjustment by correlates, we exclusively made use of the equations and functions of the theory, we put all these aside in the adjustment by elements, in order to work only with the empirically determined functions which the normal equations represent.

The coefficients of the elements in the normal equations are, as it will be seen, arranged in a remarkably symmetrical manner, and each of them has a significance for the problem which it is easy to state.

The coefficients in the diagonal line, which are respectively multiplied by the element to which the equation particularly refers, are as sums of squares all positive, and each of them is the square of the mean error for that function of the observations in whose equation it occurs. We have for instance

$$\left[\frac{pp}{\lambda}\right] = \left[\frac{p}{\lambda}\cdot\frac{p}{\lambda}\cdot\lambda_2(o)\right] = \lambda_2\left[\frac{po}{\lambda}\right].$$

The coefficients outside the diagonal line are identical in pairs, the coefficient of x, $\left[\frac{qp}{\lambda}\right]$ in y's particular equation, is the same as the coefficient of y, $\left[\frac{pq}{\lambda}\right]$ in x's particular equation. They show immediately if some of the functions $\left[\frac{po}{\lambda}\right]$, $\left[\frac{qo}{\lambda}\right]$, ... $\left[\frac{ro}{\lambda}\right]$ should happen to be mutually free; if for instance x's function $\left[\frac{po}{\lambda}\right]$ is to be free of y's function $\left[\frac{qo}{\lambda}\right]$, we must have $\left[\frac{p}{\lambda}\cdot\frac{q}{\lambda}\cdot\lambda_2(o)\right] = \left[\frac{pq}{\lambda}\right] = 0$.

§ 52. If now the elements have been selected in such a convenient way that all these sums of the products vanish, and the normal equations consequently appear in the special form

$$\left.\begin{array}{l} \left[\frac{pp}{\lambda}\right]x = \left[\frac{po}{\lambda}\right] \\ \left[\frac{qq}{\lambda}\right]y = \left[\frac{qo}{\lambda}\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \left[\frac{rr}{\lambda}\right]z = \left[\frac{ro}{\lambda}\right] \end{array}\right\} \qquad (87)$$

then they offer us directly the solution of the problem of adjustment. The adjusted values for the elements are

$$x = \left[\frac{po}{\lambda}\right]:\left[\frac{pp}{\lambda}\right],\quad y = \left[\frac{qo}{\lambda}\right]:\left[\frac{qq}{\lambda}\right],\quad \ldots\quad z = \left[\frac{ro}{\lambda}\right]:\left[\frac{rr}{\lambda}\right], \qquad (88)$$

and the squares of the mean errors

$$\lambda_2(x) = \left[\frac{pp}{\lambda}\right]^{-1},\quad \lambda_2(y) = \left[\frac{qq}{\lambda}\right]^{-1},\quad \ldots\quad \lambda_2(z) = \left[\frac{rr}{\lambda}\right]^{-1}, \qquad (89)$$

and from these we can then compute both the adjusted value and its λ_2 for every linear function of the elements, because these are mutually free functions. In particular from the equations (85),

$$u_i = p_i x + q_i y + \cdots + r_i z,$$

we can compute the adjusted values u_i of the observations, then from (35) the squares of the mean errors $\lambda_2(u_i)$, and also the law of errors for every function of observations and elements.

§ 53. In ordinary cases a transformation of the system of elements is required. It is required for the solution of the normal equations in order to find the values of the elements; but we must remember that we have here a double problem, as it is also our object to free the transformed elements so that they may be used for determinations of the mean errors. The transformation therefore cannot be selected so arbitrarily as in analogous problems of pure mathematics; yet there is a multiplicity of possibilities, and in many special cases radical changes can lead to very beautiful solutions (see § 62). The first thing, however, is to secure a method which may be always applied; and this must be selected in such a way that the elements are eliminated one by one, so that the later computation of them is prepared, and moreover, constantly, in such a way that freedom is attained.

This can, if we commence for instance by eliminating the element x, be attained in the following way. The normal equation which particularly refers to x,

$$\left[\frac{pp}{\lambda}\right]x + \left[\frac{pq}{\lambda}\right]y + \dots + \left[\frac{pr}{\lambda}\right]z = \left[\frac{po}{\lambda}\right], \tag{90}$$

and which will be put aside to be used later on for the computation of x, is multiplied by such factors, viz. $\varphi = \left[\frac{qp}{\lambda}\right] : \left[\frac{pp}{\lambda}\right], \dots \omega = \left[\frac{rp}{\lambda}\right] : \left[\frac{pp}{\lambda}\right]$, that x vanishes when the products are respectively subtracted from the other normal equations; but it must be remembered that we are not allowed to multiply the latter by any factor. The equation for x can then be written

$$\xi = x + \varphi y + \dots + \omega z = \left[\frac{po}{\lambda}\right] : \left[\frac{pp}{\lambda}\right] \tag{91}$$

where $\lambda_2(\xi) = \left[\frac{pp}{\lambda}\right]^{-1}$.

The functions in the other equations

$$\left[\frac{qo}{\lambda}\right] - \varphi\left[\frac{po}{\lambda}\right] = \left[\frac{(q-\varphi p)o}{\lambda}\right], \quad \dots \quad \left[\frac{ro}{\lambda}\right] - \omega\left[\frac{po}{\lambda}\right] = \left[\frac{(r-\omega p)o}{\lambda}\right]$$

become, by this means, not only independent of x but also free of $\left[\frac{po}{\lambda}\right]$ or of ξ, for

$$\left[\frac{p}{\lambda} \cdot \frac{q-\varphi p}{\lambda} \cdot \lambda_2(o)\right] = \left[\frac{pq}{\lambda}\right] - \varphi\left[\frac{pp}{\lambda}\right] = 0, \text{ etc.}$$

The equations which in a double sense have been freed from x, get exactly the same characteristic functional form as the normal equations had. If we write

$$q'_i = q_i - \varphi p_i, \quad \dots \quad r'_i = r_i - \omega p_i, \tag{92}$$

so that the equations for the observations become

$$p_i \xi + q'_i y + \dots + r'_i z = u_i,$$

we not only get, as we see at once,

$$\left[\frac{q'o}{\lambda}\right] = \left[\frac{qo}{\lambda}\right] - \varphi\left[\frac{po}{\lambda}\right], \quad \dots \quad \left[\frac{r'o}{\lambda}\right] = \left[\frac{ro}{\lambda}\right] - \omega\left[\frac{po}{\lambda}\right], \tag{93}$$

but also

$$\left.\begin{array}{l} \left[\frac{q'q'}{\lambda}\right] = \left[\frac{qq}{\lambda}\right] - \varphi\left[\frac{qp}{\lambda}\right], \;\ldots\; \left[\frac{q'r'}{\lambda}\right] = \left[\frac{qr}{\lambda}\right] - \omega\left[\frac{qp}{\lambda}\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left[\frac{r'q'}{\lambda}\right] = \left[\frac{rq}{\lambda}\right] - \varphi\left[\frac{rp}{\lambda}\right], \;\ldots\; \left[\frac{r'r'}{\lambda}\right] = \left[\frac{rr}{\lambda}\right] - \omega\left[\frac{rp}{\lambda}\right] \end{array}\right\} \quad (94)$$

Hence we proceed exactly in the same way from this first stage of the transformation of the normal equations

$$\left.\begin{array}{l} \left[\frac{q'q'}{\lambda}\right]y + \cdots + \left[\frac{q'r'}{\lambda}\right]z = \left[\frac{q'o}{\lambda}\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \left[\frac{r'q'}{\lambda}\right]y + \cdots + \left[\frac{r'r'}{\lambda}\right]z = \left[\frac{r'o}{\lambda}\right] \end{array}\right\} \quad (95)$$

using, for instance, the first of them for the elimination of the element y. If

$$\omega' = \left[\frac{r'q'}{\lambda}\right] : \left[\frac{q'q'}{\lambda}\right],$$

y is replaced by

$$\eta = y + \cdots + \omega' z = \left[\frac{q'o}{\lambda}\right] : \left[\frac{q'q'}{\lambda}\right], \quad (96)$$

which is free of the element ξ, and for which we have

$$\lambda_2(\eta) = \left[\frac{q'q'}{\lambda}\right]^{-1}. \quad (97)$$

By means of ω' and corresponding coefficients we have, analogously to (93) and (94),

$$\left[\frac{r''o}{\lambda}\right] = \left[\frac{r'o}{\lambda}\right] - \omega'\left[\frac{q'o}{\lambda}\right], \;\ldots\; \left[\frac{r''r''}{\lambda}\right] = \left[\frac{r'r'}{\lambda}\right] - \omega'\left[\frac{r'q'}{\lambda}\right],$$

which are independent of any special computation of the coefficients r''.

Continuing in this way, till we have obtained a set consisting only of free functions, we find, consequently, just a system of elements, ξ, η, ζ, which possess the above-mentioned desired property, its normal equations being of the same form as (87), viz.:

$$\left.\begin{array}{l} \left[\frac{pp}{\lambda}\right]\xi \qquad\qquad = \left[\frac{po}{\lambda}\right] \\ \qquad \left[\frac{q'q'}{\lambda}\right]\eta \qquad = \left[\frac{q'o}{\lambda}\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad\qquad \left[\frac{r^{\nu}r^{\nu}}{\lambda}\right]\zeta = \left[\frac{r^{\nu}o}{\lambda}\right] \end{array}\right\} \quad (98)$$

With these elements the equations for the adjusted values of the several observations become

$$p_i \xi + q'_i \eta + \dots + r^{\nu}_i \zeta = u_i, \tag{99}$$

and for the squares of their mean errors

$$p_i^2 \left[\frac{pp}{\lambda}\right]^{-1} + q_i'^2 \left[\frac{q'q'}{\lambda}\right]^{-1} + \dots + r_i^{\nu 2} \left[\frac{r^{\nu} r^{\nu}}{\lambda}\right]^{-1} = \lambda_2(u_i). \tag{100}$$

If we want to compute adjusted values and mean errors for the original elements or functions of the same, the means of so doing is given by the equations of transformation

$$\left.\begin{array}{r} x + \varphi y + \dots + \omega z = \xi \\ y + \dots + \omega' z = \eta \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ z = \zeta \end{array}\right\} \tag{101}$$

or by (90), the first equation (95) and the last of (98), being identical with (101). For not only the original elements $x, y, \dots z$ are easily computed by these, but also the coefficients in the inverse transformation

$$\left.\begin{array}{l} x = \xi + \alpha\eta + \dots + \gamma\zeta \\ y = \quad\;\; \eta + \dots + \delta\zeta \\ z = \qquad\qquad\quad\;\; \zeta. \end{array}\right\} \tag{102}$$

Now, if F is a given linear function of $x, y, \dots z$, then by obvious numerical operations we get an expression for it,

$$F = a\xi + b\eta + \dots + d\zeta,$$

and for the square of its mean error we get

$$\lambda_2(F) = a^2\left[\frac{pp}{\lambda}\right]^{-1} + b^2\left[\frac{q'q'}{\lambda}\right] + \dots + d^2\left[\frac{r^{\nu}r^{\nu}}{\lambda}\right].$$

If for special criticism we want the computation of $\lambda_2(u_i)$ for many observations, we may take advantage of transforming the equations of observations, computing their coefficients by (92), or

$$q'_i = q_i - \varphi p_i, \; \dots \; r'_i = r_i - \omega p_i$$
$$r'' = r'_i - \omega' q'_i;$$

but we remember that $q', \dots r''$ are quite superfluous for the coefficients of (95).

§ 54. In the theory of the adjustment by elements we must not overlook the proposition concerning the computation of the minimum sum of squares for the benefit of the summary criticism as well as for checking our computation. We are able to compute the sum $\left[\frac{(o-u)^2}{\lambda}\right]$, which is to approach the value $n - m$, as soon as we have found only the elements, without being obliged to know the adjusted values for the separate observations. And this computation can be performed, not only for the legitimate adjustment, but for any values whatever of the elements. It is easiest to show this for transformed elements, $\xi_1, \eta_1, \ldots \zeta_1$. The values for the observations corresponding to these must be computed by (99)

$$p_i \xi_1 + q'_i \eta_1 + \cdots + r^{\nu}_i \zeta_1 = v_i .$$

From this we get

$$\left[\frac{(o-v)^2}{\lambda_2(o)}\right] = \left[\frac{oo}{\lambda}\right] - 2\left[\frac{po}{\lambda}\right]\xi_1 - 2\left[\frac{q'o}{\lambda}\right]\eta_1 - \cdots - 2\left[\frac{r^{\nu}o}{\lambda}\right]\zeta_1 + \left[\frac{pp}{\lambda}\right]\xi_1^2 + \left[\frac{q'q'}{\lambda}\right]\eta_1^2 + \cdots + \left[\frac{r^{\nu}r^{\nu}}{\lambda}\right]\zeta_1^2 . \quad (103)$$

If we here substitue for $\left[\frac{po}{\lambda}\right], \left[\frac{q'o}{\lambda}\right], \ldots \left[\frac{r^{\nu}o}{\lambda}\right]$ their values in terms of the elements $\xi, \eta, \ldots \zeta$, of the legitimate adjustment, we find from the equations (98)

$$\left[\frac{(o-v)^2}{\lambda_2(o)}\right] = \left[\frac{oo}{\lambda}\right] + \left[\frac{pp}{\lambda}\right]((\xi_1-\xi)^2-\xi^2) + \left[\frac{q'q'}{\lambda}\right]((\eta_1-\eta)^2-\eta^2) + \cdots + \left[\frac{r^{\nu}r^{\nu}}{\lambda}\right]((\zeta_1-\zeta)^2-\zeta^2). \quad (104)$$

It is evident from this that the condition of minimum is $\xi_1 = \xi$, $\eta_1 = \eta$, $\zeta_1 = \zeta$. The minimum sum of squares is therefore obtained only by the determination of the functions that are free of the theory, by means of their directly observed values. And for this minimum

$$\left[\frac{(o-u)^2}{\lambda_2(o)}\right] = \left[\frac{oo}{\lambda}\right] - \left[\frac{pp}{\lambda}\right]\xi^2 - \left[\frac{q'q'}{\lambda}\right]\eta^2 - \cdots - \left[\frac{r^{\nu}r^{\nu}}{\lambda}\right]\zeta^2 = \quad (105)$$

$$= \left[\frac{oo}{\lambda}\right] - \left[\frac{po}{\lambda}\right]\xi - \left[\frac{q'o}{\lambda}\right]\eta - \cdots - \left[\frac{r^{\nu}o}{\lambda}\right]\zeta = \quad (106)$$

$$= \left[\frac{oo}{\lambda}\right] - \frac{\left[\frac{po}{\lambda}\right]^2}{\left[\frac{pp}{\lambda}\right]} - \frac{\left[\frac{q'o}{\lambda}\right]^2}{\left[\frac{q'q'}{\lambda}\right]} - \cdots - \frac{\left[\frac{r^{\nu}o}{\lambda}\right]^2}{\left[\frac{r^{\nu}r^{\nu}}{\lambda}\right]} . \quad (107)$$

It deserves to be noticed that the middle one of these expressions holds good, in unchanged form, also of the original, not transformed elements and coefficients. We have

$$\left[\frac{(o-u)^2}{\lambda_2(o)}\right] = \left[\frac{oo}{\lambda}\right] - \left[\frac{po}{\lambda}\right]x - \left[\frac{qo}{\lambda}\right]y - \cdots - \left[\frac{ro}{\lambda}\right]z , \quad (108)$$

which is easily proved by substituting in (106) the values obtained from (101). The equation is particularly valuable as a check on the accuracy of our computation.

§ 55. In going through the theory of adjustment by elements here developed, it will be seen that a very essential part of the work, viz. the computation of the transformed values of the coefficients in the equations for the several observations, may nearly always be dispensed with. The sums of the squares, $\left[\frac{qq}{\lambda}\right]$, and the sums of the products, $\left[\frac{qr}{\lambda}\right]$, must be transformed; but they are in themselves sufficient for the determination of the transformations, and by their help we find values and mean errors for the elements, first the transformed ones, but indirectly also the original ones. The adjusted values $u_1 \ldots u_n$ of the observations can, consequently, also be computed without any knowledge of $q'_i \ldots r'_i \ldots r^\nu_i$. Only for the computation of $\lambda_2(u_i) \ldots \lambda_2(u_n)$, consequently for a special criticism, we cannot escape the often considerable work which is necessary for the purpose.

For the summary criticism by $\left[\frac{(o-u)^2}{\lambda_2(o)}\right] = n - m \pm \sqrt{2(n-m)}$, we can even, as we have seen, dispense with the after-computation of the several observations by means of the elements. We ought, however, to restrict the work of adjustment so far only, when the case is either very difficult or of slight importance, for this minimum sum of squares is generally computed much more sharply, and always with much greater certainty, directly by o_i, u_i, and $\lambda_2(o)$, than by the formulæ (105), (106), and (107).

Add to this, that the special criticism does not exclusively rest on $\lambda_2(u_i)$ and the scales $1 - \frac{\lambda_2(u)}{\lambda_2(o)}$, but that the very deviations $o_i - u_i$, when they are arranged according to the more or less essential circumstances of the observations, are even a main point in the criticism. Systematical errors, especially inaccuracies or defects in hypotheses and theories, will betray themselves in the surest and easiest way by the progression of the errors; regular variation in $o - u$ as a function of some circumstance, or mere absence of frequent changes of signs, will disclose errors which might remain hidden by the check according to $\Sigma \frac{(o-u)^2}{\lambda_2(o)} = \Sigma\left(1 - \frac{\lambda_2(u)}{\lambda_2(o)}\right)$; and such progression in the errors may, we know, even be used to indicate how we ought to try to improve the defective theory.

§ 56. By series of adjustment (compare Dr. J. P. Gram, Udjevningsrækker, Kjøbenhavn 1879, and Crelle's Journal vol. 94), i. e. where the theory gives the observations in the form of a series with an indeterminate (infinite) number of terms, each term being multiplied by an unknown factor, an element, and where consequently adjustment by elements must be employed, the criticism gets the special task of indicating how many (or which) terms of the series we are to include in the adjustment. Formula (107) furnishes us with the means of doing this.

$$\left[\frac{(o-u)^2}{\lambda_2(o)}\right] = \left[\frac{oo}{\lambda}\right] - \Sigma \frac{\left[\frac{r^\nu o}{\lambda}\right]^2}{\left[\frac{r^\nu r^\nu}{\lambda}\right]} = n - m \pm \sqrt{2(n-m)}.$$

For the m terms in the series, which is here indicated by Σ, correspond, each of them, to an element, consequently to one of the terms of the series of adjustment. For each term we take into this, the right side of the equation of criticism is diminished by about a unity; the result of the criticism, consequently, becomes more favourable if we leave out all the terms for which $\left[\frac{r^{\nu}o}{\lambda}\right]^2 \cdot \left[\frac{r^{\nu}r^{\nu}}{\lambda}\right]^{-1} < 1$. If we retain any terms which essentially fall under this rule, the adjustment becomes an under-adjustment; if, on the other hand, we leave out terms for which $\left[\frac{r^{\nu}o}{\lambda}\right]^2 \cdot \left[\frac{r^{\nu}r^{\nu}}{\lambda}\right]^{-1} > 1$, we make ourselves guilty of an over-adjustment.

Example 1. The five-place logarithms in a table are looked upon as mutually unbound observations for which the mean error is constantly $\sqrt{\frac{1}{12}}$ of the fifth decimal place. The "observations", log 795, log 796, log 797, log 798, log 799, log 800, log 801, log 802, log 803, log 804, and log 805, are to be adjusted as an integral function of the second degree

$$\log(800+t) = x' + y't + zt^2.$$

In order to reckon with small integral numbers, we subtract before the adjustment $2{\cdot}90309 + 0{\cdot}00054\,t$, both from the observations and from the formulæ. Taking $0{\cdot}00001$ as our unity, we have then the equations for the observations:

$$\begin{aligned}
-2 &= x - 5y + 25z \\
-2 &= x - 4y + 16z \\
-1 &= x - 3y + 9z \\
-1 &= x - 2y + 4z \\
0 &= x - 1y + 1z \\
0 &= x \\
0 &= x + 1y + 1z \\
0 &= x + 2y + 4z \\
1 &= x + 3y + 9z \\
1 &= x + 4y + 16z \\
1 &= x + 5y + 25z.
\end{aligned}$$

From this we get $\left[\frac{oo}{\lambda}\right] = 156$, and the normal equations:

$$\begin{aligned}
-36 &= 132x + 0y + 1320z \\
420 &= 0x + 1320y + 0z \\
-540 &= 1320x + 0y + 23496z.
\end{aligned}$$

The element y is consequently immediately free of x and z, but the latter must be made free of one another, which is done by multiplying the first equation by 10 and subtracting it from the third. The transformation into free functions then only requires $\xi = x + 10z$ substituted for x, and we have:

$$-36 = 132\xi,$$
$$420 = 1320y,$$
$$-180 = 10296z,$$

consequently,

$$\xi = -0{\cdot}2727, \quad \lambda_2(\xi) = 1: \ 132 = 390:51480 = {\cdot}007576$$
$$y = 0{\cdot}3182, \quad \lambda_2(y) = 1: \ 1320 = 39:51480 = {\cdot}000758$$
$$z = -0{\cdot}0175, \quad \lambda_2(z) = 1:10296 = 5:51480 = {\cdot}000097.$$

The mean error of y is consequently $\pm 0{\cdot}0275$, and that of z $\pm 0{\cdot}0099$. The element x is found by $x = \xi - 10z = -0{\cdot}0977$, to which corresponds $\lambda_2(x) = \lambda_2(\xi) + 100\lambda_2(z) = 0{\cdot}0173 = (0{\cdot}1315)^2$. For log 800 we find thus $2{\cdot}9030890 \pm 0{\cdot}0000013$, and the corresponding difference of the table is $54{\cdot}318 \pm 0{\cdot}028$.

For the sum of the squares of the deviations we have, according to (105)—(107),

$$\left[\frac{(o-u)^2}{\lambda_2(o)}\right] = 156 - 9{\cdot}82 - 133{\cdot}64 - 3{\cdot}15 = 9{\cdot}39,$$

which shows that the term of the second degree contributes somewhat to the goodness of the adjustment. This sum of squares ought, according to the number of the observations and the elements, to be $11 - 3 = 8$, with a mean uncertainty of ± 4.

The best formula for computing the adjusted values of the several observations and their mean errors is $u_t = \xi + yt + z(t^2 - 10)$, which gives:

	u	$o-u$	$(o-u)^2$		$\lambda_2(u)$	Scale
log 795	= 2·9003688	+ ·12	·0144	390 + 39 · 25 + 5 · 225 = 2490	·0484	·419
log 796	= 2·9009136	— ·36	·1296	390 + 39 · 16 + 5 · 36 = 1194	·0232	·722
log 797	= 2·9014580	+ ·20	·0400	390 + 39 · 9 + 5 · 1 = 746	·0145	·826
log 798	= 2·9020019	— ·19	·0361	390 + 39 · 4 + 5 · 36 = 726	·0141	·831
log 799	= 2·9025457	+ ·43	·1849	390 + 39 · 1 + 5 · 81 = 834	·0162	·806
log 800	= 2·9030890	+ ·10	·0100	390 + 39 · 0 + 5 · 100 = 890	·0173	·792
log 801	= 2·9036321	— ·21	·0441	390 + 39 · 1 + 5 · 81 = 834	·0162	·806
log 802	= 2·9041747	— ·47	·2209	390 + 39 · 4 + 5 · 36 = 726	·0141	·831
log 803	= 2·9047170	+ ·30	·0900	390 + 39 · 9 + 5 · 1 = 746	·0145	·826
log 804	= 2·9052590	+ ·10	·0100	390 + 39 · 16 + 5 · 36 = 1194	·0232	·722
log 805	= 2·9058006	— ·06	·0036	390 + 39 · 25 + 5 · 225 = 2490	·0484	·419
			·7836	12870		8·000

Both the checks agree: the sum of squares is $12 \times 0{\cdot}7836 = 9{\cdot}40$, and the sum of the scales is 11—3.

It ought to be noticed that the adjustment gives very accurate results throughout the greater part of the interval, with the exception of the beginning and the end. The exactness, however, is not greatest in the middle, but near the 1st and the 3rd quarter.

Example 2. A finite, periodic function of one single essential circumstance, an angle V, is supposed to be the object of observation. The theory, consequently, has the form:

$$o_v = c_0 + c_1 \cos V + s_1 \sin V + c_2 \cos 2V + s_2 \sin 2V + \ldots$$

We assume that there are n unbound, equally exact observations for a series of values of V, whose difference is constant and $= \frac{2\pi}{n}$, for instance for $V = 0, 60°, 120°, 180°, 240°, 300°$. Show that the normal equations are here originally free, and that they admit of an exceedingly simple computation of each isolated term of the periodic series.

Example 3. Determine the abscissæ for 4 points on a straight line whose mutual distances are measured equally exactly, and are unbound. (Cmp. Adjustment by Correlates, Example 3, and § 60).

Example 4. Three unbound observations must, according to theory, depend on two elements, so that

$$\begin{aligned} o_1 &= x^2, & \lambda_2(o_1) &= 1 \\ o_2 &= xy, & \lambda_2(o_2) &= \tfrac{1}{2} \\ o_3 &= y^2, & \lambda_2(o_3) &= 1. \end{aligned}$$

The theory, therefore, does not give us equations of the linear form. This may be produced in several ways, most simply by the common method of presupposing approximate values of both elements, the known a for x and b for y, and considering the corrections ξ and η to be the elements of the adjustment. We therefore put $x = a + \xi$, and $y = b + \eta$. Rejecting terms of the 2nd degree, we get the equations of the observations:

$$\begin{aligned} o_1 - a^2 &= 2a\xi \\ o_2 - ab &= b\xi + a\eta \\ o_3 - b^2 &= 2b\eta, \end{aligned}$$

where the middle equation has still double weight. The normal equations are:

$$\begin{aligned} 2a(o_1 - a^2) + 2b(o_2 - ab) &= (4a^2 + 2b^2)\xi + 2ab\eta \\ 2a(o_2 - ab) + 2b(o_3 - b^2) &= \qquad 2ab\,\xi + (4b^2 + 2a^2)\eta; \end{aligned}$$

ξ is consequently not free of η, but we find

$$2ax = o_1 + a^2 - \frac{b^2(b^2 o_1 - 2abo_2 + a^2 o_3)}{(a^2 + b^2)^2}, \quad \lambda_2(x) = \frac{a^2 + 2b^2}{4(a^2 + b^2)^2}$$

$$2by = o_3 + b^2 - \frac{a^2(b^2 o_1 - 2abo_2 + a^2 o_3)}{(a^2 + b^2)^2}, \quad \lambda_2(y) = \frac{2a^2 + b^2}{4(a^2 + b^2)^2}.$$

For the adjusted value u_2 of the middle observation we have

$$(a^2+b^2)^2 u_2 = ab^2 o_1 + (a^4+b^4) o_2 + a^2 b o_3 , \quad \lambda_2(u_2) = \frac{1}{2}\frac{a^4+b^4}{(a^2+b^2)^2}.$$

If we had transformed the elements (comp. § 62) by putting

$$\xi = a\zeta - bv$$
$$\eta = b\zeta + av ,$$

or

$$x = a(1+\zeta) - bv$$
$$y = b(1+\zeta) + av ,$$

we should have obtained free normal equations

$$2(a^2 o_1 + 2ab o_2 + b^2 o_3) - 2(a^2+b^2)^2 = 4(a^2+b^2)^2 \zeta$$
$$2(-ab o_1 + (a^2-b^2) o_2 + ab o_3) = 2(a^2+b^2)^2 v.$$

If we had placed absolute confidence in the adjusting principle of the sum of squares as a minimum, a solution might have been founded on

$$(o_1 - a^2)^2 + 2(o_2 - ab)^2 + (o_3 - b^2)^2 = \min.$$

The conditions of minimum are:

$$\frac{1}{4}\frac{d\min}{da} = (o_1 - a^2)a + (o_2 - ab)b = 0$$
$$\frac{1}{4}\frac{d\min}{db} = (o_2 - ab)a + (o_3 - b^2)b = 0.$$

The solution with respect to a and b is not very difficult. We see for instance immediately that

$$(o_1 - a^2)(o_3 - b^2) = (o_2 - ab)^2$$

or

$$o_1 o_3 - o_2^2 = b^2 o_1 - 2ab o_2 + a^2 o_3.$$

Still better is it to introduce $s^2 = a^2+b^2$, by which the equations become

$$(o_1 - s^2)a + o_2 b = 0$$
$$o_2 a + (o_3 - s^2) b = 0 ,$$

consequently,

$$s^4 - s^2(o_1+o_3) + o_1 o_3 - o_2^2 = 0$$
$$\left(s^2 - \frac{o_1+o_3}{2}\right)^2 = \left(\frac{o_1-o_3}{2}\right)^2 + o_2^2.$$

If the errors in o_1, o_2, and o_3 are not large, $o_1 o_3 - o_2^2$ must be small; one of the two values of s^2 must then be small, the other nearly equal to o_1+o_3; only the latter can be used.

Further, we get:

$$-\frac{a}{b} = \frac{o_2}{o_1 - s^2} = \frac{o_3 - s^2}{o_2}$$

$$\left(\frac{a}{b}\right)^2 = \frac{o_3 - s^2}{o_1 - s^2}$$

$$a^2 = \frac{(o_3 - s^2)s^2}{o_1 + o_3 - 2s^2}, \quad b^2 = \frac{(o_1 - s^2)s^2}{o_1 + o_3 - 2s^2}.$$

In this way we avoid guessing at approximate values (for which otherwise we should perhaps have taken $a^2 = o_1$ and $b^2 = o_3$). The values which we have here found for a^2 and b^2, and to which may be added

$$-ab = \frac{o_2 s^2}{o_1 + o_3 - 2s^2},$$

are really exact; and if we substitute them in the above normal equations, we get $\xi = 0$ and $\eta = 0$.

Even when, as in this case, the theory is not linear, it is not unusual for the sum of the squares to be a minimum. Caution, however, is necessary; particularly, it may happen that the sum of the squares becomes a maximum for the found elements, or for some of them.

We may also in another way make the equations of this example linear, namely, by considering the logarithms of o_1, o_2, o_3 as the observed quantities, and finding the logarithms of the elements from the equations which will then be linear.

$$\begin{aligned} \log o_1 &= 2 \log x \\ \log o_2 &= \log x + \log y \\ \log o_3 &= \qquad\quad 2 \log y. \end{aligned}$$

In this way we throw the difficulty over upon the squares of the mean errors. As

$$\log (z + dz) = \log z + \frac{dz}{z},$$

we may approximately take

$$\lambda_2 (\log z) = \frac{1}{z^2} \lambda_2 (z).$$

If a and b also here indicate approximate values of x and y, the weights of the 3 equations, respectively, become proportional to a^4, $2a^2b^2$, and b^4. Thus we find the normal equations

$$\begin{aligned} 2a^4 \log o_1 + 2a^2b^2 \log o_2 &= (4a^4 + 2a^2b^2) \log x + 2a^2b^2 \log y \\ 2a^2b^2 \log o_2 + 2b^4 \log o_3 &= \qquad 2a^2b^2 \log x + (4b^4 + 2a^2b^2) \log y, \end{aligned}$$

which give the simple results

$$2\log x = \log o_1 - \left(\frac{b^2}{a^2+b^2}\right)^2 \log \frac{o_1 o_2}{o_2^2}, \quad \lambda_2(\log x) = \frac{a^2+2b^2}{4a^2(a^2+b^2)^2}$$

$$2\log y = \log o_2 - \left(\frac{a^2}{a^2+b^2}\right)^2 \log \frac{o_1 o_2}{o_2^2}, \quad \lambda_2(\log y) = \frac{2a^2+b^2}{4b^2(a^2+b^2)^2}.$$

This solution agrees only approximately with the preceding one. It might seem for a moment that, in this way, we might do without the supposition of approximate values for the elements, but this is far from being the case. For the sake of the weights we must, with the same care, demand that a and x, as also b and y, agree, and we must repeat the adjustment till the squares of the mean errors get the *theoretically* correct values. And then it is only a necessary, but not a sufficient condition, that $x-a$ and $y-b$ are small. Unless the exactness of the observations is also so great that the mean errors of o_i are small in proportion to o_i itself, the laws of errors of the logarithms cannot be considered typical at the same time as those of the observations themselves.

Example 5. The co-ordinates of four points in a circle are observed with equal mean errors and without bonds: $x_1 = 20$, $y_1 = 10$; $x_2 = 16$, $y_2 = 18$; $x_3 = 3$, $y_3 = 17$; and $x_4 = 2$, $y_4 = 4$. In the adjustment for the co-ordinates a and b of the centre and the radius r, we cannot use the common form of the equations

$$(x-a)^2 + (y-b)^2 = r^2,$$

because it embraces more than *one* observed quantity besides the elements. In order to obtain the separation of the observations necessary for adjustment by elements, we must add a supplementary element, or parameter, V_i for each point, writing for instance

$$x_i = a + r\cos V_i, \quad y_i = b + r\sin V_i.$$

As the equations are not linear we must work by successive corrections $\triangle a$, $\triangle b$, $\triangle r$, $\triangle V_i$ of the elements, of which the first approximate system can be obtained by ordinary computation from 3 points. For the theoretical corrections $\triangle x_i$ and $\triangle y_i$ of the co-ordinates we get by differentiation of the above equations

$$\triangle x_i = \triangle a + \triangle r \cdot \cos V_i - \triangle V_i \cdot r\sin V_i$$

$$\triangle y_i = \triangle b + \triangle r \cdot \sin V_i + \triangle V_i \cdot r\cos V_i.$$

These equations for the observations lead us to a system of seven normal equations. By the "method of partial elimination" (§ 61) these are not difficult to solve, but here the simplicity of the problem makes it possible for us immediately to discover the artifice. We know that every transformation of equally well observed rectangular co-ordinates results in free functions. The radial and the tangential corrections

$$\triangle x_i \cos V_i + \triangle y_i \sin V_i = \triangle n_i$$

and

$$\triangle x_i \sin V_i - \triangle y_i \cos V_i = \triangle t_i$$

can, consequently, here be taken directly for the mean values of corrections of observed quantities, and as only the four equations

$$\triangle t_i = \triangle a \sin V_i - \triangle b \cos V_i - r \triangle V_i$$

contain the four corrections $\triangle V_i$ of the parameters, they can be legitimately reserved for the successive corrections of the elements. In this way

$$\triangle n_i = \triangle a \cos V_i + \triangle b \sin V_i + \triangle r$$

with equal mean errors, $\lambda_2(n) = \lambda_2(x) = \lambda_2(y)$, are the "equations for the observations" of this adjustment, and give the three normal equations:

$$\begin{aligned}
[\triangle n \cos V] &= \triangle a [\cos^2 V] + \triangle b [\cos V \sin V] + \triangle r [\cos V] \\
[\triangle n \sin V] &= \triangle a [\cos V \sin V] + \triangle b [\sin^2 V] + \triangle r [\sin V] \\
[\triangle n] &= \triangle a [\cos V] + \triangle b [\sin V] + \triangle r \cdot 4.
\end{aligned}$$

In the special case under consideration, we easily see that the first, second, and fourth point lie on the circle with $r = 10$, whose centre has the co-ordinates $a = 10$ and $b = 10$; the parameters are consequently:

$$V_1 = 0°0'0, \quad V_2 = 53°7'8, \quad V_3 = 135°0'0, \quad \text{and} \quad V_4 = 216°52'2.$$

For the third point the computed co-ordinates are: $x_3 = 2{\cdot}9290$ and $y_3 = 17{\cdot}0710$, consequently, $\triangle x_3 = +0{\cdot}0710$ and $\triangle y_3 = -0{\cdot}0710$, $\triangle t_3 = 0$, and $\triangle n_3 = -0{\cdot}1005$; all other differences $\triangle x_i = 0$ and $\triangle y_i = 0$. The "equations for the observations" are:

$$\begin{aligned}
1{\cdot}0000 \triangle a + 0{\cdot}0000 \triangle b + 1{\cdot}0000 \triangle r &= 0{\cdot}0000 \\
0{\cdot}6000 \triangle a + 0{\cdot}8000 \triangle b + 1{\cdot}0000 \triangle r &= 0{\cdot}0000 \\
-0{\cdot}7071 \triangle a + 0{\cdot}7071 \triangle b + 1{\cdot}0000 \triangle r &= -0{\cdot}1005 \\
-0{\cdot}8000 \triangle a - 0{\cdot}6000 \triangle b + 1{\cdot}0000 \triangle r &= 0{\cdot}0000.
\end{aligned}$$

		For checking
The normal equations are:		
$2{\cdot}5000 \triangle a + 0{\cdot}4600 \triangle b + 0{\cdot}0929 \triangle r$	$= +0{\cdot}0710$	$+0{\cdot}0002$
$0{\cdot}4600 \triangle a + 1{\cdot}5000 \triangle b + 0{\cdot}9071 \triangle r$	$= -0{\cdot}0710$	$0{\cdot}0000$
$R = 0{\cdot}0929 \triangle a + 0{\cdot}9071 \triangle b + 4{\cdot}0000 \triangle r$	$= -0{\cdot}1005.$	$0{\cdot}0000$
By elimination of $\triangle r$ we get		
$2{\cdot}4978 \triangle a + 0{\cdot}4390 \triangle b$	$= +0{\cdot}0733$	$+0{\cdot}0001$
$B = 0{\cdot}4390 \triangle a + 1{\cdot}2943 \triangle b$	$= -0{\cdot}0482;$	$-0{\cdot}0001$
and by eliminating $\triangle b$		
$A = +2{\cdot}3490 \triangle a$	$= +0{\cdot}0896.$	$0{\cdot}0000$

From R, B, and A we compute

$$\triangle a = +0{\cdot}0381, \quad \triangle b = -0{\cdot}0501, \quad \text{and} \quad \triangle r = -0{\cdot}01465.$$

The checks are found by substitution of these in the several equations. The 4 equations for the observations give the following adjusted values of $\triangle n_i$:

$$\triangle n_1 = +0{\cdot}0234,\ \triangle n_2 = -0{\cdot}0319,\ \triangle n_3 = -0{\cdot}0770,\ \text{and}\ \triangle n_4 = -00151:$$

the sum of squares $\left[\frac{(o-u)^2}{\lambda_2}\right]$ (here $= (8-7)\lambda_2$) is consequently

$$= (0{\cdot}0234)^2 + (0{\cdot}0319)^2 + (0{\cdot}0235)^2 + (0{\cdot}0151)^2 = 0{\cdot}00235.$$

For this, by the equation (108), we get

$$0{\cdot}01010 - 0{\cdot}00271 - 0{\cdot}00356 - 0{\cdot}00147 = 0{\cdot}00236$$

as the final check of the adjustment.

The 4 equations for $\triangle t_i$ give us

$$\triangle V_1 = +17'2,\ \triangle V_2 = +20'8,\ \triangle V_3 = -2'9,\ \text{and}\ \triangle V_4 = -21'6.$$

Thus, by addition of the found corrections to the approximate values,

$$r = 9{\cdot}98535,\quad a = 10{\cdot}0381,\quad b = 9{\cdot}9499,$$
$$V_1 = 0°17'2,\ V_2 = 58°28'6,\ V_3 = 134°57'1,\ \text{and}\ V_4 = 216°30'6,$$

we have the whole system of elements for the next approximation, if they are not the definitive values. In both cases we must compute by them the adjusted values of the co-ordinates, *according to the exact formulæ*; the resulting differences, obs.—comp., are:

Point	$\triangle x$	$\triangle y$	$\triangle n$	$\triangle t$
1	−0·0232	+0·0002	−0·0232	+0·0002
2	+0·0191	+0·0257	+0·0320	0·0000
3	+0·0166	−0·0166	−0·0234	−0·0001
4	−0·0123	−0·0090	+0·0152	0·0000.

The sum of the squares, $[(\triangle x)^2 + (\triangle y)^2] = 0{\cdot}00236$, agrees with the above value, which indicates that the approximation of this first hypothesis may have been sufficient. Indeed, the students who will try the next approximation by means of our final differences, will, in this case, find only small corrections.

From the equations A, B, and R, which express the free elements by the original bound elements, $\triangle a$, $\triangle b$, $\triangle r$, we easily compute the equations for the inverse transformation:

$$\begin{aligned}\triangle a &= 0{\cdot}4257\cdot A\\ \triangle b &= -0{\cdot}1444\cdot A + 0{\cdot}7726\cdot B\\ \triangle r &= 0{\cdot}0228\cdot A - 0{\cdot}1752\cdot B + 0{\cdot}25\cdot R.\end{aligned}$$

By these, any function of the elements for a given parameter can be expressed as a linear function of the free functions A, B, and R; and by $\lambda_2(A) = 2{\cdot}3490\,\lambda_2$, $\lambda_2(B) = 1{\cdot}2943\,\lambda_2$, and $\lambda_2(R) = 4\lambda_2$, the mean error is easily found. Thus the squares of the mean errors of the co-ordinates x and y are

$$\lambda_2(x) = \{2\cdot3490(\ \ 0\cdot4257 + 0\cdot0228\cos V)^2 + 1\cdot2943(-0\cdot1752\cos V)^2 \qquad + 4(0\cdot25\cos V)^2\}\lambda_2$$
$$\lambda_2(y) = \{2\cdot3490(-0\cdot1444 + 0\cdot0228\sin V)^2 + 1\cdot2943(\ \ 0\cdot7726 - 0\cdot1752\sin V)^2 + 4(0\cdot25\sin V)^2\}\lambda_2.$$

Only the value $\lambda_2 = 0\cdot00236$, found by the summary criticism, is here very uncertain.

XIII. SPECIAL AUXILIARY METHODS.

§ 57. We have often occasion to use the method of least squares, particularly adjustment by elements; and this sometimes requires so much work that we must try to shorten it as much as possible, even by means which are not quite lawful. Several temptations lie near enough to tempt the many who are soon tired by a somewhat lengthened computation, but not so much by looking for subtleties and short cuts. And as, moreover, the method was formerly considered the best solution — among other more or less good — not the only one that was justified under the given supposition, it is no wonder that it has come to be used in many modifications which must be regarded as unsafe or wrong. After what we have seen of the difference between free and bound functions, it will be understood that the consequences of transgressions against the method of least squares stand out much more clearly in the mean errors of the results than in their adjusted values. And as — to some extent justly — more importance is attached to getting tolerably correct values computed for the elements, than to getting a correct idea of the uncertainty, the lax morals with respect to adjustments have taken the form of an assertion to the effect that we can, within this domain, do almost as we like, without any great harm, especially if we take care that a sum of squares, either the correct one or another, becomes a minimum. This, of course, is wrong. In a text-book we should do more harm than good by stating all the artifices which even experienced computers have allowed themselves to employ, under special circumstances and in face of particularly great difficulties. Only a few auxiliary methods will be mentioned here, which are either quite correct or nearly so, when simple caution is observed.

§ 58. When methodic adjustment was first employed, large numbers of figures were used in the computations (logarithms with 7 decimal places), and people often complained of the great labour this caused; but it was regarded as an unavoidable evil, when the elements were to be determined with tolerable exactness. We can very often manage, however, to get on by means of a much simpler apparatus, if we do not seek something