Philosophical Magazine Series 5
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm16

XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves

Dr. D. J. Korteweg [a] & Dr. G. de Vries [a]
[a] University of Amsterdam

Available online: 08 May 2009

at the constant pressures of 100, 102, and 104 volts respectively consisted of specially good specimens.

In applying the rule that the economical potential difference is about the one which causes the lamp to produce 0·25 candle per watt, it is important, however, to examine 8-candle 100-volt Edison-Swan lamps when bought to see whether they are really marked "100 E.F. 8." For while the result of various purchases of 8-candle 100-volt Edison-Swan lamps during the past three years has always resulted in lamps marked "100 E.F. 8" being sent us, although the marking on the lamps was never specified by us, a recent batch of lamps that we have received contained among them certain lamps marked "100 B. 8," which not only differed in the marking but also in the filament being of a simple horse-shoe shape, and not with a loop at the top as in the case of the other lamps. And, on testing these Edison-Swan B lamps, we were surprised to find that with no one of them, when run at 100 volts, did the watts per candle exceed 3·9, and in some cases the watts per candle were as low as 3·01. We have not, however, had these B lamps for a sufficiently long time in our possession to be able to express any opinion about their life-history.

XLI. *On the Change of Form of Long Waves advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves. By* Dr. D. J. Korteweg, *Professor of Mathematics in the University of Amsterdam, and* Dr. G. de Vries*.

Introduction.

IN such excellent treatises on hydrodynamics as those of Lamb and Basset, we find that even when friction is neglected long waves in a rectangular canal must necessarily change their form as they advance, becoming steeper in front and less steep behind†. Yet since the investigations of de Boussinesq‡, Lord Rayleigh§, and St. Venant‖ on the solitary wave, there has been some cause to doubt the truth of this assertion. Indeed, if the reasons adduced were really decisive, it is difficult to see why the solitary wave should

* Communicated by the Authors.

† It seems that this opinion was expressed for the first time by Airy, "Tides and Waves," *Encyc. Metrop.* 1845.

‡ *Comptes Rendus*, 1871, vol. lxii.

§ Phil. Mag. 1876, 5th series, vol. i. p. 257.

‖ *Comptes Rendus*, 1885, vol. ci.

make an exception*; but even Lord Rayleigh and McCowan †, who have successfully and thoroughly treated the theory of this wave, do not directly contradict the statement in question. They are, as it seems to us, inclined to the opinion that the solitary wave is only stationary to a certain approximation.

It is the desire to settle this question definitively which has led us into the somewhat tedious calculations which are to be found at the end of our paper. We believe, indeed, that from them the conclusion may be drawn, that in a frictionless liquid there may exist absolutely stationary waves and that the form of their surface and the motion of the liquid below it may be expressed by means of rapidly convergent series. But, in order that these lengthy calculations might not obscure other results, which were obtained in a less elaborate way, we have postponed them to the last part of our paper.

First, then, we investigate the deformation of a system of waves of arbitrary shape but moving in one direction only, *i. e.* we consider one of the two systems of waves, starting in opposite directions in consequence of any disturbance, after their complete separation from each other. By adding to the motion of the fluid a uniform motion with velocity equal and opposite to the velocity of propagation of the waves, we may reduce the surface of such a system to approximate, but not perfect, rest.

If, then, $l+\eta$ (η being a small quantity) represent the elevation of the surface above the bottom at a horizontal distance x from the origin of coordinates, we have succeeded in deducing the equation

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}} \cdot \frac{\partial\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right)}{\partial x},$$

where α is a small but arbitrary constant, which is in close connexion with the exact velocity of the uniform motion given to the liquid, and where $\sigma = \frac{1}{3}l^3 - \frac{Tl}{\rho g}$ depends upon the depth l of the liquid, upon the capillary tension T at its surface, and upon its density ρ.

On assuming $\frac{\partial \eta}{\partial t} = 0$ we of course obtain the differential

* Though the theory of the solitary wave is duly discussed in the treatise of Basset, the inconsistency of his result with the doctrine of the necessary change of form of long waves seems not to have sufficiently attracted the attention of the author.

† Phil. Mag. 1891, 5th series, vol. xxxii.

equation for stationary waves, and it is easily shown that the well-known equation

$$\eta = h \operatorname{sech}^2 x \sqrt{\frac{h}{4\sigma}}$$

of the solitary wave is included as a particular case in the general solution of this equation. But, in referring to this kind of wave, we have to notice the result that, taking capillarity into account, a *negative* wave will become the stationary one, when the depth of the liquid is small enough.

On proceeding then to the general solution, a new type of long stationary wave is detected, the shape of the surface being determined by the equation

$$\eta = h \operatorname{cn}^2 x \sqrt{\frac{h+k}{4\sigma}} \left(\text{mod. } \mathrm{M} = \sqrt{\frac{h}{h+k}}\right).$$

We propose to attach to this type of wave the name of *cnoidal* waves (in analogy with sinusoidal waves). For $k=0$ they become identical with the solitary wave. For large values of k they bear more and more resemblance to sinusoidal waves, though their general aspect differs in this respect, that their elevations are narrower than their hollows; at least when the liquid is not too shallow, in which latter case this peculiar feature is reversed by the influence of capillarity.

For very large values of k these cnoidal waves coincide with the train of oscillatory waves of unchanging shape discovered by Stokes*, which therefore *in the theory of long waves* † constitutes a particular case of the cnoidal form. Indeed the equation ‡ obtained by Stokes, when written in our notation, becomes

$$\eta = h \cos \frac{2\pi x}{\lambda} - \frac{3h^2\lambda^2}{64\pi^2 l^3} \cos \frac{4\pi x}{\lambda};$$

but, as Sir G. Stokes remarks, in order that the method of approximation adopted by him may be legitimate, $\frac{\lambda^2 h}{l^3}$ must be a small fraction. Now, when capillarity is neglected, the wave-length λ of our cnoidal waves is equal to

$$\frac{4\mathrm{K}\sqrt{l^3}}{\sqrt{3(h+k)}},$$

* Transactions of the Cambridge Phil. Soc. vol. viii. (1847), reprinted in Stokes, Math. and Phys. Papers, vol. i. p. 197.

† Stokes' solution is more general in so far as it applies also to those cases wherein the depth of the liquid is moderate or large in respect to the wave-length.

‡ Stokes, Math. and Phys. Papers, vol. i. p. 210.

and therefore

$$\frac{\lambda^2 h}{l^3} = \frac{16K^2 h}{3(h+k)} = \frac{16}{3}M^2K^2.$$

This is a small fraction only when M, the modulus, is small, but the cnoidal waves then resemble sinusoidal waves; and it is obvious that in this case the equation of their surface may be developed in a rapidly convergent Fourier-series, of which Sir G. Stokes has given the first two terms.

After some more discussion about these cnoidal waves, concerning their velocity of propagation and the motion of the particles of fluid below their surface, we proceed to a closer examination of the deformation of long waves. To this effect we apply the equation for $\frac{\partial \eta}{\partial t}$ to various types of non-stationary waves, and it will appear that, though sinusoidal waves become steeper in front when advancing, other types of waves may behave otherwise.

I. *The Formula for* $\frac{d\eta}{dt}$.

In our investigations (in accordance with the method used by Lord Rayleigh, Phil. Mag. 1876, vol. i. p. 257, whose paper has been of great influence on our researches), we start from the supposition that the horizontal and vertical u and v of the fluid may be expressed by rapidly convergent series of the form

$$u = f + yf_1 + y^2f_2 + \dots$$
$$v = y\phi_1 + y^2\phi_2 + \dots$$

where y represents the height of a particle above the bottom of the canal, and where $f, f_1, \dots \phi_1, \phi_2, \dots$ are functions of x and t. Of course the validity of this assumption must be proved later on by the fact that series of this description can be found satisfying all the conditions of the problem.

From one of these conditions, viz., the incompressibility of the liquid, which is expressed by $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, we may deduce

$$\phi_n = -\frac{1}{n}\frac{\partial f_{n-1}}{\partial x},$$

and from another, viz., the absence of rotation in the fluid, expressed by $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$:—

$$f_1 = 0 \; ; \; f_n = \frac{1}{n} \frac{\partial \phi_{n-1}}{\partial x} = -\frac{1}{n(n-1)} \frac{\partial^2 f_{n-2}}{\partial x^2}.$$

In this manner we obtain the following set of equations :—

$$u = f - \frac{1}{2} y^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{24} y^4 \frac{\partial^4 f}{\partial x^4} - \ldots \quad . \quad . \quad . \quad (1)$$

$$v = -y \frac{\partial f}{\partial x} + \frac{1}{6} y^3 \frac{\partial^3 f}{\partial x^3} - \frac{1}{120} y^5 \frac{\partial^5 f}{\partial x^5} + \ldots \quad . \quad (2)$$

and, moreover, if ϕ be the velocity potential and ψ the stream-function :—

$$\phi = \int f \partial x - \frac{1}{2} y^2 \frac{\partial f}{\partial x} + \frac{1}{24} y^4 \frac{\partial^3 f}{\partial x^3} - \ldots \quad . \quad . \quad . (3)$$

$$\psi = yf - \frac{1}{6} y^3 \frac{\partial^2 f}{\partial x^2} + \frac{1}{120} y^5 \frac{\partial^4 f}{\partial x^4} - \ldots \quad . \quad . \quad . \quad (4)$$

which set of equations satisfies for the interior of the fluid all the conditions of the problem, whilst at the same time it is easy to see that for *long* waves these series are rapidly convergent. Indeed, for such waves the state of motion changes slowly with x, and therefore the successive differential-quotients with respect to this variable of all functions referring, as f does, to the state of motion, must rapidly decrease.

Passing now to the conditions at the boundary, let p_1 (a constant) be the atmospheric pressure, p_1' the pressure at a point below the surface where the capillary forces cease to act, and T the surface-tension. We then have, distinguishing here and elsewhere by the suffix ($_1$) those quantities which refer to the surface,

$$p_1' = p_1 - \mathrm{T} \frac{\partial^2 y_1}{\partial x^2} \; ;$$

but, according to a well-known equation of hydrodynamics,

$$\frac{p_1'}{\rho} = \chi(t) - \frac{\partial \phi_1}{\partial t} - \tfrac{1}{2}(u_1^2 + v_1^2) - g y_1,$$

therefore

$$\frac{p_1}{\rho} = \chi(t) - \frac{d\phi_1}{dt} - \tfrac{1}{2}(u_1^2 + v_1^2) - g y_1 + \frac{\mathrm{T}}{\rho} \frac{\partial^2 y_1}{\partial x^2} = \mathrm{L} - g y_1 + \mathrm{M} y_1^2$$

$$+ \mathrm{N} y_1^4 + \mathrm{P} y_1^6 + \ldots + \frac{\mathrm{T}}{\rho} \frac{\partial^2 y_1}{\partial x^2}, \quad . \quad (5)$$

where

$$L=\chi(t)-\int\frac{\partial f}{\partial t}dx-\tfrac{1}{2}f^2,$$

$$M=\frac{1}{2}f\frac{\partial^2 f}{\partial x^2}+\frac{1}{2}\frac{\partial^2 f}{\partial x\partial t}-\frac{1}{2}\left(\frac{\partial f}{\partial x}\right)^2,$$

$$N=-\frac{1}{24}f\frac{\partial^4 f}{\partial x^4}-\frac{1}{8}\left(\frac{\partial^2 f}{\partial x^2}\right)+\frac{1}{6}\frac{\partial f}{\partial x}\cdot\frac{\partial^3 f}{\partial x^3}-\frac{1}{24}\frac{\partial^4 f}{\partial x^3\partial t},$$

$$P=\frac{1}{720}f\frac{\partial^6 f}{\partial x^6}+\frac{1}{48}\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^4 f}{\partial x^4}-\frac{1}{72}\left(\frac{\partial^3 f}{\partial x^3}\right)^2-\frac{1}{120}\frac{\partial f}{\partial x}\cdot\frac{\partial^5 f}{\partial x^5}+\frac{1}{720}\frac{\partial^6 f}{\partial x^5\partial t}.$$

By differentiation with respect to x equation (5) may be written

$$\frac{\partial L}{\partial x}+y_1^2\frac{\partial M}{\partial x}+y_1^4\frac{\partial N}{\partial x}+y_1^6\frac{\partial P}{\partial x}+\ldots-g\frac{\partial y_1}{\partial x}+2My_1\frac{\partial y_1}{\partial x}$$
$$+4Ny_1^3\frac{\partial y_1}{\partial x}+6Py_1^5\frac{\partial y_1}{\partial x}+\ldots+\frac{T}{\rho}\frac{\partial^3 y_1}{\partial x^3}=0. \quad . \quad (6)$$

Moreover, a second equation must hold good at the surface, viz.

$$-u_1\frac{\partial y_1}{\partial x}+v_1-\frac{\partial y_1}{\partial t}=0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In order to satisfy equations (6) and (7) by the method of successive approximations, we put $y_1=l+\eta$, $f=q_0+\beta$, where l and q_0 are supposed to be constants, and η and β small functions depending upon x and t. Dealing, then, with the fact that for long waves, whose wave-length is great in comparison with the depth of the canal, every new differentiation with respect to x gives rise to continually smaller quantities, these equations become as a *first* approximation:—

$$q_0\frac{\partial\beta}{\partial x}+\frac{\partial\beta}{\partial t}+g\frac{\partial\eta}{\partial x}=0,$$

$$q_0\frac{\partial\eta}{\partial x}+\frac{\partial\eta}{\partial t}+l\frac{\partial\beta}{\partial x}=0,$$

and are satisfied by taking

$$\frac{d\eta}{dt}=\frac{d\beta}{dt}=0\,;\ \ \beta=-\frac{q_0}{l}(\eta+\alpha),$$

and

$$q_0=\sqrt{gl}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where α is an arbitrary constant which we will suppose to be small.

It is obvious that this solution coincides with the one usually given for the case of long waves of arbitrary shape made stationary by attributing to the fluid a velocity equal and opposite to that of the waves, on the assumption that the velocity in a vertical direction may be neglected and that the horizontal velocity may be considered uniform across each section of the canal.

But, if we wish to proceed to a *second* approximation, we have to put

$$f=q_0-\frac{q_0}{l}(\eta+\alpha+\gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where γ is small compared with η and α. On substituting this in (6) and (7) and on writing out the result, rejecting all terms* which are small compared with any one of the remaining terms, we find respectively:—

$$\frac{q_0}{l}\frac{\partial\eta}{\partial t}+g\frac{\partial\gamma}{\partial x}-\frac{g}{l}(\eta+\alpha)\frac{\partial\eta}{\partial x}-\left(\tfrac{1}{2}l^2g-\frac{T}{\rho}\right)\frac{\partial^3\eta}{\partial x^3}=0, \quad . \quad (10)$$

and

$$\frac{q_0}{l}\frac{\partial\eta}{\partial t}-g\frac{\partial\gamma}{\partial x}-\frac{g}{l}(2\eta+\alpha)\frac{\partial\eta}{\partial x}+\tfrac{1}{6}l^2g\frac{\partial^3\eta}{\partial x^3}=0. \quad . \quad . \quad (11)$$

In eliminating $\frac{\partial\gamma}{\partial x}$ from these equations, we have at last

$$\frac{d\eta}{dt}=\frac{3q_0}{2l}\frac{\partial\left(\frac{1}{2}\eta^2+\frac{2}{3}\alpha\eta+\frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right)}{\partial x} \quad . \quad . \quad . \quad (12)$$

where

$$\sigma=\tfrac{1}{3}l^3-\frac{Tl}{\rho g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This very important equation, to which we shall have frequently to revert in the course of this paper, indicates the deformation of a system of waves of arbitrary shape, but moving in one direction only. Before applying it, we may point out the close connexion between the constant α, which may still be chosen arbitrarily, and the uniform velocity given to the fluid. Indeed it is easy to see from (1) and (9) how a variation $\delta\alpha$ of the constant α corresponds to a change

* The terms for instance with $\frac{\partial\eta}{\partial x}\cdot\frac{\partial^2\eta}{\partial x^2}$ and $\left(\frac{\partial\eta}{\partial x}\right)^3$ are rejected in comparison with $\eta\frac{\partial\eta}{\partial x}$, which is retained in the equations, those with $\frac{\partial\gamma}{\partial t}$ and $\frac{\partial^3\eta}{\partial x^2\partial t}$ against $\frac{d\eta}{dt}$.

$\delta q = -\frac{q_0}{l}\delta\alpha$ in this velocity, but, on taking the variation of (12) with respect to α, we obtain

$$\delta\frac{d\eta}{dt} = \frac{q_0}{l}\,.\,\delta\alpha\,.\,\frac{\partial\eta}{\partial x} = -\delta q\,.\,\frac{\partial\eta}{\partial x},$$

which equation may be easily verified geometrically.

II. *Stationary Waves.*

For stationary waves $\frac{d\eta}{dt}$ must be zero. Therefore we have from (12)

$$\partial\left(\tfrac{1}{2}\eta^2 + \tfrac{2}{3}\alpha\eta + \tfrac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right) = 0.$$

This gives by integration

$$c_1 + \tfrac{1}{2}\eta^2 + \tfrac{2}{3}\alpha\eta + \tfrac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2} = 0\,; \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and by multiplication with $6\,d\eta$ and further integration,

$$c_2 + 6c_1\eta + \eta^3 + 2\alpha\eta^2 + \sigma\left(\frac{\partial\eta}{\partial x}\right)^2 = 0. \quad . \quad . \quad . \quad (15)$$

If now the fluid be undisturbed at infinity, and if l be taken equal to the depth which it has there, then equations (14) and (15) must be satisfied by $\eta = 0$, $\frac{\partial\eta}{\partial x} = 0$, and $\frac{\partial^2\eta}{\partial x^2} = 0$. Therefore, in *this* case c_1 and c_2 are equal to zero, and equation (15) leads to

$$\frac{\partial\eta}{\partial x} = \pm\sqrt{-\frac{\eta^2(\eta+2\alpha)}{\sigma}}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Here, before we can proceed, we have to discriminate between σ positive and σ negative. In the *first case* 2α is necessarily negative because $\frac{\partial\eta}{\partial x}$ must be real for small values of η. If, then, we put it equal to $-h$, we have

$$\frac{d\eta}{dx} = \pm\sqrt{\frac{1}{\sigma}}\,.\,\eta\,.\,\sqrt{h-\eta}:$$

from which, supposing x to be zero for $\eta = h$, we easily obtain the well-known equation of the *positive* solitary wave, viz.:—

$$\eta = h\,\mathrm{sech}^2\,.\,x\sqrt{\frac{h}{4\sigma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In the *second case* 2α must be positive. In putting it equal to h, and in substituting $-\eta'$ for η, we have from (16)

$$\frac{\partial\eta'}{\partial x}=\pm\sqrt{\frac{1}{-\sigma}}\cdot\eta'\sqrt{h-\eta'},$$

or, by integration,

$$\eta=-\eta'=-h\operatorname{sech}^2 x\sqrt{\frac{h}{-4\sigma}}.$$

This is the equation of a *negative* solitary wave, and we are able now to draw the conclusion that whenever σ is negative; that is whenever the depth of the liquid is less than $\sqrt{\frac{3T}{g\rho}}$, the stationary wave is a negative one. For water at 20° C. this limiting depth is equal to 0·47 cm. ($T=72$, $g=981$, $\rho=0{\cdot}998$ B.A.U.).

Now, for a further discussion of equation (15), we drop the assumption that the fluid is undisturbed at infinity. If then l be taken equal to the smallest depth of the liquid, we must have $\frac{\partial\eta}{\partial x}=0$ for $\eta=0$, and therefore in virtue of (15) $c_2=0$. On supposing then σ positive*, c_1 must be negative in order that $\frac{\partial\eta}{\partial x}$ may be real for small positive values of η, but then the equation

$$\eta^2+2\alpha\eta+6c_1=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

has a positive root h and a negative $-k$, and we may get from (15)

$$\frac{\partial\eta}{\partial x}=\pm\sqrt{\frac{1}{\sigma}\eta(h-\eta)(k+\eta)}. \quad . \quad . \quad . \qquad (19)$$

By substitution in this equation (19) of $\eta=h\cos^2\chi$ and by integration, we find

$$\eta=h\operatorname{cn}^2 x\sqrt{\frac{h+k}{4\sigma}}\left(M=\sqrt{\frac{h}{h+k}}\right), \quad . \quad . \qquad (20)$$

* When σ negative, let then l be equal to the *greatest* depth. On substituting $\sigma=-\sigma'$, $\eta=-\eta'$ we have again c_1 negative,

$$\left(\frac{d\eta'}{dx}\right)^2=\frac{1}{\sigma'}\eta'(h-\eta')(k+\eta'),$$

and, finally,

$$\eta=-\eta'=-h\operatorname{cn}^2 x\sqrt{\frac{h+k}{4\sigma'}},$$

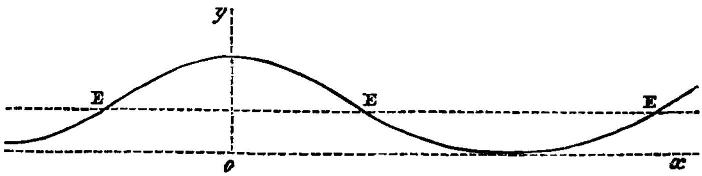
where h and $-k$ are the roots of $\eta'^2-2\alpha\eta+6c_1=0$.

which is the equation of a train of periodic waves whose wave-length increases when k decreases.

For $k=0$ this length becomes infinite, and the equation may be shown to coincide with (17).

The following figure (fig. 1) represents such a train of stationary waves for the case in which $k=\frac{9}{16}h$, $\mathrm{M}=0{\cdot}8$.

Fig. 1.



III. *Stationary Periodic Waves (Cnoidal Waves).*

Proceeding now to a further investigation of the waves determined by equation (20), we calculate from (10) and (11) the value of γ. From these equations we get

$$\frac{d\gamma}{dx}=-\frac{1}{2l}\eta\frac{\partial\eta}{\partial x}+\left(\frac{1}{3}l^2-\frac{\mathrm{T}}{2g\rho}\right)\frac{\partial^3\eta}{\partial x^3},$$

or by integration,

$$\gamma=-\frac{1}{4l}\eta^2+\left(\frac{1}{3}l^2-\frac{\mathrm{T}}{2g\rho}\right)\frac{\partial^2\eta}{\partial x^2},$$

where the constant of integration is rejected because its retention would only have had the effect of augmenting in equation (9) the value of the arbitrary constant α.

On substituting, then, f from (9) in (1) and (2), observing that in virtue of (14)

$$\frac{\partial^2\eta}{\partial x^2}=-\frac{1}{2\sigma}(3\eta^2+4\alpha\eta+6c_1)=-\frac{1}{2\sigma}(3\eta^2-2(h-k)\eta-hk),$$

these equations are replaced by

$$u=\sqrt{gl}-\sqrt{\frac{g}{l}}\left\{\eta+\tfrac{1}{2}(k-h)-\frac{\eta^2}{4l}+\left(\frac{1}{l}+\frac{\mathrm{T}}{2g\rho\sigma}\right)\left[(h-k)\eta\right.\right.$$
$$\left.\left.+\tfrac{1}{2}kh-\tfrac{3}{2}\eta^2\right]\right\}+\frac{1}{2\sigma}\sqrt{\frac{g}{l}}\{(h-k)\eta+\tfrac{1}{2}hk-\tfrac{3}{2}\eta^2\}y^2+\ldots \quad (21)$$

$$v=\sqrt{\frac{g\eta(h-\eta)(k+\eta)}{l\sigma}}\cdot y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

When $k=0$ they determine the motion of the fluid for a solitary wave.

In the first place we now will endeavour to calculate the velocity of propagation. For the solitary wave this is simple enough. If we consider that the liquid at infinity is brought to rest when a uniform motion with a horizontal velocity

$$-q=-\sqrt{gl}\left(1+\frac{h}{2l}\right) \quad . \quad . \quad . \quad . \quad . \quad (23)$$

is added to the motion expressed by (21) and (22), it is clear that this velocity, with reversed sign, must be taken for the velocity of propagation of the solitary wave.

But for a train of oscillatory waves Sir G. Stokes has shown * that various definitions of this velocity may be given, leading at the higher order of approximation to different values. It seemed to us most rational to define it as the velocity of propagation of the wave-form when the *horizontal momentum* of the liquid has been reduced to zero by the addition of a uniform motion. This definition corresponds to the second one of Sir G. Stokes. According to it, we have to solve the equation

$$\int_0^\lambda dx \int_0^{l+\eta} (u-q)dy=0, \quad . \quad . \quad . \quad . \quad (24)$$

where q denotes the velocity of propagation, and where

$$\lambda=\frac{2K\sqrt{\sigma}}{\sqrt{h+k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

is equal to the wave-length.

If, then,

$$V=\int_0^\lambda \eta dx=4\sqrt{\frac{\sigma}{h+k}}\left\{(h+k)E(K)-kK\right\}$$

$$=\lambda\left\{(h+k)\frac{E(K)}{K}-k\right\} \quad . \quad . \quad . \quad (26)$$

denote the volume of a single wave reckoned from above its lowest point, we get from (24), retaining only such terms as are of the first order compared with η, h, and k:—

* Math. and Phys. Papers, vol. i. p. 202.

$$q=\frac{\int_0^\lambda dx\int_0^{l+\eta}u\,dy}{\int_0^\lambda dx\int_0^{l+\eta}dy}=\frac{\int_0^\lambda\left\{\sqrt{gl}-\sqrt{\frac{g}{l}}\eta-\frac{1}{2}(k-h)\sqrt{\frac{g}{l}}\right\}(l+\eta)dx}{\int_0^\lambda(l+\eta)dx}$$

$$=\frac{\sqrt{gl}\left(1-\frac{k-h}{2l}\right)l\lambda}{l\lambda+\mathrm{V}}=\sqrt{gh}\left(1-\frac{k-h}{2l}-\frac{\mathrm{V}}{l\lambda}\right)$$

$$=\sqrt{gl}\left(1+\frac{k+h}{2l}-\frac{k+h}{l}\frac{\mathrm{E(K)}}{\mathrm{K}}\right). \quad . \quad . \quad . \quad . \quad . \quad (27)$$

On subtracting this velocity from that expressed by equation (21), we obtain

$$u'=u-q=-\sqrt{\frac{g}{l}}\left(\eta+k-(k+h)\frac{\mathrm{E}(k)}{\mathrm{K}}\right)=-\sqrt{\frac{g}{l}}\left(\eta-\frac{\mathrm{V}}{l}\right); \quad (28)$$

and it is obvious at once that in this manner we have annulled the velocity of the particles for which

$$\eta=\frac{\mathrm{V}}{l}.$$

This last equation has a simple geometrical meaning. It designates those particles E (fig. 1) whose height above the bottom of the channel is equal to the height where the surface of the liquid would stand when the waves were flattened. Therefore for a first approximation we may say that the various particles of the fluid change the direction of their horizontal motion at the very moment when one of these points E is passing over them.

We now proceed to the calculation of the path of a single particle of fluid. Let x_0, y_0 denote the coordinates of such a particle at the origin of time, and $x'=x_0+\xi'$, $y'=y_0+\zeta'$ its coordinates at the time t, u' and v' its horizontal and vertical velocity at that time, $l+\eta'$ its elevation above the bottom, then we have

$$\xi'=\int_0^t u'dt=\sqrt{\frac{g}{l}}\int_0^t\left(\eta'-\frac{\mathrm{V}}{\lambda}\right)dt;$$

$$\zeta'=\int_0^t v'dt=\sqrt{\frac{g}{l}}\int_0^t y\frac{\partial\eta'}{\partial x'}dt.$$

Here η' is equal to the value of η for $x=x'+qt$; and therefore we have $dx=(u'-q)dt$, or to a first approximation $dt=-\frac{1}{q}dx=-\frac{1}{\sqrt{gl}}dx$; but then

$$\xi'=-\frac{1}{l}\int_{x_0}^{x_0+\sqrt{gl}.t}\left(\eta-\frac{V}{\lambda}\right)dx=\frac{Vt}{\lambda}\sqrt{\frac{g}{l}}-\frac{h}{l}\int_{x_0}^{x_0+\sqrt{gl}.t}\operatorname{cn}^2\frac{2Kx}{\lambda}dx.$$

Or, according to a well-known formula *,

$$\xi'=-\frac{(h+k)\lambda}{2Kl}\left[Z\left(\frac{2K(x_0+\sqrt{gl}.t)}{\lambda}\right)-Z\left(\frac{2Kx_0}{\lambda}\right)\right]. \quad . \quad (29)$$

At the same time we have

$$\zeta'=-\frac{1}{l}.y.\int_{x_0}^{x_0+\sqrt{gl}.t}\frac{\partial\eta}{\partial x}dx=-\frac{h}{l}.y.\left[\operatorname{cn}^2(x_0+\sqrt{gl}.t)\sqrt{\frac{h+k}{4\sigma}}-\operatorname{cn}^2x_0\sqrt{\frac{h+k}{4\sigma}}\right]. \quad . \quad . \quad . \quad . \quad (30)$$

Of course, as all fluid particles with the same y describe congruent paths, these formulæ may be simplified by supposing $x_0=0$.

IV. *Deformation of Non-Stationary Waves.*

In order to study the deformation of non-stationary waves, we will now apply our formula (12) to various types of waves.

Solitary Waves.—As a first example we choose a solitary wave whose surface is given by

$$\eta=h\operatorname{sech}^2px. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

According to (12), the deformation of thi s wave is expressed by

$$\frac{d\eta}{dt}=-\frac{3q_0ph}{l}(4\sigma p^2-h)\left[-\operatorname{sech}^2px+\frac{2(\alpha+2\sigma p^2)}{3(4\sigma p^2-h)}\right]\operatorname{sech}^2px.\tanh px. \quad . \quad . \quad (32)$$

But before we are able to draw any conclusion from this expression, it is necessary to separate the two parts of $\frac{d\eta}{dt}$, of

* $Z(u)=u\left(1-\frac{E(K)}{K}\right)-M^2\int_0^u\operatorname{sn}^2u.du.$ Compare, for instance, Cayley, 'An Elementary Treatise on Elliptic Functions,' 1876, ch. vi. § 187.

which the first is due to a true change of form of the wave-surface, whilst the second may be attributed to a small advancing motion of the wave, which is left after the addition of the uniform motion with velocity $q_0=\sqrt{gl}$. To this effect we have still at our disposal the quantity α, whose close connexion with the uniform motion, which we have added in order to make the wave nearly stationary, has been indicated above.

One of the best ways to obtain the desired separation is certainly to make stationary the highest point of the wave, and this is effected by fulfilling the condition

$$2(\alpha+2\sigma p^2)=3(4\sigma p^2-h),$$

or

$$\alpha=4\sigma p^2-\tfrac{3}{2}h\,;$$

for in that case equation (32) is simplified to

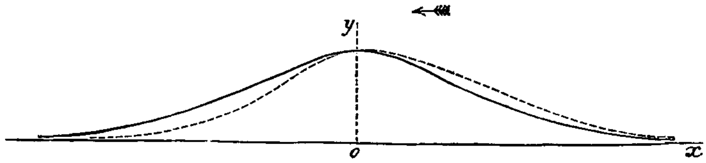
$$\frac{d\eta}{dt}=-\frac{3q_0ph}{l}(4\sigma p^2-h)\operatorname{sech}^2 px\,.\tanh^3 px\,; \quad . \quad (33)$$

and then, for $x=0$,

$$\frac{\partial\,.\frac{\partial\eta}{\partial x}}{\partial t}=\frac{\partial\,.\frac{\partial\eta}{\partial t}}{\partial x}\text{ is zero together with }\frac{\partial\eta}{\partial x}.$$

In discussing this equation (33), we see at once that a solitary wave (31) is stationary when $h=4\sigma p^2$; and this is in accordance with the equation (17) of the stationary solitary wave which we have obtained above. When $h>4\sigma p^2$, the change of form of the wave, calculated from (33), is shown by the dotted line in fig. 2.

Fig. 2.



Here the wave becomes steeper in front*, whilst for $h<4\sigma p^2$ the figure would show the opposite change of form, when, contrary to the opinion expressed by Airy and others, the wave becomes less steep in front and steeper behind.

* The left side of the figure is the front side of the wave, because the wave has been made stationary by the application of a positive velocity (*i. e.* from left to right) to the fluid.

If, now, we take account of the fact that, as may easily be inferred from (31), the wave-surface becomes steeper in proportion as p is increased, we are then justified in saying that a solitary wave which is *steeper* than the stationary one, corresponding to the same height, becomes less steep *in front* and steeper *behind*, but that its behaviour is exactly opposite when it is *less steep* than the stationary one.

Cnoidal Waves.—Applying formula (12) to the cnoidal wave,

$$\eta = h \,\mathrm{cn}^2 px, \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

we get

$$\frac{d\eta}{dt} = -\frac{3q_0 ph}{l}\left\{\frac{2[\alpha-\sigma p^2(2-4\mathrm{M}^2)]}{3(4\sigma\mathrm{M}^2p^2-h)} - \mathrm{cn}^2 . px\right\}(4\sigma\mathrm{M}^2p^2 - h)\,\mathrm{sn}\,px \,.\, \mathrm{cn}\,px \,.\, \mathrm{dn}\,px. \quad . \quad . \quad . \quad (35)$$

Supposing then

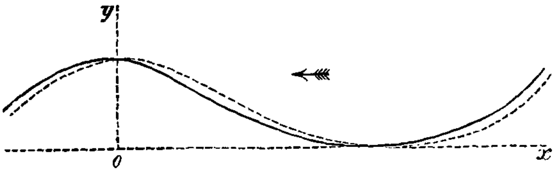
$$2[\alpha-\sigma p^2(2-4\mathrm{M}^2)] = 3(4\sigma\mathrm{M}^2p^2-h),$$

we have

$$\frac{d\eta}{dt} = -\frac{3q_0 ph}{l}(4\sigma\mathrm{M}^2p^2-h)\,\mathrm{sn}^3 px \,.\, \mathrm{cn}\,px \,.\, \mathrm{dn}\,px. \quad (36)$$

Here fig. 3 shows the change of form calculated for the case $h-4\sigma\mathrm{M}^2p^2>0$.

Fig. 3.



When $h-4\sigma\mathrm{M}^2p^2=0$, the waves are stationary in accordance with (20), whilst for $h-4\sigma\mathrm{M}^2p^2<0$ they become steeper *behind*; and this last result, since p is inversely proportional to the wave-length, may be stated by saying that cnoidal waves become less steep in front and steeper behind when, for a given modulus and a given height, their length is smaller than the one required for the stationary wave of this modulus and height.

In proportion as M is taken smaller the cnoidal waves more and more resemble sinusoidal waves. They would take the sinusoidal form for $\mathrm{M}=0$, but then an infinitely small wave-length would be required for the stationary case. For this reason sinusoidal waves may always be considered as cnoidal waves whose length is too large to be stationary, that is, they are always becoming steeper in front.

Sinusoidal Waves.—This last result is easily verified by direct application of (12) to the equation of a train of sinusoidal waves:

$$\eta = \mathrm{A} \sin \frac{2\pi x}{\lambda};$$

for, supposing

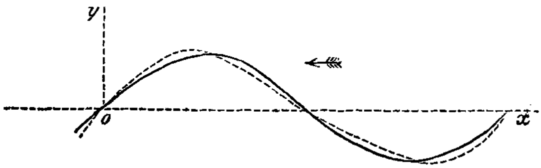
$$\alpha = \frac{2\pi^2 \sigma}{\lambda^2},$$

we obtain

$$\frac{d\eta}{dt} = \frac{3q_0 \pi \mathrm{A}^2}{2l\lambda} \sin \frac{4\pi x}{\lambda};$$

and from this the change of form indicated in fig. 4 is easily calculated.

Fig. 4.



More complicated Cases.—For the sake of curiosity, we represent by means of the following figures the change of form for some more complicated cases.

Fig. 5.

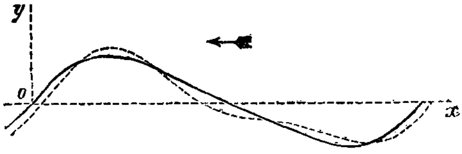


Fig. 6.

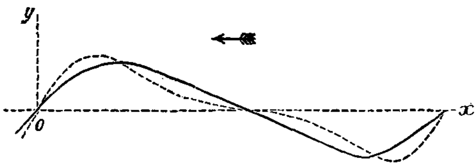
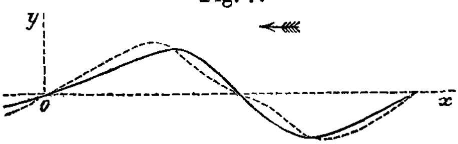


Fig. 7.



Figs. 5 and 6 refer to the equation

$$\eta = A_1 \sin\frac{2\pi x}{\lambda} + \tfrac{1}{3}A_1 \sin\frac{4\pi x}{\lambda}.$$

In fig. 5 $\frac{A_1}{l}$ is supposed to be small compared with $\left(\frac{l}{\lambda}\right)^2$, as is the case with waves of extremely small height. In fig. 6 we suppose $\left(\frac{l}{\lambda}\right)^2$ to be small in regard to $\frac{A_1}{l}$. Generally for more complicated forms of wave these two cases have to be discriminated. When there is a moderate proportionality between the two fractions the result is still more complicated.

Finally, fig. 7 refers to the equation

$$\eta = A_1 \sin\frac{2\pi x}{\lambda} - \tfrac{1}{3}A_1 \sin\frac{4\pi x}{\lambda},$$

in case that $\left(\frac{l}{\lambda}\right)^2$ is the smaller fraction.

It is worthy of remark that all these waves grow steeper in front.

V. *Calculation of the Fluid Motion for Stationary Waves to the Higher Order of Approximation.*

In order to remove every doubt as to the existence of absolutely stationary waves, we will show how by development in rapidly convergent series the state of motion of the fluid belonging to such a wave-motion may be calculated.

Expressing again the horizontal and vertical velocity of a particle by means of the series (1) and (2) which fulfil all the conditions for the interior of the fluid, we have only, neglecting capillarity, to satisfy the surface-conditions,

$$v_1 = u_1 \frac{\partial \eta}{\partial x}, \quad \ldots \ldots \ldots \quad (37)$$

and

$$u_1^2 + v_1^2 + 2g\eta = \text{constant}. \quad \ldots \ldots \quad (38)$$

For the case of cnoidal waves, which is the general one, we have found as a first approximation,

$$\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{3}{l^3}\eta(h-\eta)(k+\eta).$$

But now, to obtain higher approximations, we assume, indicating by accents differentiation with respect to x,

$$\eta'^2 = a\eta(h-\eta)(k+\eta)(1 + b\eta + c\eta^2 + \ldots), \quad \ldots \quad (39)$$

and

$$f = q + r\eta + s\eta^2 + t\eta^3 + u\eta^4 + \ldots \ldots \ldots \quad (40)$$

On writing out (39), neglecting such terms as are of a higher order than the fourth compared with η, h, and k, which latter quantities are of the same order, we obtain

$$\eta'^2 = ahk\eta + \{a(h-k)+abhk\}\eta^2 + \{-a+ab(h-k)\}\eta^3 - ab\eta^4; \quad . \quad (41)$$

and by differentiation,

$$\eta'' = \tfrac{1}{2}ahk + \{a(h-k)+abhk\}\eta + \{-\tfrac{3}{2}a+\tfrac{3}{2}ab(h-k)\}\eta^2 - 2ab\eta^3. \quad (42)$$

From (40), by successive differentiations and substitutions, retaining all terms up to the third and the $3\frac{1}{2}$th order, we deduce :—

$$f' = (r+2s\eta+3t\eta^2)\eta';$$

$$f'' = \tfrac{1}{2}arhk + \{ar(h-k)+abrhk+3ashk\}\eta$$
$$+\{-\tfrac{3}{2}ar+\tfrac{3}{2}abr(h-k)+4as(h-k)\}\eta^2 + (-2abr-5as)\eta^3;$$

$$f''' = [ar(h-k)+abrhk+3ashk$$
$$+\{-3ar+3abr(h-k)+8as(h-k)\}\eta + (-6abr-15as)\eta^2]\eta';$$

$$f^{\text{iv}} = \tfrac{1}{2}a^2rhk(h-k) + \{a^2r(h-k)^2 - \tfrac{3}{2}a^2rhk\}\eta$$
$$-\tfrac{15}{2}a^2r(h-k)\eta^2 + \tfrac{15}{2}a^2r\eta^3;$$

$$f^{\text{v}} = [a^2r(h-k)^2 - \tfrac{3}{2}a^2rhk - 15a^2r(h-k)\eta + \tfrac{45}{2}a^2r\eta^2]\eta';$$

where η' is a quantity of the order $\frac{3}{2}$.

Substituting these values in equation (1), where $y=l+\eta$, we have, retaining terms of the third order :—

$$u_1 = f - \tfrac{1}{2}l^2f'' - l\eta f'' + \tfrac{1}{24}l^4f^{\text{iv}} = q - \tfrac{1}{4}arl^2hk + \tfrac{1}{48}a^2rl^4hk(h-k)$$
$$+\{r - \tfrac{1}{2}arl^2(h-k) - \tfrac{1}{2}abrl^2hk - \tfrac{3}{2}asl^2hk - \tfrac{1}{2}arlhk$$
$$+\tfrac{1}{24}a^2rl^4(h-k)^2 - \tfrac{3}{16}a^2rl^4hk\}\eta$$
$$+\{s + \tfrac{3}{4}al^2r - \tfrac{3}{4}abrl^2(h-k) - 2asl^2(h-k)$$
$$-arl(h-k) - \tfrac{5}{16}a^2rl^4(h-k)\}\eta^2$$
$$+\{t + abrl^2 + \tfrac{5}{2}asl^2 + \tfrac{3}{2}arl + \tfrac{5}{16}a^2rl^4\}\eta^3. \quad . \quad . \quad . \quad . \quad (43)$$

We find in the same way, including terms of the $3\frac{1}{2}$th order :—

$$v_1 = -lf' - \eta f' + \tfrac{1}{6}l^3f''' + \tfrac{1}{2}l^2\eta f''' - \tfrac{1}{120}l^5f^{\text{v}}$$
$$= \eta'[-rl + \tfrac{1}{6}arl^3(h-k) + \tfrac{1}{6}abrl^3hk + \tfrac{1}{2}asl^3hk - \tfrac{1}{120}a^2rl^5(h-k)^2$$
$$+\tfrac{3}{80}a^2rl^5hk + \{-2sl - r - \tfrac{1}{2}arl^3 + \tfrac{1}{2}abrl^3(h-k)$$
$$+\tfrac{4}{3}asl^3(h-k) + \tfrac{1}{2}arl^2(h-k) + \tfrac{1}{8}a^2rl^5(h-k)\}\eta$$
$$+\{-3tl - 2s - abrl^3 - \tfrac{5}{2}asl^3 - \tfrac{3}{2}arl^2 - \tfrac{3}{16}a^2rl^5\}\eta^2]. \quad . \quad (44)$$

If now we write, in accordance with (37),

$$u_1=\frac{v_1}{\eta'}=\mathrm{A}+\mathrm{B}\eta+\mathrm{C}\eta^2+\mathrm{D}\eta^3+\ldots, \quad . \quad . \quad . \quad (45)$$

we have from (43) and (44):—

$$\mathrm{A}=q-\tfrac{1}{4}arl^2hk+\tfrac{1}{48}a^2rl^4hk(h-k)=-rl+\tfrac{1}{6}arl^3(h-k)$$
$$+\tfrac{1}{6}abrl^3hk+\tfrac{1}{2}asl^3hk-\tfrac{1}{120}a^2rl^5(h-k)^2+\tfrac{3}{80}a^2rl^5hk. \quad . \quad (46)$$

$$\mathrm{B}=r-\tfrac{1}{2}arl^2(h-k)-\tfrac{1}{2}abrl^2hk-\tfrac{3}{2}asl^2hk-\tfrac{1}{2}arlhk$$
$$+\tfrac{1}{24}a^2rl^4(h-k)^2-\tfrac{3}{16}a^2rl^4hk=-2sl-r-\tfrac{1}{2}arl^3$$
$$+\tfrac{1}{2}abrl^3(h-k)+\tfrac{4}{3}asl^3(h-k)+\tfrac{1}{2}arl^2(h-k)+\tfrac{1}{8}a^2rl^5(h-k). \quad (47)$$

$$\mathrm{C}=s+\tfrac{3}{4}al^2r-\tfrac{3}{4}abrl^2(h-k)-2asl^2(h-k)-arl(h-k)-\tfrac{5}{16}a^2rl^4(h-k)$$
$$=-3tl-2s-abrl^3-\tfrac{5}{2}asl^3-\tfrac{3}{2}arl^2-\tfrac{3}{16}a^2rl^5. \quad . \quad (48)$$

$$\mathrm{D}=t+abrl^2+\tfrac{5}{2}asl^2+\tfrac{3}{2}arl+\tfrac{5}{16}a^2rl^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Moreover, since (38) may be written in the form

$$u_1^2(1+\eta'^2)+2g\eta=(\mathrm{A}+\mathrm{B}\eta+\mathrm{C}\eta^2+\mathrm{D}\eta^3)^2(1+ahk\eta$$
$$+a(h-k)\eta^2-a\eta^3)+2g\eta=\text{constant}, \quad . \quad . \quad . \quad . \quad . \quad (50)$$

we readily obtain

$$2\mathrm{AB}+ahk\mathrm{A}^2+2g=0, \quad . \quad . \quad . \quad . \quad (51)$$

$$2\mathrm{AC}+\mathrm{B}^2+a(h-k)\mathrm{A}^2=0. \quad . \quad . \quad . \quad . \quad (52)$$

$$2\mathrm{AD}+2\mathrm{BC}-a\mathrm{A}^2=0. \quad . \quad . \quad . \quad . \quad (53)$$

From the equations (46), (47), (48), (51), (52), (53), the six quantities q, r, s, t, a, and b may be calculated, and if we had retained everywhere terms of one higher order, we might have got eight equations with eight unknown quantities, &c.

By a first approximation we readily obtain from (46)–(49):—

$$r=-\frac{q}{l}\,; \quad s=\frac{q}{l^2}+\tfrac{1}{4}aql\,; \quad t=-\frac{q}{l^3}-\tfrac{1}{3}aq+\tfrac{1}{3}abql-\tfrac{7}{48}a^2ql^3\,;$$

$$\mathrm{A}=q\,; \quad \mathrm{B}=-\frac{q}{l}\,; \quad \mathrm{C}=\frac{q}{l^2}-\tfrac{1}{2}aql\,;$$

$$\mathrm{D}=-\frac{q}{l^3}+\tfrac{2}{3}aq-\tfrac{2}{3}abql+\tfrac{1}{6}a^2ql^3\,;$$

and then from (51)–(53),

$$q^2=gl\,; \quad a=\frac{3}{l^3}\,; \quad b=\frac{3}{4l}. \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Proceeding to the second approximation, we find

$$r=-\frac{q}{l}\left(1+\frac{h-k}{2l}\right);\quad s=\frac{q}{l^2}+\tfrac{1}{4}aql+\tfrac{7}{4}\frac{q}{l^2}\cdot\frac{h-k}{l};\quad \mathrm{A}=q\,;$$

$$\mathrm{B}=-\frac{q}{l}+\frac{q}{l}\cdot\frac{h-k}{l};\ \mathrm{C}=\frac{q}{l^2}-\tfrac{1}{2}aql-\frac{19}{8}\frac{q}{l^2}\cdot\frac{h-k}{l}\,;$$

and then again from (51) and (52),

$$q^2=gl\left(1+\frac{h-k}{l}\right);\ a=\frac{3}{l^3}-\frac{15}{4}\cdot\frac{h-k}{l^4}. \quad . \quad . \quad (55)$$

Finally, a third approximation leads to :—

$$r=-\frac{q}{l}\left(1+\frac{h-k}{2l}-\frac{9}{20}\frac{(h-k)^2}{l^2}-\frac{93}{80}\frac{hk}{l^2}\right);\ \mathrm{A}=q+\tfrac{3}{4}q\cdot\frac{hk}{l^2}\,;$$

$$\mathrm{B}=-\frac{q}{l}+\frac{q}{l}\cdot\frac{h-k}{l}-\frac{21}{20}\frac{q}{l}\frac{(h-k)^2}{l^2}-\frac{12}{5}\frac{q}{l}\cdot\frac{hk}{l^2}\,;$$

$$q^2=gl\left(1+\frac{h-k}{l}-\frac{1}{20}\frac{(h-k)^2}{l^2}-\frac{33}{20}\frac{hk}{l^2}\right). \quad . \quad . \quad . \quad (56)$$

By means of these results we may now readily obtain from (1) and (2) expressions for u and v including respectively the terms of the 2^{nd} and $2\frac{1}{2}^{\text{th}}$ order.

They are :—

$$u=\sqrt{gl}\left\{\left(1+\frac{h-k}{2l}-\frac{3}{20}\frac{(h-k)^2}{l^2}-\frac{33}{40}\frac{hk}{l^2}\right)+\left(1+\frac{h-k}{l}\right)\frac{\eta}{l}\right.$$
$$\left.+\frac{7}{4}\frac{\eta^2}{l^2}+\left(\frac{3}{4}\cdot\frac{hk}{l^2}+\frac{3}{2}\frac{h-k}{l}\frac{\eta}{l}-\frac{9}{4}\frac{\eta^2}{l^2}\right)y^2\right\}. \quad . \quad . \quad (57)$$

$$v=\eta'\sqrt{\frac{g}{l}}\left\{\left(1+\frac{h-k}{l}-\frac{7}{4}\cdot\frac{\eta}{l}\right)y+\left(-\frac{h-k}{2l}+\frac{3}{2}\cdot\frac{\eta}{l}\right)y^3\right\};\ (58)$$

where

$$\eta'^2=\frac{3}{l^3}\left(1-\frac{5}{4}\frac{h-k}{l}\right)(h-\eta)(k+\eta)\left(1+\frac{3}{4}\cdot\frac{\eta}{l}\right). \quad . \quad (59)$$

VI. *Calculation of the Equation of the Surface.*

We will now show how for the equation of the surface of a stationary train of waves a more correct expression than (20) can be deduced. For this purpose we have to integrate the differential equation (39), or rather we have to prove that a series can be given which solves this equation to any desired degree of accuracy. Now such a series may be obtained in the following manner. Let

$$\eta_1 = h_1 \operatorname{cn}^2 \tfrac{1}{2}x \sqrt{a(h_1+k_1)} \left(\mathrm{M} = \sqrt{\frac{h_1}{h_1+k_1}}\right) \quad . \quad (60)$$

represent the solution of an equation

$$\eta_1'^2 = a\eta_1(h_1-\eta_1)(k_1+\eta_1), \quad . \quad . \quad . \quad . \quad (61)$$

where h_1 and k_1 have values which are slightly different from those of h and k in (39); then these values and the coefficients α, β, &c., of a series

$$\eta = \alpha\eta_1 + \beta\eta_1^2 + \gamma\eta_1^3 + \delta\eta_1^4 + \ldots \quad . \quad . \quad (62)$$

may be determined in such a way* that this series (62) satisfies the equation (39).

Indeed, substituting (62) in (39) and taking into account (61), equation (39) reduces to

$$\begin{aligned}(\alpha + 2\beta\eta_1 + 3\gamma\eta_1^2 + \ldots)^2(h_1-\eta_1)(k_1+\eta_1)& \\ = (\alpha + \beta\eta_1 + \gamma\eta_1^2 + \ldots)(h - \alpha\eta_1 - \beta\eta_1^2 - \gamma\eta_1^3 + \ldots)&(k + \alpha\eta_1 \\ + \beta\eta_1^2 + \gamma\eta_1^3 + \ldots)(1 + b\alpha\eta_1 + (b\beta + c\alpha)\eta_1^2 + \ldots)&,\end{aligned}$$

and it is only necessary to equalize the coefficients of the corresponding terms of both members of this equation.

If we retain all terms to the fourth order, we find in this way, after some reductions :—

$$\alpha h_1 k_1 - hk = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$\alpha^2(h_1-k_1) - \alpha^2(h-k) - (b\alpha^2 - 3\beta)hk = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$-\alpha^3 + \alpha^4 - (b\alpha^4 - 2\alpha^2\beta)(h-k) - (c\alpha^3 - 2b\alpha^2\beta + 8\beta^2 - 5\alpha\gamma)hk = 0 \quad (65)$$

$$-4\alpha\beta + 3\alpha^2\beta + b\alpha^4 - (c\alpha^3 + 3b\alpha^2\beta - 3\beta^2 - 4\alpha\gamma)(h-k) = 0 \quad . \quad . \quad (66)$$

$$-4\beta^2 - 6\alpha\gamma + c\alpha^4 + 4b\alpha^3\beta + 3\alpha\beta^2 + 3\alpha^2\gamma = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

To a *first* approximation these equations are satisfied by taking

$$h_1 = h\,;\ k_1 = k\,;\ \alpha = 1\,;\ \beta = b\,;\ \gamma = b^2 + \tfrac{1}{3}c \quad . \quad . \quad (68)$$

If then we substitute in (63), (64), (65), and (66)

$$h_1 = h + \epsilon,\quad k_1 = k + x,\quad \alpha = 1 + \alpha_1,\quad \beta = b + \beta_1$$

where α_1 and β_1 are quantities of the first, ϵ and x of the second order, we find from these equations by *second* approximation:—

* The coefficient a in (61) might also have been chosen slightly different in value from a in (39), but this would only have introduced an unnecessary indeterminateness in the solution.

$$\epsilon = -bhk\ ;\ \ \varpi = bhk\ ;\ \ \alpha_1 = -b(h-k)\ ;\ \ \beta_1 = (-2b^2 + \tfrac{1}{3}c)(h-k). \ . \quad (69)$$

Substituting as a *third* approximation :—

$$h_1 = h - bhk + \epsilon_1\ ;\ \ k_1 = k + bhk + \varpi_1,\ \ \alpha = 1 - b(h-k) + \alpha_2,$$

we obtain finally,

$$\epsilon_1 = \tfrac{1}{3}chk(-h+2k)\ ;\ \ \varpi_1 = \tfrac{1}{3}chk(2h-k)\ ;\ \ \alpha_2 = (b^2 - \tfrac{2}{3}c)(h^2 - hk + k^2). \quad (70)$$

Hence the equation of the surface of the waves is, including all terms of the third order :—

$$\eta = [1 - b(h-k) + (b^2 - \tfrac{2}{3}c)(h^2 - hk + k^2)]\eta_1 + [b + (-2b^2 + \tfrac{1}{3}c)(h-k)]\eta_1^2 + (b^2 + \tfrac{1}{3}c)\eta_1^3 + . \ \ . \quad (71)$$

where

$$\eta_1 = h_1 \operatorname{cn}^2 \tfrac{1}{2}x\sqrt{a(h_1+k_1)}\left(\mathrm{M} = \sqrt{\frac{h_1}{h_1+k_1}}\right). \ . \ . \quad (60)$$

$$h_1 = h - bhk + \tfrac{1}{3}chk(-h+2k)\ ;\ \ k_1 = k + bhk + \tfrac{1}{3}chk(2h-k). \quad (72)$$

Here, according to (59),

$$a = \frac{3}{l^3}\left(1 - \frac{5}{4}\frac{h-k}{l} + \ldots\right);\ \ b = \frac{3}{4l} + \ldots\ ; \ . \ . \quad (73)$$

whereas the value of c and more correct expressions for a and b could only have been obtained by means of still more tedious calculations, which we have not executed.

If we confine ourselves to that degree of approximation for which all the calculations have been effected, we may write for the equation of the wave-surface :—

$$\eta = \left[1 - \frac{3(h-k)}{4l}\right]\eta_1 + \frac{3}{4l}\eta_1^2. \ . \ . \ . \ . \quad (74)$$

$$\eta_1 = h \operatorname{cn}^2 \tfrac{1}{2}\left(1 - \frac{5(h-k)}{8l}\right)x\sqrt{\frac{3(h+k)}{l^3}}. \ . \ . \quad (75)$$

$$\mathrm{M} = \left(1 - \frac{3k}{8l}\right)\sqrt{\frac{h}{h+k}}. \ . \ . \ . \ . \ . \quad (76)$$

For the solitary wave, when $k=0$, we have *

$$\eta = \left[1 - \frac{3h}{4l}\right]\eta_1 + \frac{3}{4l}\eta_1^2. \ . \ . \ . \ . \quad (77)$$

$$\eta_1 = h \operatorname{sech}^2 \tfrac{1}{2}\left(1 - \frac{5h}{8l}\right)x\sqrt{\frac{3h}{l^3}}. \ . \ . \quad (78)$$

January 1895.

* Another close approximation of the surface-equation of this wave has been deduced by McCowan, Phil. Mag. [5] vol. xxxii. (1891), p. 48.